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Sparse Domination of the Martingale Transform

by

Michael Kutzler

B.S., University of Wisconsin-Parkside, 2016

THESIS

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science
Mathematics

The University of New Mexico

Albuquerque, New Mexico

July, 2021

Dedication

To my Mom-Brenda, Dad-Scott, Sister-Laura, and Girlfriend-Shiloh for being loving family members. I couldn't do any of this without you.

Acknowledgments

I would like to thank my advisor, Professor Cristina María Pereyra, for her help and support of the direction of my degree. She has been so busy with the chair position during the Covid-19 outbreak, yet made sure I completed my thesis. I also want to thank Professor Matthew Blair and Professor Irina Holmes for taking their time to be on my thesis committee. I know you both are very busy, and I really appreciate you taking the time to read my thesis as well as being a part of my thesis committee. Last but not least, I want to thank my friend, Son, who has always been extremely helpful and encouraging for the last 4 years whenever I have needed help. I hope he goes on to be a great professor in math or gets a great job in the industry that he surely deserves.

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Abstract

Linear operators are of huge importance in modern harmonic analysis. Many operators can be dominated by finitely many sparse operators [5]. The main result in this thesis is showing a toy operator, namely the Martingale Transform is dominated by a single sparse operator [6]. Sparse operators are based on a sparse family which is simply a subset of a dyadic grid. We also show the A_2 conjecture for the Martingale Transform which follows from the sparse domination of the Martingale Transform and the A_2 conjecture for sparse operators [3].

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Chapter 1

Introduction

1.1 Overview

As mentioned in the abstract, many operators can be bounded by finitely many sparse operators [5]. You will see that the famous A_2 conjecture for sparse operators is very important. You will see that it leads to the A_2 conjecture for the Martingale Transform. The study of pointwise sparse domination has led to the study of pointwise domination techniques by sparse-like operators and this is very powerful as well [5].

1.2 History

The following information on the history of some different types of bounds for operators, culminating up to the bounds covered in this paper is pulled from Professor Pereyra's lecture notes [5].

We have known since the 70's that the maximal function is bounded on $L^p(w)$ if

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and only if the weight w is in the Muckenhoupt A_p class (Definition C.0.2) in the appendix by [12], and a similar result holds for the Hilbert Transform by [13].

In the year 2000, Stefanie Petermichl made a huge discovery. She discovered a representation formula for the venerable Hilbert Transform as an average of dyadic shift operators over dyadic grids shown in [14]. This allowed her, and others, to reduce arguments to finding estimates for these simpler dyadic models.

After this method came the Bellman function method. This method was used to get sharp weighted inequalities. Some notable mathematicians to use this method are Fedor Nazarov, Sergei Treil, and Alexander Volberg. This method was also used and paired with sharp extrapolation by Oliver Dragičević et al [16].

Next, stopping time and median oscillation arguments led to domination by sparse operators. Some notable mathematicians to use these kinds of arguments are Tuomas Hytönen, Andrei K. Lerner, David Cruz-Uribe, José María Martell, Carlos Pérez, Michael Lacey, María Reguera, Eric Sawyer, and Ignacio Uriarte-Tuero. This led up to the famous A_2 conjecture based on a representation formula for any Calderón-Zygmund operator as an average of appropriate dyadic operators. Since then domination by sparse dyadic operators has become a huge part of modern Harmonic Analysis.

Fellow UNM alumni Oleksandra Beznosova proved the A_2 conjecture for the dyadic paraproduct in [10], and together with Hytönen's dyadic representation theorem, this led to Hytönen's proof of the full A_2 conjecture found in [11].

Dyadic models have been used in Harmonic Analysis for a long time. Averaging and sparse domination techniques have allowed researchers to transfer results from the dyadic world to the continuous world. No longer are dyadic models toy models, they can truly inform the continuous world.

1.3 Some core definitions and main theorems

Here we layout some of the groundwork for the main results in this thesis. First, we give a definition of a η -Sparse operator followed by the definition of a η -Sparse family. Given an η -Sparse family \mathcal{S} , a *sparse operator* $\mathcal{A}_{\mathcal{S}}$ is defined as follows:

$$\mathcal{A}_{\mathcal{S}}f(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \mathbb{1}_Q(x)$$

where $\langle f \rangle_Q$ denotes the mean value of the function f on Q , namely

$$\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

Here a η -sparse collection \mathcal{S} for $0 < \eta < 1$ is a collection of dyadic cubes (see definition C.0.1 in the appendix) such that for each $Q \in \mathcal{S}$ there exists pairwise disjoint measurable sets with the sparse property: $E_Q : E_Q \subset Q$ and $|E_Q| \geq \eta|Q|$. Note that these definitions make sense in \mathbb{R}^n , but we will restrict our attention to \mathbb{R} and use the terms dyadic interval and dyadic cube interchangeably. The following is one of the main theorems in this paper and was shown by David Cruz-Uribe, Chema Martell, and Carlos Pérez in [3].

Theorem 1.3.1. *Let \mathcal{S} be an η -sparse family of cubes, then for all $w \in A_2$ and $f \in L^2(w)$ the following inequality holds,*

$$\|\mathcal{A}_{\mathcal{S}}f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}$$

where $C > 0$ is independent of w , and $w \in A_2$ means a positive almost everywhere locally integrable function such that $\sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q < \infty$ where the supremum is taken over all cubes Q in \mathbb{R}^n .

The following is a statement about bounding Calderón-Zygmund operators by a finite number of sparse operators which was shown by José Manuel Conde Alonso and Guillermo Rey in [9]:

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Theorem 1.3.2. *Let T be a Calderón-Zygmund operator (Definition 3.1.2). For every compactly supported $f \in L^1(\mathbb{R}^n)$, there are finitely many η_n -sparse families, $\{\mathcal{S}_j\}$ such that for almost everywhere $x \in \mathbb{R}^n$,*

$$|Tf(x)| \leq C_{n,T} \sum_{j=1}^{N_n} \mathcal{A}_{\mathcal{S}_j}(|f|)(x)$$

where $C_{n,T}$ is a positive constant.

1.4 Road Map of the Thesis

In Chapter 2 we go over the proof of the famous A_2 conjecture for sparse operators which was proved by David Cruz-Uribe, José María Martell, and Carlos Perez in [3]. We will use this famous theorem to show the A_2 conjecture for the Martingale Transform later in the thesis.

In Chapter 3 we cover Calderón-Zygmund singular integral operators. We introduce the prototypical Calderón-Zygmund singular integral operator, the Hilbert Transform, and also list some well known boundedness properties of the Hilbert Transform.

In Chapter 4 we get to the main part of this thesis, proving the sparse domination of the Martingale Transform from Lacey [6], using Professor Pereyra's guidance and lecture notes, [5]. We first introduce the Martingale Transform, mention some brief history about some boundedness properties of the Martingale Transform, and then we prove the sparse domination of the Martingale Transform. Note that most of the work comes in the section following the proof, section 4.4, Technical Lemmas.

In Chapter 5, we finish the thesis by showing the A_2 conjecture for the Martingale Transform. Thanks to the A_2 conjecture for sparse operators, the proof is fairly short in conjunction with the sparse domination of the Martingale Transform.

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For convenience I will briefly touch on what is in each section of the appendix. In Appendix A we cover some brief definitions and facts about weighted L^p spaces. In Appendix B we cover some important definitions and facts about bounded linear operators and linear functionals including dual spaces. Some of the lemmas are essential for some of the proofs, and they are referenced when used in the proofs. In appendix C we cover Dyadic Grids, and we specifically cover the standard dyadic grid. We also define what it means for a weight to be in Muckenhoupt A_p class. In Appendix D we cover Haar functions, and the fact that they form an orthonormal basis for $L^2(\mathbb{R})$. We also cover the fact that the Martingale Transform is an isometry on $L^2(\mathbb{R})$.

Chapter 2

A_2 conjecture for sparse operators

Next we present the A_2 conjecture for Sparse Operators and the proof of it given by David Cruz-Uribe, José María Martell, and Carlos Perez in 2012 in [3] using [5] as a guide.

2.1 Useful Definitions For Proof Of A_2 Conjecture

In this section, we give some useful definitions for the proof. First we give the definition of a weight function that is critical in weighted L^p spaces.

Definition 2.1.1. *A weight w is a locally integrable, real valued function on \mathbb{R}^n that is positive almost everywhere.*

Next, we give the definition of a w mass of a cube:

Definition 2.1.2. *The w -mass of a cube Q in \mathbb{R}^n is defined as follows:*

$$w(Q) = \int_Q w(x)dx.$$

Chapter 2. A_2 conjecture for sparse operators

Here, we define the A_2 characteristic of a weight:

Definition 2.1.3. *The A_2 characteristic of a weight, w on \mathbb{R}^n is denoted $[w]_{A_2} = \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$ and we say that $w \in A_2$ iff $[w]_{A_2} < \infty$, where the supremum is taken over all cubes in \mathbb{R}^n and the averaging operator $\langle w \rangle_Q = \frac{1}{|Q|} \int_Q w(x) dx$. Note that for $n = 1$, we will use the words interval and cube interchangeably.*

Lastly, before we get into the proof, we define the weighted dyadic maximal function:

Definition 2.1.4. *The Weighted Dyadic Maximal Function with respect to a weight u , and a dyadic grid \mathcal{D} (see definition C.0.1 in the appendix) on \mathbb{R} is defined by*

$$M_u^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}, x \in Q} \frac{1}{u(Q)} \int_Q |f(y)| u(y) dy.$$

Remark 2.1.1. *Note that $M_u^{\mathcal{D}}$ is bounded on $L^2(u)$, in other words, for all $f \in L^2(u)$, $\|M_u^{\mathcal{D}}\|_{L^2(u)} \leq C \|f\|_{L^2(u)}$ for some $C > 0$. In fact, $C = 2$ works by Lemma B.0.5 in the appendix.*

2.2 Proof of the A_2 Conjecture for Sparse Operators

Here is the beautiful proof of the A_2 conjecture from David Cruz-Uribe, José María Martell, and Carlos Perez in [3]:

Theorem 2.2.1. *Let \mathcal{S} be an η -sparse family of cubes, then for all $w \in A_2$ and $f \in L^2(w)$ the following inequality holds,*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^2(w)} \leq C [w]_{A_2} \|f\|_{L^2(w)} \tag{2.1}$$

where $C > 0$ is independent of w .

Chapter 2. A_2 conjecture for sparse operators

Proof. For $w \in A_2$ and \mathcal{S} a η -sparse family, showing (2.1) is equivalent to showing for all $f \in L^2(w)$ and $g \in L^2(w^{-1})$ where $\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$, the inner product on \mathbb{R} :

$$|\langle \mathcal{A}_{\mathcal{S}} f, g \rangle| \lesssim [w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})} \quad (2.2)$$

by duality as

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^2(w)} = \sup \left\{ \left| \int_{\mathbb{R}} \mathcal{A}_{\mathcal{S}} f h w dx \right| : \|h w\|_{L^2(w^{-1})} = 1 \right\}$$

by Lemma B.0.4, and

$$\begin{aligned} \sup \left\{ \left| \int_{\mathbb{R}} \mathcal{A}_{\mathcal{S}} f h w dx \right| : \|h w\|_{L^2(w^{-1})} = 1 \right\} &\leq C[w]_{A_2} \|f\|_{L^2(w)} \|h w\|_{L^2(w^{-1})} \\ &= C[w]_{A_2} \|f\|_{L^2(w)} \end{aligned}$$

assuming (2.2) and the fact that $\|h w\|_{L^2(w^{-1})} = 1$. Thus, we proceed by showing (2.2).

$$\begin{aligned} |\langle \mathcal{A}_{\mathcal{S}} f, g \rangle| &= \left| \int_{\mathbb{R}} \left(\sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y) dy \right) \mathbb{1}_Q(x) \right) \overline{g(x)} dx \right| \\ &\leq \int_{\mathbb{R}} \left| \left(\sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y) dy \right) \mathbb{1}_Q(x) \right) \overline{g(x)} \right| dx \end{aligned}$$

by the triangle inequality for integrals

$$\begin{aligned} &= \int_{\mathbb{R}} \left| \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y) dy \right) \mathbb{1}_Q(x) \right| |g(x)| dx \\ &\leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{S}} \left| \frac{1}{|Q|} \int_Q f(y) dy \right| \mathbb{1}_Q(x) |g(x)| dx \\ &\leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \mathbb{1}_Q(x) |g(x)| \frac{|Q|}{|Q|} dx \end{aligned}$$

by the triangle inequality for sums and again for integrals

$$= \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \left(\int_{\mathbb{R}} \mathbb{1}_Q(x) |g(x)| \frac{|Q|}{|Q|} dx \right)$$

Chapter 2. A_2 conjecture for sparse operators

by Fubini-Tonelli

$$\begin{aligned}
&= \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) |Q| \left(\frac{1}{|Q|} \int_Q |g(x)| dx \right) \\
&= \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q |Q| \langle |g| \rangle_Q \\
&= \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q |Q|
\end{aligned}$$

by a little algebra

$$\leq \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q \frac{1}{\eta} |E_Q|$$

as \mathcal{S} is η -sparse using the sparse property ($|E_Q| \geq \eta|Q|$)

$$\begin{aligned}
&= \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q \frac{1}{\eta} \int_{E_Q} w^{\frac{1}{2}}(x) w^{-\frac{1}{2}}(x) dx \\
&= \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q \frac{1}{\eta} \int_{E_Q} |w^{\frac{1}{2}}(x) w^{-\frac{1}{2}}(x)| dx
\end{aligned}$$

as $|E_Q| = \int_{E_Q} 1 dx = \int_{E_Q} w(x) w^{-1}(x) dx$ and weight functions are almost everywhere positive

$$\leq \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q \frac{1}{\eta} \|w^{\frac{1}{2}}\|_{L^2(E_Q)} \|w^{-\frac{1}{2}}\|_{L^2(E_Q)}$$

by Cauchy-Schwarz since $\|w^{\frac{1}{2}}\|_{L^2(E_Q)} = \left(\int_{E_Q} (w^{\frac{1}{2}}(x))^2 dx \right)^{\frac{1}{2}}$, and similarly for $\|w^{-\frac{1}{2}}\|_{L^2(E_Q)}$,

$$\begin{aligned}
&= \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} \langle w \rangle_Q \langle w^{-1} \rangle_Q \frac{1}{\eta} (w(E_Q))^{\frac{1}{2}} (w^{-1}(E_Q))^{\frac{1}{2}} \\
&\leq \frac{[w]_{A_2}}{\eta} \sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q}{\langle w^{-1} \rangle_Q} \frac{\langle |g| w w^{-1} \rangle_Q}{\langle w \rangle_Q} (w(E_Q))^{\frac{1}{2}} (w^{-1}(E_Q))^{\frac{1}{2}}
\end{aligned}$$

as $[w]_{A_2} = \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$

$$\leq \frac{[w]_{A_2}}{\eta} \left(\sum_{Q \in \mathcal{S}} \frac{\langle |f| w w^{-1} \rangle_Q^2}{\langle w^{-1} \rangle_Q^2} (w^{-1}(E_Q)) \right)^{\frac{1}{2}} \left(\sum_{Q \in \mathcal{S}} \frac{\langle |g| w w^{-1} \rangle_Q^2}{\langle w \rangle_Q^2} (w(E_Q)) \right)^{\frac{1}{2}}$$

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again by Cauchy-Schwarz

$$\leq \frac{[w]_{A_2}}{\eta} \left(\sum_{Q \in \mathcal{S}} \int_{E_Q} \frac{\langle |f|ww^{-1} \rangle_Q^2}{\langle w^{-1} \rangle_Q^2} w^{-1}(x) dx \right)^{\frac{1}{2}} \left(\sum_{Q \in \mathcal{S}} \int_{E_Q} \frac{\langle |g|ww^{-1} \rangle_Q^2}{\langle w \rangle_Q^2} w(x) dx \right)^{\frac{1}{2}}$$

where we used the fact that $w^{-1}(E_Q) = \int_{E_Q} w^{-1}(x) dx$, $w(E_Q) = \int_{E_Q} w(x) dx$, and linearity of the integral.

$$\leq \frac{[w]_{A_2}}{\eta} \left(\sum_{Q \in \mathcal{S}} \|M_{w^{-1}}^{\mathcal{D}}(fw)\|_{L^2(w^{-1}, E_Q)}^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \mathcal{S}} \|M_w^{\mathcal{D}}(gw^{-1})\|_{L^2(w, E_Q)}^2 \right)^{\frac{1}{2}}$$

by Lemma 2.3.1

$$\leq \frac{[w]_{A_2}}{\eta} \|M_{w^{-1}}^{\mathcal{D}}(fw)\|_{L^2(w^{-1})} \|M_w^{\mathcal{D}}(gw^{-1})\|_{L^2(w)}$$

as E_Q are disjoint as \mathcal{S} is η -sparse

$$\leq \frac{[w]_{A_2}}{\eta} 2 \|fw\|_{L^2(w^{-1})} 2 \|gw^{-1}\|_{L^2(w)}$$

using the boundedness property of the maximal function in Remark 2.1.1

$$= C[w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}$$

by Lemma B.0.3 in the appendix. □

2.3 Technical Lemma

In this section we give a useful technical lemma for the proof:

Lemma 2.3.1. $|\frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q}|^2 v(E_Q) \leq \int_{E_Q} |M_v^{\mathcal{D}} h(x)|^2 v(x) dx$ where $Q \in \mathcal{S}$ and \mathcal{S} is an η -sparse family.

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Proof. For all $x \in Q$,

$$\begin{aligned}
 \frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q} &= \frac{\frac{1}{|Q|} \int_Q |h(y)|v(y)dy}{\frac{1}{|Q|} \int_Q v(y)dy} \\
 &\leq \sup_{Q' \in \mathcal{D}, x \in Q'} \frac{\frac{1}{|Q'|} \int_{Q'} |h(y)|v(y)dy}{\frac{1}{|Q'|} \int_{Q'} v(y)dy} \\
 &= \sup_{Q' \in \mathcal{D}, x \in Q'} \frac{1}{v(Q')} \int_{Q'} |h(y)|v(y)dy \\
 &= M_v^{\mathcal{D}} h(x)
 \end{aligned}$$

thus,

$$\left| \frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q} \right|^2 \leq |M_v^{\mathcal{D}} h(x)|^2$$

for all $x \in Q$, hence

$$\int_{E_Q} \left| \frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q} \right|^2 v(x)dx \leq \int_{E_Q} |M_v^{\mathcal{D}} h(x)|^2 v(x)dx$$

and $\left| \frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q} \right|^2 \int_{E_Q} v(x)dx = \left| \frac{\langle |h|v \rangle_Q}{\langle v \rangle_Q} \right|^2 v(E_Q)$ which concludes the proof. \square

Chapter 3

Calderón-Zygmund Singular Integral Operators

Calderón-Zygmund singular integral operators are very popular in modern harmonic analysis. It has been shown that Calderón-Zygmund singular integral operators can be bounded by finitely many sparse operators [9] [8] [2] [6]. Clearly, since the A_2 conjecture holds for sparse operators, such a domination of Calderón-Zygmund singular integral operators by finitely many sparse operators will imply the full A_2 conjecture for Calderón-Zygmund singular integral operators.

3.1 Definitions

Again, let's start off the section with some useful definitions. A couple of good books for Calderón-Zygmund singular integral operators are [17], and [15].

Definition 3.1.1. *A one dimensional standard Calderón-Zygmund Kernel $K(x, y)$*

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is a function on $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that:

$$|K(x, y)| \leq \frac{C}{|x - y|} \quad (3.1)$$

$$|K(x, y) - K(x, z)| \leq C \frac{|y - z|^\delta}{|x - y|^{1+\delta}} \text{ when } |x - y| > 2|y - z| \quad (3.2)$$

$$|K(x, y) - K(w, y)| \leq C \frac{|x - w|^\delta}{|x - y|^{1+\delta}} \text{ when } |x - y| > 2|x - w| \quad (3.3)$$

for some $\delta > 0$.

This leads to the following definition:

Definition 3.1.2. A Calderón-Zygmund singular integral operator $Tf(x)$ is integration of f against a standard Calderón-Zygmund Kernel in the principal value sense:

$$Tf(x) = \lim_{\epsilon \rightarrow 0} \int_{|x-y|>\epsilon} K(x, y)f(y)dy.$$

Next, we define the Hilbert Transform and show that the Hilbert kernel is indeed a Calderón-Zygmund Kernel in the next section:

Definition 3.1.3. The Hilbert transform on \mathbb{R} corresponds to the kernel

$$K(x, y) = \frac{1}{x-y} \frac{1}{\pi}. \text{ More precisely, } Hf(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} dy.$$

3.2 Prototypical Calderón-Zygmund Singular Integral Operator

Many mathematicians consider the Hilbert Transform to be the prototypical Calderón-Zygmund singular integral operator.

Here, we show that the Hilbert kernel, $K(x, y) = \frac{1}{x-y} \frac{1}{\pi}$ for $x \neq y$, $x, y \in \mathbb{R}$ is a standard Calderón-Zygmund Kernel and hence a Calderón-Zygmund singular integral operator:

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Proof. To show (3.1), we just use $C = \frac{1}{\pi}$:

$$\left| \frac{1}{\pi} \frac{1}{x-y} \right| \leq \frac{1}{\pi} \left| \frac{1}{x-y} \right|.$$

To show (3.2), we break up into 2 cases: $|x-z| \geq |x-y|$ and $|x-z| < |x-y|$. For $|x-z| \geq |x-y|$ we have

$$\begin{aligned} \frac{1}{\pi} \left| \frac{1}{x-y} - \frac{1}{x-z} \right| &= \frac{1}{\pi} \frac{|y-z|}{|x-y||x-z|} \\ &\leq \frac{1}{\pi} \frac{|y-z|}{|x-y||x-y|} \\ &= \frac{1}{\pi} \frac{|y-z|}{|x-y|^2} \end{aligned}$$

Thus, $C = \frac{1}{\pi}$ and $\delta = 1$ works for this case. For $|x-z| < |x-y|$ we use the fact that $|x-y| > 2|y-z|$ and we let $\delta = 1$ and $C = 2$. We need to show that:

$$\frac{1}{\pi} \frac{|y-z|}{|x-y||x-z|} \leq 2 \frac{|y-z|}{|x-y|^2}$$

but this is equivalent to

$$\frac{1}{\pi} \frac{|x-y|}{|x-z|} \leq 2$$

which is equivalent to

$$\frac{1}{\pi} |x-y| \leq 2|x-z|. \tag{3.4}$$

Thus, we will show (3.4) given that $|x-y| \geq 2|y-z|$:

$$\begin{aligned} 2|x-z| &= 2|x-y+y-z| \\ &= 2|(x-y)-(z-y)| \\ &\geq 2||x-y|-|z-y|| \end{aligned}$$

by the reverse triangle inequality

$$\begin{aligned} &= 2||x-y|-|y-z|| \\ &\geq 2(|x-y|-|y-z|) \\ &= |x-y|+|x-y|-2|y-z| \\ &= |x-y|+C_0 \end{aligned}$$

where $C_0 > 0$ as $|x - y| \geq 2|y - z|$,

$$\geq \frac{1}{\pi}|x - y|.$$

Lastly, (3.3) is shown similarly to (3.2), and we can choose $C = 2$ and $\delta = 1$ to satisfy (3.1), (3.2), and (3.3). \square

3.3 Calderón-Zygmund Singular Integral Operator Example

In this section we get a feel for how the Hilbert Transform and, hence, a Calderón-Zygmund singular integral operator, behaves with a brief example of the Hilbert Transform acting on the characteristic function of $[0, 1]$. For $x \notin [0, 1]$,

$$\begin{aligned} H\chi_{[0,1]}(x) &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|x-y|>\epsilon} \frac{\chi_{[0,1]}(y)}{x-y} dy \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{1}{x-y} dy \\ &= \frac{-1}{\pi} \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{1}{y-x} dy \\ &= \lim_{\epsilon \rightarrow 0} \frac{-1}{\pi} \ln|1-x| + \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \ln|x| \\ &= \frac{1}{\pi} \ln \left| \frac{x}{1-x} \right|. \end{aligned}$$

For $x \in [0, 1]$,

$$\begin{aligned} H\chi_{[0,1]}(x) &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_0^{x-\epsilon} \frac{1}{x-y} dy + \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{x+\epsilon}^1 \frac{1}{x-y} dy \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} (-\ln|\epsilon| + \ln|x| - \ln|1-x| + \ln|\epsilon|) \\ &= \frac{1}{\pi} \ln \left| \frac{x}{1-x} \right|. \end{aligned}$$

Although $H\chi_{[0,1]}(x) \in L^2(\mathbb{R})$, as the Hilbert Transform is an isometry on $L^2(\mathbb{R})$ (Lemma D.0.1 in the appendix), $H\chi_{[0,1]}(x) \notin L^1(\mathbb{R})$, as the decay is not fast enough at $-\infty$ or ∞ .

3.4 Boundedness Properties of the Hilbert Transform

The following section provides information on some different types of bounds of the Hilbert Transform. For more details please see [5] Section 2.2.

Frigyez Riesz was able to find a qualitative bound for the Hilbert transform for $1 < p < \infty$ back in 1927:

Theorem 3.4.1. *(Riesz 1927) H is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, i.e. there exists a constant $C_p > 0$, that depends on p , such that $\|Hf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}$.*

Although the Hilbert Transform is not bounded on $L^1(\mathbb{R})$ as was mentioned in the previous section, Kolmogorov was able to show that H is a weak type (1,1) operator:

Theorem 3.4.2. *(Kolmogorov 1927) H is a weak type (1,1) operator, i.e. there exists a constant $C > 0$ such that $|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R})}$ for all $f \in L^1(\mathbb{R})$.*

In 1973 Richard Hunt, Benjamin Muckenhoupt and Richard Wheeden found a quantitative bound for H on $L^p(w)$:

Theorem 3.4.3. *(Hunt-Muckenhoupt-Wheeden 1973) H is bounded on $L^p(w)$ for $1 < p < \infty$ for all w in Muckenhoupt A_p class (Definition C.0.2 in the appendix), i.e. there exists a constant $C_p(w) > 0$ that depends on p and the weight w such that $\|Hf\|_{L^p(w)} \leq C_p(w) \|f\|_{L^p(w)}$.*

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In 2007 Stefanie Petermichl found a quantitative bound for H on $L^p(w)$:

Theorem 3.4.4. *(Petermichl 2007) Given $1 < p < \infty$ there exists a constant $C_p > 0$ that depends on p such that for all w in Muckenhoupt A_p class (Definition C.0.2 in the appendix), and for all $f \in L^p(w)$, $\|Hf\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$ where $[w]_{A_p}$ is defined in (Definition C.0.2 in the appendix).*

In 2012 Tuomas Hytönen extended Stefanie Petermichl's result to general Calderón-Zygmund operators:

Theorem 3.4.5. *(Hytönen 2012) Given $1 < p < \infty$, w in Muckenhoupt A_p class (Definition C.0.2 in the appendix), and T a Calderón-Zygmund operator, there exist a constant $C_{p,T} > 0$, that depends on p and T such that for all $f \in L^p(w)$, $\|Tf\|_{L^p(w)} \leq C_{p,T} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$ where $[w]_{A_p}$ is defined in (Definition C.0.2 in the appendix).*

In this thesis we will focus on the case $p = 2$ and prove the A_2 -conjecture for the Martingale Transform, a dyadic toy model for the Hilbert Transform. The Martingale Transform is defined for $f \in L^2(w)$ by

$$T_\sigma f = \sum_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I,$$

where \mathcal{D} denotes a dyadic grid (Definition C.0.1 in the appendix) and $\{h_I\}_{I \in \mathcal{D}}$ is the Haar system (an orthonormal basis in $L^2(\mathbb{R})$, see Theorem D.0.1 in the appendix). We will show that there is a constant $C > 0$ such that for all $w \in A_2$, $f \in L^2(w)$,

$$\|T_\sigma f\|_{L^2(w)} \leq C [w]_{A_2} \|f\|_{L^2(w)}$$

where $[w]_{A_2} = \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty$ and this supremum is taken over all intervals $I \subset \mathbb{R}$.

Chapter 4

Sparse Domination of the Martingale Transform

In this section we show the sparse domination of the Martingale Transform [6], [5]. The Martingale transform is a good toy model operator for Calderón-Zygmund singular integral operators [5]. Thus, we can get a similar feel for what it is like to bound a Calderón-Zygmund singular integral operator by sparse operators.

4.1 Martingale Transform

In this section we define the Martingale Transform and mention some of its bounds. The Martingale Transform is an isometry on $L^2(\mathbb{R})$ which follows from Parseval (see Lemma D.0.1 in the appendix). On $L^p(w)$ the Martingale Transform is bounded as was shown by Donald L. Burkholder in 1984. The A_2 conjecture for the Martingale Transform was proved using Bellman functions in 2001 by Janine Wittwer. Here we show it using sparse domination in combination with the A_2 theorem for sparse operators.

Definition 4.1.1. *The Martingale Transform is defined as follows on $L^p(\mathbb{R})$:*

$$T_\sigma f(x) = \sum_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I(x)$$

where σ_I is equal to ± 1 , \mathcal{D} is a dyadic grid (defined in Appendix C.0.1), and $h_I(x)$ is the Haar function associated to the interval I as described in appendix D.0.1.

4.2 Useful Definitions and Remarks

This section lays out some definitions and remarks used in the proof of the sparse domination of the Martingale transform.

Definition 4.2.1. *A linear operator T on $L^1(\mathbb{R})$ is said to be of weak type $(1, 1)$ if there exists a constant $C > 0$ such that:*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R} : T(f(x)) > \lambda\}| \leq C \|f\|_{L^1(\mathbb{R})}.$$

Definition 4.2.2. *The sharp truncation of the Martingale Transform is defined as follows:*

$$T_\sigma^\# f = \sup_{I' \in \mathcal{D}} \left| \sum_{I \in \mathcal{D}, I' \subsetneq I} \sigma_I \langle f, h_I \rangle h_I \right|.$$

Remark 4.2.1. *This sharp truncation is of weak type $(1, 1)$ by a celebrated result of Donald L. Burkholder in [7].*

Definition 4.2.3. *We define the parent \tilde{I} of I where $I \in \mathcal{D}$ to be the unique $\tilde{I} \in \mathcal{D}, I \subset \tilde{I}$ with $|\tilde{I}| = 2|I|$.*

Definition 4.2.4. *Given a set F in \mathbb{R} that is a union of dyadic intervals, a collection of maximal dyadic intervals \mathcal{E} is defined to be those dyadic intervals contained in F whose parents are not contained in F . In set notation:*

$$\mathcal{E} = \{J \in \mathcal{D} : J \subset F, \tilde{J} \not\subset F\}.$$

Note that by definition \mathcal{E} is a disjoint family of dyadic intervals and $F = \bigcup_{J \in \mathcal{E}} J$.

Definition 4.2.5. For I a dyadic interval we define

$$T_\sigma^I f = \sigma_I \langle f \rangle_I \mathbb{1}_I + \sum_{J: J \subset I, J \in \mathcal{D}} \sigma_J \langle f, h_J \rangle h_J.$$

Definition 4.2.6. The Hardy-Littlewood Maximal Function denoted $Mf(x)$ of a locally integrable function f is defined as follows:

$$Mf(x) = \sup_{I: x \in I} \frac{1}{|I|} \int_I |f(t)| dt$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$.

Remark 4.2.2. Note that the Hardy-Littlewood Maximal Function is known to be of type $(1,1)$ (this is in [4]), and the same is true for the Dyadic Maximal Function, $M^{\mathcal{D}}f(x)$. Further, Stephen M. Buckley showed the quantitative estimate in [18]:

$$\|Mf\|_{L^p(w)} \leq C[w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.$$

Definition 4.2.7. $\langle f \rangle_I = \frac{1}{|I|} \int_I f(x) dx$.

Definition 4.2.8. Define $P_j f(x) = \langle f \rangle_I$ where I is in the j th generation of \mathcal{D} , namely \mathcal{D}_j , and $x \in I$.

Definition 4.2.9. Let $Q_j f(x) = P_{j+1} f(x) - P_j f(x)$.

4.3 Proof Of Sparse Domination Of Martingale Transform

Finally, we arrive at the proof of the sparse domination of the Martingale Transform. We use [6],[5] as a guide.

Theorem 4.3.1. For \mathcal{D} a dyadic grid and $f \in L^1(\mathbb{R})$ supported on I_0 there exists a $\frac{1}{2}$ -sparse family $\mathcal{S} \subset \mathcal{D}$ with $I_0 \subset \mathcal{S}$ such that

$$|\mathbb{1}_{I_0} T_\sigma f| \leq C \mathcal{A}_{\mathcal{S}}(|f|).$$

This pointwise bound holds almost everywhere.

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Proof. Without loss of generality assume that $\int_{I_0} |f(x)| dx > 0$ (and f is supported on I_0). Note that both the Maximal Function is of weak type (1,1) (Definition 4.2.1) from Remark 4.2.2 and also the Truncated Martingale Transform is of type (1,1) by Remark 4.2.1. For any dyadic interval $I \subset I_0$, we define

$$F_I = \{x \in I : \max\{M^{\mathcal{D}}f(x), T_{\sigma}^{\#}f(x)\} > \frac{1}{2}C_0\langle |f| \rangle_I\}. \quad (4.1)$$

Then by Lemma 4.4.1 we have:

$$|F_I| \leq \frac{1}{2}|I|. \quad (4.2)$$

Note that if $x \in F_I$ then either $M^{\mathcal{D}}f(x) > \frac{1}{2}C_0\langle |f| \rangle_I$, or $T_{\sigma}^{\#}f(x) > \frac{1}{2}C_0\langle |f| \rangle_I$. This means there is $J = J_x \in \mathcal{D}$ such that either $\langle |f| \rangle_J > \frac{1}{2}C_0\langle |f| \rangle_I$, or $\sum_{K:K \supseteq J} \sigma_K \langle f, h_K \rangle h_k(x) > \frac{1}{2}C_0\langle |f| \rangle_I$. This in turn implies $M^{\mathcal{D}}f(y) > \frac{1}{2}C_0\langle |f| \rangle_I$, or $\sum_{K:K \supseteq J} \sigma_K \langle f, h_K \rangle h_k(y) > \frac{1}{2}C_0\langle |f| \rangle_I$ for all $y \in J$. Hence $x \in J \cap I \subset F_I$ since $I, J \in \mathcal{D}$, and $J \cap I \neq \emptyset$, (4.2) implies that it must be that $J \cap I = J \subset F_I$. Therefore $F_I = \bigcup_{x \in F_I} J_x$. We define \mathcal{E}_I to be the collection of maximal dyadic intervals (Definition 4.2.4) contained in F_I . Note that \mathcal{E}_I is a necessarily pairwise disjoint collection of intervals by definition. Thus, if we let $\mathcal{S}_0 = \{I_0\}$, $\mathcal{S}_1 = \mathcal{E}_{I_0} = \bigcup_{I \in \mathcal{S}_0} \mathcal{E}_I$, $\mathcal{S}_2 = \bigcup_{I \in \mathcal{S}_1} \mathcal{E}_I, \dots, \mathcal{S}_n = \bigcup_{I \in \mathcal{S}_{n-1}} \mathcal{E}_I$, then each \mathcal{S}_j is a pairwise disjoint family of dyadic intervals. Now we define $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{S}_n$, $E_J = J \setminus F_J$. For $J \in \mathcal{S}$, as $E_J \cap F_J = \emptyset$, $|J| = |E_J| + |F_J|$, hence by (4.2) $|E_J| > \frac{|J|}{2}$. Thus as $E_J \subset J$, and are pairwise disjoint by construction, we have that \mathcal{S} is a $\frac{1}{2}$ -sparse family (defined in Section 1.3). Next, Lemma 4.4.4 shows that for all $n \geq 1$:

$$|T_{\sigma}f(x)\mathbb{1}_{I_0}(x)|$$

is less than or equal to

$$(1 + C_0) \left[\langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \dots + \sum_{I_1 \in \mathcal{E}_{I_0}} \dots \sum_{I_n \in \mathcal{E}_{I_{n-1}}} |T_{\sigma}^{I_n} f(x)| \right].$$

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If x is in a finite number of F_{I_n} , say for $1 \leq n \leq N$, then the “remainder” term:

$$\sum_{I_1 \in \mathcal{E}_{I_0}} \cdots \sum_{I_N \in \mathcal{E}_{I_{N-1}}} |T_\sigma^{I_N} f(x)|$$

is zero for N sufficiently large because the “remainder” term is supported on the union of F_I for $I \in \mathcal{E}_{N-1}$, but x is no longer in this union. Thus, we have

$$|T_\sigma f(x) \mathbb{1}_{I_0}(x)|$$

is less than or equal to

$$(1 + C_0) \left[\langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \cdots + \sum_{I_1 \in \mathcal{E}_{I_0}} \cdots \sum_{I_N \in \mathcal{E}_{I_{N-1}}} \langle |f| \rangle_{I_N} \mathbb{1}_{I_N}(x) \right]$$

which equals

$$\begin{aligned} (1 + C_0) \sum_{j=0}^N \sum_{I \in \mathcal{S}_j} \langle |f| \rangle_I \mathbb{1}_I(x) &\leq (1 + C_0) \sum_{j=0}^{\infty} \sum_{I \in \mathcal{S}_j} \langle |f| \rangle_I \mathbb{1}_I(x) \\ &= (1 + C_0) \mathcal{A}_S f(x). \end{aligned}$$

Now if $x \in F_{I_n}$ for infinitely many $n \in \mathbb{N}$, then $x \in \bigcap_{n=0}^{\infty} F_n$, where

$F_0 = F_{I_0} \subset I_0, F_1 = \bigcup_{I \in \mathcal{S}_1} F_I \subset F_0, \dots, F_n = \bigcup_{I \in \mathcal{S}_n} F_I \subset F_{n-1}$, then the remainder term is not necessarily zero, but as $I_0 \supset F_0 \supset F_1 \supset F_2 \supset \dots$ we have

$|\bigcap_{n=0}^{\infty} F_n| = \lim_{n \rightarrow \infty} |F_n| \leq \lim_{n \rightarrow \infty} \frac{1}{2^n} |I_0| = 0$, and we are left with a set of measure zero. Hence, this pointwise bound holds almost everywhere. \square

4.4 Technical Lemmas

The following are some technical lemmas used in the proof of the sparse domination of the Martingale Transform.

Lemma 4.4.1. *Let \mathcal{D} be a Dyadic Grid. Let $I_0 \in \mathcal{D}$, and T_1, T_2 be weak type $(1, 1)$ operators (Definition 4.2.1). Further, let*

$$F_{I_0}(C) = \{x \in I_0 : \max\{|T_1 f(x)|, |T_2 f(x)|\} > \frac{1}{2} C \langle |f| \rangle_{I_0}\}$$

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where C is some positive constant and f is supported on I_0 . Then there exist $C_0 > 0$ independent of f and I_0 such that

$$|F_{I_0}(C_0)| \leq \frac{1}{2}|I_0|.$$

Proof. Assume not. Then for all $C_0 > 0$, $|F_{I_0}(C_0)| > \frac{1}{2}|I_0|$. Hence,

$$|\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| > \frac{1}{4}|I_0|$$

or

$$|\{x \in I_0 : T_2 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| > \frac{1}{4}|I_0|.$$

Because if not, for $x \in F_{I_0}$ we have

$$\max\{T_1 f(x), T_2 f(x)\} > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}$$

if and only if

$$T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}$$

or

$$T_2 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}.$$

Thus, as

$$|F_{I_0}(C_0)| = |\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\} \cup \{x \in I_0 : T_2 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}|$$

which is less than or equal to

$$|\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| + |\{x \in I_0 : T_2 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}|$$

which is less than or equal to

$$\frac{1}{4}|I_0| + \frac{1}{4}|I_0| = \frac{1}{2}|I_0|$$

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by subadditivity of measure which is a contradiction. Thus, without loss of generality let's assume that

$$|\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| > \frac{1}{4}|I_0|.$$

Then as $T_1 f(x)$ is weak type (1,1) for some $C > 0$,

$$\frac{1}{2}C_0 \langle |f| \rangle_{I_0} |\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| \leq C \|f\|_{L^1(\mathbb{R})}$$

hence

$$\begin{aligned} |\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| &\leq \frac{C \|f\|_{L^1(\mathbb{R})}}{\frac{1}{2}C_0 \langle |f| \rangle_{I_0}} \\ &= \frac{2C \|f\|_{L^1(\mathbb{R})}}{C_0 \frac{1}{|I_0|} \int_{I_0} |f|} \\ &= \frac{2C |I_0| \|f\|_{L^1(\mathbb{R})}}{C_0 \int_{I_0} |f|} \\ &= \frac{2C |I_0|}{C_0} \end{aligned}$$

as f is supported on I_0 . Now we can just choose C_0 large enough, specifically $C_0 \geq 8C$ so that

$$|\{x \in I_0 : T_1 f(x) > \frac{1}{2}C_0 \langle |f| \rangle_{I_0}\}| \leq \frac{1}{4}|I_0|$$

which is a contradiction. Note that C_0 is independent of f and I_0 . □

Lemma 4.4.2. For $f \in L^1(\mathbb{R})$, f supported on I_0 where I_0 is a dyadic interval,

$$|T_\sigma^I f(x)| \leq |T_\sigma(f \mathbb{1}_I)(x)| \mathbb{1}_I(x) + \langle |f| \rangle_I \mathbb{1}_I(x)$$

where I is a dyadic interval.

Proof. Recall that $T_\sigma^I f = \sum_{J: J \subset I} \sigma_J \langle f, h_J \rangle h_J + \sigma_I \langle f \rangle_I \mathbb{1}_I$. First, observe that using the nested property of the dyadic intervals ($I, J \in \mathcal{D}$, then $I \subsetneq J$, $I \supset J$, or $I \cap J = \emptyset$, see Lemma 9.25 in [1]).

$$T_\sigma(f \mathbb{1}_I)(x) \mathbb{1}_I(x)$$

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equals

$$\mathbb{1}_I \left[\sum_{J:J \subset I} \sigma_J \langle f \mathbb{1}_I, h_J \rangle h_J + \sum_{J:J \supseteq I} \sigma_J \langle f \mathbb{1}_I, h_J \rangle h_J + \sum_{J:J \cap I = \emptyset} \sigma_J \langle f \mathbb{1}_I, h_J \rangle h_J \right]$$

which, since $\langle f \mathbb{1}_I, h_J \rangle = 0$ whenever $J \cap I = \emptyset$ equals

$$\sum_{J:J \subset I} \sigma_J \langle f \mathbb{1}_I, h_J \rangle h_J + \mathbb{1}_I \sum_{J:J \supseteq I} \sigma_J \langle f \mathbb{1}_I, h_J \rangle h_J$$

which equals

$$\sum_{J:J \subset I} \sigma_J \langle f, h_J \rangle h_J + \sigma_{\tilde{I}} \langle f \mathbb{1}_I, h_{\tilde{I}} \rangle h_{\tilde{I}} \mathbb{1}_I + \mathbb{1}_I \sum_{n=2}^{\infty} \sigma_{I_n} \langle f \mathbb{1}_I, h_{I_n} \rangle h_{I_n}$$

where I_n is the n th elder relative of I i.e. $I_1 = \tilde{I}, I_2 = \tilde{\tilde{I}}$, and more generally, $I_n = \tilde{\tilde{\tilde{I}}}_{n-1}$. Further as

$$\begin{aligned} \langle f \mathbb{1}_I, h_{I_n} \rangle h_{I_n}(x) \mathbb{1}_I(x) &= \left(\int_{I_n} f \mathbb{1}_I h_{I_n} \right) h_{I_n}(x) \mathbb{1}_I(x) \\ &= \left(\pm \frac{1}{|I_n|^{\frac{1}{2}}} \right) \left(\pm \frac{1}{|I_n|^{\frac{1}{2}}} \right) \left(\int_{I_n} f \mathbb{1}_I \right) \mathbb{1}_I(x) \\ &= \left(\frac{1}{|I_n|} \int_I f \right) \mathbb{1}_I(x) \end{aligned}$$

as $I \subset I_n$, hence h_{I_n} only takes on its values in I_{n_i} or I_{n_r} . Thus,

$\langle f \mathbb{1}_I, h_{I_n} \rangle h_{I_n} \mathbb{1}_I = \frac{|I|}{|I_n|} \left(\frac{1}{|I|} \int_I f \right) \mathbb{1}_I = \frac{1}{2^n} \langle f \rangle_I \mathbb{1}_I$. Hence,

$$\begin{aligned} T_{\sigma}(f \mathbb{1}_I) \mathbb{1}_I &= \sum_{J:J \subset I} \sigma_J \langle f, h_J \rangle h_J + \frac{1}{2} \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I + \sum_{n=2}^{\infty} \frac{1}{2^n} \sigma_{I_n} \langle f \rangle_I \mathbb{1}_I \\ &= \sum_{J:J \subset I} \sigma_J \langle f, h_J \rangle h_J + \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I \left[\frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{\sigma_{I_n}}{\sigma_{\tilde{I}}} \right] \\ &= T_{\sigma}^I f + \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I \left[-\frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{\sigma_{I_n}}{\sigma_{\tilde{I}}} \right] \end{aligned}$$

and as

$$\left| -\frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{\sigma_{I_n}}{\sigma_{\tilde{I}}} \right| \leq 1$$

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since $\frac{\sigma_{I_n}}{\sigma_I} = \pm 1$ we have

$$|T_\sigma^I f(x)| \leq |T_\sigma(f\mathbb{1}_I)(x)\mathbb{1}_I(x)| + \langle |f| \rangle_I \mathbb{1}_I(x),$$

by the triangle inequality. □

Lemma 4.4.3. *If $x \in I_0$, then*

$$|T_\sigma f(x)| \leq C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0} + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)|$$

where \mathcal{E}_{I_0} is the set of maximal dyadic intervals (Definition 4.2.4) in F_{I_0} and $F_{I_0} = \{x \in I_0 : \max\{M^{\mathcal{D}} f(x), T_\sigma^\# f(x)\} > \frac{1}{2} C_0 \langle |f| \rangle_{I_0}\}$.

Proof. Case 1: $x \in E_{I_0} = I_0 \setminus F_{I_0}$: There is a unique $I_n \in \mathcal{D}_n$ such that $x \in I_n$, and

$$Q_n f(x) = P_{n+1} f(x) - P_n f(x) = \langle f, h_{I_n} \rangle_{I_n} h_{I_n}(x)$$

by Lemma 9.35 in [1], and further

$$\langle f, h_{I_n} \rangle_{I_n} h_{I_n}(x) = \sum_{I \in \mathcal{D}_n} \langle f, h_I \rangle h_I(x).$$

Also,

$$\sum_{n=k}^{n=N} (P_{n+1} f(x) - P_n f(x)) = P_{N+1} f(x) - P_k f(x)$$

as the sum is telescoping. Hence,

$$\begin{aligned} \sum_{n=k}^{n=\infty} (P_{n+1} f(x) - P_n f(x)) &= \lim_{N \rightarrow \infty} (P_{N+1} f(x) - P_k f(x)) \\ &= f(x) - P_k f(x) \end{aligned}$$

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by the Lebesgue Differentiation Theorem almost everywhere. Also,

$$\begin{aligned}
 \sum_{n=-\infty}^{n=k-1} (P_{n+1}f(x) - P_n f(x)) &= \lim_{N \rightarrow -\infty} (-P_N f(x) + P_k f(x)) \\
 &= \left(\lim_{N \rightarrow -\infty} -\frac{1}{|I_N|} \int_{I_N} f(x) dx \right) + P_k f(x) \\
 &= \left(\lim_{N \rightarrow -\infty} -\frac{1}{|I_N|} \int_{I_0} f(x) dx \right) + P_k f(x)
 \end{aligned}$$

as f is supported on I_0 . Note that

$$\left| \int_{I_0} f(x) dx \right| \leq \left(\int_{I_0} 1^2 dx \right)^{\frac{1}{2}} \left(\int_{I_0} |f(x)|^2 dx \right)^{\frac{1}{2}} \leq |I_0|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})} = C'$$

by Cauchy-Schwarz. Thus,

$$\begin{aligned}
 \sum_{n=-\infty}^{n=k-1} (P_{n+1}f(x) - P_n f(x)) &= \left(\lim_{N \rightarrow -\infty} \frac{1}{|I_N|} C \right) + P_k f(x) \\
 &= P_k f(x)
 \end{aligned}$$

and we have that

$$f(x) = \sum_{n=-\infty}^{n=\infty} (P_{n+1}f(x) - P_n f(x))$$

almost everywhere. Note that

$$\langle T_\sigma f, h_J \rangle_J = \sigma_J \langle f, h_J \rangle$$

and

$$\begin{aligned}
 T_\sigma^\# f(x) &= \sup_{I' \in \mathcal{D}, x \in I'} \left| \sum_{I \in \mathcal{D}, I' \subset I} \sigma_I \langle f, h_I \rangle h_I \right| \\
 &= \sup_{k \in \mathbb{Z}} \left| \sum_{n=-\infty}^{n=k-1} P_{n+1} T_\sigma f(x) - P_n T_\sigma f(x) \right| \\
 &= \sup_{k \in \mathbb{Z}} |P_k T_\sigma f(x)|
 \end{aligned}$$

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where the last equality came from the same telescoping sum and limiting argument used above. Thus,

$$|T_\sigma f(x)| = \left| \sum_{n=-\infty}^{n=\infty} P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x) \right|$$

which equals

$$\left| \sum_{n=-\infty}^{n=k-1} (P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x)) + \sum_{n=k}^{n=\infty} (P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x)) \right|$$

which is equal to

$$\left| P_kT_\sigma f(x) + \sum_{n=k}^{n=\infty} (P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x)) \right|$$

which is less than or equal to

$$\sup_{k \in \mathbb{Z}} \left| P_kT_\sigma f(x) \right| + \left| \sum_{n=k}^{n=\infty} (P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x)) \right|$$

which equals

$$\left| T_\sigma^\# f(x) \right| + \left| \sum_{n=k}^{n=\infty} (P_{n+1}T_\sigma f(x) - P_nT_\sigma f(x)) \right|$$

which is equal to

$$|T_\sigma^\# f(x)| + \left| \lim_{n \rightarrow \infty} (P_nT_\sigma f(x)) - P_kT_\sigma f(x) \right|$$

as the sum is telescoping, hence for all $k \in \mathbb{Z}$ and for all $x \in E_{I_0}$

$$|T_\sigma f(x)| \leq |T_\sigma^\# f(x)| + |T_\sigma f(x) - P_kT_\sigma f(x)|.$$

Given $\epsilon > 0$, we can find K sufficiently large so that $P_kT_\sigma f(x)$ is ϵ -close to $T_\sigma f(x)$ i.e. $|T_\sigma f(x) - P_kT_\sigma f(x)| < \epsilon$ for all $k \geq K$ and we can do this for all $k \in \mathbb{Z}$ by the Lebesgue Differentiation Theorem almost everywhere. Hence, for almost every $x \in E_{I_0}$

$$|T_\sigma f(x)| \leq |T_\sigma^\# f(x)| + \epsilon$$

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for k large, and letting ϵ run to zero we have

$$|T_\sigma f(x)| \leq |T_\sigma^\# f(x)|.$$

Thus as $x \in E_{I_0}$, we have that

$$|T_\sigma^\# f(x)| \leq \frac{1}{2} C_0 \langle |f| \rangle_{I_0}$$

and therefore $|T_\sigma f(x)| \leq \frac{1}{2} C_0 \langle |f| \rangle_{I_0}$ for almost every $x \in E_{I_0}$.

Case 2: $x \in F_{I_0}$:

If $x \in F_{I_0}$ then there must exist a unique $I \in \mathcal{E}_{I_0}$ such that $x \in I$, and

$$\begin{aligned} |T_\sigma f(x)| &= \left| \sum_{J \in \mathcal{D}} \sigma_J \langle f, h_J \rangle h_J(x) \right| \\ &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \cap \bar{I} = \emptyset} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \subset \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| \\ &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \subset \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| \end{aligned}$$

as $x \in I$. Hence,

$$\begin{aligned} |T_\sigma f(x)| &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \subset I} \sigma_J \langle f, h_J \rangle h_J(x) + \sigma_{\bar{I}} \langle f, h_{\bar{I}} \rangle h_{\bar{I}}(x) \right| \\ &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \subset I} \sigma_J \langle f, h_J \rangle h_J(x) + (\langle f \rangle_I - \langle f \rangle_{\bar{I}}) \sigma_{\bar{I}} \right| \end{aligned}$$

by Lemma 9.35 in [1]. Thus,

$$\begin{aligned} |T_\sigma f(x)| &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) + T_\sigma^I f(x) - \sigma_{\bar{I}} \langle f \rangle_{\bar{I}} \right| \\ &\leq \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| + |T_\sigma^I f(x)| + |-\sigma_{\bar{I}} \langle f \rangle_{\bar{I}}| \\ &= \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| + |T_\sigma^I f(x)| + |\langle f \rangle_{\bar{I}}| \\ &\leq \left| \sum_{J \not\supseteq \bar{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| + |T_\sigma^I f(x)| + \langle |f| \rangle_{\bar{I}} \end{aligned}$$

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by the triangle inequality for numbers. Further, for all $y \in \tilde{I}$

$$\begin{aligned} \left| \sum_{J \supseteq \tilde{I}} \sigma_J \langle f, h_J \rangle h_J(x) \right| &\leq \sup_{I' \in \mathcal{D}} \left| \sum_{I'' \supset I'} \sigma_{I''} \langle f, h_{I''} \rangle h_{I''}(y) \right| \\ &= T_\sigma^\sharp f(y) \end{aligned}$$

and

$$\langle |f| \rangle_{\tilde{I}} \leq M^{\mathcal{D}} f(y).$$

Hence,

$$\begin{aligned} |T_\sigma f(x)| &\leq T_\sigma^\sharp f(y) + |T_\sigma^I f(x)| + M^{\mathcal{D}} f(y) \\ &\leq T_\sigma^\sharp f(y) + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)| + M^{\mathcal{D}} f(y) \end{aligned}$$

for all $x \in I \in \mathcal{E}_{I_0}$ and for all $y \in \tilde{I}$. Also, as I is maximal there must exist a $y_0 \in \tilde{I}$ with $y_0 \notin F_{I_0}$, otherwise I would not be maximal. Thus, we finally arrive at:

$$\begin{aligned} |T_\sigma f(x)| &\leq T_\sigma^\sharp f(y_0) + M^{\mathcal{D}} f(y_0) + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)| \\ &\leq C_0 \langle |f| \rangle_{I_0} + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)| \end{aligned}$$

where the last inequality follows from the definition of F_{I_0} , in particular, since $y_0 \notin F_{I_0}$, $T_\sigma^\sharp f(x) \leq \frac{C_0}{2} \langle |f| \rangle_{I_0}$ and $M^{\mathcal{D}} f(x) \leq \frac{C_0}{2} \langle |f| \rangle_{I_0}$. This proves the lemma for almost every $x \in I_0$. \square

Lemma 4.4.4. $|T_\sigma f(x)| \mathbb{1}_{I_0}(x) \leq (1 + C_0) \left[\langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \dots + \sum_{I_1 \in \mathcal{E}_{I_0}} \dots \sum_{I_{n-1} \in \mathcal{E}_{I_{n-2}}} \langle |f| \rangle_{I_{n-2}} \mathbb{1}_{I_{n-2}}(x) \right] + \sum_{I_1 \in \mathcal{E}_{I_0}} \dots \sum_{I_n \in \mathcal{E}_{I_{n-1}}} |T_\sigma^{I_n} f(x)|$

Proof. By Lemma 4.4.3 we have

$$|T_\sigma f(x)| \mathbb{1}_{I_0}(x) \leq C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} |T_\sigma^{I_1} f(x)|$$

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which is less than or equal to

$$C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} |T_\sigma(f \mathbb{1}_{I_1})(x)| \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x)$$

by Lemma 4.4.2. Thus, by Lemma 4.4.3 we have

$$|T_\sigma f(x)| \mathbb{1}_{I_0}(x)$$

is less than or equal to

$$C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \left[C_0 \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_2 \in \mathcal{E}_{I_1}} |T_\sigma^{I_2}(f \mathbb{1}_{I_2})(x)| \right],$$

which equals

$$C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} (1 + C_0) \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \sum_{I_2 \in \mathcal{E}_{I_1}} |T_\sigma^{I_2} f(x)|,$$

which is less than or equal to

$$C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} (1 + C_0) \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \sum_{I_2 \in \mathcal{E}_{I_1}} |T_\sigma(f \mathbb{1}_{I_2})(x)| \mathbb{1}_{I_2}(x) + \langle |f| \rangle_{I_2} \mathbb{1}_{I_2}(x),$$

again by Lemma 4.4.2. Thus, by Lemma 4.4.3 we have $|T_\sigma f(x)|$ is less than or equal to

$$C_0 \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} (1 + C_0) \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \sum_{I_2 \in \mathcal{E}_{I_1}} \left[C_0 \langle |f| \rangle_{I_2} \mathbb{1}_{I_2}(x) + \sum_{I_3 \in \mathcal{E}_{I_2}} |T_\sigma^{I_3} f(x)| \right]$$

which is less than or equal to:

$$(1 + C_0) \langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} (1 + C_0) \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \sum_{I_2 \in \mathcal{E}_{I_1}} (1 + C_0) \langle |f| \rangle_{I_2} \mathbb{1}_{I_2}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \sum_{I_2 \in \mathcal{E}_{I_1}} \sum_{I_3 \in \mathcal{E}_{I_2}} |T_\sigma^{I_3} f(x)|.$$

Hence, after n iterations this is less than or equal to :

$$(1 + C_0) \left[\langle |f| \rangle_{I_0} \mathbb{1}_{I_0}(x) + \sum_{I_1 \in \mathcal{E}_{I_0}} \langle |f| \rangle_{I_1} \mathbb{1}_{I_1}(x) + \dots + \sum_{I_1 \in \mathcal{E}_{I_0}} \dots \sum_{I_{n-1} \in \mathcal{E}_{I_{n-2}}} \langle |f| \rangle_{I_{n-2}} \mathbb{1}_{I_{n-2}}(x) \right] + \sum_{I_1 \in \mathcal{E}_{I_0}} \dots \sum_{I_n \in \mathcal{E}_{I_{n-1}}} |T_\sigma^{I_n} f(x)|. \quad \square$$

Chapter 5

A_2 Conjecture for the Martingale Transform

In this chapter, we present the A_2 conjecture for the Martingale Transform. Namely, we are going to show that for f in $L^2(w)$, the Martingale Transform is bounded above by the product of a constant independent of w and the A_2 characteristic of w . Here we see the importance of the A_2 conjecture for Sparse Operators.

5.1 Ingredients for the proof

In this section we state the A_2 conjecture for the Martingale Transform and remind the reader of the main ingredients used for the proof presented in the next section.

Theorem 5.1.1. (*A_2 conjecture for the Martingale Transform*) For $w \in A_2$ (Definition 2.1.3 in Chapter 2), there exist $C > 0$, independent of w , such that for all $f \in L^2(w)$, we have:

$$\|T_\sigma f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}.$$

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Theorem 5.1.2. (*A_2 conjecture for sparse operators*) Let \mathcal{S} be an η -sparse family of cubes, then for all $w \in A_2$ and $f \in L^2(w)$ the following inequality holds,

$$\|\mathcal{A}_{\mathcal{S}}f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$$

where $C > 0$ is independent of w , and $w \in A_2$ means a positive almost everywhere locally integrable function such that $\sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q < \infty$ where the supremum is taken over all cubes Q in \mathbb{R}^n (here we are restricting to \mathbb{R}).

Theorem 5.1.3. For \mathcal{D} a dyadic grid and $f \in L^1(\mathbb{R})$ supported on I_0 there exists a $\frac{1}{2}$ -sparse family $\mathcal{S} \subset \mathcal{D}$ with $I_0 \subset \mathcal{S}$ such that

$$|\mathbb{1}_{I_0}T_{\sigma}f| \leq C\mathcal{A}_{\mathcal{S}}(|f|).$$

This pointwise bound holds almost everywhere.

5.2 Proof of the A_2 Conjecture for the Martingale Transform

Theorem 5.2.1. (*A_2 conjecture for the Martingale Transform*) For $w \in A_2$ (Definition 2.1.3), there exist $C > 0$, independent of w , such that for all $f \in L^2(w)$, we have:

$$\|T_{\sigma}f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}.$$

Proof. Let's start off by defining

$$f_n^+(x) = f(x)\mathbb{1}_{(x \in [0, 2^n])}(x),$$

$$f_n^-(x) = f(x)\mathbb{1}_{(x \in [-2^n, 0])}(x),$$

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$$f_n(x) = f(x)\mathbb{1}_{(x \in [-2^n, 2^n])}(x).$$

Note that $f_n = f_n^- + f_n^+$. Then, by the linearity of the Martingale Transform, we have:

$$\mathbb{1}_{[-2^n, 2^n]}(x)T_\sigma(f_n(x)) = \mathbb{1}_{[-2^n, 2^n]}(x)(T_\sigma f_n^-(x) + T_\sigma f_n^+(x)).$$

By definition, $T_\sigma f_n^-$ is supported on $(-\infty, 0)$, and $T_\sigma f_n^+$ is supported on $[0, \infty)$, thus

$$\mathbb{1}_{[-2^n, 2^n]}(x)T_\sigma f_n(x) = \mathbb{1}_{[-2^n, 0)}(x)T_\sigma f_n^-(x) + \mathbb{1}_{[0, 2^n)}(x)T_\sigma f_n^+(x).$$

Hence by some algebra and linearity of the integral

$$\begin{aligned} \|\mathbb{1}_{[-2^n, 2^n]}T_\sigma f_n\|_{L^2(w)}^2 &= \|\mathbb{1}_{[-2^n, 0)}T_\sigma f_n^-\|_{L^2(w)}^2 + \|\mathbb{1}_{[0, 2^n)}T_\sigma f_n^+\|_{L^2(w)}^2 \\ &\lesssim \|\mathcal{A}_S^- f_n^-\|_{L^2(w)} + \|\mathcal{A}_S^+ f_n^+\|_{L^2(w)} \end{aligned}$$

by Theorem 4.3.1, the sparse domination of the Martingale Transform, and then applying the A_2 conjecture for sparse operators, this is less than or equal to

$$\begin{aligned} C^2[w]_{A_2}^2 (\|f_n^-\|_{L^2(w)}^2 + \|f_n^+\|_{L^2(w)}^2) &= C^2[w]_{A_2}^2 \|f_n\|_{L^2(w)}^2 \\ &\leq C^2[w]_{A_2}^2 \|f\|_{L^2(w)}^2 \end{aligned}$$

where we used a little algebra like we did previously for the equality. Thus,

$$\|\mathbb{1}_{[-2^n, 2^n]}T_\sigma f_n\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)} \tag{5.1}$$

where this constant is independent of w as this is the constant C is from the A_2 conjecture for sparse operators. Next, we show that $T_\sigma f_n$ converges to $T_\sigma f$ in $L^2(w)$. By linearity of of the Martingale Transform, we have

$$\|T_\sigma f_n - T_\sigma f\|_{L^2(w)} = \|T_\sigma(f_n - f)\|_{L^2(w)}$$

and as the Martingale Transform is a bounded operator, this is less than or equal to

$$C\|f_n - f\|_{L^2(w)}.$$

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Hence, letting n run to infinity:

$$\lim_{n \rightarrow \infty} \|T_\sigma f_n - T_\sigma f\|_{L^2(w)} \leq \lim_{n \rightarrow \infty} C \|f_n - f\|_{L^2(w)} = 0 \quad (5.2)$$

by the dominated convergence theorem. (f_n is dominated by f which is in $L^2(w)$).

By the triangle inequality we have:

$$\|T_\sigma f\|_{L^2(w)}$$

is less than or equal to

$$\|T_\sigma f - \mathbb{1}_{[-2^n, 2^n]} T_\sigma f\|_{L^2(w)} + \|\mathbb{1}_{[-2^n, 2^n]} T_\sigma f - \mathbb{1}_{[-2^n, 2^n]} T_\sigma f_n\|_{L^2(w)} + \|\mathbb{1}_{[-2^n, 2^n]} T_\sigma f_n\|_{L^2(w)},$$

which equals

$$\|T_\sigma f - \mathbb{1}_{[-2^n, 2^n]} T_\sigma f\|_{L^2(w)} + \|\mathbb{1}_{[-2^n, 2^n]} (T_\sigma f - T_\sigma f_n)\|_{L^2(w)} + \|\mathbb{1}_{[-2^n, 2^n]} T_\sigma f_n\|_{L^2(w)},$$

and by Equation (5.1), this is less than or equal to:

$$\|T_\sigma f - \mathbb{1}_{[-2^n, 2^n]} T_\sigma f\|_{L^2(w)} + \|T_\sigma f - T_\sigma f_n\|_{L^2(w)} + C[w]_{A_2} \|f\|_{L^2(w)}.$$

Thus letting n run to infinity, we have

$$\|T_\sigma f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}$$

by Inequality (5.2). □

Appendices

Appendix A

L_p Spaces

In this section of the appendix we cover some brief definitions and facts about weighted L^p spaces.

Definition A.0.1. For $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we call $q = \frac{p}{p-1}$ the conjugate exponent to p .

Definition A.0.2. For a measure space (X, μ) we define $L^p(X, \mu)$ to be

$$\left\{ f : \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} < \infty \right\}$$

where f is a measurable complex valued function defined on X , and μ is a measure.

We will focus on the case $X = \mathbb{R}$ and $d\mu = w dx$ where w is a weight (Definition 2.1.1).

Remark A.0.1. $\left(\int_{\mathbb{R}} |f|^p w \right)^{\frac{1}{p}}$ yields a norm denoted $\|f\|_{L^p(w)}$, hence $L^p(w)$ is a normed vector space. Also, observe that $L^p(\mathbb{R})$ corresponds to $w = 1$.

Appendix B

Bounded Operators, Linear Functionals, and Dual Spaces

In this section of the appendix we cover some important definitions and facts about bounded linear operators and linear functionals including dual spaces. Some of the lemmas are essential for some of the proofs, and they are referenced when used in the proofs.

Definition B.0.1. *For a linear operator $T : X \mapsto Y$, where X and Y are normed vector spaces, we say that T is bounded if there exists a positive constant C such that $\|T(x)\|_Y \leq C\|x\|_X$.*

We will focus on the case $X = Y = L^p(w)$.

Definition B.0.2. *A linear functional (over \mathbb{C}) is a linear map from a vector space to \mathbb{C} .*

Definition B.0.3. *The Dual Space of $L^p(w)$ denoted $(L^p(w))^*$ is the space of all bounded linear functionals on $L^p(w)$ i.e*

$$\{T : L^p(w) \mapsto \mathbb{C}, T \text{ linear}, |T(f)| \leq C\|f\|_{L^p(w)}\}.$$

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Definition B.0.4. The standard norm we will put on $(L^p(w))^*$ is called the operator norm of $(L^p(w))^*$ denoted $\|T\|$ for $T \in (L^p(w))^*$ and it is defined as

$$\begin{aligned}\|T\| &= \sup\{|Tf| : \|f\|_{L^p(w)} = 1\} \\ &= \sup\left\{\frac{|Tf|}{\|f\|_{L^p(w)}} : f \neq 0\right\} \\ &= \inf\{C : |Tf| \leq C\|f\|_{L^p(w)} \forall f \in L^p(w)\}.\end{aligned}$$

See [4] p.154.

Definition B.0.5. For $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we call $q = \frac{p}{p-1}$ the conjugate exponent to p .

Lemma B.0.1. If p and q are conjugate exponents, then $T(f) = \int_{\mathbb{R}} fg$ defines a bounded linear functional on $L^p(\mathbb{R})$ for every g in $L^q(\mathbb{R})$ by Holder's Inequality:

$$|T(f)| = \left| \int_{\mathbb{R}} fg \right| \leq \int_{\mathbb{R}} |fg| \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

.

The next theorem is well known and we take it from [4], p.190, Thm 6.15.

Theorem B.0.1. If p, q are conjugate exponents then for every $T \in (L^p(w))^*$ there exists a $g \in L^q(w)$ such that $T(f) = \int_{\mathbb{R}} fgw$ for every $f \in L^p(w)$. Moreover, $\|T\|_{(L^p(w))^*} = \|g\|_{L^q(w)}$.

Remark B.0.1. Note that by definition of the operator norm (Definition B.0.4 in the appendix), every $h \in L^q(w)$ defines a linear functional on $L^p(w)$ which we will denote T_h , and $T_h(f) = \int_{\mathbb{R}} fhwdx$ has operator norm $\|T_h\|_{(L^p(w))^*} = \|h\|_{L^q(w)}$.

Lemma B.0.2. For a weight w , $g = hw \in L^2(w^{-1})$ if and only if $h = gw^{-1} \in L^2(w)$. Further, $\|g\|_{L^2(w^{-1})} = \|h\|_{L^2(w)}$.

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Proof. Assume $\|g\|_{L^2(w^{-1})}^2 = \int_{\mathbb{R}} |hw|^2 w^{-1} < \infty$. Then

$$\begin{aligned} \int_{\mathbb{R}} |hw|^2 w^{-1} &= \int_{\mathbb{R}} |h|^2 w^2 w^{-1} \\ &= \int_{\mathbb{R}} |h|^2 w \\ &= \|h\|_{L^2(w)}^2 < \infty \end{aligned}$$

so that $h \in L^2(w)$, and further $\|g\|_{L^2(w^{-1})} = \|h\|_{L^2(w)}$. The proof that $g \in L^2(w^{-1})$ if $h \in L^2(w)$ is similar. \square

Lemma B.0.3. $g \in L^2(w)$ if and only if $gw \in L^2(w^{-1})$ and further $\|g\|_{L^2(w)} = \|gw\|_{L^2(w^{-1})}$.

Proof. Assume $\|g\|_{L^2(w)}^2 = \int_{\mathbb{R}} |g|^2 w dx < \infty$. Then

$$\begin{aligned} \int_{\mathbb{R}} |g|^2 w dx &= \int_{\mathbb{R}} |gw|^2 w^{-1} dx \\ &= \|gw\|_{L^2(w^{-1})}^2 < \infty \end{aligned}$$

so that $gw \in L^2(w^{-1})$, and further $\|g\|_{L^2(w)} = \|gw\|_{L^2(w^{-1})}$. The other direction for containment is similar. \square

The following lemma is true for any bounded operator T on $L^2(w)$. We will state it for the sparse operator $T = \mathcal{A}_{\mathcal{S}}$.

Lemma B.0.4. $\|\mathcal{A}_{\mathcal{S}} f\|_{L^2(w)} = \sup \left\{ \left| \int_{\mathbb{R}} \mathcal{A}_{\mathcal{S}} f h w dx \right| : \|hw\|_{L^2(w^{-1})} = 1 \right\}$.

Proof. For $f \in L^2(w)$, $\mathcal{A}_{\mathcal{S}} f \in L^2(w)$, since $\mathcal{A}_{\mathcal{S}}$ is a bounded operator on $L^2(w)$, $T_{\mathcal{A}_{\mathcal{S}} f}$ defines a linear functional on $L^2(w)$, and by Remark B.0.1 in the appendix, has

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operator norm $\|T_{\mathcal{A}Sf}\|_{(L^2(w))^*} = \|\mathcal{A}Sf\|_{L^2(w)}$. Further,

$$\begin{aligned} \|T_{\mathcal{A}Sf}\|_{(L^2(w))^*} &= \sup \left\{ \left| T_{\mathcal{A}Sf}(h) \right| : \|h\|_{L^2(w)} = 1 \right\} \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \mathcal{A}Sf h w dx \right| : \|h\|_{L^2(w)} = 1 \right\} \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \mathcal{A}Sf h w dx \right| : \|hw\|_{L^2(w^{-1})} = 1 \right\} \end{aligned}$$

where we used the Definition B.0.4 (in the appendix) of the Operator Norm in the first equality and Lemma B.0.3 in the last equality. \square

Lemma B.0.5. $\|M_u^{\mathcal{D}}\|_{L^p(u)} \leq q\|f\|_{L^p(u)}$, where p and q are conjugate exponents (see Definition A.0.1 in the appendix). This estimate is in [5], p.19. Here, $M_u^{\mathcal{D}}$ denotes the weighted dyadic maximal function (Definition 2.1.4 in Chapter 2).

Appendix C

Dyadic Grids and Weights

In this section of the appendix we cover Dyadic Grids, and we specifically cover the standard dyadic grid. We also define what it means for a weight to be in Muckenhoupt A_p class.

Definition C.0.1. *A dyadic grid, \mathcal{D} on \mathbb{R} is a collection of intervals $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$ organized in generations \mathcal{D}_j with each generation partitioning \mathbb{R} such that:*

1. *If $I, J \in \mathcal{D}$ then $I \cap J = \emptyset$, $I \subseteq J$, or $J \subset I$.*
2. *Given $I \in \mathcal{D}_j$ there are two disjoint dyadic intervals that we call the left and right children such that $I = I_l \cup I_r$, and $|I_l| = |I_r| = \frac{|I|}{2}$.*

See [1] for a more detailed description.

Remark C.0.1. *The standard dyadic grid on \mathbb{R} consists of the following intervals:*

$$I_{j,k} = [k2^{-j}, (k+1)2^{-j})$$

where $j, k \in \mathbb{Z}$ and $\mathcal{D}_j = \{I_{j,k} : k \in \mathbb{Z}\}$ forms a partition on \mathbb{R} for each $j \in \mathbb{Z}$. If $I \in \mathcal{D}_j$, $|I| = 2^{-j}$, thus as j runs to ∞ the intervals shrink, and as j runs to $-\infty$ the

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intervals get larger. Note that we can separate $\mathcal{D} = \mathcal{D}^- \cup \mathcal{D}^+$, where \mathcal{D}^- consists of intervals to the left of zero, and \mathcal{D}^+ consists of intervals to the right of zero. Further, zero is always an endpoint and never an interior point for any dyadic interval in the standard grid.

Definition C.0.2. A weight w (definition 2.1.1 in Chapter 2) is in Muckenhoupt A_p class for $1 < p < \infty$ if and only if

$$[w]_{A_p} = \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w^{\frac{-1}{p-1}}(x) dx \right)^{p-1}$$

is finite.

Appendix D

Haar Functions

In this section of the appendix we cover Haar functions, and the fact that they form an orthonormal basis for $L^2(\mathbb{R})$. We also cover the fact that the Martingale Transform is an isometry on $L^2(\mathbb{R})$.

Definition D.0.1. *The Haar Function denoted $h_I(x)$ is defined as follows:*

$$h_I(x) = \frac{1}{|I|^{\frac{1}{2}}}(\mathbb{1}_{I_r}(x) - \mathbb{1}_{I_l}(x))$$

where $I = [a, b)$, $I_l = [a, \frac{a+b}{2})$, and $I_r = [\frac{a+b}{2}, b)$.

Theorem D.0.1. *(Haar 1910's) The Haar functions indexed on the dyadic intervals $\{h_I\}_{I \in \mathcal{D}}$ form an orthonormal basis of $L^2(\mathbb{R})$. Therefore, for all $f \in L^2(\mathbb{R})$, $f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I$, and $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2$, where $\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$, the inner product on $L^2(\mathbb{R})$.*

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Lemma D.0.1. *The Martingale Transform $T_\sigma f = \sum_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I(x)$, $\sigma_I = \pm 1$ is a bounded operator on $L^2(\mathbb{R})$, moreover it is an isometry.*

Proof. By definition, $\langle T_\sigma f, h_I \rangle = \sigma_I \langle f, h_I \rangle$. Thus, by Parseval twice,

$$\|T_\sigma f\|_{L^2(\mathbb{R})}^2 = \sum_{I \in \mathcal{D}} |\langle T_\sigma f, h_I \rangle|^2 = \sum_{I \in \mathcal{D}} |\sigma_I|^2 |\langle f, h_I \rangle|^2 = \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2 = \|f\|_{L^2(\mathbb{R})}^2.$$

□

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