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Electronic Technique For Resolving Ambiguities Of Sinusoidal Function Transducers.

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ELECTRONIC TECHNIQUE FOR RESOLVING AMBIGUITIES OF SINUSOIDAL FUNCTION TRANSDUCERS

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INTRODUCTION

A circular function transducer is a device whose output is a circular trigonometric (sinusoidal) function of the input to the device. The sinusoidal nature of this class of transducer's output results in identical responses for many admissible inputs. Consequently, an ambiguity exists as to which of the many possible input values produce any particular output value. Even a history of the output from an initial reference input becomes ambiguous once the output completes one cycle of the sinusoid. To be a generally useful instrumentation component, the ambiguity of circular function transducers must be resolved.

Circular function transducers are currently used in several applications, because other transducers are impractical or otherwise not applicable. One of the most widely used circular function transducers is the optical interferometer. The optical interferometer produces intensity [1]* changes of light which are circular functions of the optical properties of the paths traveled by the light beams. The outputs of other interferometers, such as the microwave interferometers, are also the circular functions of their inputs.

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*Numbers in brackets indicate the literature references at the end of this paper.

Another transducer with circular function characteristics has been recently reported by Wunsch and Erteza [2]. To measure high-voltage pulses, Wunsch and Erteza apply the pulse to the electro-optic Kerr cell, through which linearly polarized light is passed. The polarization of the light leaving the Kerr cell is a circular function of the voltage applied to the cell. In this application, as in most others, the true independent variable is time because the parameter measured is usually dynamic rather than static. The determination of whether an initially known input has in fact begun to decrease or increase at a certain later time can be extremely difficult. A simple recording of the circular function transducer's output is not an adequate solution.

A better understanding of circular function ambiguities may be gained by a detailed examination of a hypothetical transducer's circular function (Figure 1).

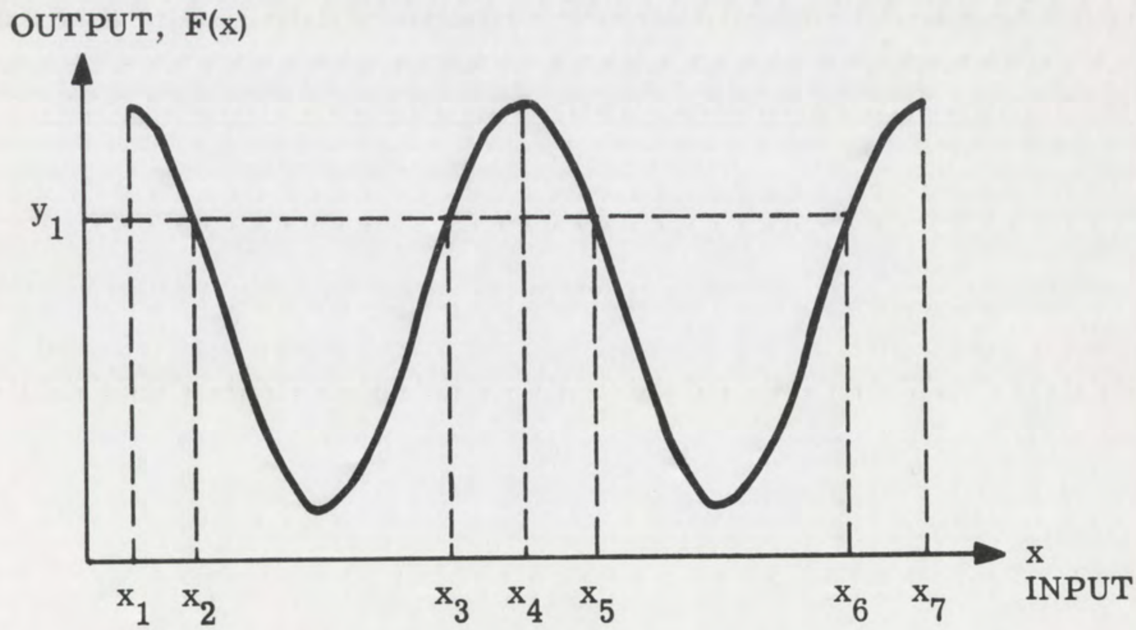

Figure 1. Hypothetical Circular Transducer Function

The fact that the output $F(x) = y_1$ can be caused by a number of inputs--specifically x_2, x_3, x_5 and x_6 (Figure 1)-- illustrates the fundamental ambiguity of the circular function. If the input value is known to lie between x_1 and x_4, either of two inputs (an ambiguity of two) is possible which can produce an output of y_1. Taking x_1 as a reference point and letting the input vary between x_1 and x_4, the ambiguity is at most two until x_4 is reached. After the input reaches x_4, an increase or an equivalent decrease in the input value will produce identical output records and consequently the ambiguity increases to four or less until F(x) reaches another peak value. After F(x) comes to the second peak output value; x_1, x_4, or x_7; the maximum ambiguity becomes eight. If the input's direction of passage (increasing or decreasing of the input) through the values x_1, x_4, x_7, etc., is known, the total ambiguity can be held to two by recording the direction of each such passage. The ambiguity may be completely resolved if this direction information is available and recorded continuously. By combining the time record of the output with a record of direction changes in the input, the complete time dependence of the input function is established.

One purpose of this paper is to develop a mathematical model for an electronic technique which will resolve the ambiguities of circular function transducers. Experimental confirmation of the technique is provided by reporting the resolution of ambiguities in a circular function transducer. A particular optical interferometer in which all conventional methods for resolving ambiguity in optical interferometers are inapplicable has been selected as the circular function transducer. Ordinarily, ambiguity in interferometers is resolved with a quadrature system by constructing, in effect, two interferometers which simultaneously measure the optical pathlength of interest. The sinusoidal output of one of the interferometers is phase shifted ±90 degrees from the sinusoidal output of the other interferometer. In conventional optical interferometers with independent stable light sources, the usual techniques of producing the 90-degree phase difference involve two reference beams and two measurement beams. One of these beam pairs has identical pathlengths whereas the other differs in length by odd multiples of $\lambda/4$ between the pair.

The transducer used in the experimental demonstration of the technique is the Laser Feedback Interferometer [3] schematically illustrated in Figure 2.

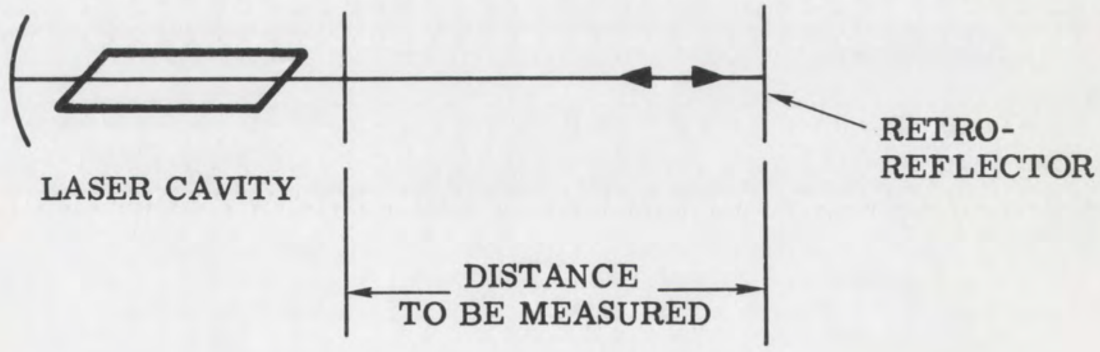

Figure 2. Schematic Illustration of Laser Feedback Interferometer

Unlike other types, this interferometer does not have a stable intensity light source. The light intensity of the laser is controlled or determined by the amount of feedback of its own light from a retro-reflector. The laser output is a coherent beam of light and the intensity and phase angle of the reflected light into the laser determine the amount of feedback. The net in-phase contribution of the reflected light into the laser determines the output of the transducer.

The diagram in Figure 3 helps explain why conventional quadrature techniques are not applicable to the Laser Feedback Interferometer.

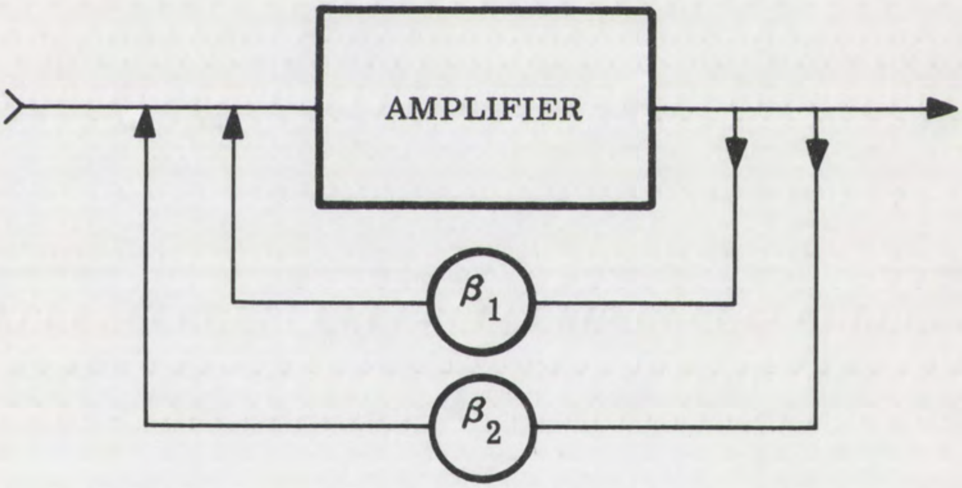

Figure 3. Amplifier with Feedback Analog of Laser Feedback Interferometer

A laser is essentially an amplifier with sufficient feedback, β_1, to become an optical oscillator. In the Laser Feedback Interferometer, additional feedback, β_2, is provided by the retro-reflector. The two feedback loops (β_1 and β_2) in effect result in a single feedback function and produce a single phase oscillation at the output of the amplifier. Similarly, if additional feedback loops were introduced in an attempt to produce a quadrature system, all loops would still add to a single resultant feedback function. The Laser Feedback Interferometer constitutes an important example where conventional quadrature techniques are not applicable. The electronic technique presented in this paper, however, is a more general solution to the problem of ambiguity resolution in circular transducers. This technique is applicable to situations where conventional quadrature techniques are applicable as well as to those where they are not.

AMBIGUITY RESOLUTION IN SINUSOIDAL FUNCTION TRANSDUCERS

The ambiguity problem arises in instrumentation whenever a circular* (sinusoidal) function transducer is encountered. For this reason circular function transducers have been in the past used only when absolutely necessary. A solution to the problem of circular function ambiguity is of particular interest in areas of instrumentation.

Past efforts to remove the ambiguity of circular functions have been centered around the type of quadrature systems mentioned previously. The term "quadrature" comes from a mathematical definition which states that the quadrature of a function is the integral of the function itself. The indefinite integral of a circular function is the same function phase shifted by 90 degrees. Figure 4 illustrates the two outputs of a quadrature system. Here the circular function $F_2(x)$ is just $F_1(x)$ with a phase shift of -90 degrees.

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*In the discussions which follow, the term circular function implies sinusoidal functions in particular, although the theory applies generally to periodic functions with bounded first derivative.

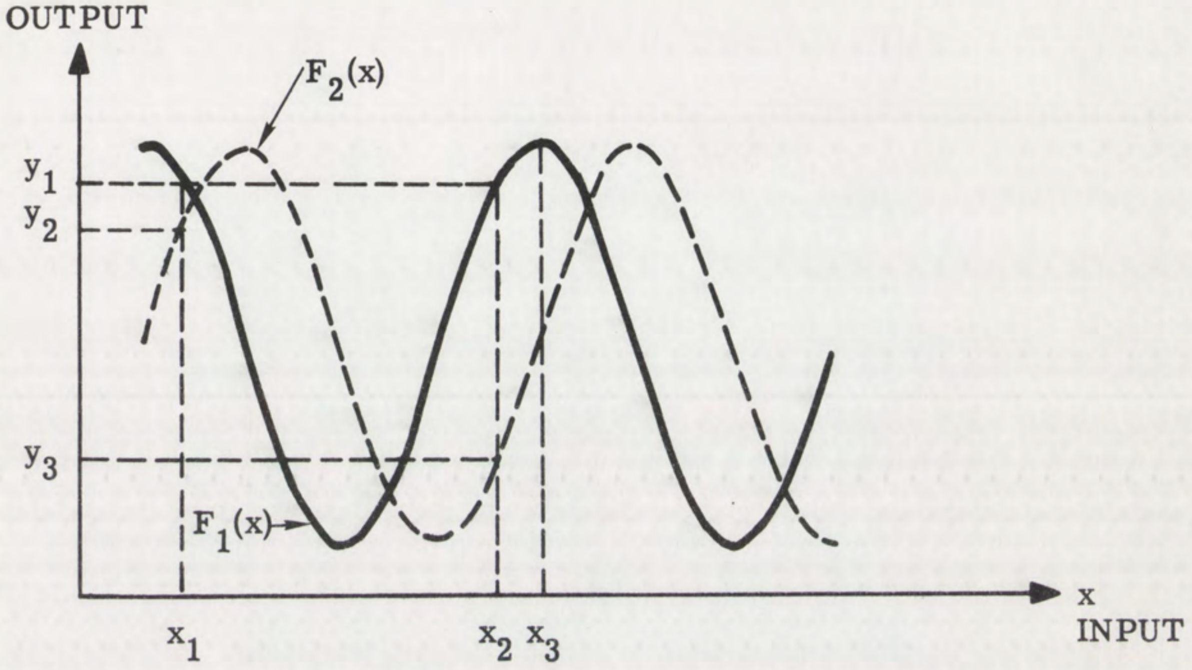

Figure 4. Quadrature Circular Functions

The quadrature system's ability to resolve ambiguity is explained by a discussion of Figure 4. The output value y_1 for $F_1(x)$ may be caused by either input value x_1 or x_2. However, if the input is x_1, the value of $F_2(x)$ is y_2, while an input of x_2 produces an output of y_3 for $F_2(x)$. From the figure it is apparent that for each input value within a cycle (of $F_1(x)$ or $F_2(x)$) the values of $F_1(x)$ and $F_2(x)$ are a unique pair. The information provided by the combination of the two outputs enables the quadrature system to determine the input uniquely.

The remaining problem is to ascertain the appropriate cycle or period of $F_1(x)$ to which the inputs currently belong. A convenient point for defining the beginning and end of a period of $F_1(x)$ is the peak amplitude such as the one that occurs at x_3 in Figure 4. Therefore, if the input is at x_3 and decreases, $F_1(x)$ is in a certain cycle; on the other hand, if the input increases, $F_1(x)$ is in a different cycle (until it reaches another peak value). The problem of determining the current period of $F_1(x)$ may be resolved by continuously determining the direction of change of the input from a reference such as x_3. The distance of the present input value from x_3 (or any other reference) can be resolved to within an integer number of cycles. This information is obtained by cumulatively adding the number of cycles completed in one direction and subtracting the cumulative number of cycles completed in the opposite direction. The necessary directional information is, however, already contained in the two quadrature-signals as will be explained below. If the input increases uniformly with time, the sinusoidal functions $F_1(x)$ and $F_2(x)$ have a phase relationship such that the former leads the latter by 90 degrees, whereas if the input decreases uniformly the phase relationship reverses. The input's direction of change is thus indicated by the unique sign of the phase difference (±90 degrees) between $F_1(x)$ and $F_2(x)$.

Since the principle of quadrature systems has been discussed, a brief description of conventional ways of obtaining quadrature signals in interferometers is in order. One of the most common methods of obtaining quadrature signals in optical interferometers is to pass a portion of the measurement beam through a phase retardation plate. This retarded beam is combined with the reference beam to produce a fringe pattern. Similarly the unretarded portion of the measurement beam is also combined with the reference to produce a separate fringe pattern. The phase delay of the retarded portion is adjusted so that the two fringe patterns achieved are in quadrature [4]. The two patterns are sensed by separate photodetectors which individually produce a sine function and its quadrature, a cosine function of the measurement distance [5]. A variation of the above scheme is to delay a portion of the reference beam rather than the measurement beam in producing the desired interference patterns. Both of these systems employ two reference beams or two measurement beams to generate the quadrature signals. As stated previously, these schemes prove to be incompatible with the Laser Feedback Interferometer configuration.

The fundamental concept behind the circular function resolution technique presented in this paper is that the first derivative of a sinusoidal function is a sinusoidal function. In fact, the derivative is in quadrature with the original sinusoid. To illustrate this:

$$G_1(x) \equiv \cos x$$

$$G_2(x) \equiv \frac{d}{dx}\Big[G_1(x)\Big] = \frac{d}{dx}\Big[\cos x\Big]$$

but

$$\frac{d}{dx}[\cos x] \equiv -\sin x.$$

However

$$-\sin x \equiv \cos(x + 90°)$$

$$\therefore \quad G_2(x) = \cos(x + 90°) = G_1(x + 90°)$$

The first derivative of a function is precisely the expression of that function's slope at every point with respect to the variable of differentiation. At any point of interest, $F_2(x)$ (Figure 4) is the value of $F_1(x)$'s slope except for a multiplying constant. Thus a system which provides the transducer's slope at each input value necessarily produces the desired quadrature information.

The slope information of a linear transducer is easily obtainable by applying a small periodic variation to the transducer's input (see Figure 5). From Figure 5a it is seen that a perturbation to the input

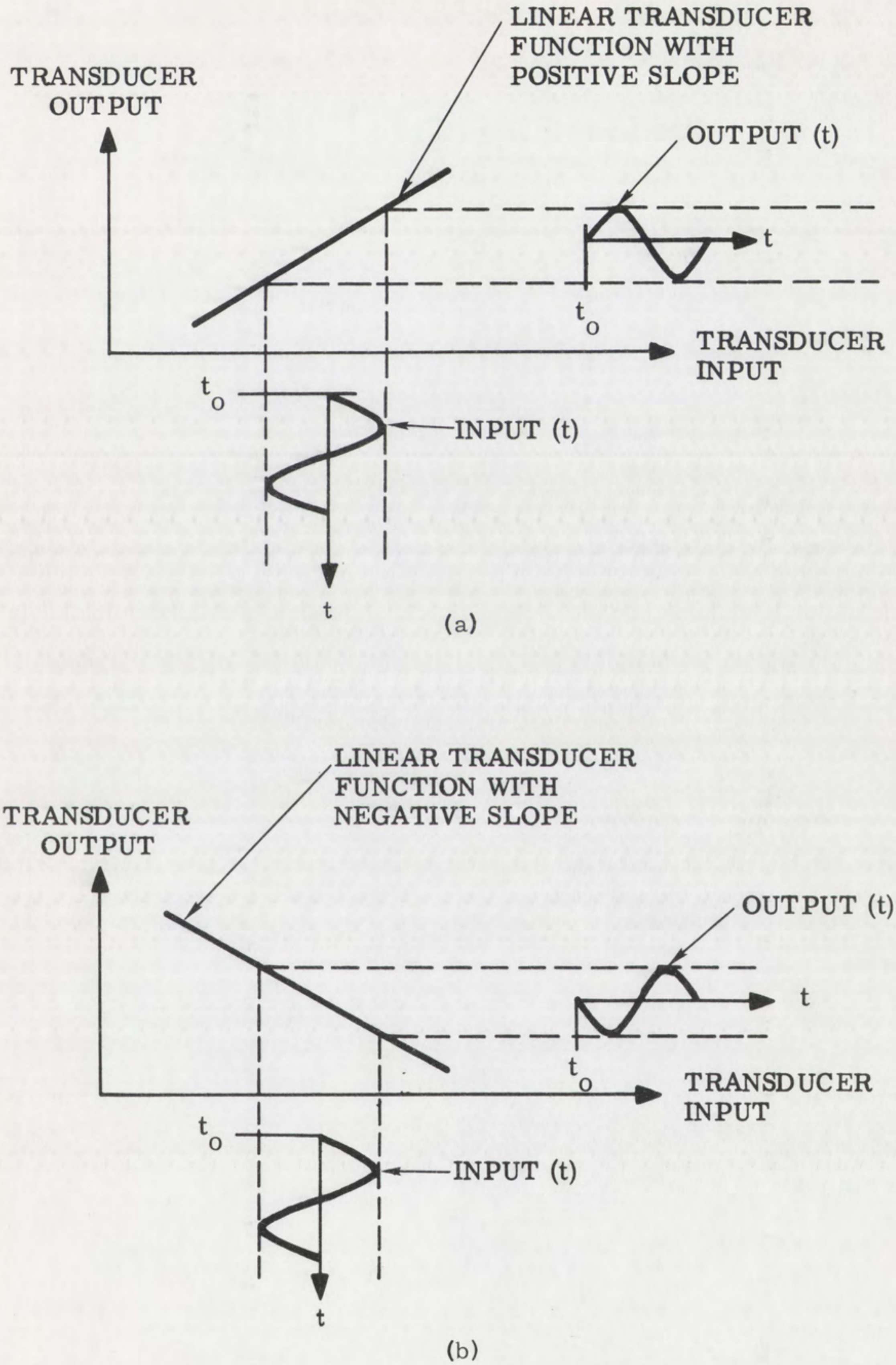

Figure 5. Slope Characteristics of Linear Transducer Functions

of the transducer with a positive slope will produce an output component with the same frequency and phase as the periodic input. Also, the slope's magnitude is the amplitude ratio of the output to the input; i. e., the greater or smaller the slope, the greater or smaller, respectively, the output for a constant input amplitude. When the slope has the same magnitude but a negative sign as in Figure 5b, the output has the same amplitude as before but is 180 degrees out of phase with respect to the input. To summarize the foregoing: the transducer's slope can be determined by inducing a small periodic variation centered on a particular value to be measured (transducer input). The measurement of the slope at that value is contained in the amplitude and phase of the output's component having the identical fundamental period as that of the small perturbation at the input.

This technique of acquiring the transducer's slope is not restricted to linear devices. If the amount of perturbation induced is sufficiently small so that the function is approximately linear over the variation, intuitively, the technique should be applicable to many functional forms with bounded first derivative. That the technique is definitely applicable to circular functions will be proven in another section of this paper. The supporting mathematical theory to be given later shows that, for circular functions, the perturbation need not be restricted to an approximately linear region of the response.

APPLICATION OF TECHNIQUE TO INTERFEROMETERS

The technique, discussed earlier, for obtaining the quadrature information by determining the slope is readily applicable to interferometers. For the sake of simplicity, only optical interferometers will be discussed with the understanding that other interferometers may be treated in the same manner. The parameter of interest in an optical interferometer is the optical pathlength of the measurement beam. The hypothetical transducer response or output function introduced in Figure 1 may easily be identified as that of an optical interferometer in which x is the optical pathlength of the measurement beam and $F(x)$ is the intensity of the light at the photo detector. The peaks of the curve are caused by constructive phase addition of light (interference), while the troughs are created by destructive interference. The difference in optical pathlength required to change the output from a peak to a trough is one-half wavelength of the monochromatic source. This optical pathlength is determined by the physical pathlength and the refractive index of the medium comprising the path. The optical interferometer may be used to measure the change in either of these physical properties. A periodic variation of either property may be used to implement the quadrature system since the effect is the same in either case.

A variation of the index of refraction in any portion of the optical path can be easily produced by the electro-optic Pockel's Effect or by actually changing the density of a region. The Pockel's Effect (index of refraction a linear function of electric field) may be utilized by inserting a potassium dihydrogen phosphate (KDP) crystal [6] or similar material into a portion of the optical path and applying a periodic voltage to it. The variation of density may be achieved by placing a confined quantity of compressible gas in the optical path and applying a periodic pressure.

The physical pathlengths of an optical interferometer are easily varied by mechanical means. Motion along the optical path of a surface which reflects the beam introduces a proportional variation of the optical distance to be measured. In some situations, it may be more convenient to vibrate a reflector in the reference beam instead. This vibration causes a change in the relative phase between the reference and measurement beams, resulting in an apparent variation of the measurement beam pathlength.

Another way to obtain a change in the phase difference between the reference and measurement beams is through motion of a beam splitter common to both. This motion should be perpendicular to the plane of the beam splitter. As a result, the motion changes the length of the transmitted beam while holding the reflected beam's length constant.

The methods of inducing either an effective or real variation in the physical pathlength of the measurement beam are applicable, with certain restrictions, to the Laser Feedback Interferometer. These restrictions center around the fact that the reference beam has zero optical pathlength. Only a reference phase exists in the light reflected back into the laser cavity from the laser output mirror. Direct variation of the measurement may be achieved by three means, each caused by a motion parallel to the optical path of:

1. The external mirror
2. The entire laser cavity
3. The laser output mirror

Index of refraction changes may also be induced to vary the measurement beam pathlength. If changes in the wavelength can be tolerated, the optical properties of the laser cavity may be varied. Thus the reference phase may be changed by vibration of either, or both, of the reflectors. Likewise, the refractive index of a region within the cavity may be perturbed. A change in the laser's wavelength of operation may also cause a change in the laser's output intensity [7]. This change in intensity must be very small or the electronic signal from the interferometer will contain an appreciable error.

MATHEMATICAL THEORY OF SINUSOIDAL FUNCTION TRANSDUCER SLOPE DETECTION

Consider the transducer response curve (sinusoidal function) and the induced variation ($\Delta x \cos 2\pi f_1 t$) about the input value $x = x_1$ as illustrated in Figure 6.

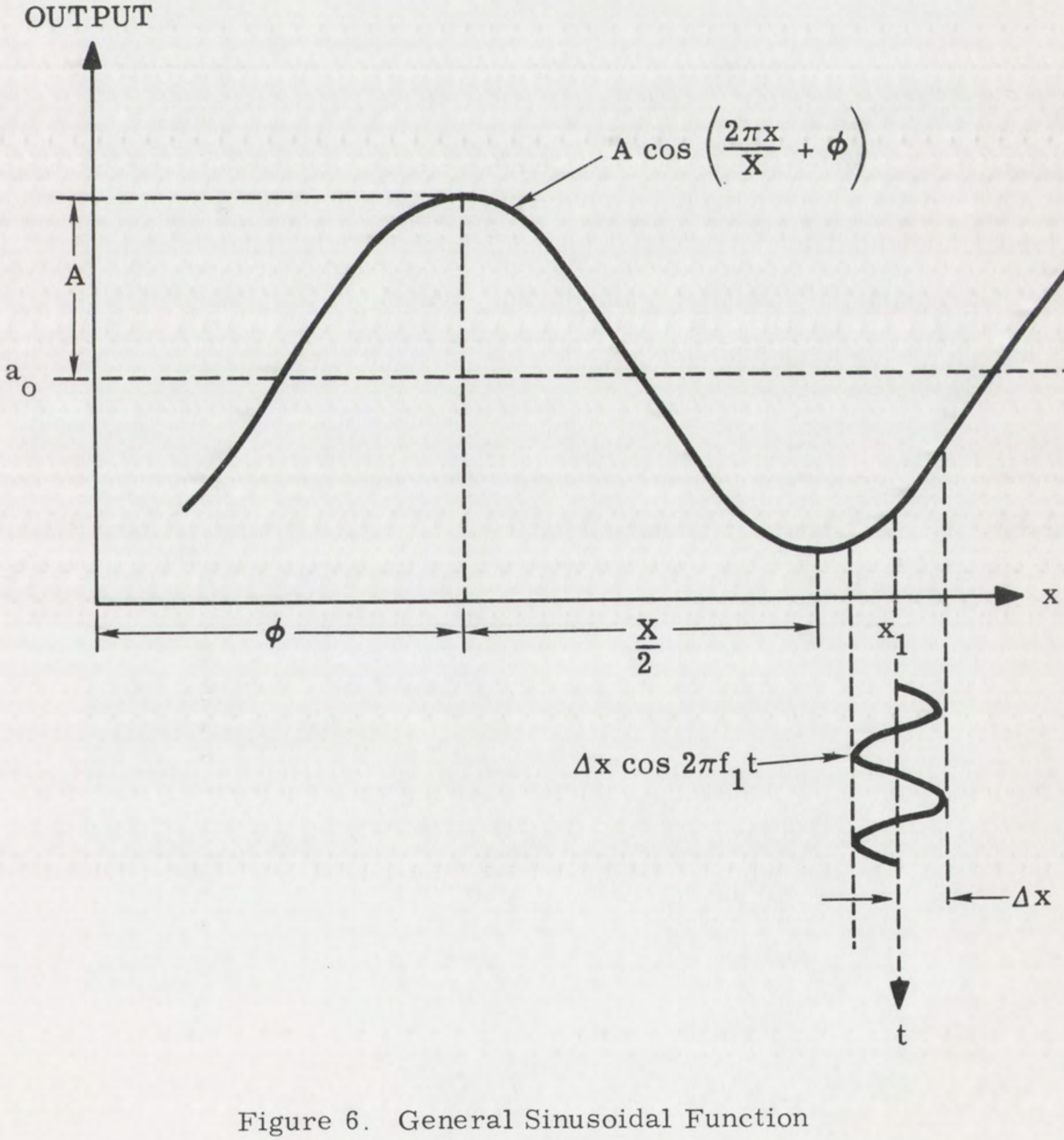

Figure 6. General Sinusoidal Function

The output of the transducer as a function of x and time (t) is in general:

$$F(x, t) = a_0 + A \cos \left(\frac{2\pi x}{X} + \phi + \frac{2\pi \Delta x}{X} \cos 2\pi f_1 t\right).$$

This is the classical phase modulation equation of communication theory [8].

Rewriting in exponential form:

$$F(x, t) = a_0 + \mathscr{Re}\, A \left[e^{i\left(\frac{2\pi x}{X} + \phi + \frac{2\pi\Delta x}{X} \cos 2\pi f_1 t\right)} \right]$$

or

$$F(x, t) = a_0 + \mathscr{Re}\, A \left[e^{i\left(\frac{2\pi x}{X} + \phi\right)} e^{i\left(\frac{2\pi\Delta x}{X} \cos 2\pi f_1 t\right)} \right].$$

An expansion of $e^{i\left(\frac{2\pi\Delta x}{X} \cos 2\pi f_1 t\right)}$ in a Fourier Series may now be made and results in (see [8], page 1078, Equation A. 1. 48a):

$$e^{i\left(\frac{2\pi\Delta x}{X} \cos 2\pi f_1 t\right)} = \sum_{m=0}^{\infty} i^m \epsilon_m J_m\left(\frac{2\pi\Delta x}{X}\right) \cos (m 2\pi f_1 t).$$

where ϵ_m is the Neumann factor:

$$\epsilon_0 \equiv 1, \quad \epsilon_m \equiv 2 \text{ for } m \neq 0$$

and $J_m\left(\frac{2\pi\Delta x}{X}\right)$ is the Bessel function of the first kind.

Substituting this result back into F(x, t)

$$F(x, t) = a_0 + \mathscr{Re}\, A \left[e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=0}^{\infty} i^m \epsilon_m J_m\left(\frac{2\pi\Delta x}{X}\right) \cos (m 2\pi f_1 t) \right].$$

Writing out the first two terms of the summation separately

$$F(x,t) = a_0 + \mathcal{R}e\, A \left[e^{i\left(\frac{2\pi x}{X} + \phi\right)} J_0\left(\frac{2\pi \Delta x}{X}\right) \right.$$

$$+ 2i\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} J_1\left(\frac{2\pi \Delta x}{X}\right) \cos 2\pi f_1 t$$

$$\left. + 2\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi \Delta x}{X}\right) \cos m2\pi f_1 t \right].$$

Expanding the complex exponentials of the first terms:

$$F(x,t) = a_0 + \mathcal{R}e\, A \left[\left\{ \cos\left(\frac{2\pi x}{X} + \phi\right) + i \sin\left(\frac{2\pi x}{X} + \phi\right) \right\} J_0\left(\frac{2\pi \Delta x}{X}\right) \right.$$

$$+ 2i \left\{ \cos\left(\frac{2\pi x}{X} + \phi\right) + i \sin\left(\frac{2\pi x}{X} + \phi\right) \right\} J_1\left(\frac{2\pi \Delta x}{X}\right) \cos 2\pi f_1 t$$

$$\left. + 2\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi \Delta x}{X}\right) \cos m2\pi f_1 t \right].$$

Taking the real components of the first two terms within the brackets:

$$F(x,t) = a_0 + A\, J_0\left(\frac{2\pi \Delta x}{X}\right) \cos\left(\frac{2\pi x}{X} + \phi\right)$$

$$- 2AJ_1\left(\frac{2\pi \Delta x}{X}\right) \sin\left(\frac{2\pi x}{X} + \phi\right) \cos 2\pi f_1 t$$

$$+ \mathcal{R}e \left[2A\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi \Delta x}{X}\right) \cos m2\pi f_1 t \right]. \qquad (1)$$

For convenience:

$$F_1(x) \equiv a_0 + AJ_0\left(\frac{2\pi\Delta x}{X}\right) \cos\left(\frac{2\pi x}{X} + \phi\right)$$

$$F_2(x, t) \equiv -2AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left(\frac{2\pi x}{X} + \phi\right) \cos 2\pi f_1 t$$

$$F_3(x, t) \equiv \mathcal{R}e\, A\left[2\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi\Delta x}{X}\right) \cos m2\pi f_1 t\right]$$

so that

$$F(x, t) = F_1(x) + F_2(x, t) + F_3(x, t).$$

Since Δx will be a constant, $J_0\left(\frac{2\pi\Delta x}{X}\right)$ and $J_1\left(\frac{2\pi\Delta x}{X}\right)$ will be constant for all x and t. Thus $F_1(x)$ is the same function as the original transducer function for the parameter x, except for a different amplitude in the $\cos\left(\frac{2\pi x}{X} + \phi\right)$ term.

Examining $F_2(x, t)$, it is seen that the amplitude of the f_1 frequency component is

$$-2J_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left(\frac{2\pi x}{X} + \phi\right) \equiv F_4(x)$$

which is in quadrature with

$$F_1(x) = a_0 + AJ_0\left(\frac{2\pi\Delta x}{X}\right) \cos\left(\frac{2\pi x}{X} + \phi\right)$$

for the parameter x. These two signals, $F_1(x)$ and $F_4(x)$, when considered together uniquely define every point on the transducer curve within a range of X.

If Δx is small, such that $\frac{2\pi \Delta x}{X} \leq 0.1$ radians, the following approximations [9] may be applied:

$$J_p(x) \approx \frac{1}{2^p p!} x^p .$$

Therefore:

$$F_1(x) \approx a_0 + A \cos\left(\frac{2\pi x}{X} + \phi\right) , \text{ (The original transducer function);}$$

$$F_4(x) \approx A \frac{2\pi \Delta x}{X} \sin\left(\frac{2\pi x}{X} + \phi\right) .$$

Otherwise, the size of Δx must be carefully selected to avoid values which will cause either $J_0\left(\frac{2\pi \Delta x}{X}\right)$ or $J_1\left(\frac{2\pi \Delta x}{X}\right)$ to be zero and thus nullify one of the quadrature signals. Also Δx may be chosen to maximize one of the two quadrature signals as when $\Delta x \approx 0.295$ X produces the maximum amplitude for the $\sin\left(\frac{2\pi x}{X} + \phi\right)$ term. A value of $\Delta x = 0$ maximizes the amplitude of the $\cos\left(\frac{2\pi x}{X} + \phi\right)$ term but results in $J_1\left(\frac{2\pi \Delta x}{X}\right) \equiv 0$ thus making the total $\sin\left(\frac{2\pi x}{X} + \phi\right)$ term to be zero. If $\Delta x \approx 0.142$ X, the amplitudes of the two quadrature signals are equal.

Another consideration in selecting a value for Δx may be the reduction of the amplitudes of the second and higher harmonics of f_1. This is accomplished by making $\frac{2\pi \Delta x}{X}$ either very small or the root of a particular Bessel function. The required processing of these harmonics will become apparent when F(x, t) is applied to an electronic mixer with the other input being $\cos 2\pi f_1 t$. The output of this mixer is:

$$F_5(x,t) \equiv F(x,t)\cos 2\pi f_1 t \equiv [F_1(x) + F_2(x,t) + F_3(x,t)]\cos 2\pi f_1 t$$

$$= a_0 \cos 2\pi f_1 t + AJ_0\left(\frac{2\pi\Delta x}{X}\right)\cos\left(\frac{2\pi x}{X} + \phi\right)\cos 2\pi f_1 t$$

$$-2AJ_1\left(\frac{2\pi\Delta x}{X}\right)\sin\left(\frac{2\pi x}{X} + \phi\right)\cos 2\pi f_1 t \cos 2\pi f_1 t$$

$$+ \mathcal{R}e\, A\left[2\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi\Delta x}{X}\right)\right.$$

$$\left. \cdot \cos m2\pi f_1 t \cos 2\pi f_1 t\right] . \qquad (2)$$

The term of primary interest is

$$F_2(x,t)\cos 2\pi f_1 t = -2AJ_1\left(\frac{2\pi\Delta x}{X}\right)\sin\left(\frac{2\pi x}{X} + \phi\right)\cos 2\pi f_1 t \cos 2\pi f_1 t$$

By trignometric identity:

$$F_2(x,t)\cos 2\pi f_1 t = -AJ_1\left(\frac{2\pi\Delta x}{X}\right)\sin\left(\frac{2\pi x}{X} + \phi\right)\left[\cos 4\pi f_1 t + \cos(0)\right]$$

$$= -AJ_1\left(\frac{2\pi\Delta x}{X}\right)\sin\left(\frac{2\pi x}{X} + \phi\right)$$

$$-AJ_1\left(\frac{2\pi\Delta x}{X}\right)\sin\left(\frac{2\pi x}{X} + \phi\right)\cos 4\pi f_1 t . \qquad (3)$$

This contains the required information plus an undesired second harmonic term. The first two terms and the second harmonic term of Equation 2 produce undesirable outputs at the fundamental frequency. These frequencies may be removed by passing the mixer output through a low pass filter. This filter should be designed with a cutoff frequency below f_1

and attenuation characteristics at f_1 sufficient to reduce the fundamental frequency terms to an acceptable value. The resulting question is, "What is the optimum cutoff frequency in terms of system bandwidth?" If x is not a function of time the cutoff frequency could be anywhere below f_1. However, x is generally a function of time (x = x(t)) and this sets the necessary bandwidth of the system as well as the fundamental frequency f_1.

For the purposes of illustration: let $x = x(t) = X\left[\frac{1}{2} f_1 + \Delta f\right] t$ and when inserted into Equation 2,

$$
\begin{aligned}
F_5(t) &= F(t) \cos 2\pi f_1 t = F[x(t), t] \cos 2\pi f_1 t \\
&= a_0 \cos 2\pi f_1 t + AJ_0\left(\frac{2\pi\Delta x}{X}\right) \cos(\pi f_1 t + 2\pi\Delta f t + \phi) \cos 2\pi f_1 t \\
&\quad - 2AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin(\pi f_1 t + 2\pi\Delta f t + \phi) \cos 2\pi f_1 t \cos 2\pi f_1 t \\
&\quad + \mathcal{R}e\, A\left[2\, e^{i[2\pi t(1/2\, f_1 + \Delta f) + \phi]} \sum_{m=2}^{\infty} i^m J_m\left(\frac{2\pi\Delta x}{X}\right)\right. \\
&\quad \left. \cdot \cos m 2\pi f_1 t \cos 2\pi f_1 t\right] .
\end{aligned}
\tag{4}
$$

Expanding the third term,

$$
\begin{aligned}
F_2(t) \cos 2\pi f_1 t &= -2AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left[2\pi t\left(\frac{1}{2} f_1 + \Delta f\right) + \phi\right] \cos 2\pi f_1 t \cos 2\pi f_1 t \\
&= -AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left[2\pi t\left(\frac{1}{2} f_1 + \Delta f\right) + \phi\right] \\
&\quad - AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left[2\pi t\left(\frac{1}{2} f_1 + \Delta f\right) + \phi\right] \cos 2(2\pi f_1 t) .
\end{aligned}
$$

The first term of this equation is the quadrature information and has a frequency $\frac{1}{2} f_1 + \Delta f$. The low pass filter should have a cutoff frequency above $\frac{1}{2} f_1 + \Delta f$ in order for this information to be passed. However, if the second term of Equation 4 is expanded

$$AJ_0\left(\frac{2\pi\Delta x}{X}\right) \cos(\pi f_1 t + 2\pi\Delta f t + \phi) \cos 2\pi f_1 t$$

$$= \frac{1}{2} AJ_0\left(\frac{2\pi\Delta x}{X}\right) \left\{ \cos(3\pi f_1 t + 2\pi\Delta f t + \phi) + \cos\left[- \pi f_1 t + 2\pi\Delta f t + \phi\right]\right\}$$

this term makes a contribution at a frequency of $\frac{1}{2} f_1 - \Delta f$ which is lower in frequency than the quadrature signal at $\frac{1}{2} f_1 + \Delta f$. A low pass filter with the cutoff frequency mentioned will pass both of these frequencies and thus give an erroneous total output. To prevent this situation from occurring, f_1 should be more than twice the largest value of the time derivative of $\frac{x(t)}{X}$. The low pass filter should now have a cutoff frequency of just above $\frac{1}{2} f_1$ for maximum usable bandwidth.

Another approach to the removal of undesired information in the mixer output is to pass the raw signal through a band pass filter prior to the mixer. This filter should pass frequencies around f_1 with cutoffs between f_1 and zero frequency and between f_1 and $2f_1$. This removes the zero frequency and all harmonic terms from Equation 1, leaving

$$F'(x,t) = -2AJ_1\left(\frac{2\pi\Delta x}{X}\right)\ \sin\left(\frac{2\pi x}{X} + \phi\right)\ \cos 2\pi f_1 t = F_2(x,t)$$

If x is now a function of time, $x = x(t)$; particularly if $x(t) = Xt\left[\frac{1}{2}f_1 + \Delta f\right]$, $F_2(x,t)$ becomes

$$F_2(x,t) = -AJ_1\left(\frac{2\pi\Delta x}{X}\right)\ \Big[\sin(3\pi f_1 t + 2\pi\Delta f t + \phi)$$

$$+ \sin(-\pi f_1 t + 2\pi\Delta f t + \phi)\Big].$$

Since this function, $F_2(x,t)$, is desired information, the band pass should be broad enough to encompass the frequency range from $\frac{1}{2}f_1 - \Delta f$ to $\frac{3}{2}f_1 + \Delta f$. However, expansion of the terms of $F(x,t)$ for $x(t) = Xt\left(\frac{1}{2}f_1 + \Delta f\right)$ shows that $AJ_0\left(\frac{2\pi\Delta x}{X}\right)\cos\left(\frac{2\pi x(t)}{X} + \phi\right)$ contributes at a frequency $\frac{1}{2}f_1 + \Delta f$ which is above the lower cutoff of the filter discussed so far. In addition, $-2AJ_0\left(\frac{2\pi\Delta x}{X}\right)\cos\left(\frac{2\pi x(t)}{X} + \phi\right)\cos 2(2\pi f_1 t)$ contributes at a frequency of $3/2\ f_1 - \Delta f$, below the upper cutoff frequency of the filter. These two frequencies contain information which is not desirable at the mixer input but is within the frequency spectrum of the desired information. Preventing overlap of the desired information spectrum and the undesired information spectrum while maintaining maximum usable bandwidth is necessary. The solution is the same as before: select f_1 to be more than twice the largest value of $\frac{d}{dt}\left[\frac{x(t)}{X}\right]$. The maximum value for the time derivative of $\frac{x(t)}{X}$ is the maximum frequency in the output of the normal circular transducer. This result is interesting because it is the same as the minimum sampling frequency required to reconstruct a sampled function. Then the filter band pass can be from $1/2\ f_1$ to $3/2\ f_1$. With the characteristics of the band pass filter (which is located ahead of the mixer) determined, attention is turned to the output of the mixer, as given by

$$
\begin{aligned}
F'_5(x, t) &= F_2(x, t) \cos 2\pi f_1 t \\
&= -2AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left(\frac{2\pi x}{X} + \phi\right) \cos 2\pi f_1 t \cos 2\pi f_1 t \\
&= -AJ_1\left(\frac{2\pi\Delta x}{X}\right) \sin\left(\frac{2\pi x}{X} + \phi\right) \left[1 + \cos 2(2\pi f_1 t)\right] .
\end{aligned}
$$

The second harmonic term is not desired in the output and may be removed by a low pass filter. The bandwidth of this filter is readily apparent, since the highest time frequency variation of x(t) is $\frac{1}{2} f_1 - \delta f$. The bandwidth of low pass should be larger than $\frac{1}{2} f_1 - \delta f$, such as $\frac{1}{2} f_1$.

Now that the frequency characteristics of the "quadrature" signal processing system have been determined, the necessary frequency characteristics for processing the original signal appear quite readily. As previously shown in Equation 1, the original signal is

$$
F_1(x) = a_0 + AJ_0\left(\frac{2\pi\Delta x}{X}\right) \cos\left(\frac{2\pi x(t)}{X} + \phi\right) .
$$

This signal may be easily separated from F(x, t) by means of a low pass filter with a cutoff frequency below f_1. The required bandwidth of this filter is from f = 0 to above the highest frequency in the original information signal (chosen in the development of the quadrature signal to be $1/2\, f_1 - \delta f$. The suggested band pass is $f = 0$ to $f = 1/2\, f_1$.

While δf has not been specified to this point, it has significance in a practical application. From a theoretical standpoint, δf may be zero, since in theory all filters are flat in frequency response and have "skirts" with infinite slope. Such a filter is highly desirable, since its effects on all signals at pass band frequencies are the same. However, such a filter is not possible in a practical application. For this reason δf is chosen so that the filter is as flat as required at all information frequencies, while the cutoff frequencies are $1/2\, f_1$ and $3/2\, f_1$ in the various filters discussed previously. The slopes of the filter skirts should, of course, be great enough to provide adequate attenuation of signals at undesired frequencies.

The results of the preceding discussion may now be summarized for convenience. The maximum frequency generated by x, $\frac{d}{dt}\left[\frac{x(t)}{X}\right]\Big|_{max}$, dictates the required theoretical bandwidth of the system and the operating frequency f_1 and results in

1. $f_1 \geq 2 \left\{ \frac{d}{dt} \left[\frac{x(t)}{X}\right] \Big|_{max} \right\}$

2. $\left\{ \frac{d}{dt} \left[\frac{x(t)}{X}\right] \Big|_{max} \right\}$ < Low pass cutoff frequency.

3. $f_1 - \left\{ \frac{d}{dt} \left[\frac{x(t)}{X} \right] \bigg|_{max} \right\} >$ Band pass lower cutoff frequency.

$1/2\ f_1 \leq$ Band pass lower cutoff frequency.

4. $f_1 + \left\{ \frac{d}{dt} \left[\frac{x(t)}{X} \right] \bigg|_{max} \right\} <$ Band pass upper cutoff frequency.

$3/2\ f_1 \geq$ Band pass upper cutoff frequency.

The only restrictions on the amplitude (Δx) of the induced variation in x are that $\frac{2\pi\Delta x}{X}$ does not have a value equal to any of the roots of the zeroth and first order Bessel function of the first kind.

Throughout the preceeding discussion, the periodic variation has been of the stated form $\Delta x \cos 2\pi f_1 t$. The form $\Delta x \sin 2\pi f_1 t$ may be used just as well since the only difference is the reference point in time. Other periodic but nonsinusoidal perturbation signals may yield the desired quadrature signals. The analysis for these signals, however, is quite complicated as can be quickly shown. Let the perturbation signal be g(t) with a fundamental period, T. The Fourier Series expansion for g(t) is then:

$$g(t) = \sum_{n=0}^{\infty} (a_n \cos 2\pi f_n t + b_n \sin 2\pi f_n t); \text{ where } f_n = \frac{n}{T}$$

or

$$g(t) = \sum_{n=0}^{\infty} A_n \cos (2\pi f_n t + \beta_n); \text{ where } \beta_n = \tan^{-1} \left(- \frac{b_n}{a_n} \right)$$

and

$$A_n = \frac{a_n}{\cos \beta_n} = - \frac{b_n}{\sin \beta_n} .$$

The output of the sinusoidal transducer, with the application of this perturbation signal, is:

$$F(x, t) = a_o + A \cos\left[\frac{2\pi x}{X} + \phi + \frac{2\pi}{X} g(t)\right]$$

$$= a_o + A \cos\left[\frac{2\pi x}{X} + \phi + \frac{2\pi}{X} \sum_{n=0}^{\infty} A_n \cos(2\pi f_n t + \beta_n)\right].$$

This may be written in complex notation as

$$F(x, t) = a_o + \mathcal{Re}\left\{A\, e^{i\left[\frac{2\pi x}{X} + \phi\right]} e^{i\left[\frac{2\pi}{X} \sum_{n=0}^{\infty} A_n \cos(2\pi f_n t + \beta_n)\right]}\right\}.$$

Expanding the form $e^{i a \cos\theta}$ as before [8]

$$e^{i\left[\frac{2\pi}{X} \sum_{n=0}^{\infty} A_n \cos(2\pi f_n t + \beta_n)\right]} = \prod_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\left[i^m \epsilon_m J_m\left(\frac{2\pi A_n}{X}\right) \cos m(2\pi f_n t + \beta_n)\right]\right\}.$$

Consequently,

$$F(x, t) = a_o + \mathcal{Re}\left\langle A\, e^{i\left(\frac{2\pi x}{X} + \phi\right)} \prod_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\left[i^m \epsilon_m J_m\left(\frac{2\pi A_n}{X}\right) \cos m(2\pi f_n t + \beta_n)\right]\right\}\right\rangle.$$

Many of the cross-product terms resulting from the product notation contribute signals at the perturbation's fundamental frequency. Therefore, the quadrature information (which previously was the amplitude of this frequency) is not intuitively apparent.

ENGINEERING IMPLEMENTATION OF TECHNIQUE

The essential components needed to develop quadrature information in a general situation are illustrated (inside the dotted line) in Figure 7.

The sinusoidal generator supplies the electrical waveform to perturb the parameter to be measured. This generator's frequency is chosen in accordance with the restrictions set forth by the mathematical model of the preceding section.

The block in Figure 7 labeled "linear transducer" is not described in great detail since there will generally be a different type of transducer for each different application. The function of the transducer in every situation is to convert the sinusoidal electrical waveform at its input into a similar periodic variation of the parameter to be measured.

Once the periodic variation of the parameter is present, the output of the circular transducer contains the original signal $\left(\cos\left(\frac{2\pi x(t)}{X} + \phi\right)\right)$ plus a signal at the fundamental frequency of the variation. In addition, harmonics of the fundamental frequency are also contained in the transducer's output. All perturbation induced frequencies are removed when

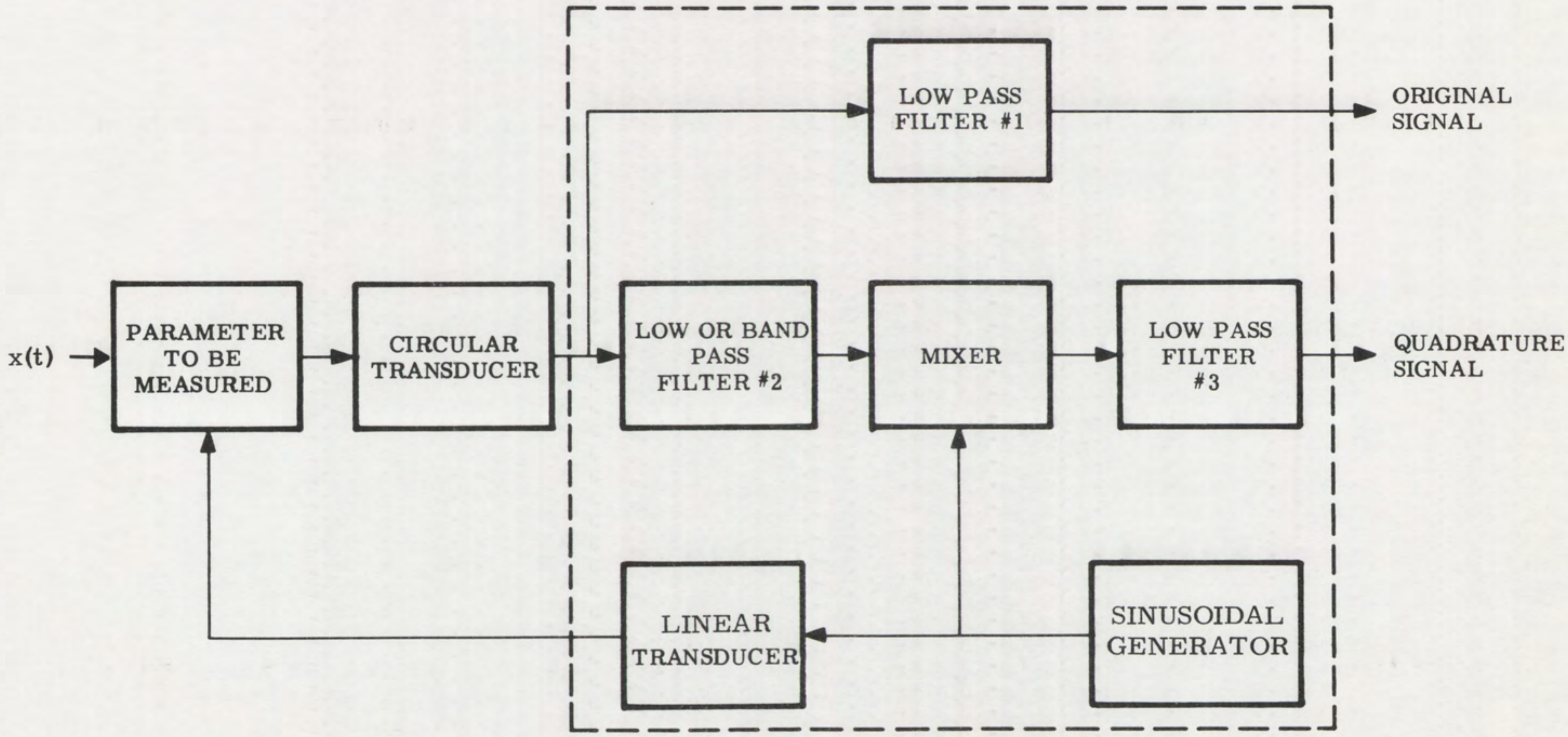

Figure 7. Electronic Block Diagram for Implementation of Resolution Technique

this output is passed through an appropriate low pass filter (Low Pass Filter #1). Such a filter should have a cutoff frequency no greater than one half the perturbation's frequency. The output of this filter is the original signal (as shown in the mathematical model) and is so indicated in Figure 7.

The desired quadrature information is the amplitude (with sign) of the output's component at the perturbation frequency. The output of the circular transducer also passes through a different filter (Low Pass Filter #2) before it reaches the mixer (multiplier). The function of this filter is to attenuate all frequencies above the fundamental to prevent the appearance in the mixer's output of undesirable information. This type of information results from the mixing of the harmonics with the other input to the mixer.

The filter cutoff frequency required to achieve the necessary rejection has been shown to be three halves the fundamental frequency or less. The filter preceding the mixer may be a band pass with a lower cutoff at one half the fundamental frequency or greater. The band pass prevents the fundamental frequency from appearing in the mixer's output, thus reducing the requirements of the final low pass filter. If the band pass presents a high impedance to the original signal, the majority of this signal's power will appear in the output of Low Pass Filter #1.

The mixer detects the amplitude of the fundamental frequency component in the output. This detection is accomplished by multiplying the filtered output of the circular transducer with the output of the sinusoidal generator. The result is the sum and difference of the frequencies in the two inputs. The sum and difference of the fundamental frequency in both inputs is twice the frequency and zero, respectively. The amplitude of the fundamental then appears as identical amplitudes of both a zero frequency or "DC" signal and a signal at twice the fundamental frequency.

The low pass filter following the mixer (Low Pass Filter #3) passes the portion of the mixer output near zero frequency. The desired quadrature information (amplitude of the fundamental before the mixer) is in this frequency range. The desirable cutoff frequency for this filter, as shown previously, is one half the fundamental frequency or less.

In a practical application of this technique, some auxiliary equipment may be required. Amplifiers may be required at some signal processing points to maintain useful signal levels. Since the amplifiers required will in general be different in every case, they have not been indicated in Figure 7.

In some applications, a phase difference may exist between the fundamental frequencies of the perturbation as they appear at the two inputs to the mixer. This phase difference should be zero and if it is not, the quadrature signal amplitude will be reduced by a multiplying constant. This constant is the cosine of the phase difference which has a value of zero for a phase difference of ±90 degrees. To remove the phase difference, a phase shifter should be added between the periodic function generator and the mixer.

DEMONSTRATION OF WORKING SYSTEM

To verify the practicality of the technique presented in the preceding sections, an experiment was conducted. The purpose of the experiment was to demonstrate the resolution of the ambiguity in an actual circular transducer, the Laser Feedback Interferometer. A component layout of the experiment appears in Figure 8.

All the equipment indicated in Figure 8 is commercially available with the exception of the low pass filter and the displacement device. Each component and its function will be discussed in detail before the experiment is described.

The quartz crystal indicated in Figures 8 and 9 has a vapor-deposited layer of gold on both faces. The gold surface which is not attached to the displacement mechanism serves as the retro-reflector for the Laser Feedback Interferometer. The desired periodic variation of the optical path length is produced by applying the output of the audio oscillator (see Figure 8), through the voltage step-up transformer, to the two gold faces of the crystal. The application of a periodic voltage to the two faces causes a periodic contraction and expansion of the crystal in a direction normal to the two faces. One face is firmly attached to the massive translator hardware and the other face (the retro-reflector)

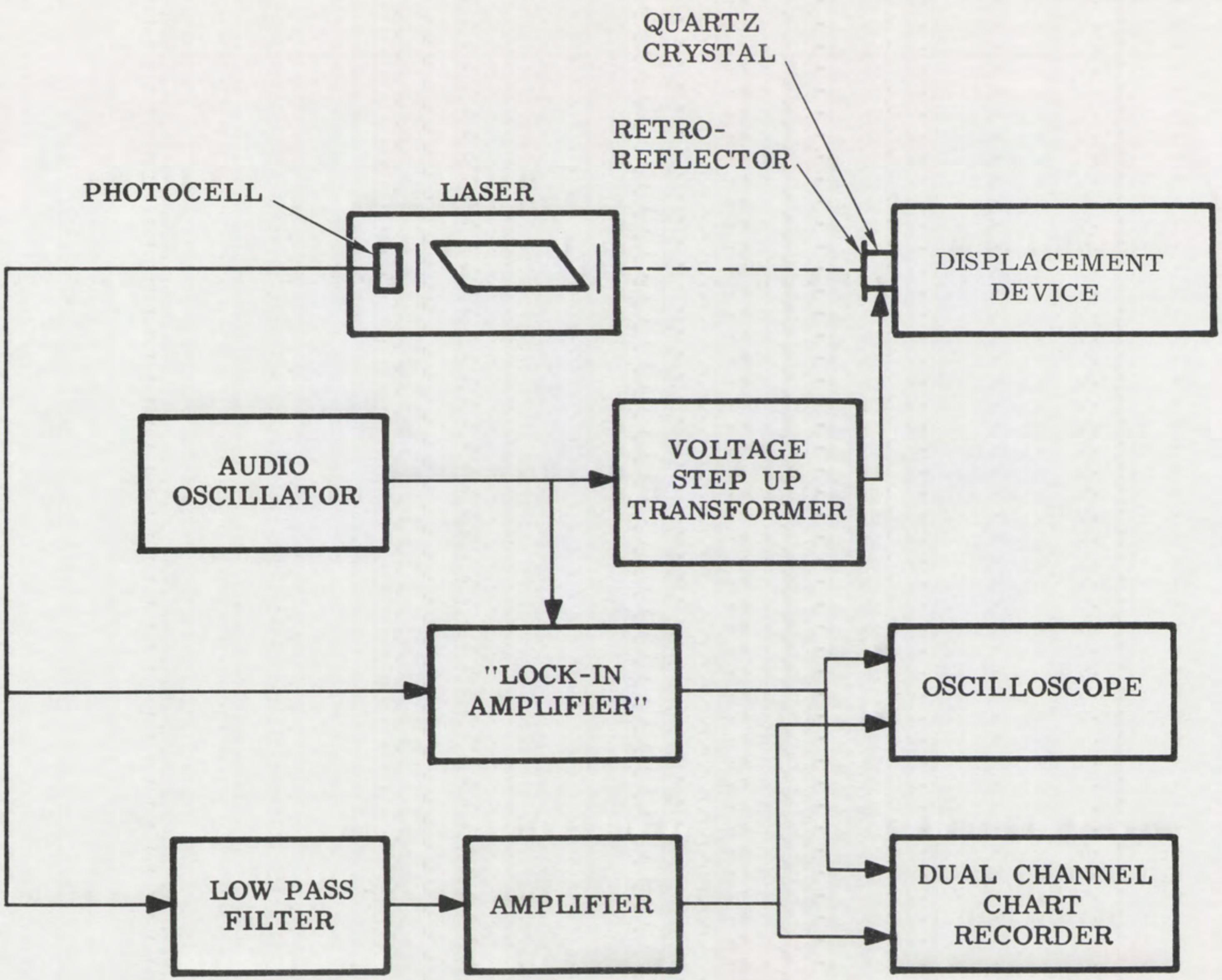

Figure 8. Equipment Block Diagram for Resolving Ambiguities of Laser Feedback Interferometer

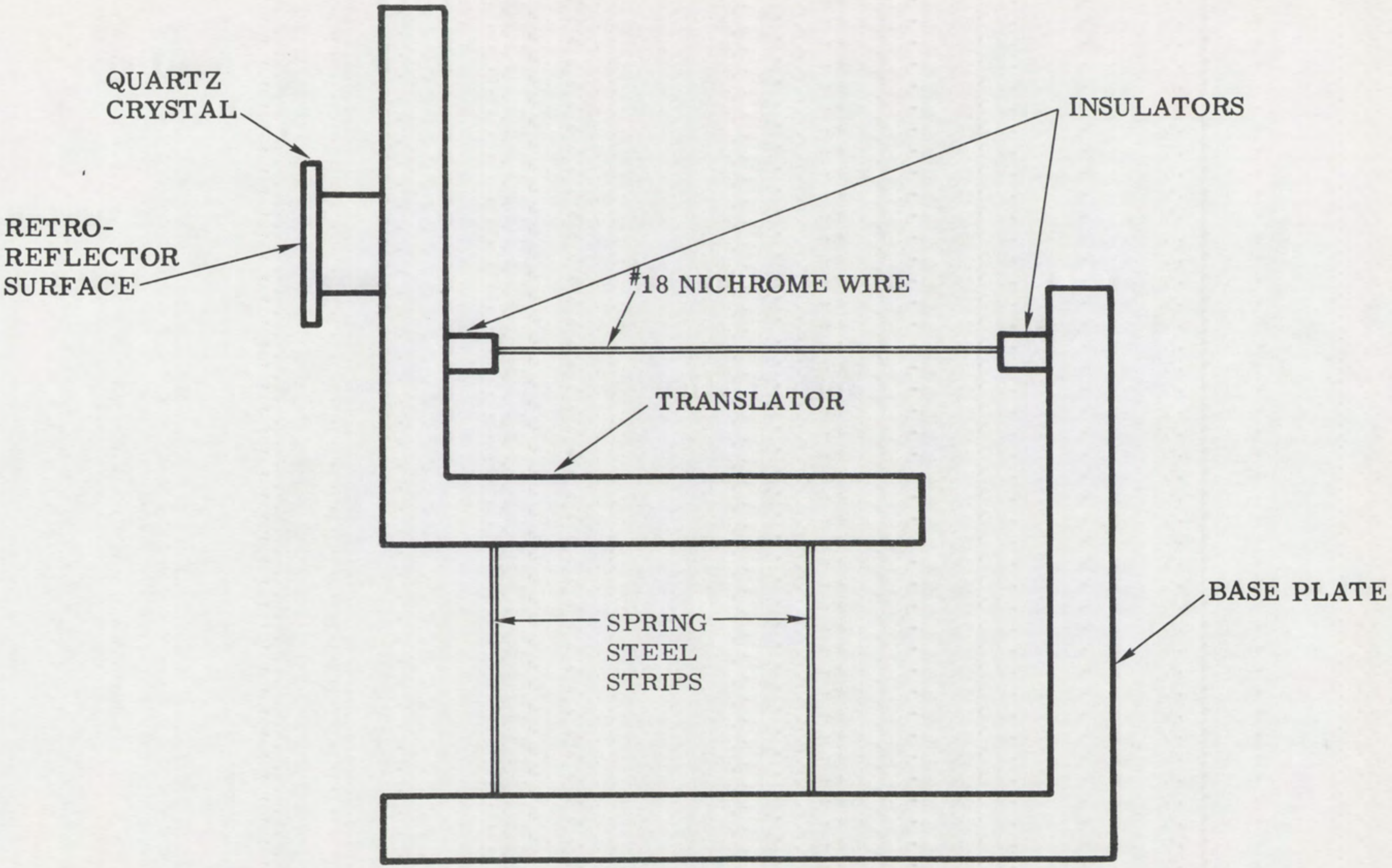

Figure 9. Schematic Drawing of Displacement Device

is free to move. The crystal's contraction and expension cause the retro-reflector to move periodically along the optical axis of the interferometer. The quartz crystal then becomes the transducer shown in Figure 7.

The changes in optical path length created by the vibrating retro-reflector and displacement motion appear in the laser as changes in light intensity. For the stability of the emission wavelength, a Spectra-Physics Model 119, c-w Neon-Helium gas laser was used, and adjusted to operate at the center of the Lamb dip [7]. This model laser has a photocell mounted behind the rear mirror of the laser cavity. The photocell senses the intensity of the laser from the amount of light that leaks through the rear cavity reflector. The laser, retro-reflector, and photocell form the circular function transducer of Figure 7. The distance between the laser output mirror and the retro-reflector is the parameter to be measured.

The displacement device was constructed to provide a small motion in a known direction along the optical axis of the Laser Feedback Interferometer. A schematic drawing of the displacement device and the retro-reflector mounted to it is given in Figure 9. The actual experimental device and retro-reflector are shown in the photograph in Figure 10.

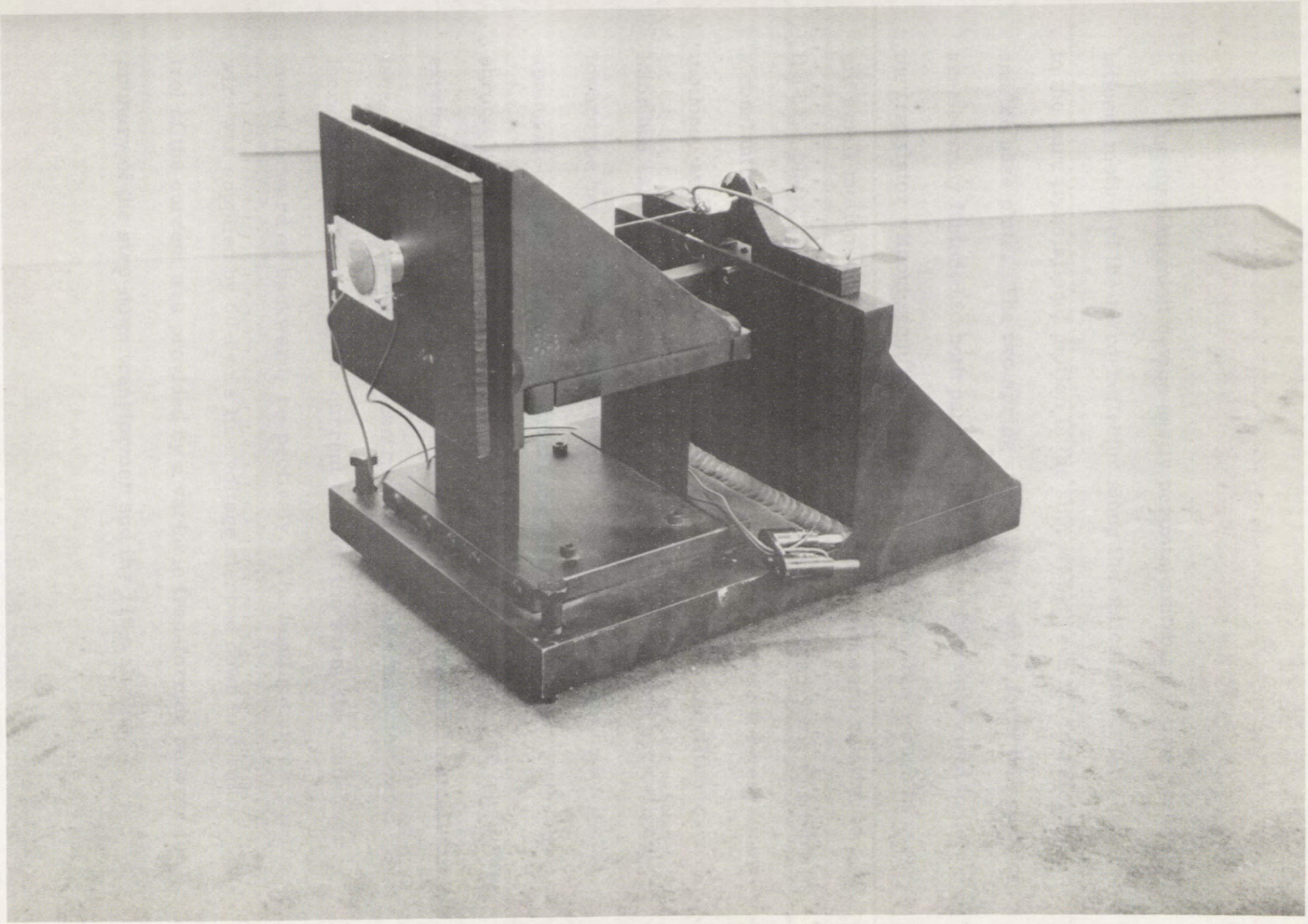

Figure 10. Translator and Retro-Reflector Assembly

The displacement device easily permits controllable pure translational motion of the retro-reflector along the optical path. The key to the pure translational motion is the support between the translator and the base plate. The two spring steel strips, which form this support, are securely fastened to both parts. They are sufficiently rigid to hold the translator without significant sagging but are flexible enough to permit small motions of the translator forward and backward relative to the base plate. The two metal strips constrain the translator to maintain a constant angular relation with the base. Consequently, for very small motions, as were required by this experiment, the motion of the translator has essentially no rotation. The translator is pulled back slightly toward the base plate so that the nichrome wire is held in tension by the spring steel strips. When electrical current is passed through the nichrome wire, the wire is heated and increases in length. The longitudinal expansion of the wire allows the translator to move forward a small distance. As the wire cools, its contraction pulls the translator back. By increasing or decreasing the current through the wire, the translator is moved forward or backward, respectively. The heating current for the wire was supplied by 60-cycle line voltage stepped down to 5 volts. Control of the current was provided by a variable transformer between the primary of the step-down transformer and the 110-volt line.

The electrical output of the photocell contains both the original and quadrature information. The quadrature information processing was carried out completely by a "lock-in amplifier," (Model HR-8) manufactured by Princeton Applied Research. The "lock-in amplifier" embodies the following components shown in Figure 7 and discussed in the preceding section:

1. Band pass filter prior to mixer
2. Mixer
3. Low pass filter following mixer
4. Phase shifter for the input from the oscillator to the mixer

The "lock-in amplifier" takes the composite signal from the photocell and detects the fundamental frequency's amplitude--the quadrature information. The output of the "lock-in amplifier" is displayed by applying it to the horizontal deflection plates of an oscilloscope (Tektronix Model 555) and one channel of the chart recorder (see Figure 8), a Sanborn Model 320.

The only remaining task is the extraction of the original information from the photocell. The original information is obtained by passing the photocell output through a low pass filter. In this case, a simple resistive-capacitive circuit with a cutoff frequency of 30 Hertz was used.

This cutoff frequency was selected to match the maximum cutoff frequency of the output low pass filter in the "lock-in amplifier." The frequency content of the original information was therefore required to be held below 30 Hertz during the experiment. The output of the low pass filter was amplified by a Tektronix Type D preamplifier in conjunction with a Tektronix Model 127 power supply. The output of the preamplifier was applied to the vertical deflection plates of the oscilloscope and the second channel of the chart recorder. The purpose of the oscilloscope was to provide a conventional quadrature system display. Such a display produced a circular trace each time the circular function passed through one complete cycle. A permanent record of the original and quadrature signals during the experiment was provided by the dual-channel chart record.

The purpose of the experiment was to verify the technique outlined previously. Consequently, the exclusion of other sources of variation in the optical data length was required. The two major sources of the undesired variations were vibrations of the optical components due to building and support vibrations and air turbulence along the optical path. To combat the vibration problem, the optical components, laser and retroreflector, were arranged on a granite surface plate (see Figure 11).

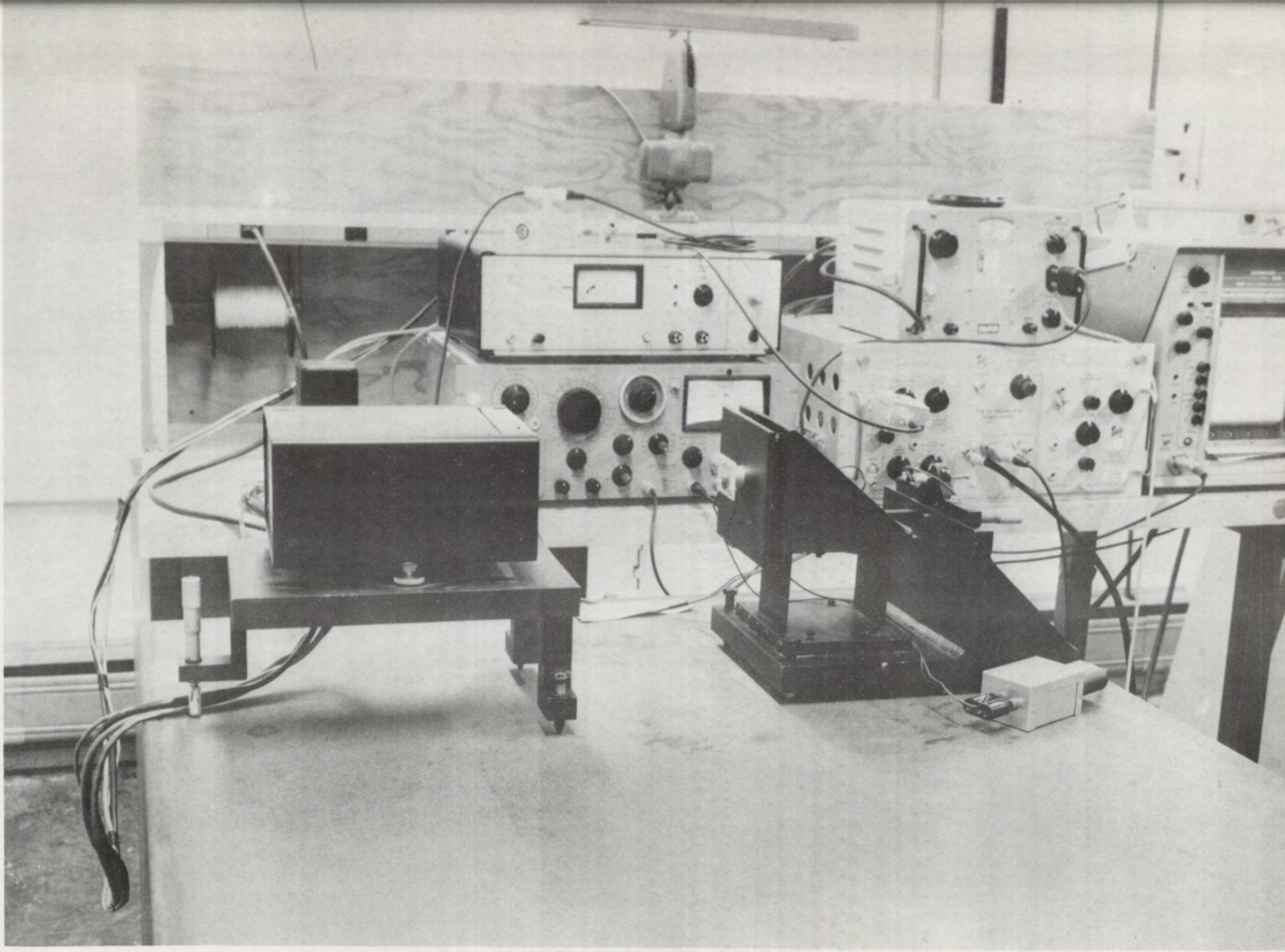

Figure 11. Optical Component Arrangement on Surface Plate

Vibration-absorbing material was located between the plate's supporting legs and the floor. The entire optical system was covered with a large transparent plastic box to eliminate the air turbulence of the room along the beam path (see Figure 12). All experiment components were connected as illustrated in Figure 8. The center of the "lock-in amplifier's" band pass filter was tuned to the frequency of the audio oscillator (400 Hertz).

By operating the displacement device, the elliptical Lissajous type display on the oscilloscope was used to align the retro-reflector. This method was employed because the amplitude of the original signal (ellipse vertical size) is a direct indication of the amount of feedback modulation occurring in the laser. The horizontal dimension was maximized by adjusting (within the "lock-in amplifier") the phase of the oscillator signal applied to the mixer. The gains of the horizontal and vertical amplifiers were adjusted to give a circular rather than elliptical trace as the translator was moved. To demonstrate qualitatively the technique's resolution ability, the original and quadrature signals were recorded and are shown in Figure 13. The chart speed was set to 5 mm per second, while the translator was alternately moved in one direction for a period of a few seconds and then in the other for a similar period.

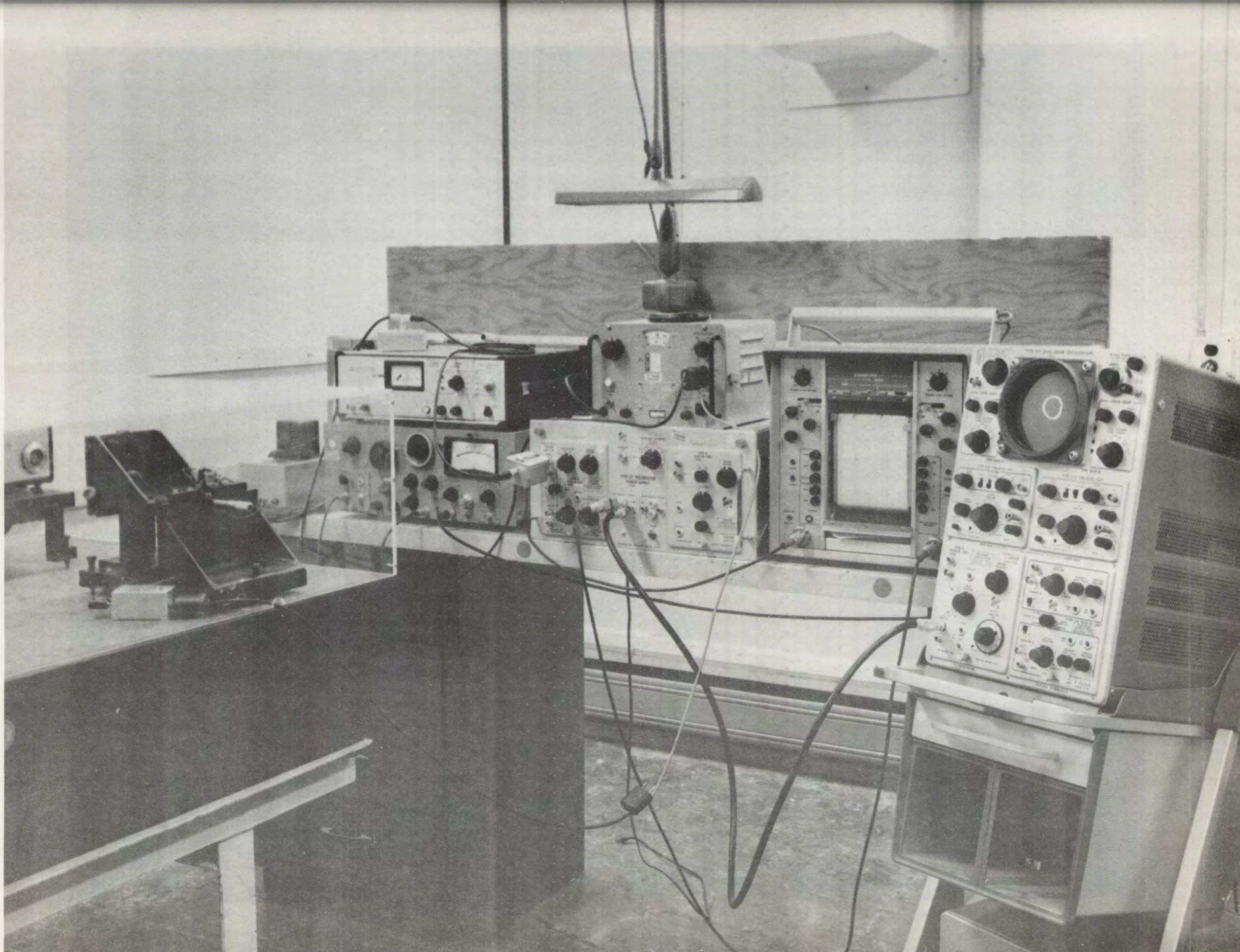

Figure 12. Complete Experiment Assembly

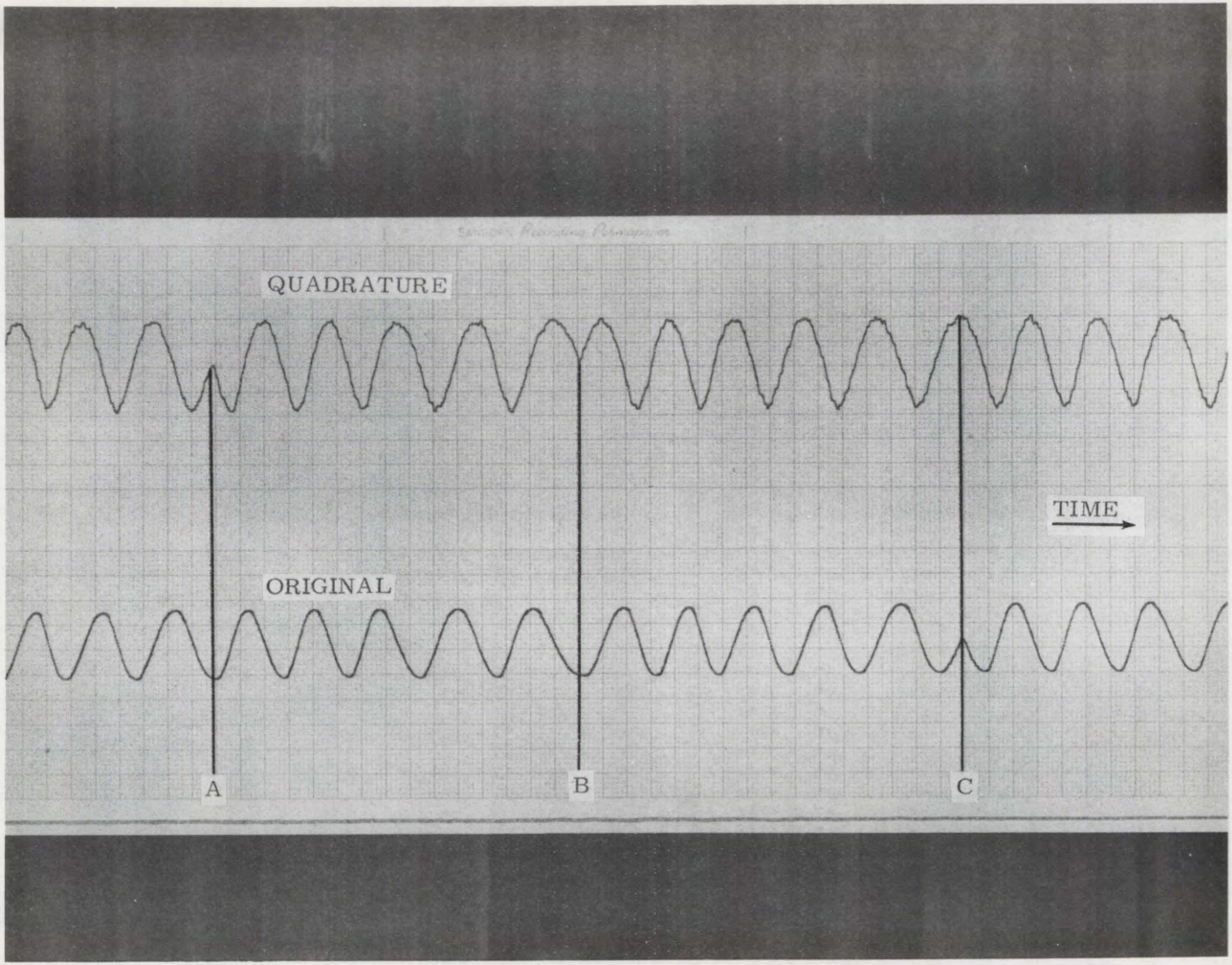

Figure 13. Experimental Record Demonstrating Resolution of Circular Function Ambiguity

Examination of Figure 13 shows that while the distance between the laser and retro-reflector is decreasing (prior to A) the quadrature signal is 90 degrees ahead of the original signal. Although the distance begins increasing at point A in the record, this change of direction is apparent in the quadrature signal but not in the original signal. After the distance begins increasing, the quadrature signal is 90 degrees behind the original signal. At point B the direction of the translator was reversed a second time. Again the change in direction is apparent in the quadrature information, but not in the original information. The quadrature signal leads the original signal by 90 degrees again after point B, as the distance decreases. The third reversal of direction, at point C, is apparent in the original signal but not in the quadrature signal. The oscilloscope spot motion during the period of the chart record was:

Prior to A - Counterclockwise

Between A and B - Clockwise

Between B and C - Counterclockwise

After C - Clockwise

Each point on the circular trace represents a unique input value to the circular transducer because one complete circle of the oscilloscope trace corresponds exactly to one cycle of the circular function. Based on the oscilloscope display and the chart record, the conclusion is easily reached that the technique does indeed remove the ambiguities of the circular transducer.

To substantiate quantitatively the mathematical analysis of the technique (which was presented in a previous section), data were taken on the amplitudes of the original and quadrature signals as well as the amplitude of the retro-reflector motion. Since the retro-reflector perturbation was caused by a periodic voltage applied to crystalline quartz, the amplitude of that motion is the piezoelectric constant d_{11} times the amplitude of the applied voltage. The value of d_{11} for quartz is reported by Bechmann [10] to be 2.31×10^{-12} meters/volt. The voltage applied to the crystal was measured to be 170 volts zero to peak. The amplitude of the retro-reflector's motion is then

$$\Delta x = d_{11} \times \text{voltage applied}$$

$$= 2.31 \times 10^{-12} \frac{\text{meters}}{\text{volt}} \times 170 \text{ volts}$$

$$= 3.93 \times 10^{-10} \text{ meters}$$

$$= 3.93 \text{ Å}$$

From the mathematical analysis of the technique, the magnitude ratio of the original to quadrature signal amplitudes is seen to be

$$\text{Ratio} = \frac{\text{Original Signal Amplitude}}{\text{Quadrature Signal Amplitude}}$$

$$= \frac{A\,J_0\left(\frac{2\pi\Delta x}{X}\right)}{2A\,J_1\left(\frac{2\pi\Delta x}{X}\right)} = \frac{J_0\left(\frac{2\pi\Delta x}{X}\right)}{2J_1\left(\frac{2\pi\Delta x}{X}\right)}$$

For the Laser Feedback Interferometer X is one half the operating wavelength of the laser or 3164 Å. If the amplitude ratio of the experimentally measured original and quadrature signals is used, the above relationship may be solved for the only unknown, Δx. The two values of Δx may then be compared. The purpose of this operation is to demonstrate that the experimental results quantitatively support the mathematical analysis.

To obtain the needed amplitude data, both the original and quadrature signal processors were calibrated from the calibrator signal of the "lock-in amplifier." The "lock-in amplifier" gain was determined to be 5.0×10^4 while the composite gain of low pass filter and preamplifier (original signal) was 1.64. The two channels of the chart recorder were calibrated from the "lock-in" calibrator also and set to sensitivities of 1 volt/cm for the quadrature trace and 10 millivolts/cm for the original signal. The "lock-in amplifier's" output is scaled to read the RMS value of the input amplitude. For this reason, the output value of the "lock-in amplifier" must be scaled up by 1.414 to convert the RMS output to the required peak value output. The complete sensitivity of the quadrature signal processing system (recorder chart deflection for photocell output) is then

$$\frac{1 \text{ volt/cm}}{5 \times 10^{+4}} \times 1.414 = 28.28 \times 10^{-6} \frac{\text{volts}}{\text{cm}} .$$

Similarly the overall gain of the original signal system is

$$\frac{10 \text{ millivolts/cm}}{1.64} = 6.1 \text{ millivolts/cm.}$$

The recording of Figure 14 was made with the calibrated system. Immediately upon completion of the recording, the system calibration was checked and found to be unchanged.

From Figure 14, the peak-to-peak amplitude of the recording is seen to be 0.85 cm or the original signal amplitude is

$$0.85 \times 6.1 \frac{\text{millivolts}}{\text{cm}} = 5.18 \text{ millivolts.}$$

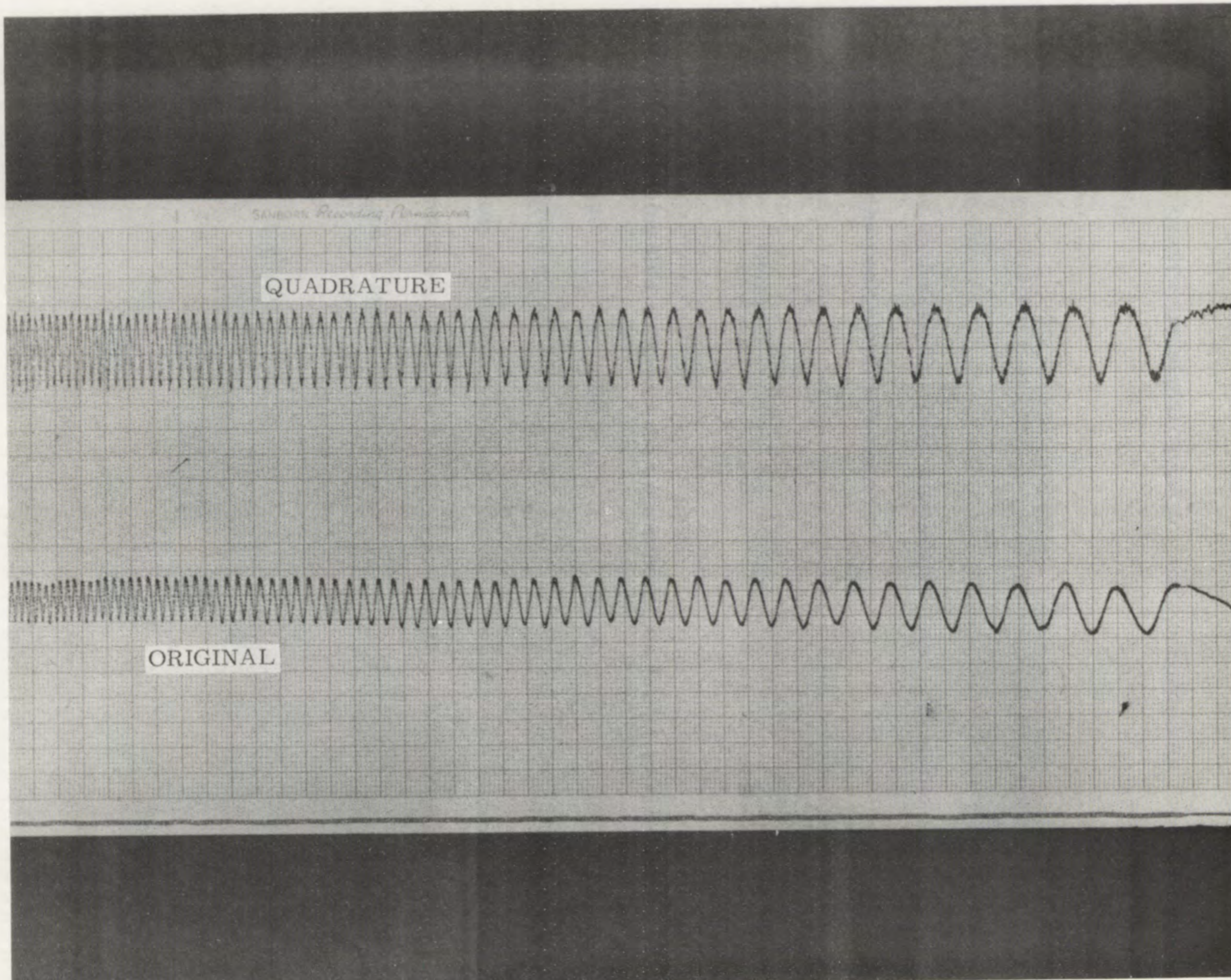

Figure 14. Calibrated Record of Quadrature Signals

Also the quadrature signal, peak to peak, recorded as 1.4 centimeters, is equal to

$$1.4 \text{ cm} \times 28.28 \frac{10^{-6} \text{ volts}}{\text{cm}} = 39.59 \times 10^{-6} \text{ volts.}$$

The ratio of the peak-to-peak amplitudes is of course the same as the ratio of the peak amplitudes and is

$$\text{Ratio} = \frac{5.18 \times 10^{-3}}{39.59 \times 10^{-6}} = 130.8.$$

Using this ratio and the approximations for $J_p(x)$ given by Hildebrand [9], the value of Δx in the argument of the Bessel functions may be determined. The approximations are justified since they are satisfactory for arguments less than 0.1 radian and a computation of the argument based on $\Delta x = 3.93$ Å is less than 0.010 radian. The approximation is stated as

$$J_p(z) \approx \frac{z^p}{2^p \, p!}.$$

Thus the ratio expression is

$$\text{Ratio} = \frac{J_0\left(\frac{2\pi\Delta x}{X}\right)}{2J_1\left(\frac{2\pi\Delta x}{X}\right)} \approx \frac{1}{2\pi\,\frac{\Delta x}{X}} .$$

Since the experimental ratio is 130. 8,

$$\Delta x \approx X/2\,\pi(130.8) = \frac{3164\ \text{Å}}{821.8} = 3.84\ \text{Å}.$$

This value of Δx is less than 4 percent different from the Δx calculated above to be 3. 93 Å. This small error is certainly within the accuracy limits of the measuring equipment used. Consequently the mathematical analysis is in agreement with the experimental results.

CONCLUSIONS

This paper has developed the mathematical model for a technique which resolves the ambiguity in the output of a sinusoidal transducer. The model was based on determining the slope of the transducer's function by introducing a sinusoidal perturbation at the input of the transducer. The quadrature (slope) information has been shown to appear in the transducer's output as the amplitude and phase of the perturbation's frequency. In considering the practical application of the technique, the mathematical analysis has shown that the perturbation frequency must be at least twice the maximum time derivative of the normal input to the transducer. Further practical considerations, such as bandwidths of filters, have also been carried out.

The equipment required to implement the technique in a general situation has been deduced from the mathematical model. The characteristics and operation of each necessary system component have been discussed with regard to this model. The complete system utilizing these components has been schematically presented in Figure 7.

Experimental verification of the technique has been reported. This verification was based on the resolution of directional ambiguity as demonstrated by the discussion of the data from Figure 13. Resolution of the ambiguity was possible because the technique produced the necessary quadrature information. The experiment also provided quantitative support for the mathematical model by introducing the data from Figure 14 into the expressions developed for the quadrature signals. This procedure resulted in a calculated value of 3.84Å for the retro-reflector's perturbation amplitude. This calculation is within 4 percent of the amplitude (3.93Å) computed from the voltage applied to the quartz crystal and the physical constants of quartz. Since the experiment was conducted utilizing a sinusoidal transducer (Laser Feedback Interferometer) for which no conventional technique was applicable, the electronic technique was concluded to be a more general solution to the ambiguity problem.

The electronic resolution technique makes possible the application of the Laser Feedback Interferometer (LFI) to seismic strain detection. This is significant because the LFI is the simplest interferometer known which could be used in this application. Because of this simplicity, no elaborate underground chambers are required; only a simple bore hole with a retro-reflector attached to the end is required. The laser and the small amount of necessary electronics would be located at the open end of the bore hole. Similarly, the LFI can replace the more complicated interferometers currently used as measuring machines for machine tools. The technique should also find applications in instrumentation concerned with phase and polarization measurements.

During the experiment reported, a definite need for a nonexisting piece of equipment was apparent. The desired device would linearize, possibly with electronics alone, the circular result (as appears on an oscilloscope) of the combined quadrature signals. Such a device would be useful in all known quadrature systems because its output would be, identically, the circular transducer's input time dependence. This one piece of hardware would, hopefully, reduce the total amount of equipment required.

Future analytical and experimental work may lead to the use of nonsinusoidal periodic perturbation signals as discussed in the mathematical model presentation. Likewise, further analytical and experimental studies will be required to extend the technique to nonsinusoidal transducers which have ambiguous outputs.

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