Neutrosophic Sets and Systems

Volume 36  Article 6

9-23-2020

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Neutrosophic $\aleph$–interior ideals in semigroups

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Abstract: We define the concepts of neutrosophic $\aleph$-interior ideal and neutrosophic $\aleph$–characteristic interior ideal structures of a semigroup. We infer different types of semigroups using neutrosophic $\aleph$-interior ideal structures. We also show that the intersection of neutrosophic $\aleph$-interior ideals and the union of neutrosophic $\aleph$-interior ideals is also a neutrosophic $\aleph$-interior ideal.

Keywords: Semi group, neutrosophic $\aleph$–ideals, neutrosophic $\aleph$-interior ideals, neutrosophic $\aleph$–product.

1. Introduction

Nowadays, the theory of uncertainty plays a vital role to manage different issues relating to modelling engineering problems, networking, real-life problem relating to decision making and so on. In 1965, Zadeh[24] introduced the idea of fuzzy sets for modelling vague concepts in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as Intuitionistic fuzzy set. Also, from his viewpoint, there are two degrees of freedom in the real world, one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset.

Smarandache generalized fuzzy set and intuitionistic fuzzy set, and named as neutrosophic set (see [4, 7, 8, 14, 19, 22-23]). These sets are characterized by a truth membership function, an indeterminacy membership function and a falsity membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. A Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as the result of sports games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [3, 11, 12, 16-18]).

For more details on neutrosophic set theory, the readers visit the website http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In [2], Abdel Basset et al. designed a framework to manage scheduling problems using neutrosophic theory. As the concept of time-cost tradeoffs and deterministic project scheduling disagree with the real situation, some data were changed during the implementation process. Here fuzzy scheduling and time-cost tradeoffs models assumed only truth-membership functions dealing
with uncertainties of the project and their activities duration which were unable to treat indeterminacy and inconsistency.

In [6], Abdel Basset et al. evaluated the performance of smart disaster response systems under uncertainty. In [5], Abdel Basset et al. introduced different hybrid neutrosophic multi-criteria decision-making framework for professional selection that employed a collection of neutrosophic analytical network process and order preference by similarity to the ideal solution under bipolar neutrosophic numbers.

In [21], Prakasam Muralikrishna1 et al. presented the characterization of MBJ – Neutrosophic δ – Ideal of δ – algebra. They analyzed homomorphic image, pre–image, cartesian product and related results, and these concepts were explored to other substructures of a δ – algebra. In [9], Chalapathi et al. constructed certain Neutrosophic Boolean rings, introduced Neutrosophic complement elements and mainly obtained some properties satisfied by the Neutrosophic complement elements of Neutrosophic Boolean rings.

In [14], M. Khan et al. presented the notion of neutrosophic ℵ-subsemigroup in semigroup and explored several properties. In [11], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups and introduced the concept of the characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [10], B. Elavarasan et al. introduced the notion of neutrosophic ℵ-ideal in semigroup and explored its properties. Also, the conditions for neutrosophic ℵ-structure to be neutrosophic ℵ-ideal are given, and discussed the idea of characteristic neutrosophic ℵ-structure in semigroups and obtained several properties. In [20], we have introduced and discussed several properties of neutrosophic ℵ-product and the intersection of neutrosophic ℵ-ideals were identical for regular semigroups. In this paper, we define and discuss the concepts of neutrosophic ℵ-interior ideal and neutrosophic ℵ-characteristic interior ideal structures of a semigroup.

Throughout this paper, X denotes a semigroup. Now, we present the important definitions of semigroup that we need in sequel.

Recall that for any $X_1, X_2 \subseteq X$, $X_1X_2 = \{ab | a \in X_1 \text{ and } b \in X_2\}$, multiplication of $X_1$ and $X_2$.

Let $X$ be a semigroup and $\emptyset \neq X_1 \subseteq X$. Then

(i) $X_1$ is known as subsemigroup if $X_1X_1 \subseteq X_1$.

(ii) A subsemigroup $X_1$ is known as left (resp., right) ideal if $X_1X \subseteq X_1$ (resp., $XX_1 \subseteq X_1$).

(iii) $X_1$ is known as ideal if $X_1$ is both a left and a right ideal.

(iv) $X$ is known as left (resp., right) regular if for each $r \in X$, there exists $i \in X$ such that $r = ir^2$ (resp., $r = r^2i$) [13].

(v) $X$ is known as regular if for each $b_1 \in X$, there exists $i \in X$ such that $b_1 = b_1ib_1$.

(vi) $X$ is known as intra-regular if for each $x_1 \in X$, there exist $i, j \in X$ such that $x_1 = ix_1j$ [15].

2. Definitions of neutrosophic ℵ - structures

We present definitions of neutrosophic ℵ-structures namely neutrosophic ℵ-subsemigroup, neutrosophic ℵ-ideal, neutrosophic ℵ-interior ideal of a semigroup $X$. 

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The set of all the functions from $X$ to $[-1, 0]$ is denoted by $\mathcal{Z}(X, [-1, 0])$. We call that an element of $\mathcal{Z}(X, [-1, 0])$ a $\mathcal{K}$-function on $X$. A $\mathcal{K}$-structure means an ordered pair $(X, g)$ where $X$ is a neutrosophic sets and $g$ is a neutrosophic function on $X$.

**Definition 2.1.**[14] A neutrosophic $\mathcal{K}$-structure of $X$ is defined to be the structure:

$$X_M := \frac{x}{(T_M, I_M, F_M)} = \left\{ r \in X \mid r \in X \right\},$$

where $T_M$, $I_M$, and $F_M$ are the negative truth, negative indeterminacy and negative falsity membership functions on $X$ ($\mathcal{K}$-functions).

It is evident that $-3 \leq T_M(r) + I_M(r) + F_M(r) \leq 0$ for all $r \in X$.

**Definition 2.2.**[14] A neutrosophic $\mathcal{K}$-structure $X_M$ of $X$ is called a neutrosophic $\mathcal{K}$-subsemigroup of $X$ if the following assertion is valid:

$$\forall g_i, h_j \in X \left( T_M(g_i, h_j) \leq T_M(g_i) \lor T_M(h_j) \lor I_M(g_i) \land I_M(h_j) \land F_M(g_i, h_j) \leq F_M(g_i) \lor F_M(h_j) \right).$$

Let $X_M$ be a neutrosophic $\mathcal{K}$-structure and $y, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq y + \delta + \varepsilon \leq 0$. Consider the sets:

$$T_M^r = \{ r_i \in X | T_M(r_i) \leq y \},$$

$$I_M^r = \{ r_i \in X | I_M(r_i) \geq \delta \},$$

$$F_M^r = \{ r_i \in X | F_M(r_i) \leq \varepsilon \}.$$  

The set $X_M(y, \delta, \varepsilon) := \{ r_i \in X | T_M(r_i) \leq y, I_M(r_i) \geq \delta, F_M(r_i) \leq \varepsilon \}$ is known as $(y, \delta, \varepsilon)$-level set of $X_M$. It is easy to observe that $X_M(y, \delta, \varepsilon) = T_M^r \cap I_M^r \cap F_M^r$.

**Definition 2.3.**[10] A neutrosophic $\mathcal{K}$-structure $X_M$ of $X$ is called a neutrosophic $\mathcal{K}$-ideal (resp., right) ideal of $X$ if

$$\forall g_i, h_j \in X \left( T_M(g_i, h_j) \leq T_M(h_j) \lor T_M(g_i) \lor I_M(g_i, h_j) \lor I_M(h_j) \lor F_M(g_i, h_j) \leq F_M(g_i) \lor F_M(h_j) \right).$$

$X_M$ is neutrosophic $\mathcal{K}$-ideal of $X$ if $X_M$ is neutrosophic $\mathcal{K}$-left and $\mathcal{K}$-right ideal of $X$.

**Definition 2.4.** A neutrosophic $\mathcal{K}$-subsemigroup $X_M$ of $X$ is known as a neutrosophic $\mathcal{K}$-interior ideal if

$$\forall x, a, y \in X \left( T_M(xa) \leq T_M(a) \lor I_M(xa) \leq I_M(a) \lor F_M(xa) \leq F_M(a) \right).$$

It is easy to observe that every neutrosophic $\mathcal{K}$-ideal is neutrosophic $\mathcal{K}$-interior ideal, but neutrosophic $\mathcal{K}$-interior ideal need not be a neutrosophic $\mathcal{K}$-ideal, as shown by an example.

**Example 2.5.** Let $X$ be the set of all non-negative integers except 1. Then $X$ is a semigroup with usual multiplication.

Let $X_M = \begin{cases} \theta & \text{if } x \in (-0.9, -0.1, 0.7), \\ 2 & \text{if } x \in (-0.4, -0.6, -0.5), \\ 5 & \text{if } x \in (-0.3, -0.8, -0.3), \\ 10 & \text{if } x \in (-0.7, -0.4, -0.6), \\ \text{otherwise} & \end{cases}$. Then $X_M$ is neutrosophic $\mathcal{K}$-interior ideal, but not neutrosophic $\mathcal{K}$-ideal with $T_M(2.5) = -0.3 \leq T_M(2)$.

**Definition 2.6.**[14] For any $E \in X$, the characteristic neutrosophic $\mathcal{K}$-structure is defined as

$$X_E(X_M) = \frac{X}{(X_E(T_M^r), X_E(I_M^r), X_E(F_M^r))}$$

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where

\[
X_E(T)_M: X \to [-1, 0], \ r \to \begin{cases} 1 & \text{if } r \in E \\ 0 & \text{otherwise} \end{cases},
\]

\[
X_E(I)_M: X \to [-1, 0], \ r \to \begin{cases} 0 & \text{if } r \in E \\ 1 & \text{otherwise} \end{cases},
\]

\[
X_E(F)_M: X \to [-1, 0], \ r \to \begin{cases} 1 & \text{if } r \in E \\ 0 & \text{otherwise}. \end{cases}
\]

**Definition 2.7.**[14] Let \( X_N := \frac{x}{(T_N, I_N, F_N)} \) and \( X_M := \frac{x}{(T_M, I_M, F_M)} \) be neutrosophic \( \aleph \)—structures of \( X \). Then

(i) \( X_N \) is called a neutrosophic \( \aleph \)—substructure of \( X_M \), denote by \( X_M \subseteq X_N \), if \( T_M(r) \geq T_N(r), I_M(r) \leq I_N(r), F_M(r) \geq F_N(r) \) for all \( r \in X \).

(ii) If \( X_N \subseteq X_M \) and \( X_M \subseteq X_N \), then we say that \( X_N = X_M \).

(iii) The neutrosophic \( \aleph \)—product of \( X_N \) and \( X_M \) is defined to be a neutrosophic \( \aleph \)—structure of \( X \),

\[
X_N \odot X_M := \frac{x}{(T_{N \odot M}, I_{N \odot M}, F_{N \odot M})} = \left\{ h \in X \right\},
\]

where

\[
(T_{N \odot M}, I_{N \odot M}, F_{N \odot M}) = \left\{ \begin{cases} \bigwedge_{h=rs} (T_N(r) \lor T_M(s)) & \text{if } \exists r, s \in X \text{ such that } h = rs \\ 0 & \text{otherwise}, \end{cases} \right. \]

\[
(I_{N \odot M}, I_{N \odot M}) = \left\{ \begin{cases} \bigvee_{h=rs} (I_N(r) \land I_M(s)) & \text{if } \exists u, v \in X \text{ such that } h = rs \\ -1 & \text{otherwise}, \end{cases} \right. \]

\[
(F_{N \odot M}, F_{N \odot M}) = \left\{ \begin{cases} \bigwedge_{h=rs} (F_N(r) \lor F_M(s)) & \text{if } \exists u, v \in X \text{ such that } h = rs \\ 0 & \text{otherwise}. \end{cases} \right. \]

For \( i \in X \), the element \( \frac{i}{(T_{N \odot M}(i), I_{N \odot M}(i), F_{N \odot M}(i))} \) is simply denoted by \( (X_N \odot X_M)(i) = (T_{N \odot M}(i), I_{N \odot M}(i), F_{N \odot M}(i)). \)

(iii) The union of \( X_N \) and \( X_M \), a neutrosophic \( \aleph \)—structure over \( X \) is defined as

\[
X_N \uplus X_M = X_{N \uplus M} = \left( X; T_{N \uplus M}, I_{N \uplus M}, F_{N \uplus M} \right),
\]

where

\[
(T_{N \uplus M}, I_{N \uplus M}, F_{N \uplus M}) = \left\{ \begin{cases} T_N(h_i) \lor T_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \\ I_N(h_i) \land I_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \\ F_N(h_i) \lor F_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \end{cases} \right. \]

(iv) The intersection of \( X_N \) and \( X_M \), a neutrosophic \( \aleph \)—structure over \( X \) is defined as

\[
X_N \cap X_M = X_{N \cap M} = \left( X; T_{N \cap M}, I_{N \cap M}, F_{N \cap M} \right),
\]

where

\[
(T_{N \cap M}, I_{N \cap M}, F_{N \cap M}) = \left\{ \begin{cases} T_N(h_i) \land T_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \\ I_N(h_i) \lor I_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \\ F_N(h_i) \land F_M(h_i) & \text{if } h_i \in X \text{ such that } h_i = rs \end{cases} \right. \]

3. Neutrosophic \( \aleph \)—interior ideals

We study different properties of neutrosophic \( \aleph \)—interior ideals of \( X \). It is evident that neutrosophic \( \aleph \)—ideal is a neutrosophic \( \aleph \)—interior ideal of \( X \), but not the converse. Further, for a regular and for an intra-regular semigroup, every neutrosophic \( \aleph \)—interior ideal is neutrosophic \( \aleph \)—ideal.
All throughout this part, we consider $X_M$ and $X_N$ are neutrosophic $\mathbb{K}$–structures of $X$.

**Theorem 3.1.** For any $L \subseteq X$, the equivalent assertions are:

(i) $L$ is an interior ideal,

(ii) The characteristic neutrosophic $\mathbb{K}$–structure $\chi_L(X_N)$ is a neutrosophic $\mathbb{K}$–interior ideal.

**Proof:** Suppose $L$ is an interior ideal and let $x, a, y \in X$.

If $a \in L$, then $xay \in L$, so $\chi_L(T)_N(xay) = -1 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) = 0 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) = -1 = \chi_L(F)_N(a)$.

If $a \notin L$, then $\chi_L(T)_N(xay) \leq 0 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) \geq -1 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) \leq 0 = \chi_L(F)_N(a)$.

Therefore $\chi_L(X_N)$ is a neutrosophic $\mathbb{K}$–interior ideal.

Conversely, assume that $\chi_L(X_N)$ is a neutrosophic $\mathbb{K}$–interior ideal. Let $u \in L$ and $x, y \in X$. Then

$$\chi_L(T)_N(xay) \leq \chi_L(T)_N(u) = -1,$$

$$\chi_L(I)_N(xay) \geq \chi_L(I)_N(u) = 0,$$

$$\chi_L(F)_N(xay) \leq \chi_L(F)_N(u) = -1 .$$

So $xay \in L$. □

**Theorem 3.2.** If $X_M$ and $X_N$ are neutrosophic $\mathbb{K}$–interior ideals, then $X_{M\cap N}$ is neutrosophic $\mathbb{K}$–interior ideal.

**Proof:** Let $X_M$ and $X_N$ be neutrosophic $\mathbb{K}$–interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cap N}(rst) = T_M(rst) \cap T_N(rst) \leq T_M(s) \cap T_N(s) = T_{M\cap N}(s),$$

$$I_{M\cap N}(rst) = I_M(rst) \cap I_N(rst) \geq I_M(s) \cap I_N(s) = I_{M\cap N}(s),$$

$$F_{M\cap N}(rst) = F_M(rst) \cap F_N(rst) \leq F_M(s) \cap F_N(s) = F_{M\cap N}(s).$$

Therefore $X_{M\cap N}$ is neutrosophic $\mathbb{K}$–interior ideal. □

**Corollary 3.3.** The arbitrary intersection of neutrosophic $\mathbb{K}$–interior ideals is a neutrosophic $\mathbb{K}$–interior ideal.

**Theorem 3.4.** If $X_M$ and $X_N$ are neutrosophic $\mathbb{K}$–interior ideals, then $X_{M\cup N}$ is neutrosophic $\mathbb{K}$–interior ideal.

**Proof:** Let $X_M$ and $X_N$ be neutrosophic $\mathbb{K}$–interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cup N}(rst) = T_M(rst) \cup T_N(rst) \leq T_M(s) \cup T_N(s) = T_{M\cup N}(s),$$

$$I_{M\cup N}(rst) = I_M(rst) \cup I_N(rst) \geq I_M(s) \cup I_N(s) = I_{M\cup N}(s),$$

$$F_{M\cup N}(rst) = F_M(rst) \cup F_N(rst) \leq F_M(s) \cup F_N(s) = F_{M\cup N}(s).$$

Therefore $X_{M\cup N}$ is neutrosophic $\mathbb{K}$–interior ideal. □

**Corollary 3.5.** The arbitrary union of neutrosophic $\mathbb{K}$–interior ideals is neutrosophic $\mathbb{K}$–interior ideal.

**Theorem 3.6.** Let $X$ be a regular semigroup. If $X_M$ is neutrosophic $\mathbb{K}$–interior ideal, then $X_M$ is neutrosophic $\mathbb{K}$–ideal.
Proof: Assume that $X_M$ is an interior ideal, and let $u, v \in X$. As $X$ is regular and $u \in X$, there exists $r \in X$ such that $u = uru$. Now, $T_M(uv) = T_M(uruv) \leq T_M(u)$, $I_M(uv) = I_M(uruv) \geq I_M(u)$ and $F_M(uv) = F_M(uruv) \leq F_M(u)$. Therefore $X_M$ is neutrosophic $\kappa -$ right ideal.

Similarly, we can show that $X_M$ is neutrosophic $\kappa -$ left ideal and hence $X_M$ is neutrosophic $\kappa -$ ideal. $\square$

Theorem 3.7. Let $X$ be an intra-regular semigroup. If $X_M$ is neutrosophic $\kappa -$ interior ideal, then $X_M$ is neutrosophic $\kappa -$ ideal.

Proof: Suppose that $X_M$ is neutrosophic $\kappa -$ interior ideal, and let $u, v \in X$. As $X$ is intra regular and $u \in X$, there exist $s, t \in S$ such that $u = su^2t$. Now,

$$T_M(uv) = T_M(su^2tv) \leq T_M(u),$$
$$I_M(uv) = I_M(su^2tv) \geq I_M(u)$$
$$F_M(uv) = F_M(su^2tv) \leq F_M(u).$$

Therefore $X_M$ is neutrosophic $\kappa -$ right ideal. Similarly, we can show that $X_M$ is neutrosophic $\kappa -$ left ideal and hence $X_M$ is neutrosophic $\kappa -$ ideal. $\square$

Definition 3.8. A semigroup $X$ is left simple (resp., right simple) if it does not contain any proper left ideal (resp., right ideal) of $X$. A semigroup $X$ is simple if it does not contain any proper ideal of $X$.

Definition 3.9. A semigroup $X$ is said to be neutrosophic $\kappa -$simple if every neutrosophic $\kappa -$ ideal is a constant function

i.e., for every neutrosophic $\kappa -$ ideal $X_M$ of $X$, we have $T_M(i) = T_M(j)$, $I_M(i) = I_M(j)$ and $F_M(i) = F_M(j)$ for all $i, j \in X$.

Notation 3.10. If $X$ is a semigroup and $s \in X$, we define a subset, denoted by $I_s$ as follows:

$$I_s = \{ i \in X \mid T_N(i) \geq T_N(s), \ I_N(i) \geq I_N(s) \ \text{and} \ F_N(i) \leq F_N(s) \}.$$

Proposition 3.11. If $X_N$ is neutrosophic $\kappa -$ right (resp., $\kappa -$ left, $\kappa -$ ideal) ideal, then $I_s$ is right (resp., left, ideal) ideal for every $s \in X$.

Proof: Let $s \in X$. Then it is clear that $s \neq I_s \subseteq X$. Let $u \in I_s$ and $x \in X$. Then $ux \in I_x$. Indeed; Since $X_N$ is neutrosophic $\kappa -$ right ideal and $u, x \in X$, we get $T_N(ux) \leq T_N(u), I_N(ux) \geq I_N(u)$ and $F_N(ux) \leq F_N(u)$. Since $u \in I_s$, we get $T_N(u) \leq T_N(s), I_N(u) \geq I_N(s)$ and $F_N(u) \leq F_N(s)$ which imply $ux \in I_s$. Therefore $I_s$ is a right ideal for every $s \in X$. $\square$

Theorem 3.12.[4] For any $L \subseteq X$, the equivalent assertions are:

(i) $L$ is left (resp., right) ideal,

(ii) Characteristic neutrosophic $\kappa -$structure $\chi L(X_N)$ is neutrosophic $\kappa -$ left (resp., right) ideal.

Theorem 3.13. Let $X$ be a semigroup. Then $X$ is simple if and only if $X$ is neutrosophic $\kappa -$simple.
Proof: Suppose $X$ is simple. Let $X_M$ be a neutrosophic $\mathcal{K}$-ideal and $u, v \in X$. Then by Proposition 3.11, $I_u$ is an ideal of $X$. As $X$ is simple, we have $I_u = X$. Since $v \in I_u$, we have $T_M(v) \leq T_M(u)$, $I_M(v) \geq I_M(u)$ and $F_M(v) \leq F_M(u)$.

Similarly, we can prove that $T_M(u) \leq T_M(v)$, $I_M(u) \geq I_M(v)$ and $F_M(u) \leq F_M(v)$. So we have $T_M(u) = T_M(v)$, $I_M(u) = I_M(v)$ and $F_M(u) = F_M(v)$. Hence $X$ is neutrosophic $\mathcal{K}$-simple.

Conversely, assume that $X$ is neutrosophic $\mathcal{K}$-simple and $I$ is an ideal of $X$. Then by Theorem 3.12, $\chi(X_I)$ is a neutrosophic $\mathcal{K}$-ideal. We now claim that $X = I$. Let $w \in X$. Since $X$ is neutrosophic $\mathcal{K}$-simple, we have $\chi(X_I)(w) = \chi(X_I)(y)$ for every $y \in X$. In particular, we have $\chi(T_N(w) = \chi(T_N)(d) = -1, \chi(I_N(w) = \chi(I_N)(d) = 0$ and $\chi(F_N(w) = \chi(F_N)(d) = -1$ for any $d \in I$ which implies $w \in I$. Thus $X \subseteq I$ and hence $X = I$. □

Lemma 3.14. Let $X$ be a semigroup. Then $X$ is simple if and only for every $t \in X$, we have $X = XtX$.

Proof: Suppose $X$ is simple and let $t \in X$. Then $X(tX) \subseteq XtX$ and $(XtX)X \subseteq XtX$ imply that $XtX$ is an ideal. Since $X$ is simple, we have $XtX = X$.

Conversely, let $P$ be an ideal and let $a \in P$. Then $X = XaX, XaX \subseteq XPX \subseteq P$ which implies $P = X$. Therefore $X$ is simple. □

Theorem 3.15. Suppose $X$ is a semigroup. Then $X$ is simple if and only every neutrosophic $\mathcal{K}$-ideal of $X$ is a constant function.

Proof: Suppose $X$ is simple and $s, t \in X$. Let $X_N$ be neutrosophic $\mathcal{K}$-ideal. Then by Lemma 3.14, we get $X = XsX = XtX$. As $s \in XsX$, we have $s = atb$ for $a, b \in X$. Since $X_N$ is neutrosophic $\mathcal{K}$-ideal, we have $T_N(s) = T_N(atb) \leq T_N(t), I_N(s) = I_N(atb) \geq I_N(t)$ and $F_N(s) = F_N(atb) \leq F_N(t)$. Similarly, we can prove that $T_N(t) \leq T_N(s), I_N(t) \geq I_N(s)$ and $F_N(t) \leq F_N(s)$. So $X_N$ is a constant function.

Conversely, suppose $X_N$ is neutrosophic $\mathcal{K}$-ideal. Then $X_N$ is neutrosophic $\mathcal{K}$-ideal. By hypothesis, $X_N$ is a constant function and so $X_N$ is neutrosophic $\mathcal{K}$-simple. By Theorem 3.13, $X$ is simple. □

Theorem 3.16. Let $X_M$ be neutrosophic $\mathcal{K}$-structure and let $\gamma, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \gamma + \delta + \varepsilon \leq 0$. If $X_M$ is neutrosophic $\mathcal{K}$-ideal, then $(\gamma, \delta, \varepsilon)$-level set of $X_M$ is neutrosophic $\mathcal{K}$-ideal whenever $X_M(\gamma, \delta, \varepsilon) \neq \emptyset$.

Proof: Suppose $X_M(\gamma, \delta, \varepsilon) \neq \emptyset$ for $\gamma, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \gamma + \delta + \varepsilon \leq 0$.

Let $X_M$ be a neutrosophic $\mathcal{K}$-ideal and let $u, v, w \in X_M(\gamma, \delta, \varepsilon)$. Then $T_M(uvw) \leq T_M(v) \leq \alpha, I_M(uvw) \geq I_M(v) \geq \beta$ and $F_M(uvw) \leq F_M(v) \leq \gamma$ which imply $uvw \in X_M(\alpha, \beta, \gamma)$. Therefore $X_M(\gamma, \delta, \varepsilon)$ is a neutrosophic $\mathcal{K}$-ideal of $X$. □

Theorem 3.17. Let $X_N$ be neutrosophic $\mathcal{K}$-structure with $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $T^a_N, I^b_N$ and $F^c_N$ are ideal sets, then $X_N$ is neutrosophic $\mathcal{K}$-ideal of $X$ whenever it is non-empty.

Proof: Suppose that for $a, b, c \in X$ with $T_N(abc) > T_N(b)$. Then $T_N(abc) > T_N(b)$ for some $t_a \in [-1, 0]$. So $b \in T^a_N(b)$ but $abc \in T^a_N(b)$, a contradiction. Thus $T_N(abc) \leq T_N(b)$. □
Suppose that for \( a, b, c \in X \) with \( I_N(abc) < I_N(b) \). Then \( I_N(abc) < t_a \leq I_N(b) \) for some \( t_a \in [-1, 0) \). So \( b \in I_N^a(b) \) but \( abc \notin I_N^a(b) \), a contradiction. Thus \( I_N(abc) \geq I_N(b) \).

Suppose that for \( a, b, c \in X \) with \( F_N(abc) > F_N(b) \). Then \( F_N(abc) > t_a \geq F_N(b) \) for some \( t_a \in [-1, 0) \). So \( b \in F_N^a(b) \) but \( abc \notin F_N^a(b) \), a contradiction. Thus \( F_N(abc) \leq F_N(b) \).

Thus \( X_N \) is neutrosophic \( K \) – interior ideal.

\[ \square \]

**Theorem 3.18.** Let \( X_M \) be neutrosophic \( K \) – structure over \( X \). Then the equivalent assertions are:

(i) \( X_M \) is neutrosophic \( K \) – interior ideal,

(ii) \( X_N \circ X_M \circ X_N \subseteq X_M \) for any neutrosophic \( K \) – structure \( X_N \).

**Proof:** Suppose \( X_M \) is neutrosophic \( K \) – interior ideal. Let \( x \in X \). For any \( u, v, w \in X \) such that \( x = uvw \). Then \( T_M(x) = T_M(uvw) \leq T_M(v) \leq T_M(u) \lor T_M(v) \lor T_N(w) \) which implies \( T_M(x) \leq T_{\mathcal{N}(M \circ N)}(x) \). Otherwise \( x \neq uvw \). Then \( T_M(x) = 0 = T_{\mathcal{N}(M \circ N)}(x) \). Similarly, we can prove that \( I_M(x) \geq I_{\mathcal{N}(M \circ N)}(x) \) and \( F_M(x) \leq F_{\mathcal{N}(M \circ N)}(x) \). Thus \( X_N \circ X_M \circ X_N \subseteq X_M \).

Conversely, assume that \( X_N \circ X_M \circ X_N \subseteq X_M \) for any neutrosophic \( K \) – structure \( X_N \).

Let \( u, v, w \in X \). If \( x = uvw \), then

\[
T_M(uvw) = T_M(x) \leq (\chi_X(T)_N \circ T_M \circ \chi_X(T)_N)(x) = \bigwedge_{x=uvw} \{ \chi_X(T)_N(u) \lor T_M(v) \lor \chi_X(T)_N(w) \}
\]

\[
= \bigwedge_{x=uvw} \{ \chi_X(T)_N(u) \lor (T_M(v) \lor \chi_X(T)_N(w)) \}
\]

\[
\leq \chi_X(T)_N(u) \lor T_M(v) \lor \chi_X(T)_N(w) = T_M(v),
\]

\[ I_M(uvw) = I_M(x) \leq (\chi_X(I)_N \circ I_M \circ \chi_X(I)_N)(x) = \bigvee_{x=uvw} \{ \chi_X(I)_N \circ I_M(r) \land \chi_X(I)_N(w) \}
\]

\[
= \bigvee_{x=uvw} \{ \chi_X(I)_N \circ I_M(u) \land \chi_X(I)_N(w) \}
\]

\[
\geq \chi_X(I)_N(u) \land I_M(v) \land \chi_X(I)_N(w) = I_M(v),
\]

and

\[
F_M(uvw) = F_M(x) \leq (\chi_X(F)_N \circ F_M \circ \chi_X(F)_N)(x) = \bigwedge_{x=uvw} \{ \chi_X(F)_N \circ F_M(r) \lor \chi_X(F)_N(w) \}
\]

\[
= \bigwedge_{x=uvw} \{ \chi_X(F)_N \circ F_M(U) \lor \chi_X(F)_N(w) \}
\]

\[
\leq \chi_X(F)_N(u) \lor F_M(v) \lor \chi_X(F)_N(w) = F_M(v).
\]

Therefore \( X_M \) is neutrosophic \( K \) – interior ideal.

\[ \square \]

**Notation 3.19.** Let \( X \) and \( Z \) be semigroups. A mapping \( g : X \to Z \) is said to be a homomorphism if \( g(uv) = g(u)g(v) \) for all \( u, v \in X \). Throughout this remaining section, we denote \( \text{Aut}(X) \), the set of all automorphisms of \( X \).

**Definition 3.20.** An interior ideal \( J \) of a semigroup \( X \) is called a characteristic interior ideal if \( h(J) = J \) for all \( h \in \text{Aut}(X) \).
Definition 3.21. Let $X$ be a semigroup. A neutrosophic $\kappa -$ interior ideal $X_M$ is called neutrosophic $\kappa -$ characteristic interior ideal if $T_N(h(u)) = T_N(u)$, $I_N(h(u)) = I_N(u)$ and $F_N(h(u)) = F_N(u)$ for all $u \in X$ and all $h \in Aut(X)$.

Theorem 3.22. For any $L \subseteq X$, the equivalent assertions are:

(i) $L$ is characteristic interior ideal,

(ii) The characteristic neutrosophic $\kappa -$ structure $\chi_L(X_M)$ is neutrosophic $\kappa -$ characteristic interior ideal.

Proof: Suppose $L$ is characteristic interior ideal and let $x \in X$. Then by Theorem 3.1, $\chi_L(X_M)$ is neutrosophic $\kappa -$ interior ideal. If $x \in L$, then $\chi_L(T_M(x)) = -1$, $\chi_L(I_M(x)) = 0$, and $\chi_L(F_M(x)) = -1$. Now, for any $h \in Aut(X)$, $h(x) \in h(L) = L$ which implies $\chi_L(T_M(h(x))) = -1$, $\chi_L(I_M(h(x))) = 0$, and $\chi_L(F_M(h(x))) = -1$. If $x \notin L$, then $\chi_L(T_M(x)) = 0$, $\chi_L(I_M(x)) = -1$, and $\chi_L(F_M(x)) = 0$. Now, for any $h \in Aut(X)$, $h(x) \in h(L)$ which implies $\chi_L(T_M(h(x))) = 0$, $\chi_L(I_M(h(x))) = -1$, and $\chi_L(F_M(h(x))) = 0$. Thus $\chi_L(T_M(h(x))) = \chi_L(T_M(x)), \chi_L(I_M(h(x))) = \chi_L(I_M(x))$, and $\chi_L(F_M(h(x))) = \chi_L(F_M(x))$ for all $x \in X$ and hence $\chi_L(X_M)$ is neutrosophic $\kappa -$ characteristic interior ideal.

Conversely, assume that $\chi_L(X_M)$ is neutrosophic $\kappa -$ characteristic interior ideal. Then by Theorem 3.1, $L$ is an interior ideal. Now, let $h \in Aut(X)$ and $x \in L$. Then $\chi_L(T_M(x)) = -1$, $\chi_L(I_M(x)) = 0$, and $\chi_L(F_M(x)) = -1$. Since $\chi_L(X_M)$ is neutrosophic $\kappa -$ characteristic interior ideal, we have $\chi_L(T_M(h(x))) = \chi_L(T_M(x)), \chi_L(I_M(h(x))) = \chi_L(I_M(x))$, and $\chi_L(F_M(h(x))) = \chi_L(F_M(x))$ which imply $h(x) \in L$. So $h(L) \subseteq L$ for all $h \in Aut(X)$. Again, since $h \in Aut(X)$ and $x \in L$, there exists $y \in L$ such that $h(y) = x$.

Suppose that $y \notin L$. Then $\chi_L(T_M(y)) = 0$, $\chi_L(I_M(y)) = 1$, and $\chi_L(F_M(y)) = 0$. Since $\chi_L(T_M(h(y))) = \chi_L(T_M(y)), \chi_L(I_M(h(y))) = \chi_L(I_M(y))$, and $\chi_L(F_M(h(y))) = \chi_L(F_M(y))$, we get $\chi_L(T_M(h(y))) = 0$, $\chi_L(I_M(h(y))) = -1$, and $\chi_L(F_M(h(y))) = 0$ which imply $h(y) \notin L$, a contradiction. So $y \in L$ i.e., $h(y) \in L$. Thus $L \subseteq h(L)$ for all $h \in Aut(X)$ and hence $L$ is characteristic interior ideal.

Theorem 3.23. For a semigroup $X$, the equivalent statements are:

(i) $X$ is intra-regular,

(ii) For any neutrosophic $\kappa -$ interior ideal $X_M$, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: (i) $\Rightarrow$ (ii) Suppose $X$ is intra-regular, and $X_M$ is neutrosophic $\kappa -$ interior ideal and $w \in X$.

Then there exist $r, s \in X$ such that $w = rw^2s$. Now $T_M(w) = T_M(rw^2s) \leq T_M(w^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2), I_M(w) = I_M(rw^2s) \geq I_M(w^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(rw^2s) \leq F_M(w^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

(ii) $\Rightarrow$ (i) Let (ii) holds and $s \in X$. Then $I(s^2)$ is an ideal of $X$. By Theorem 3.5 of [4], $\chi_{I(s^2)}(X_M)$ is neutrosophic $\kappa -$ ideal. By assumption, $\chi_{I(s^2)}(X_M)(s) = \chi_{I(s^2)}(X_M)(s^2)$. Since $\chi_{I(s^2)}(T_M(s^2)) = -1 = \chi_{I(s^2)}(F_M(s^2))$ and $\chi_{I(s^2)}(I_M(s^2)) = 0$, we get $\chi_{I(s^2)}(T_M(s)) = -1 = \chi_{I(s^2)}(F_M(s))$ and $\chi_{I(s^2)}(I_M(s^2)) = 0$ which imply $s \in I(s^2)$. Hence $X$ is intra-regular.

Theorem 3.24. For a semigroup $X$, the equivalent statements are:

(i) $X$ is left (resp., right) regular,

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(ii) For any neutrosophic $\mathbb{R} -$interior ideal $X_M$, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

**Proof:** (i) $\Rightarrow$ (ii) Let $X$ be left regular. Then there exists $y \in X$ such that $w = yw^2$. Let $X_M$ be a neutrosophic $\mathbb{R} -$interior ideal. Then $T_M(w) = T_M(yw^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$. $I_M(w) = I_M(yw^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(yw^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

(ii) $\Rightarrow$ (i) Suppose (ii) holds and let $X_M$ be neutrosophic $\mathbb{R} -$interior ideal. Then for any $w \in X$, $X_{L(w^2)}(T)_M(w) = X_{L(w^2)}(T)_M(w^2) = -1$, $X_{L(w^2)}(I)_M(w) = X_{L(w^2)}(I)_M(w^2) = 0$ and $X_{L(w^2)}(F)_M(w) = X_{L(w^2)}(F)_M(w^2) = -1$ which imply $w \in L(w^2)$. Thus $X$ is left regular. □

**Conclusions**

In this paper, we have introduced the concepts of neutrosophic $\mathbb{R} -$interior ideals and neutrosophic $\mathbb{R} -$characteristic interior ideals in semigroups and studied their properties, and characterized regular and intra-regular semigroups using neutrosophic $\mathbb{R} -$interior ideal structures. We have also shown that $R$ is a characteristic interior ideal if and only if the characteristic neutrosophic $\mathbb{R} -$structure $\chi_R(X_R)$ is neutrosophic $\mathbb{R} -$characteristic interior ideal. In future, we will define neutrosophic $\mathbb{R} -$prime ideals in semigroups and study their properties.

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Received: May 7, 2020. Accepted: September 23, 2020