Non Bayesian Conditioning and Deconditioning

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***Abstract*—In this paper, we present a Non-Bayesian conditioning rule for belief revision. This rule is truly Non-Bayesian in the sense that it doesn't satisfy the common adopted principle that when a prior belief is Bayesian, after conditioning by X, $Bel(X|X)$ must be equal to one. Our new conditioning rule for belief revision is based on the proportional conflict redistribution rule of combination developed in DSmT (Dezert-Smarandache Theory) which abandons Bayes' conditioning principle. Such Non-Bayesian conditioning allows to take into account judiciously the level of conflict between the prior belief available and the conditional evidence. We also introduce the deconditioning problem and show that this problem admits a unique solution in the case of Bayesian prior; a solution which is not possible to obtain when classical Shafer and Bayes conditioning rules are used. Several simple examples are also presented to compare the results between this new Non-Bayesian conditioning and the classical one.**

Keywords: Belief functions, conditioning, deconditioning, probability, DST, DSmT, Bayes rule.

I. Introduction

The question of the updating of probabilities and beliefs has yielded, and still yields, passionate philosophical and mathematical debates [3], [6], [7], [9], [12], [13], [17], [20], [22] in the scientific community and it arises from the different interpretations of probabilities. Such question has been reinforced by the emergence of the possibility and the evidence theories in the eighties [4], [16] for dealing with uncertain information. We cannot browse in details here all the different authors' opinions [1], [2], [8], [10], [14], [15] on this important question but we suggest the reader to start with Dubois & Prade survey [5]. In this paper, we propose a true Non-Bayesian rule of combination which doesn't satisfy the well-adopted Bayes principle stating that $P(X|X) = 1$ (or $Bel(X|X) = 1$ when working with belief functions). We show that by abandoning such Bayes principle, one can take into account more efficiently in the conditioning process the level of the existing conflict between the prior evidence and the new conditional evidence. We show also that the full deconditioning is possible in some specific cases. Our approach is based on belief functions and the Proportional Conflict Redistribution (mainly PCR5) rule of combination developed in Dezert-Smarandache Theory (DSmT) framework [18]. Why we use PCR5 here? Because PCR5 is very efficient to combine conflicting sources of evidences[1] and because Dempster's rule often considered as a generalization of Bayes rule is actually not deconditionable (see examples in the sequel), contrariwise to PCR5, that's why we utilize PCR5. This paper is organized as follows. In section II, we briefly recall Dempster's rule of combination and Shafer's Conditioning Rule (SCR) proposed in Dempster-Shafer Theory (DST) of belief functions [16]. In section III, we introduce a new Non-Bayesian conditioning rule and show its difference with respect to SCR. In section IV, we introduce the dual problem, called the deconditioning problem. Some examples are given in section V with concluding remarks in section VI.

II. Shafer's conditioning rule

In DST, a normalized basic belief assignment (bba) $m(.)$ is defined as a mapping from the power set 2^Θ of the finite discrete frame of discernment Θ into $[0,1]$ such that $m(\emptyset) = 0$ and $\sum_{X\in 2^\Theta} m(X) = 1$. Belief and plausibility functions are in one-to-one correspondence with $m(.)$ and are respectively defined by $Bel(X) = \sum_{Z\in 2^\Theta, Z\subseteq X} m(Z)$ and $Pl(X) = \sum_{Z\in 2^\Theta, Z\cap X\neq 0} m(Z)$. They are usually interpreted as lower and upper bounds of a unknown measure of subjective probability $P(.)$, i.e. $Bel(X) \leq P(X) \leq Pl(X)$ for any X. In DST, the combination of two independent sources of evidence characterized by $m_1(.)$ and $m_2(.)$ is done using Dempster's rule as follows[2]:

$$m_{DS}(X) = \frac{\sum_{\substack{X_1,X_2\in 2^\Theta\\ X_1\cap X_2 = X}} m_1(X_1)m_2(X_2)}{1 - \sum_{\substack{X_1,X_2\in 2^\Theta\\ X_1\cap X_2=\emptyset}} m_1(X_1)m_2(X_2)} \tag{1}$$

Shafer's conditioning rule[3] (SCR) is obtained as the result of Dempster's combination of the given prior bba $m_1(.)$ with the conditional evidence, say Y represented by a source $m_2(.)$ only focused on Y, that is such that $m_2(Y) = 1$. In other words, $m(X|Y) = m_{DS}(X) = (m_1 \oplus m_2)(X)$ using $m_2(Y) = 1$ and where $\oplus$ symbol denotes here Dempster's

[1]Due to space limitation, we do not present, nor justify again PCR5 w.r.t. other rules since this has been widely explained in the literature with many examples and discussions, see for example [18], Vol. 2. and our web page.

[2]assuming that the numerator is not zero (the sources are not in total conflict).

[3]also called Dempster's conditioning by Glenn Shafer in [16].

fusion rule (1). It can be shown [16] that the conditional belief and the plausibility are given by[4]:

$$Bel(X|Y) = \sum_{\substack{Z\in 2^\Theta \\ Z\subseteq X}} m_{DS}(Z|Y) = \frac{Bel_1(X\cup\bar{Y}) - Bel_1(\bar{Y})}{1-Bel_1(\bar{Y})} \tag{2}$$

$$Pl(X|Y) = \sum_{\substack{Z\in 2^\Theta \\ Z\cap X\neq\emptyset}} m_{DS}(Z|Y) = \frac{Pl_1(X\cap Y)}{Pl_1(Y)} \tag{3}$$

When the belief is Bayesian[5], i.e. $Bel(.|Y) = Pl(.|Y) = P(.|Y)$, SCR reduces to classical conditional probability definition (Bayes formula), that is $P(X|Y) = P(X\cap Y)/P(Y)$, with $P(.) = m_1(.)$. Note that when $Y = X$ and as soon as $Bel(\bar{X}) < 1$, one always gets from (2), $Bel(X|X) = 1$ because $Bel_1(X\cup\bar{Y}) = Bel_1(X\cup\bar{X}) = Bel_1(\Theta) = 1$. For Bayesian belief, this implies $P(X|X) = 1$ for any X such that $P_1(X) > 0$, which we call *Bayes principle*. Other alternatives have been proposed in the literature [8], [15], [21], but almost all of them satisfy Bayes principle and they are all somehow extensions/generalization of Bayes rule. A true Non-Bayesian conditioning (called weak conditioning) was however introduced by Planchet in 1989 in [14] but it didn't bring sufficient interest because Bayes principle is generally considered as the best solution for probability updating based on different arguments for supporting such idea. Such considerations didn't dissuade us to abandon Bayes principle and to explore new Non-Bayesian ways for belief updating, as Planchet did in nineties. We will show in next section why Non-Bayesian conditioning can be interesting.

III. A Non Bayesian Conditioning Rule

Before presenting our Non Bayesian Conditioning Rule, it is important to recall briefly the Proportional Conflict Redistribution Rule no. 5 (PCR5) which has been proposed as a serious alternative of Dempster's rule [16] in Dezert-Smarandache Theory (DSmT) [18] for dealing with conflicting belief functions. In this paper, we assume working in the same fusion space as Glenn Shafer, i.e. on the power set 2^Θ of the finite frame of discernment Θ made of exhaustive and exclusive elements.

A. PCR5 rule of combination

Definition: Let's $m_1(.)$ and $m_2(.)$ be two independent[6] bba's, then the PCR5 rule of combination is defined as follows (see [18], Vol. 2 for details, justification and examples) when working in power set 2^Θ: $m_{PCR5}(\emptyset) = 0$ and $\forall X \in 2^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(X) = \sum_{\substack{X_1,X_2\in 2^\Theta \\ X_1\cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{X_2\in 2^\Theta \\ X_2\cap X=\emptyset}} [\frac{m_1(X)^2 m_2(X_2)}{m_1(X)+m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X)+m_1(X_2)}] \tag{4}$$

[4] $\bar{Y}$ denotes the complement of Y in the frame Θ.

[5] the focal elements of $m_1(.|Y)$ are singletons only.

[6] i.e. each source provides its bba independently of the other sources.

All fractions in (4) having zero denominators are discarded. The extension and a variant of (4) (called PCR6) for combining $s > 2$ sources and for working in other fusion spaces is presented in details in [18]. Basically, in PCR5 the partial conflicting masses are redistributed proportionally to the masses of the elements which are involved in the partial conflict only, so that the specificity of the information is entirely preserved through this fusion process. It has been clearly shown in [18], Vol. 3, chap. 1 that Smets' rule[7] is not so useful, nor cogent because it doesn't respond to new information in a global or in a sequential fusion process. Indeed, very quickly Smets fusion result commits the full of mass of belief to the empty set!!! In applications, some ad-hoc numerical techniques must be used to circumvent this serious drawback. Such problem doesn't occur with PCR5 rule. By construction, other well-known rules like Dubois & Prade, or Yager's rule, and contrariwise to PCR5, increase the non-specificity of the result.

Properties of PCR5:

- (P0): PCR5 rule is not associative, but it is quasi-associative (see [18], Vol. 2).
- (P1): PCR5 Fusion of two non Bayesian bba's is a non Bayesian bba.
  Example: Consider $\Theta = \{A, B, C\}$ with Shafer's model and with the two non Bayesian bba's $m_1(.)$ and $m_2(.)$ given in Table I. The PCR5 fusion result (rounded at the fourth decimal) is given in the right column of the Table I. One sees that $m_{PCR5}(.)$ in a non Bayesian bba since some of its focal elements are not singletons.

Table I
PCR5 fusion of two non Bayesian bba's.

Focal Elem.	$m_1(.)$	$m_2(.)$	$m_{PCR5}(.)$
A	0.1	0.2	0.3850
B	0.2	0.1	0.1586
C	0.1	0.2	0.1990
$A\cup B$	0.3	0	0.0360
$A\cup C$	0	0.5	0.2214
$A\cup B\cup C$	0.3	0	0

- (P2): PCR5 Fusion of a Bayesian bba with a non Bayesian bba is a non Bayesian bba in general[8].
  Example: Consider $\Theta = \{A, B, C\}$ with Shafer's model and Bayesian and a non Bayesian bba's $m_1(.)$ and $m_2(.)$ to combine as given in Table II. The PCR5 fusion result is given in the right column of the Table II. One sees that $m_{PCR5}(.)$ is a non Bayesian bba since some of its focal elements are not singletons.
  This property is in opposition with Dempster's rule property (see Theorem 3.7 p. 67 in [16]) which states that if Bel_1 is Bayesian and if Bel_1 and Bel_2 are combinable, then Dempster's rule provides always a Bayesian belief function. The result of Dempster's rule noted $m_{DS}(.)$ for

[7] i.e. the non normalized Dempster's rule.

[8] In some cases, it happens that Bayesian $\oplus$ Non-Bayesian = Bayesian. For example, with $\Theta = \{A, B, C\}$, Shafer's model, $m_1(A) = 0.3$, $m_1(B) = 0.7$ and $m_2(A) = 0.1$, $m_2(B) = 0.2$, $m_2(C) = 0.4$ and $m_2(A\cup B) = 0.3$, one gets $m_{PCR5}(A) = 0.2162$, $m_{PCR5}(B) = 0.6134$ and $m_{PCR5}(C) = 0.1704$ which is a Bayesian bba.

Table II
PCR5 FUSION OF BAYESIAN AND NON BAYESIAN BBA'S.

Focal Elem.	$m_1(.)$	$m_2(.)$	$m_{DS}(.)$	$m_{PCR5}(.)$
A	0.1	0	0.0833	0.0642
B	0.2	0.3	0.1000	0.1941
C	0.7	0.2	0.8167	0.6703
$A \cup C$	0	0.5	0	0.0714

this example is given in Table II for convenience. This is the major difference between PCR5 and Dempster's rule, not to mention the management of conflicting information in the fusion process of course.
In summary, and using $\oplus$ symbol to denote the generic fusion process, one has

– **With Dempster's rule** :

$$\text{Bayesian} \oplus \text{Non-Bayesian} = \text{Bayesian}$$

– **With PCR5 rule**:

$$\text{Bayesian} \oplus \text{Non-Bayesian} = \text{Non-Bayesian (in general)}$$

- (P3): PCR5 Fusion of two Bayesian bba's is a Bayesian bba (see [18], Vol. 2, pp. 43–45 for proof).

  Example: $\Theta = \{A, B, C\}$ with Shafer's model and let's consider Bayesian bba's given in the next Table. The result of PCR5 fusion rule is given in the right column of Table III. One sees that $m_{PCR5}(.)$ is Bayesian since its focal elements are singletons of the fusion space 2^Θ.

Table III
PCR5 FUSION OF TWO BAYESIAN BBA'S.

Focal Elem.	$m_1(.)$	$m_2(.)$	$m_{DS}(.)$	$m_{PCR5}(.)$
A	0.1	0.4	0.0870	0.2037
B	0.2	0	0	0.0567
C	0.7	0.6	0.9130	0.7396

B. A true Non Bayesian conditioning rule

Here[9] we follow the footprints of Glenn Shafer in the sense that we consider the conditioning as the result of the fusion of any prior mass $m_1(.)$ defined on 2^Θ with the bba $m_2(.)$ focused on the conditional event $Y \neq \emptyset$, i.e. $m_2(Y) = 1$. We however replace Dempster's rule by the more efficient[10] Proportional Conflict Redistribution rule # 5 (PCR5) given by (4) proposed in DSmT [18]. This new conditioning rule is not Bayesian and we use the symbol $\|$ (parallel) instead of classical symbol $|$ to avoid confusion in notations. Let's give the expression of $m(X \parallel Y)$ resulting of the PCR5 fusion of any prior bba $m_1(.)$ with $m_2(.)$ focused on Y. Applying (4):

$$m(X \parallel Y) = S_1^{\text{PCR5}}(X,Y) + S_2^{\text{PCR5}}(X,Y) + S_3^{\text{PCR5}}(X,Y) \quad (5)$$

with

$$S_1^{\text{PCR5}}(X,Y) \triangleq \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) \quad (6)$$

$$S_2^{\text{PCR5}}(X,Y) \triangleq m_1(X)^2 \sum_{\substack{X_2 \in 2^\Theta \\ X \cap X_2 = \emptyset}} \frac{m_2(X_2)}{m_1(X) + m_2(X_2)} \quad (7)$$

$$S_3^{\text{PCR5}}(X,Y) \triangleq m_2(X)^2 \sum_{\substack{X_2 \in 2^\Theta \\ X \cap X_2 = \emptyset}} \frac{m_1(X_2)}{m_2(X) + m_1(X_2)} \quad (8)$$

where $m_2(Y) = 1$ for a given $Y \neq \emptyset$.

Since Y is the single focal element of $m_2(.)$, the term $S_1^{\text{PCR5}}(X,Y)$ in (5) is given by $\sum_{\substack{X_1 \in 2^\Theta \\ X_1 \cap Y = X}} m_1(X_1)$, the term $S_2^{\text{PCR5}}(X,Y)$ equals $\delta(X \cap Y = \emptyset) \cdot \frac{m_1(X)^2}{1+m_1(X)}$, and the term $S_3^{\text{PCR5}}(X,Y)$ can be expressed depending on the value of X with respect to the conditioning term Y:

- If $X \neq Y$ then $m_2(X \neq Y) = 0$ (by definition), and thus $S_3^{\text{PCR5}}(X,Y) = 0$.
- If $X = Y$ then $m_2(X = Y) = 1$ (by definition), and thus $S_3^{\text{PCR5}}(X,Y) = \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap Y = \emptyset}} \frac{m_1(X_2)}{1+m_1(X_2)}$

Finally, $S_3^{\text{PCR5}}(X,Y)$ can be written as

$$S_3^{\text{PCR5}}(X,Y) = \underbrace{\delta(X \neq Y) \cdot 0}_{0} + \delta(X = Y) \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap Y = \emptyset}} \frac{m_1(X_2)}{1 + m_1(X_2)}$$
$$= \delta(X = Y) \cdot \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap Y = \emptyset}} \frac{m_1(X_2)}{1 + m_1(X_2)}$$

Finally, $m(X \parallel Y)$ for $X \neq \emptyset$ and $Y \neq \emptyset$ are given by

$$m(X \parallel Y) = \sum_{\substack{X_1 \in 2^\Theta \\ X_1 \cap Y = X}} m_1(X_1) + \delta(X \cap Y = \emptyset) \cdot \frac{m_1(X)^2}{1 + m_1(X)}$$
$$+ \delta(X = Y) \cdot \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap Y = \emptyset}} \frac{m_1(X_2)}{1 + m_1(X_2)} \quad (9)$$

$m(\emptyset \parallel Y \neq \emptyset) = 0$ by definition, since PCR5 fusion doesn't commit mass on the empty set. $m(X \parallel \emptyset)$ is kept undefined[11] since it doesn't make sense to revise a bba by an impossible event. Based on the classical definitions of $Bel(.)$ and $Pl(.)$ functions [16], one has:

$$Bel(X \parallel Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \subseteq X}} m(Z \parallel Y) \quad (10)$$

$$Pl(X \parallel Y) = \sum_{\substack{Z \in 2^\Theta \\ Z \cap X \neq \emptyset}} m(Z \parallel Y) \quad (11)$$

The "true" unknown (non Bayesian) conditional subjective probability, denoted $P(X||Y)$, must satisfy

$$Bel(X \parallel Y) \leq P(X||Y) \leq Pl(X||Y) \quad (12)$$

[9] More sophisticated conditioning rules have been proposed in [18], Vol. 2.

[10] It deals better with partial conflicts than other rules unlike Dempster's rule, it does not increase the non-specificity of the result unlike Dubois & Prade or Yager's rule, and it does respond to new information unlike Smets rule.

[11] One could also define $m(\emptyset \parallel \emptyset) = 1$ and $m(X \neq \emptyset \parallel \emptyset) = 0$ which however would not be a normal bba.

$P(X||Y)$ can be seen as an imprecise probability and used within IPT (Imprecise Probability Theory) [23] if necessary, or can be approximated from $m(.||Y)$ using some probabilistic transforms, typically the pignistic transform [19] or the DSmP transform [18] (Vol.3, Chap. 3). The search for direct close-form expressions of $Bel(X \parallel Y)$ and $Pl(X \parallel Y)$ from $Bel_1(.)$ and $Pl_1(.)$ appears to be an open difficult problem.

IV. Deconditioning

In the previous section we have proposed a new non Bayesian conditioning rule based on PCR5. This rule follows Shafer's idea except that we use PCR5 instead of Dempster's rule because we have shown the better efficiency of PCR5 to deal with conflicting information w.r.t. other rules. In this section, we also show the great benefit of such PCR5 rule for the deconditioning problem. The belief conditioning problem consists in finding a way to update any prior belief function ($Bel(.)$, $Pl()$ or $m(.)$) with a new information related with the (belief of) occurrence in a given conditional proposition of the fusion space, say Y, in order to get a new belief function called conditional belief function. The deconditioning problem is the inverse (dual) problem of conditioning. It consists to retrieve the prior belief function from a given posterior/conditional belief function. Deconditioning has not been investigated in deep so far in the literature (to the knowledge of the authors) since is is usually considered as impossible to achieve[12], it may present great interest for applications in advanced information systems when only a posterior belief is available (say provided by an human or an AI-expert system), but for some reason we need to compute a new conditioning belief based on a different conditional hypothesis. This motivates our research for developing deconditioning techniques. Since $Bel(.)$, $Pl()$ are in one-to-one correspondence with the basic belief assignment (bba) mass $m(.)$, we focus our analysis on the deconditioning of the conditional bba. More simply stated, we want to see if for any given conditional bba $m(.||Y)$ we can compute $m_1(.)$ such that $m(.||Y) = PCR5(m_1(.), m_2(.))$ with $m_2(Y) = 1$ and where $PCR5(m_1(.), m_2(.))$ denotes the PCR5 fusion of $m_1(.)$ with $m_2(.)$. Let's examine the two distinct cases for the deconditiong problem depending on the (Bayesian or non-Bayesian) nature of the prior $m_1(.)$.

- **Case of Bayesian prior** $m_1(.)$: Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n \geq 2$, Shafer's model, where all θ_i are singletons. Let $m_1 : \Theta \mapsto [0, 1]$ be a Bayesian bba/mass. In that case, the deconditioning problem admits a unique solution and we can always compute $m_1(.)$ from $m(.||Y)$ but two distinct cases must be analyzed depending on the cardinality of the conditional term Y.

  Case 1: When Y is a singleton, i.e. $|Y| = 1$. Suppose $m_2(Y) = 1$, with $Y = \theta_{j_0}$, for $j_0 \in \{1, 2, \dots, n\}$, where j_0 is fixed. Since the bba's $m_1(.)$ and $m_2(.)$ are both Bayesian in this case, $m(.||Y)$ is also a Bayesian bba (property P3), therefore $m(\theta_i||Y) = a_i$, where all $a_i \in [0, 1]$ with $\sum_{i=1}^{n} a_i = 1$. How to find $m_1(.)$ such that $m(.||Y) = PCR5(m_1(.), m_2(.))$? Let's denote $m_1(\theta_i) = x_i$, where all $x_i \in [0, 1]$ and $\sum_{i=1}^{n} x_i = 1$. We need to find all these x_i. We now combine $m_1(.)$ with $m_2(.)$ using PCR5 fusion rule. We transfer x_i, for $\forall i \neq j_0$, to θ_i and θ_{j_0} proportionally with respect to their corresponding masses, x_i and 1 respectively: $\frac{w_{\theta_i}}{x_i} = \frac{w_{\theta_{j_0}}}{1} = \frac{x_i}{x_i+1}$ whence $w_{\theta_i} = \frac{x_i^2}{x_i+1}$ and $w_{\theta_{j_0}} = \frac{x_i}{x_i+1}$, while $\alpha_{j_0} = x_{j_0} + \sum_{\substack{i=1\\i\neq j_0}}^{n} \frac{x_i}{x_i+1}$ or $\alpha_{j_0} = 1 - \sum_{\substack{i=1\\i\neq j_0}}^{n} \frac{x_i^2}{x_i+1}$. Since we need to find all unknowns x_i, $i = 1, \dots, n$, we need to solve $\frac{x_i^2}{x_i+1} = a_i$, for $i \neq j_0$ for x_i; since $\alpha_{j_0} = x_{j_0} + \sum_{\substack{i=1\\i\neq j_0}}^{n} \frac{x_i}{x_i+1} = a_{j_0}$, we get $x_{j_0} = a_{j_0} - \sum_{\substack{i=1\\i\neq j_0}}^{n} \frac{x_i}{x_i+1} = 1 - \sum_{\substack{i=1\\i\neq j_0}}^{n} x_i$.

  Case 2: When Y is not a singleton, i.e. $|Y| > 1$ (Y can be a partial or total ignorance). Suppose $m_2(Y) = 1$, with $Y = \theta_{j_1} \cup \theta_{j_2} \cup \ldots \cup \theta_{j_p}$, where all j_1, j_2, …, j_p are different and they belong to $\{1, 2, \dots, n\}$, $2 \leq p \leq n$. We keep the same notations for $m(.||Y)$ and Bayesian $m_1(.)$. The set $\{j_1, j_2, \dots, j_p\}$ is denoted J for notation convenience. Similarly, using PCR5 rule we transfer x_i, $\forall i \notin J$, to x_i and to the ignorance $Y = \theta_{j_1} \cup \ldots \cup \theta_{j_p}$ proportionally with respect to x_i and 1 respectively (as done in case 1). So, x_i for $i \notin J$ is found from solving the equation $\frac{x_i^2}{x_i+1} = a_i$, which gives[13] $x_i = (a_i + \sqrt{a_i^2 + 4a_i})/2$; and $x_{j_r} = a_{j_r}$ for $r \in \{1, 2, \dots, p\}$.
- **Case of Non-Bayesian prior** $m_1(.)$:
  Unfortunately, when $m_1(.)$ is Non-Bayesian, the (PCR5-based) deconditioning problem doesn't admit one unique solution in general (see the example 2.1 in the next section). But the method used to decondition PCR5 when $m_1(.)$ is Bayesian can be generalized for $m_1(.)$ non-Bayesian in the following way: 1) We need to know the focal elements of $m_1(.)$, then we denote the masses of these elements by say x_1, x_2, …, x_n; 2)Then we combine using the conjunctive rule $m_1(.)$ with $m_2(Y) = 1$, where Y can be a singleton or an ignorance; 3) Afterwards, we use PCR5 rule and we get some results like: $f_i(x_1, ..., x_n)$ for each element, where $i = 1, 2, \dots$. Since we know the results of PCR5 as $m(.||Y) = a_i$ for each focal element, then we form a system of non-linear equations: $f_i(x_1, x_2, ..., x_n) = a_i$ and we need to solve it. Such systems of equations however can admit several solutions. We can select a solution satisfying an additional criterion like by example the minimum (or the maximum) of specificity depending of the kind of Non-Bayesian prior we need to use.

V. Examples

A. Example 1: Conditioning of a Bayesian prior belief

Let's consider $\Theta = \{A, B, C\}$, Shafer's model, and the prior bba's $m_1(.)$ and $m_1'(.)$ given in Table IV and the conditional evidence $Y = A \cup B$.

[12]This truly happens when classical Bayes conditioning is used.

[13]The solution $x_i = (a_i - \sqrt{a_i^2 + 4a_i})/2$ must be discarded since it is negative and cannot be considered as a mass of belief.

Table IV
BAYESIAN PRIORS (INPUTS).

Focal Elem.	m_1	$m_1'(.)$
A	0.49	0.01
B	0.49	0.01
C	0.02	0.98

The significance of having two cases in the Bayesian prior case is straighforward. We just want to show that two different priors can yield to the same posterior bba with Bayes/SCR rule and thus we cannot retrieve these two distinct priors cases from the posterior bba. We show that the total deconditioning is possible however when using our non-Bayesian conditioning rule. SCR and PCR5-based conditioning of $m_1(.)$ and $m_1'(.)$ are given[14] in Table V. One sees that SCR of the two distinct bba's $m_1(.)$ and $m_1'(.)$ yield the same posterior/conditional bba $m(.|Y)$ which means that in this very simple Bayesian prior case, the deconditioning of $m(.|Y)$ is impossible to obtain since at least two solutions[15] for the prior beliefs are admissible. The results provided by PCR5-based conditioning makes more sense in authors' point of view since it better takes into account the degree of conflicting information in the conditioning process. One sees that two distinct Bayesian priors yield two distinct posterior bba's with PCR5-based conditioning. If one examines the belief and plausibility functions, one gets, using notation $\Delta(.|Y) = [Bel(.|Y), Pl(.|Y)]$, $\Delta'(.|Y) = [Bel'(.|Y), Pl'(.|Y)]$, $\Delta(.||Y) = [Bel(.||Y), Pl(.||Y)]$ and $\Delta'(.||Y) = [Bel'(.||Y), Pl'(.||Y)]$:

Table V
CONDITIONAL BBA'S.

Focal Elem.	$m(.\|Y)$	$m'(.\|Y)$	$m(.\|\|Y)$	$m'(.\|\|Y)$
A	0.5	0.5	0.4900	0.0100
B	0.5	0.5	0.4900	0.0100
C	0	0	0.00039215	0.48505051
$A \cup B$	0	0	0.01960785	0.49494949

Table VI
CONDITIONAL LOWER AND UPPER BOUNDS OF CONDITIONAL PROBABILITIES

2^Θ	$\Delta(.\|Y) = \Delta'(.\|Y)$	$\Delta(.\|\|Y)$	$\Delta'(.\|\|Y)$
$\emptyset$	[0,0]	[0,0]	[0,0]
A	[0.5,0.5]	[0.4900, 0.5096]	[0.0100, 0.5050]
B	[0.5,0.5]	[0.4900, 0.5096]	[0.0100, 0.5050]
C	[0,0]	[0.0004, 0.0004]	[0.4850,0.4850]
$Y = A \cup B$	[1,1]	[0.9996,0.9996]	[0.5150,0.5150]
$A \cup C$	[0.5,0.5]	[0.4904, 0.5100]	[0.4950, 0.9900]
$B \cup C$	[0.5,0.5]	[0.4904, 0.5100]	[0.4950, 0.9900]
$A \cup B \cup C$	[1,1]	[1,1]	[1,1]

The interval $\Delta(.|Y)$ corresponds to lower and upper bounds of conditional subjective probabilities $P(.|Y)$ and $\Delta(.||Y)$ corresponds to lower and upper bounds of $P(.||Y)$ (similarly for $\Delta'(.|Y)$ and $\Delta'(.||Y)$). From the Table VI, one sees that the property P2 is verified and we get an imprecise conditional probability. One sees that contrariwise to SCR (equivalent to Bayes rule in this case), one gets $Bel(Y||Y) < 1$ and also $Pl(Y||Y) < 1$. $\Delta(.||Y)$ and $\Delta'(.||Y)$ are very different because priors were also very different. This is an appealing property. If one approximates[16] the conditional probability by the mid-value of their lower and upper bounds[17], one gets values given in Table VII.

[14]Due to space limitation constraints, the verification is left to the reader.
[15]Actually an infinite number of solutions exists.

Table VII
CONDITIONAL APPROXIMATE SUBJECTIVE PROBABILITIES.

2^Θ	$P(.\|Y) = P'(.\|Y)$	$P(.\|\|Y)$	$P'(.\|\|Y)$
$\emptyset$	0	0	0
A	0.5	0.4998	0.2575
B	0.5	0.4998	0.2575
C	0	0.0004	0.4850
$Y = A \cup B$	1	0.9996	0.5150
$A \cup C$	0.5	0.5002	0.7425
$B \cup C$	0.5	0.5002	0.7425
$A \cup B \cup C$	1	1	1

When the conditioning hypothesis supports the prior belief (as for $m_1(.)$ and $m_2(.)$ which are in low conflict) the PCR5-based conditioning reacts as SCR (as Bayes rule when dealing with Bayesian priors) and $P(X.||Y)$ is very close to $P(.|Y)$. When the prior and the conditional evidences are highly conflicting (i.e. like $m_1'(.)$ and $m_2(.)$, PCR5-based conditioning rule is much more prudent than Shafer's rule and that's why it allows the possibility to have $P(Y||Y) < 1$. Such property doesn't violate the fundamental axioms (nonnegativity, unity and additivity) of Kolmogorov axiomatic theory of probabilities and this can be verified easily in our example. In applications, it is much better to preserve all available information and to work directly with conditional bba's whenever possible rather than with approximate subjective conditional probabilities.

The deconditioning of the posterior bba's $m(. \parallel Y)$ given in the Table V is done using the principle described in section IV (when $m_1(.)$ is assumed Bayesian and for case 2). We denote the unknowns $m_1(A) = x_1$, $m_1(B) = x_2$ and $m_1(C) = x_3$. Since $Y = A \cup B$ and $J = \{1, 2\}$, we solve the following system of equations (with the constraint $x_i \in [0, 1]$): $x_1 = a_1 = 0.49$, $x_2 = a_2 = 0.49$ and $x_3^2/(x_3 + 1) = a_3 = 0.00039215$. Therefore, one gets after deconditioning $m_1(A) = 0.49$, $m_1(B) = 0.49$ and $m_1(C) = 0.02$. Similarly, the deconditioning of $m'(. \parallel Y)$ given in the Table V yields $m_1'(A) = 0.01$, $m_1'(B) = 0.01$ and $m_1'(C) = 0.98$.

Note that, contrarywise to Bayes or to Jeffrey's rules [8], [11], [21], it is possible to update the prior opinion about an event A even if $P(A) = 0$ using this Non-Bayesian rule. For example, let's consider $\Theta = \{A, B, C\}$, Shafer's model and the prior Bayesian mass $m_1(A) = 0$, $m_1(B) = 0.3$ and $m_1(C) = 0.7$, i.e. $Bel_1(A) = P_1(A) = Pl(A) = 0$. Assume that the conditional evidence is $Y = A \cup B$, then one gets with SCR $m(B|A \cup B) = 1$ and with PCR5-based conditioning $m(B \parallel A \cup B) = 0.30$, $m(A \cup B \parallel A \cup B) = 0.41176$ and $m(C \parallel A \cup B) = 0.28824$, which means that $P(A|A \cup B) = 0$ with SCR/Bayes rule (i.e. no update on A), whereas $[Bel(A \parallel A \cup B), Pl(A \parallel A \cup B)] = [0, 0.41176]$, $[Bel(B \parallel$

[16]When the lower bound is equal to the upper bound, one gets the exact probability value.
[17]More sophisticated transformations could be used instead as explained in [18], Vol. 3.

$A \cup B), Pl(B \parallel A \cup B)] = [0.30, 0.71176]$ and $[Bel(C \parallel A \cup B), Pl(C \parallel A \cup B)] = [0.28823, 0.28823]$, that is $P(A \parallel A \cup B) \in [0, 0.41176]$. Typically, if one approximates $P(. \parallel A \cup B)$ by the mid-value of its lower and upper bounds, one will obtain $P(A \parallel A \cup B) = 0.20588$ (i.e. a true update of the prior probability of A), $P(B \parallel A \cup B) = 0.50588$ and $P(C \parallel A \cup B) = 0.28824$.

B. Example 2: Conditioning of a Non-Bayesian prior belief

Example 2.1: Let's consider now $\Theta = \{A, B, C\}$, Shafer's model, the conditioning hypothesis $Y = A \cup B$ and the following Non-Bayesian priors:

Table VIII
NON-BAYESIAN PRIORS (INPUTS).

Focal Elem.	m_1	$m'_1(.)$
A	0.20	0.20
B	0.30	0.30
C	0.10	0.10
$A \cup B$	0.25	0.15
$A \cup B \cup C$	0.15	0.25

The conflict between $m_1(.)$ and $m_2(Y) = 1$ and between $m'_1(.)$ and $m_2(Y) = 1$ is 0.10 in both cases. The results of the conditioning are given in Table IX. One sees that when distinct priors are Non-Bayesian, it can happen that PCR5-based conditioning rule yields also the same posterior bba's. This result shows that in general with Non-Bayesian priors the PCR5-based deconditioning cannot provide a unique solution, unless extra information and/constraints on the prior belief are specified as shown in the next example.

Table IX
CONDITIONAL BBA'S.

Focal Elem.	$m(.\vert Y)$	$m'(.\vert Y)$	$m(. \parallel Y)$	$m'(. \parallel Y)$
A	0.222	0.222	0.20	0.20
B	0.333	0.333	0.30	0.30
C	0	0	0.01	0.01
$A \cup B$	0.445	0.445	0.49	0.49

Example 2.2: Let's consider now $\Theta = \{A, B, C, D\}$, Shafer's model, the conditional evidence $Y = C \cup D$ and the posterior bba $m(. \parallel C \cup D)$ given in the right column of the table below:

Table X
CONDITIONAL BBA'S.

Focal Elem.	$m_1(.)$	$m_{PCR5}(.)$	$m(. \parallel A)$
A	x_1	$\frac{x_1^2}{1+x_1}$	0.0333
B	x_2	$\frac{x_2^2}{1+x_2}$	0.1667
$C \cup D$	x_3	$x_3 + \frac{x_1}{1+x_1} + \frac{x_2}{1+x_2}$	0.8000

If we assume that the focal elements of the prior bba $m_1(.)$ are the same as for the posterior bba $m(. \parallel C \cup D)$, then with such extra assumption, the deconditioning problem admits a unique solution which is obtained by solving the system of three equations according to Table X; that is $\frac{x_1^2}{1+x_1} = 0.0333$, whence $x_1 \approx 0.2$; $\frac{x_2^2}{1+x_2} = 0.1667$, whence $x_2 \approx 0.5$; $x_3 + \frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} = 0.8000$; whence $x_1 \approx 0.3$. Therefore, the deconditioning of $m(. \parallel C \cup D)$ provides the unique Non-Bayesian solution $m_1(A) = 0.2$, $m_1(B) = 0.5$ and $m_1(C \cup D) = 0.3$.

VI. CONCLUSIONS

In this paper, we have proposed a new Non-Bayesian conditioning rule (denoted $\parallel$) based on the Proportional Conflict Redistribution (PCR) rule of combination developed in DSmT framework. This new conditioning rule offers the advantage to take fully into account the level of conflict between the prior and the conditional evidences for updating belief functions. It is truly Non-Bayesian since it doesn't satisfy Bayes principle because it allows $P(X \parallel X)$ or $Bel(X \parallel X)$ to be less than one. We have also shown that this approach allows to solve the deconditioning (dual) problem for the class of Bayesian priors. More investigations on the deconditioning problem of Non-Bayesian priors need to be done and comparisons of this new rule with respect to the main alternatives of Bayes rule proposed in the literature (typically Jeffrey's rule and its extensions, Planchet's rule, etc) will be presented in details in a forthcoming publication.

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