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Neutrosophic Generalized Homeomorphism

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Abstract: The idea of neutrosophic generalized homeomorphism is presented in neutrosophic topological spaces. In addition to this, neutrosophic generalized closed and open mappings are also presented. At the same time, their characterizations are discussed by establishing their related attributes.

Keywords: GN-closed set, GN-closed map, GN-open map, Neutrosophic g-homeomorphism, Neutrosophic g*-homeomorphism.

1. Introduction

Neutrosophic sets were initially established as a generality of intuitionistic fuzzy sets [10] by Smarandache [18] such that the membership, the non-membership, and the indeterminacy degrees are considered. In analogy with more unsure philosophy, the neutrosophic set discharge agreement with an indeterminacy condition. The neutrosophic conception has a broad scope of real-time requests in the fields of [1-9] Artificial Intelligence, Computer Science, Information Systems, Decision Making, Uncertainty assessments of linear time-cost tradeoffs, Applied Mathematics, and solving the supply chain problem. Salama et al. [15, 16] adapted the notion of the neutrosophic set to operate in neutrosophic topological spaces (NTSs in short) and pioneered generalized neutrosophic set and topological spaces. In [11], generalized neutrosophic closed set (in short, GNCS) is defined and using this generalized neutrosophic continuous (GN-continuous), and generalized neutrosophic irresolute (in short, GN-irresolute) functions are defined. Recently in [12, 13], the perception of generalized α-contra continuous and neutrosophic almost α-contra-continuous functions are introduced. Parimala M et al. [14] introduced and studied the thought of Neutrosophic homeomorphism and Neutrosophic αψ homeomorphism in Neutrosophic topological spaces. This paper aspires to overly enunciate the thought of neutrosophic generalized homeomorphism (in short, neutrosophic g-homeomorphism) in NTSs by utilizing GN-continuous function and study some of their properties. We have also provided the idea of generalized neutrosophic closed and open mappings by establishing some of their characterizations. Besides, neutrosophic g*-homeomorphism is also presented and establish its relation with the neutrosophic g-homeomorphism.

2. Preliminaries

Definition 2.1 [15]: A neutrosophic topology (in short,N-topology) on $X \neq \emptyset$ is a family $\xi$ of N-sets in $X$ satisfying the laws given below:

(i) $0_N, 1_N \in \xi,$
(ii) \( W_1 \cap W_2 \in \xi \) being \( W_1, W_2 \in \xi \),
(iii) \( \bigcup W_l \in \xi \) for arbitrary family \( \{ W_l | l \in A \} \subseteq \xi \).

In this situation the ordered pair \( (X, \xi) \) or simply \( X \) is termed as NTS and each NS in \( \xi \) is named as neutrosophic open set (in short, NOS). The complement \( \Lambda \) of an N-open set \( \Lambda \) in \( X \) is known as neutrosophic closed set (briefly, NCS) in \( X \).

**Definition 2.2** [15]: Let \( \Lambda \) be an NS in an NTS \( (X, \xi) \). Thereupon

(i) \( N\text{nt}(\Lambda) = \bigcup \{ G | G \text{ is a NOS in } X \text{ and } G \subseteq \Lambda \} \) is termed as neutrosophic interior (in brief \( N\text{nt} \)) of \( \Lambda \);

(ii) \( N\text{cl}(\Lambda) = \bigcap \{ G | G \text{ is an NCS in } X \text{ and } G \supseteq \Lambda \} \) is termed as neutrosophic closure (shortly \( N\text{cl} \)) of \( \Lambda \).

**Definition 2.3** [11]: Allow \( (X, \xi) \) be a NTS. A NS \( \Lambda \) in \( (X, \xi) \) is termed as generalized neutrosophic closed set (in short GNCS) if \( N\text{cl}(\Lambda) \subseteq \Gamma \) whenever \( \Lambda \subseteq \Gamma \) and \( \Gamma \) is a NOS. The complement of a GNCS is generalized neutrosophic open set (in short GNOS).

**Definition 2.4** [11]: Let \( (X, \xi) \) be NTS and \( B \) be a NS in \( X \). Then neutrosophic generalized closure is defined as, \( GN\text{cl}(B) = \bigcap \{ G | G \text{ is a GNCS in } X \text{ and } B \subseteq G \} \).

**Definition 2.5** [11, 17]: A map \( \eta : X \to Y \) is said to be

(i) neutrosophic closed (in short, NC-map) if the image of every NCS in \( X \) is a NCS in \( Y \).

(ii) neutrosophic continuous (in short, N-continuous) if inverse image of every NCS in \( Y \) is a NCS in \( X \).

(iii) generalized neutrosophic continuous (in short, GN-continuous) if inverse image of every NCS in \( Y \) is a GNCS in \( X \).

(iv) generalized neutrosophic irresolute (in short, GN-irresolute) if inverse image of every GNCS in \( Y \) is a GNCS in \( X \).

**Definition 2.6** [14]: A bijection \( g : X \to Y \) is called a neutrosophic homeomorphism if \( g \) and \( g^{-1} \) are neutrosophic continuous.

### 3. Neutrosophic Generalized Homeomorphism

**Definition 3.1**: A bijection \( \eta : X \to Y \) is named as neutrosophic generalized homeomorphism (in short neutrosophic g-homeomorphism) if \( \eta \) and \( \eta^{-1} \) are GN-continuous.

**Proposition 3.2**: Every neutrosophic homeomorphism is a neutrosophic g-homeomorphism.

**Proof**: Consider a bijection mapping \( \eta : X \to Y \) be a neutrosophic homeomorphism, in which \( \eta \) as well as \( \eta^{-1} \) are N-continuous. We have each N-continuous mapping is GN-continuous, so \( \eta \) and \( \eta^{-1} \) are GN-continuous. Hence, \( \eta \) is neutrosophic g-homeomorphism.
Remark 3.3: The next illustration makes clear that the opposite of the above proposition is not valid.

Example 3.4: Let \( X = \{ p, q, r \}, \; \xi = \{ 0_N, '\mathcal{A}_1', '\mathcal{A}_2', '\mathcal{A}_3', '\mathcal{A}_4', 1_N \} \) be a N-topology on \( X \).
\[
\mathcal{A}_1 = (x, (0.2, 0.1, 0.1), (0.2, 0.1, 0.1), (0.3, 0.5, 0.5)), \quad \mathcal{A}_2 = (x, (0.1, 0.2, 0.2), (0.4, 0.3, 0.3), (0.3, 0.3, 0.3)), \\
\mathcal{A}_3 = (x, (0.2, 0.2, 0.2), (0.2, 0.1, 0.1), (0.3, 0.3, 0.3)), \quad \mathcal{A}_4 = (x, (0.1, 0.1, 0.1), (0.4, 0.3, 0.3), (0.3, 0.5, 0.5)),
\]
and let \( Y = \{ p, q, r \}, \; \sigma = \{ 0_N, '\mathcal{B}_1', '\mathcal{B}_2', '\mathcal{B}_3', '\mathcal{B}_4', 1_N \} \) be a neutrosophic topology on \( Y \).
\[
\mathcal{B}_1 = (y, (0.3, 0.3, 0.3), (0.2, 0.1, 0.1), (0.2, 0.2, 0.2)), \quad \mathcal{B}_2 = (y, (0.2, 0.2, 0.2), (0.1, 0.1, 0.1), (0.3, 0.3, 0.3)), \\
\mathcal{B}_3 = (y, (0.3, 0.3, 0.3), (0.1, 0.1, 0.1), (0.2, 0.1, 0.1)), \quad \mathcal{B}_4 = (y, (0.2, 0.2, 0.2), (0.2, 0.1, 0.1), (0.3, 0.3, 0.3)).
\]
Define \( \eta: (X, \xi) \rightarrow (Y, \sigma) \) by \( \eta(p) = p, \eta(q) = q \) and \( \eta(r) = r \). Then \( \eta \) is neutrosophic g-homeomorphism but not neutrosophic homeomorphism.

Definition 3.5: A mapping \( \eta: X \rightarrow Y \) is generalized neutrosophic closed (in short, GNC-map) if the image \( \eta(Q) \) is GNCS in \( Y \) for every NCS \( Q \) in \( X \).

Definition 3.6: A mapping \( \eta: X \rightarrow Y \) is generalized neutrosophic open (in short, GNO-map) if the image \( \eta(R) \) is GNOS in \( Y \) for every NOS \( R \) in \( X \).

Proposition 3.7: Every NC-mapping is a GNC-mapping.
Proof: Consider \( \eta: X \rightarrow Y \) is a NC-mapping, so as \( Q \) is an NCS in \( X \). As \( \eta \) is NC-mapping, \( \eta(Q) \) is NCS in \( Y \). Since each NCS is GNCS. Therefore, \( \eta(Q) \) is a GNCS in \( Y \). Hence, \( \eta \) is GNC-mapping.

Remark 3.8: The opposite of the above proposition is not valid as indicated.

Example 3.9: Let \( X = \{ p, q, r \}, \; \xi = \{ 0_N, '\mathcal{A}_1', '\mathcal{A}_2', '\mathcal{A}_3', '\mathcal{A}_4', 1_N \} \) be a N-topology on \( X \).
\[
\mathcal{A}_1 = (x, (0.2, 0.1, 0.1), (0.2, 0.1, 0.1), (0.3, 0.5, 0.5)), \quad \mathcal{A}_2 = (x, (0.1, 0.2, 0.2), (0.4, 0.3, 0.3), (0.3, 0.3, 0.3)), \\
\mathcal{A}_3 = (x, (0.2, 0.2, 0.2), (0.2, 0.1, 0.1), (0.3, 0.3, 0.3)), \quad \mathcal{A}_4 = (x, (0.1, 0.1, 0.1), (0.4, 0.3, 0.3), (0.3, 0.5, 0.5)),
\]
and let \( Y = \{ p, q, r \}, \; \sigma = \{ 0_N, '\mathcal{B}_1', '\mathcal{B}_2', '\mathcal{B}_3', '\mathcal{B}_4', 1_N \} \) be a neutrosophic topology on \( Y \).
\[
\mathcal{B}_1 = (y, (0.3, 0.3, 0.3), (0.2, 0.1, 0.1), (0.2, 0.2, 0.2)), \quad \mathcal{B}_2 = (y, (0.2, 0.2, 0.2), (0.1, 0.1, 0.1), (0.3, 0.3, 0.3)), \\
\mathcal{B}_3 = (y, (0.3, 0.3, 0.3), (0.1, 0.1, 0.1), (0.2, 0.1, 0.1)), \quad \mathcal{B}_4 = (y, (0.2, 0.2, 0.2), (0.2, 0.1, 0.1), (0.3, 0.3, 0.3)).
\]
Define \( \eta: (X, \xi) \rightarrow (Y, \sigma) \) by \( \eta(p) = p, \eta(q) = q \) and \( \eta(r) = r \). Then \( \eta \) is GNC-mapping but not NC-mapping.

Proposition 3.10: A map \( \eta: X \rightarrow Y \) is a GNC-mapping if the image of each NOS in \( X \) is GNOS in \( Y \).
Proof: Let \( R \) be a NOS in \( X \). Hence \( \overline{R} \) is a NCS in \( X \). As \( \eta \) is GNC-mapping, \( \eta(\overline{R}) \) is a GNCS in \( Y \).
Since \( \eta(\overline{R}) = \overline{\eta(R)} \), \( \eta(R) \) is a GNOS in \( Y \).

Proposition 3.11: Let \( \eta: X \rightarrow Y \) be a bijective mapping, then the next assertions are same:
(i) \( \eta \) is GNO-mapping.
(ii) \( \eta \) is GNC-mapping.
(iii) \( \eta^{-1} \) is GN-continuous.
Proof: (i) → (ii). Suppose that \( \eta \) is GNO-mapping. Then, \( P \) is a NOS in \( X \), then image \( \eta(P) \) is GNOS in \( Y \). Here, \( P \) is NCS in \( X \), then \( X - P \) is a NOS in \( X \). By prediction, \( \eta(X - P) \) is a GNOS in \( Y \). Hence, \( Y - \eta(X - P) \) is a GNCS in \( Y \). Hence, \( \eta \) is a GNC-mapping.

(ii) → (iii). Let \( R \) be an NCS in \( X \). By (ii), \( \eta(R) \) is GNCS in \( Y \). Therefore, \( \eta(R) = (\eta^{-1})^{-1}(R) \), so \( \eta^{-1} \) is a GNCS in \( Y \). Hence, \( \eta^{-1} \) is a GNC-continuous.

(iii) → (ii). Let \( Q \) be a NOS in \( X \). By (iii), \( (\eta^{-1})^{-1}(Q) = \eta(Q) \) is GNO-mapping.

Proposition 3.12: Let \( \eta : (X, \xi) \to (Y, \sigma) \) be a bijective mapping. If \( \eta \) is GN-continuous, thereupon the declarations are identical:
(i) \( \eta \) is GNC-mapping.
(ii) \( \eta \) is GNO-mapping.
(iii) \( \eta^{-1} \) is neutrosophic g-homeomorphism.

Proof: (i) → (ii). Presume that \( \eta \) is bijective as well as a GNC-mapping. So, \( \eta^{-1} \) is a GN-continuous mapping. As we have every NOS is GNOS in \( Y \). Hence, \( \eta \) is GNO-mapping.

(ii) → (iii). Consider a bijective NO-mapping \( \eta \). Furthermore, \( \eta^{-1} \) is a GN-continuous mapping. Accordingly, \( \eta \) and \( \eta^{-1} \) are GN-continuous. Hence, \( \eta \) is neutrosophic g-homeomorphism.

(iii) → (i). Let \( \eta \) be neutrosophic g-homeomorphism, then \( \eta \) and \( \eta^{-1} \) are GN-continuous. As each NCS in \( X \) is a GNCS in \( Y \), therefore \( \eta \) is a GNC-mapping.

Definition 3.13 [19]: Let \( (X, \xi) \) be an NTS said to be a as neutrosophic-T_{1/2} (in short N-T_{1/2}) space if every GNCS is NCS in \( X \).

Proposition 3.14: Let \( \eta : (X, \xi) \to (Y, \sigma) \) be neutrosophic g-homeomorphism, then \( \eta \) is neutrosophic homoemorphism if \( X \) and \( Y \) are N-T_{1/2} space.

Proof: Consider that \( D \) is an NCS in \( Y \), then \( \eta^{-1}(D) \) is a GNCS in \( X \) due to the assumption. Since \( X \) is a N-T_{1/2} space, \( \eta^{-1}(D) \) is NCS in \( X \) . Then, \( \eta \) is GN-continuous. By hypothesis \( \eta^{-1} \) is GN-continuous. Let \( H \) be a NCS in \( X \). \( (\eta^{-1})^{-1}(H) = \eta(H) \) is a NCS in \( Y \), by preassumption. As \( Y \) is a N-T_{1/2} space, \( \eta(H) \) is a NCS in \( Y \). Hence, \( \eta^{-1} \) is N-continuous. Therefore, \( \eta \) is a neutrosophic homoemorphism.

Proposition 3.15: Let \( \eta : X \to Y \) and \( \mu : Y \to Z \) be GNC-mappings where \( X \) and \( Z \) are NTSs and \( Y \) is N-T_{1/2} space, then \( (\mu \circ \eta) \) is GNC-mapping.

Proof: Let \( R \) be a NCS in \( X \). As \( \eta \) is GNC-map and \( \eta(R) \) is a GNCS in \( Y \), by assumption, \( \eta(R) \) is a NCS in \( Y \). Since \( \mu \) is GNC-map, then \( \mu(\eta(R)) \) is a GNCS in \( X \) and \( Z \). Therefore, \( (\mu \circ \eta)(R) \) is a GNC-map.

Proposition 3.16: Let \( \mu : X \to Y \) and \( \lambda : Y \to Z \) be NTSs, then the following hold:
(i) If \( (\lambda \circ \mu) \) is GNO-map and \( \mu \) is N-continuous, then \( \lambda \) is GNO-map.
(ii) If \( (\lambda \circ \mu) \) is GNO-map and \( \mu \) is GN-continuous, then \( \lambda \) is GNO-map.
\textbf{Proof:} (i) Let $K$ be NOS in $Y$. Then, $\mu^{-1}(K)$ is a NOS in $X$. Since $(\lambda \circ \mu)$ GNO-map and $(\lambda \circ \mu)\mu^{-1}(K) = \lambda\left(\mu(\mu^{-1}(K))\right) = \lambda(K)$ is GN-open in $Z$, hence $\lambda$ is GN-open map.

(ii) Let $K$ be NOS in $X$. Then, $\lambda\left(\mu(K)\right)$ is a NOS in $Z$. Hence, $\lambda^{-1}(\lambda(\mu(K))) = \mu(K)$ is GNOS in $Y$. Therefore $\mu$ is GNO-map.

4. Neutrosophic $g^*$-Homeomorphism

\textbf{Definition 4.1:} A bijection $\mu : X \to Y$ is called neutrosophic $g^*$-homeomorphism if $\mu$ and $\mu^{-1}$ are GN-irresolute mappings.

\textbf{Proposition 4.2:} Every neutrosophic $g^*$-homeomorphism is a neutrosophic $g$-homeomorphism.

\textbf{Proof:} A map $\mu$ is a neutrosophic $g^*$-homeomorphism. Predict that $K$ is a NCS in $Y$. So it is a GNCS in $Y$. By presumption, $\mu^{-1}(K)$ is a GNCS in $X$. Accordingly, $\mu$ is GN-continuous mapping. Therefore, $\mu$ and $\mu^{-1}$ are GN-continuous mappings. Hence, $\mu$ is a neutrosophic $g$-homeomorphism.

\textbf{Remark 4.3:} The example is given to show that the reverse of the above proposition is not possible.

\textbf{Example 4.4:} Let $X = \{p, q, r\}$, $\xi = \{0_n, A_1, A_2, A_3, A_4, 1_n\}$ be a N-topology on $X$.

$A_1 = (x, (0,2,0,1,0,1), (0,2,0,1,0,1), (0,3,0,5,0,5)), A_2 = (x, (0,1,0,2,0,2), (0,4,0,3,0,3), (0,3,0,3,0,3))$,

$A_3 = (x, (0,2,0,2,0,2), (0,2,0,1,0,1), (0,3,0,3,0,3)), A_4 = (x, (0,1,0,1,0,1), (0,4,0,3,0,3), (0,3,0,5,0,5))$,

and let $Y = \{p, q, r\}, \sigma = \{0_n, B_1, B_2, B_3, B_4, 1_n\}$ be a neutrosophic topology on $Y$.

$B_1 = (y, (0,3,0,3,0,3), (0,2,0,1,0,1), (0,2,0,2,0,2)), B_2 = (y, (0,2,0,2,0,2), (0,1,0,1,0,1), (0,3,0,3,0,3)), B_3 = (y, (0,3,0,3,0,3), (0,1,0,1,0,1), (0,2,0,1,0,1), B_4 = (y, (0,2,0,2,0,2), (0,2,0,1,0,1), (0,3,0,3,0,3)).$

Define $\eta : (X, \xi) \to (Y, \sigma)$ by $\eta(p) = p, \eta(q) = q$ and $\eta(r) = r$. Then $\eta$ is neutrosophic g-homeomorphism but not neutrosophic $g^*$-homeomorphism.

\textbf{Proposition 4.5:} If $\mu : X \to Y$ and $\lambda : Y \to Z$ are neutrosophic $g^*$-homeomorphisms, then $(\lambda \circ \mu)$ is a neutrosophic $g^*$-homeomorphism.

\textbf{Proof:} Consider $\mu$ and $\lambda$ as neutrosophic $g^*$-homeomorphisms. Predict $K$ is a GNCS in $Z$. Thereupon, by the presumption, $\lambda^{-1}(K)$ is a GNCS in $Y$. Hence, by hypothesis, $\mu^{-1}(\lambda^{-1}(K))$ is a GNCS in $X$. Hence, $(\lambda \circ \mu)$ is a GN-irresolute mapping. Now, consider $H$ be a GNCS in $X$. Then, by the presumption, $\mu(H)$ is a GNCS in $Y$. So, by hypothesis, $\lambda(\mu(H))$ is a GNCS in $Z$. This implies that $(\lambda \circ \mu)$ is a GN-irresolute mapping. Therefore, $(\lambda \circ \mu)$ is neutrosophic $g^*$-homeomorphism.

\textbf{Proposition 4.6:} If $\mu : X \to Y$ is a neutrosophic $g^*$-homeomorphism, then $NGcl(\mu^{-1}(K)) = \mu^{-1}(NGcl(K))$ for each NS $K$ in $Y$.

\textbf{Proof:} As $\mu$ is neutrosophic $g^*$-homeomorphism, then $\mu$ is GN-irresolute mapping. Let $K$ be a NS in $Y$. Clearly, $NGcl(K)$ is GNCS in $X$. This proves that $GNcl(K)$ is GNCS in $X$. Since $\mu^{-1}(K) \subseteq NGcl(K)$, it follows that $\mu^{-1}(NGcl(K)) = NGcl(\mu^{-1}(K))$.
\( \mu^{-1}(\text{GNcl}(K)) \), then \( \text{GNcl}(\mu^{-1}(K)) \subseteq \text{GNcl}\left(\mu^{-1}(\text{GNcl}(K))\right) = \mu^{-1}(\text{GNcl}(K)) \). Therefore, \( \text{GNcl}(\mu^{-1}(K)) \subseteq \mu^{-1}(\text{GNcl}(K)). \)

Let \( \mu \) be neutrosophic g*-homeomorphism. \( \mu^{-1} \) is a GN-irresolute mapping. Consider NS \( \mu^{-1}(K) \) in \( X \), which implies that \( \text{GNcl}(\mu^{-1}(K)) \) is GNCS in \( X \). Therefore, \( \text{GNcl}(\mu^{-1}(K)) \) is a GNCS in \( X \). This implies that \( (\mu^{-1})^{-1}(\text{GNcl}(\mu^{-1}(K))) = \mu(\text{GNcl}(\mu^{-1}(K))) \) is a GNCS in \( Y \). This proves that \( K = (\mu^{-1})^{-1}(\mu^{-1}(K)) \subseteq (\mu^{-1})^{-1}\left(\text{GNcl}(\mu^{-1}(K))\right) = \mu(\text{GNcl}(\mu^{-1}(K))) \), since \( \mu^{-1} \) is GN-irresolute mapping. Hence, \( \mu^{-1}(\text{GNcl}(K)) \subseteq \mu^{-1}\left(\mu\left(\text{GNcl}(\mu^{-1}(K))\right)\right) = \text{GNcl}(\mu^{-1}(K)). \)

That is, \( \mu^{-1}(\text{GNcl}(K)) \subseteq \text{GNcl}(\mu^{-1}(K)). \) Hence, \( \text{GNcl}(\mu^{-1}(K)) = \mu^{-1}(\text{GNcl}(K)) \).

5. Conclusions

We have introduced neutrosophic generalized homeomorphism in neutrosophic topological space using GN-continuous functions. Some characterizations have been provided to illustrate how far topological structures are conserved by the new neutrosophic notion defined. Furthermore, neutrosophic g*-homeomorphism, neutrosophic generalized open and closed mappings are also studied. The study demonstrated neutrosophic g*-homeomorphisms and also proved some of their related attributes. Also, the relation between generalized neutrosophic closed mappings and other existed Neutrosophic closed mappings in Neutrosophic topological spaces were established and derived some of their related attributes. Examples are given wherever necessary.

In future, we can carry out the further research on neutrosophic g-compactness, neutrosophic g-connectedness and neutrosophic almost g-contra continuous functions.

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