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AN ANALYSIS OF THE SIGNAL PROCESSING IN A
RADIO ALTIMETER UTILIZING RANDOM ANGLE MODULATION

By
Donald F. Post

A Dissertation
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy in Electrical Engineering

The University of New Mexico

1966

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This dissertation, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of The University of New Mexico in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

DEAN

9-30-66

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AN ANALYSIS OF THE SIGNAL ~~AND~~ PROCESSING IN A RADIO ALTIMETER
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Committee

CHAIRMAN

403735

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TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGMENT	v
CHAPTER I INTRODUCTION	1
CHAPTER II STATISTICS OF THE L.O. SIGNAL	7
1. Distribution Density of Amplitude	8
2. Correlation Function and Spectrum	11
CHAPTER III STATISTICS OF THE RECEIVED PROCESS	21
1. Specification of the Received Process	21
2. Amplitude Distribution Density	22
3. Joint Distribution Density for the Modulated Wave	24
4. The Correlation Function and Spectrum	29
CHAPTER IV THE EXPECTED VALUE AND CORRELATION FUNCTION OF THE MULTIPLIER OUTPUT	31
1. Expected Value of the Multiplier Output	31
2. Signal-Cross-Signal Correlation Function When $\{S(t)\}$ and $\{X(t)\}$ are Correlated	33
3. Signal-Cross-Noise Correlation Function	42
4. Correlation Function in the Interval Where $\{S(t)\}$ and $\{X(t)\}$ are Uncorrelated	43
CHAPTER V THE EXPECTED VALUE AND ENERGY DENSITY SPECTRUM OF THE BANDPASS FILTER OUTPUT	44
1. The Average Covariance Function at the Multiplier Output	45
2. Average Energy Density Out of the Bandpass Filter	50
3. Expected Value at the Bandpass Filter Output	53

TABLE OF CONTENTS (cont.)

	<u>Page</u>
CHAPTER VI THE AVERAGE SPECTRUM AND EXPECTED VALUE AT THE OUTPUT OF THE SQUARE-LAW DEVICE	56
1. Average Energy Density Spectrum and Covariance Function at the Output	57
2. Expected Value at the Output	63
CHAPTER VII THE EXPECTED VALUE AND AVERAGE VARIANCE OF THE DECISION VOLTAGE	66
1. The Expected Value	66
2. The Average Variance	69
3. Application to Calculation of Probability of a Correct Decision	71
CHAPTER VIII NUMERICAL RESULTS	75
CHAPTER IX SUMMARY AND CONCLUSIONS	94
1. Limitations of the Analysis	94
2. Areas for Further Study	95
APPENDIX I SPECTRAL THEORY OF A LIMITED CLASS OF NONSTATIONARY RANDOM PROCESSES	97
1. General Discussion	97
2. The Spectral Representation	98
3. The Effect of a Linear Filter on Nonstationary Processes	104
BIBLIOGRAPHY	107

LIST OF ILLUSTRATIONS

<u>Figure</u>		<u>Page</u>
1	Mathematical Block Diagram of the Altimeter	2
2	Illustration of the Assumed Geometry	3
3	The Distribution Density Function of $\Phi = \phi_o + \psi(t)$	12
4	Amplitude Distribution Density of the Normalized Signal Process	12
5	Relative Power Density Spectrum of an RF Carrier Frequency Modulated by RC Filtered White Noise for Several Values of the Parameter $\gamma = \frac{D_o^2 \sigma^2}{2t_f}$	19
6	Computed D.F. at Bandpass Filter Output; $\tau = 0$	78
7	Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0$	79
8	Computed D.F. at Bandpass Filter Output; $\tau = 0.3 \times 10^{-8}$ sec	80
9	Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.3 \times 10^{-8}$ sec	81
10	Computed D.F. at Bandpass Filter Output; $\tau = 0.6 \times 10^{-8}$ sec	82
11	Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.6 \times 10^{-8}$ sec	83
12	Computed D.F. at Bandpass Filter Output; $\tau = 0.9 \times 10^{-8}$ sec	84
13	Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.9 \times 10^{-8}$ sec	85

LIST OF ILLUSTRATIONS (cont.)

<u>Figure</u>		<u>Page</u>
14	Computed D.F. at Low-Pass Filter Output; $\tau = 0$	86
15	Computed D.D. at Low-Pass Filter Output; $\tau = 0$	87
16	Computed D.F. at Low-Pass Filter Output; $\tau = 0.3 \times 10^{-8}$ sec	88
17	Computed D.D. at Low-Pass Filter Output; $\tau = 0.3 \times 10^{-8}$ sec	89
18	Computed D.F. at Low-Pass Filter Output; $\tau = 0.6 \times 10^{-8}$ sec	90
19	Computed D.D. at Low-Pass Filter Output; $\tau = 0.6 \times 10^{-8}$ sec	91
20	Computed D.F. at Low-Pass Filter Output; $\tau = 0.9 \times 10^{-8}$ sec	92
21	Computed D.D. at Low-Pass Filter Output; $\tau = 0.9 \times 10^{-8}$ sec	93

CHAPTER I

INTRODUCTION

This paper presents an analysis of the signal processing in a continuous wave, fixed-range, radio altimeter which utilizes random angle modulation, i.e., the modulating function is a sample function from a random process. The mathematical model of the altimeter is shown in block diagram form in Figure 1, and the physical situation in which the altimeter operates is illustrated in Figure 2. As shown in Figure 2, the altimeter is assumed to be on the vehicle "V" which has a relative velocity v with respect to the reflecting plane P. The velocity vector might have any orientation, but for simplicity we shall assume that it is colinear with the unit vector $\vec{i}_y$. We assume that the altimeter is turned on when "V" reaches an altitude $h_0 + \Delta h$, so that the first reflected signal reaches "V" at altitude h_0 . It is then required to provide an indication when "V" reaches altitude h_1 . Regardless of whether or not the altimeter functions properly, we assume that it ceases to operate when "V" reaches the plane P. Thus we consider only the interval of time required for "V" to travel from h_0 to the plane P.

Referring to Figure 1, the transmitted signal is a sample function of a random process. A delayed and attenuated version of the transmitted signal is used as a local oscillator (L.O.) signal. The reflected signal, which has a variable delay, is multiplied by the L.O. signal, and this product is then processed in a bandpass filter, square-law device, and low-pass filter to provide a decision voltage $E_d(t)$. When the relative delay between the reflected and L.O. signals is greater than the correlation time of the transmitted process, the expected value and variance of the decision voltage is a function of the input noise and the uncorrelated received process, i.e., the input to the signal processing circuits is random noise; however, when the relative

Figure 1. Mathematical Block Diagram of the Altimeter.

Note: Amplifiers have been omitted since, if they are not saturated, they represent a constant multiplication.

Figure 2. Illustration of the Assumed Geometry.

delay becomes less than the correlation time of the transmitted process, the expected value has an additional increase due to the correlation between the L.O. and reflected signal processes. This additional increase in the expected value differentiates the decision voltage in the region about h_1 from the decision voltage in a region about any other altitude, and it is the basis on which a decision is made.

The decision circuit is assumed to be a voltage comparator which compares a fixed threshold voltage E_{th} with $E_d(t)$; when $E_d(t)$ exceeds E_{th} , the decision circuit indicates that altitude h_1 has been reached. In order to maintain a small, and constant, probability that the decision circuit gives a false range indication, the noise intensity out of the multiplier must remain constant. We shall assume for purposes of false-alarm calculations that an automatic gain control (AGC) circuit holds the noise output at a constant intensity, but does not reduce the signal input when the L.O. and reflected signal process are correlated; however, the derivations can be carried through for nonstationary noise with very little increase in mathematical complexity. This is equivalent to assuming that the time constant of the AGC circuit is long enough so that the receiver gain remains essentially constant in one decision interval. (We define a decision interval as the time available to make a decision when the L.O. and reflected processes are correlated. This is a function of vehicle velocity and the allowable error.) We also assume that the AGC holds the receiver gain at a level such that none of the elements of the receiver are saturated when signal plus noise is present at the input.

The analysis consists of the following:

- a. Derivation of expressions for the distribution density (d.d.) and joint d.d. of amplitude of the L.O. and reflected processes. These derivations are valid for either a constant or random multiplier in the expression for these processes.

b. Derivation of expressions for the spectrum and correlation function of the received and L.O. processes when the modulating process is gaussian distributed. Again, these equations are valid for either constant or random multipliers in the expressions for these processes, provided that the random amplitude fluctuations are independent of the random phase changes.

c. Derivation of a functional expression for the correlation function and expected value at the output of the multiplier when the modulating process is gaussian.

d. Development of a method for handling the nonstationary processes which are inherent in this system.

e. Derivation of equations for the expected value and variance of the decision voltage for a set of conditions which approximates the operation of an actual system and application of these equations to the calculation of system performance. In these derivations, terms which would make a very small contribution to the total variance were ignored; thus the resulting expression for the variance is not exact, but it is adequate for practical system calculations.

In order to obtain concrete results, the derivation of the expressions for the expected value and variance of the output of the low-pass filter were carried through for specific impulse response functions for the bandpass and low-pass filters, and for specific forms of the covariance functions of the modulating process and the noise process. The conditions assumed approximate those which we might find in an actual system, except that the multiplying constant in the received process is assumed to constant; however, no difficulty will be encountered in applying the method to a random multiplier provided that the correlation function of the modulating function is known, and providing the random amplitude functions are independent of the random phase fluctuations.

The form assumed for the covariance function of the noise process corresponds to the case where the noise has the same spectrum as the L.O. process, but the method used in the analysis is applicable to any noise process, either stationary or nonstationary. The only other assumptions made about the noise are that it is additive and independent of the signal process.

The application of the results of the analysis of the specific system to the calculation of the probability of a correct decision as a function of input signal-to-noise ratio, while maintaining a given false-alarm probability, is explained and illustrated by a specific example. It is here that the assumption of constant input noise intensity becomes important, since a constant false-alarm probability cannot be maintained with a varying noise. The calculations of system performance are based on the assumption that the decision voltage is gaussian distributed, so that it is completely described by its expected value, variance, and correlation function. The basis of this assumption is an extended version of the central limit theorem [5], which states that the distribution of a sum of sets of random variables, where the sets are independent, but the random variables within a given set are not, is asymptotically gaussian. If the time constant of the low-pass filter is much greater than the time constant of the bandpass filter (a situation which is commonly encountered when there is velocity uncertainty), we can infer that the distribution of the decision voltage is approximately gaussian. The distribution cannot be exactly gaussian because the output of the filter is always positive; however, if the expected value is much greater than the standard deviation, the approximation can be very good.

A limited simulation of the system was carried out on a digital computer, and the results are reported in Chapter VIII. These data illustrate the change in the distribution functions at the output of the low-pass and bandpass filters as a function of position within the correlation interval. Also shown is the percent error between the actual distribution and a gaussian distribution with the same mean and variance.

CHAPTER II

STATISTICS OF THE L.O. SIGNAL

As shown in Figure 1, the local oscillator signal is a delayed version of the transmitted signal. Since the delay of the L.O. signal depends on the preset altitude, we shall assume, without loss of generality, that the delay is zero, so that it has the form

$$s(t) = A_s \cos[\omega_c t - \phi_0 - \psi(t)] \quad (\text{II.1})$$

where A_s is a constant amplitude, $\omega_c = 2\pi f_c$ is the carrier frequency, $\psi(t)$ is a sample function of a zero mean, stationary, gaussian distributed random process, and ϕ_0 is a random variable denoting the initial carrier phase. If we want to consider frequency modulation as a special case of angle modulation, the random phase factor can be assumed to be derived from a process $\{\eta(t)\}$, where $\{\eta(t)\} = \left\{ \frac{d}{dt} \psi(t) \right\}$. [We shall use the standard notation $\{X(t)\}$ for the random process whose sample functions are $x_\zeta(t)$.]

In this chapter we shall derive a general expression for the d.d. of the amplitude of the L.O. signal, as well as the covariance function and spectrum. (The background material required for the calculation of the covariance function may be found in Chapters 8 and 14 of [10]*.)

* In this report, a number in brackets will designate one of the numbered references in the Bibliography.

1. Distribution Density of Amplitude

Define the random variable Γ by

$$\Gamma = \frac{S}{A_s} \quad (\text{II.2})$$

where S is the random variable denoting the amplitude of the modulated function. We wish to calculate the d.d. of Γ , which we denote by $f_\Gamma(x)$. If θ is limited to the interval $[-\pi, \pi]$, and $y = \cos \theta$, the expression for $f_y(y)$ is

$$f_y(y) = \frac{2f_\theta(\cos^{-1}y)}{(1-y^2)^{1/2}} \quad (\text{II.3})$$

hence, if we can transform the range of the phase in (II.1) to the interval $[-\pi, \pi]$, and compute the d.d. of the transformed function, the problem is solved.

Let us define a random variable Φ by

$$\Phi = \phi_0 + \Psi \quad (\text{II.4})$$

where ϕ_0 and Ψ are the random phases in (II.1). Since ϕ_0 and Ψ are statistically independent, the d.d. of Φ is given by

$$f_\Phi(\theta) = \int_{-\infty}^{\infty} f_{\phi_0}(\theta - x) f_\Psi(x) dx$$

which becomes

$$f_{\Phi}(\theta) = \frac{1}{2\pi} \int_{\theta-\pi}^{\theta+\pi} f_{\Psi}(x) dx \quad (II.5)$$

since ϕ_0 is by hypothesis uniformly distributed on the interval $[-\pi, \pi]$. We now define $\Phi_1 = \omega_c t - \Phi$. The d.d. of Φ_1 is then

$$f_{\Phi_1}(\theta, t) = \frac{1}{2\pi} \int_{\omega_c t + \theta - \pi}^{\omega_c t + \theta + \pi} f_{\Psi}(x) dx. \quad (II.6)$$

In order to compute the d.d. of $\hat{\Phi}$, where $\hat{\Phi}$ is the random variable Φ_1 transformed into the interval $[-\pi, \pi]$, we divide the domain θ into intervals of length 2π , where the n^{th} interval is $[(2n-1)\pi, (2n+1)\pi]$. Then define $\theta_n = \theta \mp 2n\pi$, where $-\pi \leq \theta \leq \pi$. The d.d. of Φ_{1n} , i.e., the random variable Φ_1 in the n^{th} interval, is then

$$f_{\Phi_{1n}}(\theta_n, t) = \frac{1}{2\pi} \int_{\omega_c t + \theta_n - \pi}^{\omega_c t + \theta_n + \pi} f_{\Psi}(x) dx \quad (II.7)$$

and

$$f_{\hat{\Phi}}(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f_{\Phi_{1n}}(\theta_n, t) \quad (II.8)$$

or, in terms of the distribution function of $\{\Psi(t)\}$,

$$f_{\hat{\Phi}}(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[F_{\Phi_{1n}}(\omega_c t + \theta_n + \pi) - F_{\Phi_{1n}}(\omega_c t + \theta_n - \pi) \right] . \quad (II.9)$$

A little thought will convince us that (II.8) and (II.9) give the same result for any value of t ; hence $\{\hat{\Phi}(t)\}$ is first-order stationary. Since this is the case, we may as well evaluate these equations at $t = 0$. [We are assuming here that the process is initiated at $t = t_0$; $t_0 < 0$.] We then have

$$f_{\hat{\Phi}}(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{\theta_n - \pi}^{\theta_n + \pi} f_{\Psi}(x) dx \quad (II.10)$$

or

$$f_{\hat{\Phi}}(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[F_{\Psi}(\theta_n + \pi) - F_{\Psi}(\theta_n - \pi) \right] . \quad (II.11)$$

The above expressions are applicable for any d.d. $f_{\Psi}(x)$. For the case under consideration here, $f_{\Psi}(x)$ has the form

$$f_{\Psi}(x) = \frac{1}{\sigma_{\Psi} \sqrt{2\pi}} e^{-x^2 / 2\sigma_{\Psi}^2} .$$

Plots of $f_{\hat{\Phi}}(\theta)$, using this d.d. for values of $\sigma_{\psi} = \pi$ and 10π , are shown in Figure 3. We thus observe that the phase is uniformly distributed over $[-\pi, \pi]$, so that the d.d. of Γ is given by (II.3). While this result has been shown here only for the case of a gaussian distribution of $\{\Psi(t)\}$, it seems intuitively obvious that the result would hold for any smooth d.d. of infinite extent which is symmetric about the origin. Plots of $f_{\Gamma}(x)$ for the same values of σ_{ψ} used in plotting Figure 3 are shown in Figure 4.

2. Correlation Function and Spectrum

The correlation function of the L.O. process is given by

$$R_s(t_1, t_2) = A_s^2 E \left\{ \cos \left[\omega_c t_1 - \phi_o - \psi_1 \right] \cos \left[\omega_c t_2 - \phi_o - \psi_2 \right] \right\} \quad (\text{II.12})$$

where $\psi_1 = \psi(t_1)$, and $\psi_2 = \psi(t_2)$. Expanding the cosine product into the cosines of the sum and difference of the angles, we obtain

$$R_s(t_1, t_2) = \frac{A_s^2}{2} E \left\{ \cos \left[\omega_c (t_1 - t_2) - \psi_1 + \psi_2 \right] + \cos \left[\omega_c (t_1 + t_2) - 2\phi_o - \psi_1 - \psi_2 \right] \right\}$$

Assuming that the distribution of $\psi_1 - \psi_2$ is symmetric about zero, we have*

$$R_s(t_1, t_2) = \frac{A_s^2}{2} \text{Re} \left[e^{-i\omega_c (t_2 - t_1)} E \left\{ e^{i(-\psi_1 + \psi_2)} \right\} \right]$$

* If θ is uniformly distributed, $E\{\cos \theta\} = 0$, so that any cosine term which contains ϕ_o will have zero expected value.

Figure 3. Computed Distribution Density Function of the Random Phase $\hat{\phi}$ (Values computed for $\sigma = \pi$ and $\sigma = 10\pi$)

Figure 4. Amplitude Distribution Density of the Normalized Signal Process.

or

$$R_s(t_1, t_2) = \frac{A^2}{2} \operatorname{Re} \left[e^{-i\omega_c(t_2 - t_1)} C_\Psi(-1, 1; t_1, t_2) \right] \quad (\text{II.13})$$

where $C_x(\xi_1, \xi_2; t_1, t_2)$ is the joint characteristic function of $\{X(t)\}$, and "Re" denotes "the real part of." Since $\{\psi(t)\}$ is stationary, the joint characteristic function depends only on $(t_2 - t_1)$ as does $e^{-i\omega_c(t_2 - t_1)}$; hence $R_s(t_1, t_2)$ may be written

$$R_s(t_1, t_2) = R_s(t_2 - t_1) = R_s(\tau) = \frac{A^2}{2} \operatorname{Re} \left[e^{-i\omega_c \tau} C_\Psi(-1, 1; \tau) \right] \quad (\text{II.14})$$

showing that $\{S(t)\}$ is also second-order stationary. If we define $\xi = -\psi_1 + \psi_2$, (II.14) can be written in terms of the first-order characteristic function of $\{\xi\}$; however, this does not reduce the labor required in the computations, so we will be content with (II.14).

Equation (II.14) is a general expression for the correlation function of an angle modulated carrier, and holds for all distributions of $\{\Psi(t)\}$ for which $E_{\sin} \{\psi_1 - \psi_2\} = 0$; unfortunately, the class of distributions for which the joint characteristic function can be written in closed form is rather limited. For the case considered here, $\{\Psi(t)\}$ is gaussian distributed and the joint characteristic function is given by [12]

$$C_\Psi(\xi_1, \xi_2; \tau) = \exp \left\{ -\frac{1}{2} Q(\xi_1, \xi_2) \right\} \quad (\text{II.15})$$

where

$$Q(\xi_1, \xi_2) = \sum_{i,j=1}^2 \mu_{ij} \xi_i \xi_j$$

and

$$\mu_{ij} = E\{\psi_i \psi_j\}$$

hence

$$\begin{aligned} R_s(\tau) &= \frac{A_s^2}{2} \operatorname{Re} \left[\exp \left\{ -i\omega_c \tau - \frac{1}{2} \sum_{i,j=1}^2 \xi_i \xi_j \mu_{ij} \right\} \right] \\ &= \frac{A_s^2}{2} \cos \omega_c \tau \exp \left\{ -[K_\psi(0) - K_\psi(\tau)] \right\} \end{aligned} \quad (\text{II.16})$$

We can write (II.16) in terms of the spectrum of $\{\Psi(t)\}$ by utilizing the Wiener-Khinchin theorem to replace $K_\psi(\tau)$ by

$$K_\psi(\tau) = 2 \int_0^\infty W_\psi(f) \cos 2\pi f \tau df$$

hence, $R_s(\tau)$ may be written as

$$R_s(\tau) = \frac{A_s^2}{2} \cos \omega_c \tau \exp \left\{ -2 \int_0^\infty W_\psi(f) [1 - \cos 2\pi f \tau] df \right\}. \quad (\text{II.17})$$

For the special case of frequency modulation, where $\eta(t) = d/dt \psi(t)$, the expression for the correlation function is

$$R_s(\tau) = \frac{A_s^2}{2} \cos \omega_c \tau \exp \left\{ -2 \int_0^\infty W_\eta(f) [1 - \cos 2\pi f \tau] \frac{df}{(2\pi f)^2} \right\}. \quad (\text{II.18})$$

The power density spectrum of the angle modulated process is obtained by applying the Wiener-Khinchin theorem to (II.17); hence

$$W_S(f) = A_s^2 \int_0^\infty \cos 2\pi f_c \tau \cos 2\pi f \tau \exp \{-2\Omega(\tau)\} d\tau \quad (\text{II.19})$$

where $\Omega(\tau) = \int_0^\infty W_\psi(f) [1 - \cos 2\pi f \tau] df$. For the case of frequency modulation the same expression applies except that here

$$\Omega(\tau) = \int_0^\infty \frac{W_\eta(f)}{(2\pi f)^2} [1 - \cos 2\pi f \tau] df.$$

As an example of the application of (II.19), assume that $\{\Psi(t)\}$ is obtained by passing stationary, gaussian distributed, white noise through an RC low-pass amplifier with gain D. Then, since $W_{\Psi}(f) = |H(f)|^2 W_i(f)$ where $H(f)$ is the amplifier system function, and $W_i(f)$ is the input spectrum,

$$W_{\Psi}(f) = \frac{D^2 N_o a^2}{a^2 + (2\pi f)^2} \quad (\text{II.20})$$

where $a = 1/t_f$, with t_f the time constant of the filter, and N_o is the amplitude of the white noise spectrum. Substituting (II.20) into the expression for $\Omega(\tau)$, we have

$$\Omega(\tau) = \int_0^{\infty} \frac{D^2 N_o a^2 [1 - \cos 2\pi f \tau]}{a^2 + (2\pi f)^2} df = \int_0^{\infty} \frac{D^2 N_o a^2 \sin^2 \pi f \tau}{a^2 + (2\pi f)^2} df$$

and so

$$\Omega(\tau) = \frac{D^2 N_o a}{4} (1 - e^{-a\tau}) \quad (\text{II.21})$$

Substituting (II.21) in (II.19), and replacing the product of cosines by the cosine of the sum and difference of the angles we have

$$W_S(f) = \frac{A_s^2}{2} \int_0^{\infty} \left\{ \cos [2\pi \tau (f - f_c)] + \cos [2\pi \tau (f + f_c)] \right\} \exp \left\{ - \frac{D^2 N_o a}{2} (1 - e^{-a\tau}) \right\} d\tau \quad (\text{II.22})$$

In the case of frequency modulation the corresponding expression is

$$W_S(f) = \frac{A_s^2}{2} \int_0^{\infty} \left\{ \cos \left[2\pi\tau (f - f_c) \right] + \cos \left[2\pi\tau (f + f_c) \right] \right\} \exp \left\{ -\frac{D^2 N_o}{2} \left(\tau - t_f + t_f e^{-\tau/t_f} \right) \right\} d\tau . \quad (\text{II.23})$$

Equation (II.22) is an exact expression for $W_S(f)$; however, the term involving $\cos \left[2\pi\tau (f + f_c) \right]$ will contribute little to the integral, so that

$$W_S(f') \approx \frac{A_s^2}{2} \int_0^{\infty} \cos(2\pi\tau f') \exp \left\{ -\frac{D^2 N_o a}{2} (1 - e^{-a\tau}) \right\} d\tau$$

where $f' = f - f_c$. In many cases of interest, the exponential factor in the above integral is already approaching zero for values of $\tau \ll t_f$; hence, for all values of τ of interest, we can approximate $e^{-a\tau}$ by

$$e^{-a\tau} \approx 1 - a\tau$$

so that

$$W_S(f') \approx \frac{A_s^2}{2} \int_0^{\infty} \cos 2\pi f' \tau \exp \left\{ -\frac{a^2 D^2 N_o \tau}{2} \right\} d\tau . \quad (\text{II.24})$$

Evaluating (II.24), we obtain

$$W_S(f') \approx \frac{2}{a^2 D^2 N_o + 4 \frac{\omega'^2}{a^2 D^2 N_o}} ; \quad \omega' = 2\pi f' . \quad (\text{II.25})$$

The corresponding expression for the frequency modulated process is

$$W_S(f') \approx \frac{A^2}{2D} \sqrt{\frac{\pi t_f}{N_o}} \exp \left\{ - \frac{\omega'^2 t_f}{D^2 N_o} \right\} . \quad (\text{II.26})$$

where we have approximated $e^{-a\tau}$ by $1 - a\tau + \frac{a^2 \tau^2}{2}$.

We see therefore that, under these conditions, phase modulation yields a rational spectrum, while frequency modulation yields a gaussian spectrum.

To illustrate the effect of the parameter $\gamma = \frac{D^2 N_o}{2t_f}$ on the spectrum, plots of (II.23) for various values of γ are shown in Figure 5. Note the spectral spreading as γ increases.

In the derivation of the equations for the mean and variance of the decision voltage, we assume that the correlation function of the modulating process has the form

$$R(\tau) = \sigma^2 \left(1 - \frac{|\tau|}{\tau_c} \right) \quad (\text{II.27})$$

where τ_c is the effective correlation time. This is, of course, the familiar triangular function. Strictly speaking, this process could not occur in a

Figure 5. Relative Power Density Spectrum of an RF Carrier Frequency Modulated by RC Filtered White Noise for Several Values of the Parameter

$$\gamma = \frac{D_o^2 \sigma^2}{2t_f}$$

frequency modulation system since the mean square derivative of a process with this correlation function does not exist; however, for an FM system, we can assume that the correlation function has the form of (II.27) except in the neighborhood of the origin, where it has the form $R(\tau) = (\sigma^2/\tau_c) e^{-\tau^2/\tau_c^2}$. Then the derivative process does exist. This is just an artifice to get around the mathematical inconvenience of an undefined derivative process.

The spectrum of the angle modulated waveform for this case is easily obtained, since, from (II.16), the correlation function has the form

$$R(\tau) = \frac{A_s^2}{2} \cos \omega_c \tau e^{-\frac{\sigma^2 |\tau|}{\tau_c}} \quad (\text{II.28})$$

hence, taking the transform of (II.28), the spectrum of the transmitted process is

$$W_S(f) = \frac{A_s^2}{4} \left[\frac{1}{\left(\frac{\alpha}{2}\right)^2 + (\omega - \omega_c)^2} \right] + \left[\frac{1}{\left(\frac{\alpha}{2}\right)^2 + (\omega + \omega_c)^2} \right] \quad (\text{II.29})$$

where $\alpha = 2\sigma^2/\tau_c$. Hence, we observe that we have a rational spectrum.

CHAPTER III

Statistics of the Received Process

In this chapter we shall specify the form of the received process, and derive the statistics of interest. In particular we shall derive expressions for the d.d. of amplitude, the joint d.d., and the correlation function and density spectrum. The d.d., and the joint d.d., of amplitude will be derived for both the "point target" case and the "scattering target" case; however, the specific equations for the correlation function and spectrum are derived only for the "point target" case. In the case of random amplitude perturbations of the reflection process, only the general equations are derived.

1. Specification of the Received Process

We shall consider a received process which may either have the same form as the transmitted process, or may be a version of the transmitted signal with random perturbations. Since there is a relative velocity between the transmitter and receiver, the delay between the L.O. signal and the received signal is a function of time. In the case of random perturbations, we shall assume that the random fluctuations occur only in the amplitude, and that these fluctuations are independent of the random phase. Under these conditions, the forms of the received signal in the two cases are:

- a. Received signal of same form as L.O. signal:

$$X(t) = A_r(t) \cos \left\{ \omega_c [t - \tau(t)] - \phi_0 - \psi[t - \tau(t)] \right\} \quad (\text{III.1})$$

where $A_r(t)$ is a deterministic function of time, and $\tau(t)$ is the time dependent delay.

b. Received signal perturbed by random fluctuations:

$$X(t) = A_r(t) \cos \left\{ \omega_c [t - \tau(t)] - \phi_0 - \psi[t - \tau(t)] \right\} \quad (\text{III.2})$$

where $A_r(t)$ is a sample function of a stationary random process.

In both of the above equations, $\tau(t)$ has the form, neglecting relativistic effects,

$$\tau(t) = t_0 - \lambda t ; t_0 = \text{initial delay at time } t_1 ; \lambda = \frac{2v}{c} . \quad (\text{III.3})$$

Note that we are to consider the received process only in the interval from $t = t_0$ to $t = \frac{t_0 + \tau_c}{\lambda}$, where τ_c is the effective correlation time of the transmitted process, although the equations for the correlation function and spectrum are derived on the basis of sample functions of infinite length. The changes caused by truncating the sample functions will be included in the subsequent analysis.

2. Amplitude Distribution Density

The amplitude d.d. of the received process is obtained in the same manner as the d.d. of the L.O. signal; hence, in the case of (III.1) we have

$$X(t) = A_1 \left[\cos \omega_c (t + \lambda t - t_0) - \phi_0 - \psi(t + \lambda t - t_0) \right] \quad (\text{III.4})$$

and the d.d. of $\hat{\Phi}$, where $\hat{\Phi}$ is the mapping of $\Phi = \phi_0 + \Psi$ into the interval $[-\pi, \pi]$, is

$$f_{\hat{\Phi}}(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{\omega_c t' + \theta_n - \pi}^{\omega_c t' + \theta_n + \pi} f_{\Psi}(y) dy \quad (\text{III.5})$$

where $\theta_n = \theta \pm 2n\pi$; $-\pi \leq \theta \leq \pi$, and $t' = t + \lambda t - t_0$. We observe again that (III.5) is the same for every value of t' , and evaluate at $t' = 0$, giving

$$f_{\hat{\Phi}}(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{\theta_n - \pi}^{\theta_n + \pi} f_{\Psi}(y) dy. \quad (\text{III.6})$$

The d.d. of $\Gamma = x/A_r$ is then given by

$$f_{\Gamma}(r) = \frac{2f_{\hat{\Phi}}(\cos^{-1} r)}{(1 - r^2)^{1/2}}. \quad (\text{III.7})$$

In the case of (III.2) the situation is somewhat different; here we have the product of two random variables, and we wish to find the d.d. of the product. It is shown in [12] that if $Z = XY$, where X and Y are random variables, the d.d. of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{xy}(x, z/x) dx. \quad (\text{III.8})$$

For this case x and y are independent, so that (III.8) may be written

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_y(z/x) f_x(x) dx . \quad (\text{III.9})$$

Applying (III.9) to our problem, we have

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{|\gamma|} f_A\left(\frac{x}{\gamma}\right) f_T(\gamma) d\gamma$$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{|\gamma|} f_A(x/\gamma) \left[\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{f_{\theta_n}(\cos^{-1} \gamma)}{(1 - \gamma^2)^{1/2}} \right] d\gamma \quad (\text{III.10})$$

where

$$f_{\theta_n}(\theta_n) = \int_{\theta_n - \pi}^{\theta_n + \pi} f_{\Psi}(y) dy .$$

Given f_A and f_{θ} , we can evaluate the expression for the d.d. of the amplitude of the return process, although in general it cannot be evaluated in closed form.

3. Joint Distribution Density for the Modulated Wave

It is of interest to calculate the joint d.d. for the case in which the return signal is of the same form as the transmitted signal; the joint density

in the case in which the received signal has random amplitude variations can then be obtained by an extension of the results of Section 2. It is shown in [12] that if x and y are random variables, and new random variables are defined by

$$\left. \begin{aligned} z &= g(x, y) \\ w &= h(x, y) \end{aligned} \right\} \quad (\text{III.11})$$

the joint density $f_{ZW}(z, w)$ is given by

$$f_{ZW}(z, w) = \frac{f_{xy}(x_1, y_1)}{|J(x_1, y_1)|} + \dots + \frac{f_{xy}(x_n, y_n)}{|J(x_n, y_n)|} + \dots$$

where (x_n, y_n) are number pairs which are real solutions of $z = g(x, y)$ and $w = h(x, y)$, and $J(x, y)$ is defined by

$$J(x, y) = \begin{vmatrix} \frac{\partial g(x, y)}{\partial x} & \frac{\partial g(x, y)}{\partial y} \\ \frac{\partial h(x, y)}{\partial x} & \frac{\partial h(x, y)}{\partial y} \end{vmatrix}$$

Applying this result to the present problem, we obtain

$$f_{X_1, X_2}(x_1, x_2) = \sum_{n=-\infty}^{\infty} \frac{f_{\Phi_1, \Phi_2}(\cos^{-1} x_1, \cos^{-1} x_2)}{|\sin \theta_{1n} \sin \theta_{2n}|} \quad (\text{III.12})$$

where $f_{\Phi_1\Phi_2}(\phi_1, \phi_2)$ is the joint density function of the random phase. Hence, if an expression for $f_{\Phi_1\Phi_2}$ can be found, the required joint density is specified.

In order to compute the joint density of the random phase, define

$$\Phi(t') = \omega_c(t - t_0) - \phi_0 - \psi(t - t_0) = \Gamma(t') - \psi(t')$$

and consider

$$F_{\Phi_1\Phi_2|\Gamma_1} = P[\Phi_1 < \phi_1, \Phi_2 < \phi_2 | \Gamma_1 = \gamma_1] = P[\psi_1 < \Gamma_1 - \phi_1, \psi_2 < \Gamma_1 + \omega_c\tau - \phi_2 | \Gamma_1 = \gamma_1].$$

Now

$$F_{\Phi_1\Phi_2}(\phi_1, \phi_2) = \int_{-\infty}^{\infty} F_{\Phi_1\Phi_2|\Gamma_1} dF_{\Gamma_1} = \int_{-\infty}^{\infty} F_{\Phi_1\Phi_2|\Gamma_1} f_{\Gamma_1}(\gamma_1) d\gamma_1 \quad (\text{III.13})$$

but

$$\begin{aligned} F_{\Phi_1\Phi_2|\Gamma_1} &= F_{\psi_1\psi_2|\Gamma_1}(\gamma_1 - \phi_1, \gamma_1 + \omega_c\tau - \phi_2 | \gamma_1) \\ &= \int_{-\infty}^{\gamma_1 - \phi_1} \int_{-\infty}^{\gamma_1 + \omega_c\tau - \phi_2} f_{\psi_1\psi_2}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

so that

$$F_{\Phi_1\Phi_2}(\phi_1, \phi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma_1 - \phi_1} \int_{-\infty}^{\gamma_1 + \omega_c\tau - \phi_2} f_{\psi_1\psi_2}(x_1, x_2) f_{\Gamma_1}(\gamma_1) dx_1 dx_2 d\gamma_1.$$

The joint d.d. is then given by .

$$\frac{\partial^2 F(\phi_1, \phi_2)}{\partial \phi_2 \partial \phi_1}$$

so that

$$f_{\phi_1 \phi_2}(\phi_1, \phi_2) = \int_{-\infty}^{\infty} f_{\Psi_1 \Psi_2}(\phi_1 - \gamma_1, \phi_2 - \omega_c \tau - \gamma_1) f_{\Gamma_1}(\gamma_1) d\gamma_1 \quad (\text{III.14})$$

where $\tau = t'_2 - t'_1$.

When $\{\Psi(t)\}$ is gaussian distributed, and $\{\Gamma_1(t')\}$ is uniformly distributed on $[\omega_c t'_1 - \pi, \omega_c t'_1 + \pi]$, (III.14) becomes

$$f_{\phi_1 \phi_2}(\phi_1, \phi_2) = \int_{\omega_c t'_1 - \pi}^{\omega_c t'_1 + \pi} \frac{1}{2\pi\sigma^2(1-r^2)^{1/2}} \exp \left\{ \frac{-1}{2\sigma^2(1-r^2)} \left[(\phi_1 - \gamma_1)^2 - 2r(\phi_1 - \gamma_1)(\phi_2 - \omega_c \tau - \gamma_1) + (\phi_2 - \omega_c \tau - \gamma_1)^2 \right] \right\} d\gamma_1$$

where r is the correlation coefficient of $\{\Psi(t)\}$. Expanding the exponent,

collecting the terms in γ_1 and completing the square, and observing that the joint density is stationary, we obtain

$$\begin{aligned}
 f_{\phi_1 \phi_2}(\phi_1, \phi_2) &= \frac{1}{2\pi\sigma^2(1-r^2)^{1/2}} \\
 &\cdot \exp \left\{ \frac{-1}{2\sigma^2(1-r^2)} \left[\phi_1^2 + \phi_2^2 + (\omega_c \tau + 2r\phi_1)(\omega_c \tau - 2\phi_2) \right] \right\} \\
 &\cdot \exp \left\{ \frac{1-r}{2\sigma^2(1-r^2)} \left[\phi_1 \left(\frac{1+r}{1-r} \right) + \phi_2 \left(\frac{1+r}{1-r} \right) - \omega_c \tau \right]^2 \right\} \\
 &\cdot \int_{-\pi}^{\pi} \exp \left\{ \frac{-1}{\sigma^2(1-r)} \left(\gamma_1 - \frac{1}{2} \left[\phi_1 \left(\frac{1+r}{1-r} \right) + \phi_2 \left(\frac{1+r}{1-r} \right) - \omega_c \tau \right] \right)^2 \right\} dy_1.
 \end{aligned}
 \tag{III.15}$$

Substitution of (III.15) into (III.12) yields the expression for the joint amplitude density of the reflected process. If we could evaluate the resulting equation, the correlation function of the reflected process could be calculated directly; however, in the case of a gaussian distribution of $\{\psi(t)\}$, the correlation function of the process is more easily derived by the methods of Chapter II. For distributions other than gaussian, evaluation of (III.12), after substitution of (III.15), and direct computation of $E\{X_1 X_2\}$ may be the best way of determining the correlation function.

4. The Correlation Function and Spectrum

The correlation function and spectrum of the received process can be obtained directly from the corresponding expressions for the transmitted process in the case in which the received process has the same form as the transmitted process; thus we have

$$\left. \begin{aligned} R_x(\tau) &= \frac{A_r^2}{2} \cos \omega' \tau e^{-(K_\psi(0) - K_\psi(\tau + \lambda \tau))} \\ W_x(f) &= A_r^2 \int_0^\infty \cos 2\pi f' \tau \cos 2\pi f \tau e^{-2\Omega(\tau)} d\tau \end{aligned} \right\} \quad (\text{III.16})$$

where $\Omega(\tau) = \int_0^\infty W_\psi(f) [1 - \cos 2\pi f \tau] df$, and $f' = f_c - f_d$. Note that these functions differ from the expressions for the transmitted process only in the shift of the center frequency due to the relative velocity, i.e., the doppler shift, and the factor $\lambda \tau$ in the argument of the covariance function of the modulating process.

In the case in which the received process has random amplitude perturbations, the expression for the correlation function is

$$R_x(\tau) = E\{A_1 \cos \Phi_1 A_2 \cos \Phi_2\} \quad (\text{III.17})$$

where A_i, Φ_i are the values of $\{A(t)\}$ and $\{\Phi(t)\}$ at time t_i . Since A and Φ are statistically independent we have

$$R_x(\tau) = E\{A_1 A_2\} E\{\cos \Phi_1 \cos \Phi_2\} = R_A(\tau) R_\Phi(\tau) \quad (\text{III.18})$$

and the spectrum is

$$\Phi_x(f) = \int_{-\infty}^{\infty} \Phi_A(f - \zeta) \Phi_r(\zeta) d\zeta . \quad (\text{III.19})$$

Hence, the random perturbations spread the spectrum of the return signal, with the amount of spreading depending on the shape and duration of the correlation of $\{A(t)\}$. If the correlation function of $\{A(t)\}$ varies much more slowly than that of the transmitted process, the spreading is negligible. Given the form of $\Phi_A(f)$ the spectrum can be determined from (III.18). [We are assuming that $A_r(t)$ is a stationary process.]

CHAPTER IV

THE EXPECTED VALUE AND CORRELATION FUNCTION OF THE MULTIPLIER OUTPUT

The output of the multiplier, when both signal and noise are present at the input, has the form

$$y(t) = s(t)[x(t) + n(t)] \quad (\text{IV.1})$$

where $\{S(t)\}$ is the L.O. process, $\{X(t)\}$ is the received process, and $\{N(t)\}$ is a noise process. In this chapter we shall calculate the expected value and the correlation function of the multiplier output for sample functions which extend over an infinite time interval. The effect of truncating the sample functions after a finite time is discussed in Appendix I.

1. Expected Value of the Multiplier Output

The expected value of the multiplier output is

$$E\{y(t)\} = E\{s(t)x(t)\} + E\{s(t)n(t)\} \quad (\text{IV.2})$$

where $E\{s(t)x(t)\}$ is the expected value of the signal-cross-signal (SXS) term, and $E\{s(t)n(t)\}$ is the expected value of the signal-cross-noise (SXN) term.

For the SXS term we have

$$E\{s(t)x(t)\} = E\left\{A_r A_s \cos\left[\omega_c t - \phi_o - \psi(t)\right] \cdot \cos\left[\omega_c (t + \lambda t - t_o) - \phi_o - \psi(t + \lambda t - t_o)\right]\right\}$$

where A_s is the constant amplitude of the transmitted process, and A_r is the amplitude of the received process. A_r may, or may not, be a random variable. If it is a random variable, we assume that it is independent of the random phase; thus, expanding the cosine product into a sum of cosines, and noting that the expected value of the cosine of the sum is zero, we have

$$E\{s(t)x(t)\} = E\left\{\frac{A_r A_s}{2}\right\} E\left\{\cos\left[(t_0 - \lambda t)\omega_c - \psi(t) + \psi(t + \lambda t - t_0)\right]\right\}$$

which may be written

$$E\{s(t)x(t)\} = E\left\{\frac{A_r A_s}{2}\right\} \operatorname{Re}\left[e^{-i\omega_c(\lambda t - t_0)} E\left\{e^{+i[\psi(t) - \psi(t + \lambda t - t_0)]}\right\}\right]. \quad (\text{IV.3})$$

But the expected value of the exponential is the joint characteristic function of $\{\Psi(t)\}$, which, since $\{\Psi(t)\}$ is assumed to be gaussian, is

$$C_{\Psi}(1, -1; \lambda t - t_0) = e^{-\left[\sigma^2 - K_{\Psi}(-\lambda t + t_0)\right]} \quad (\text{IV.4})$$

where $K_{\Psi}(t_0 - \lambda t)$ is the covariance function of $\{\Psi(t)\}$. In this study we shall assume that the covariance function of $\{\Psi(t)\}$ has the form

$$\left. \begin{aligned} K_{\Psi}(\tau) &= \sigma^2 \left(1 - \frac{|\tau|}{\tau_c}\right); \quad 0 \leq |\tau| \leq \tau_c \\ &= 0; \text{ otherwise.} \end{aligned} \right\} \quad (\text{IV.5})$$

This form is a reasonable approximation to the exponential function which one might expect to have in a practical application, and, at the same time, it simplifies the computations. Substituting (IV.4) and (IV.5) in (IV.3), we obtain

$$\left. \begin{aligned} E\{s(t)x(t)\} &= \frac{1}{2} E\{A_r A_s\} \cos[\omega_c t'] e^{-\frac{\sigma^2 |t'|}{\tau_c}} ; |t'| \leq \tau_c \\ &= \frac{1}{2} E\{A_r A_s\} \cos[\omega_c t'] e^{-\sigma^2} ; |t'| > \tau_c \end{aligned} \right\} \quad (IV.6)$$

where $t' = t - \lambda t$.

Since we assume that the transmitted and received processes are statistically independent of the noise process, the expected value of the sXn term is

$$E\{s(t)n(t)\} = E\{s(t)\}E\{n(t)\} = 0$$

because, by hypothesis, $E\{n(t)\} = 0$. Thus, the expected value of the multiplier output is given by (IV.6).

2. Signal-Cross-Signal Correlation Function

When $\{S(t)\}$ and $\{X(t)\}$ are Correlated

The correlation function of the multiplier output is by definition

$$R_{yy}(t_1, t_2) = E\{y(t_1)y(t_2)\} \quad (IV.7)$$

and the covariance function is, by definition,

$$K_{yy}(t_1, t_2) = R_{yy}(t_1, t_2) - \overline{y(t_1)} \overline{y(t_2)}$$

where $\overline{y(t)} = E\{y(t)\}$. From (IV.1), and the fact that the transmitted and received processes are statistically independent of the noise process, we have

$$R_{yy}(t_1, t_2) = E\{s(t_1)s(t_2)x(t_1)x(t_2)\} + R_{ss}(t_1, t_2)R_{nn}(t_1, t_2) \quad (\text{IV.8})$$

where $R_{nn}(t_1, t_2)$ is the correlation function of the noise process. We shall assume that the noise process is stationary, with a correlation function of the form

$$K_{nn}(\tau) = \frac{A_n^2}{2} \exp\left\{-\frac{\sigma_n^2 |\tau|}{\tau_c}\right\} \cos \omega_c \tau; \quad 0 \leq |\tau| \leq \infty \quad (\text{IV.9})$$

where A_n is a constant amplitude, and σ_n^2 is the variance of the modulating process for the noise; hence, since $R_{ss}(t_1, t_2)$ has been computed previously, the correlation function of $\{Y(t)\}$, when $\{X(t)\}$ and $\{S(t)\}$ are correlated, will be specified if we can compute $E\{s(t_1)s(t_2)x(t_1)x(t_2)\}$.

From the expressions for $s(t)$ and $x(t)$ we have

$$\begin{aligned}
 E\{y(t_1)y(t_2)\} &= A_s^2 E\{A_r(t_1)A_r(t_2)\} \\
 &\cdot E\left\{\cos\left[\omega_c t_1 - \phi_0 - \psi(t_1)\right]\cos\left[\omega_c t_2 - \phi_0 - \psi(t_2)\right]\right. \\
 &\cdot \cos\left[\omega_c (t_1 + \lambda t_1 - t_0) - \phi_0 - \psi(t_1 + \lambda t_1 - t_0)\right] \\
 &\cdot \left.\cos\left[\omega_c (t_2 + \lambda t_2 - t_0) - \phi_0 - \psi(t_2 + \lambda t_2 - t_0)\right]\right\} \\
 &+ R_{ss}(t_1, t_2)R_{nn}(t_1, t_2) .
 \end{aligned}$$

If we expand the cosine products and carry out the multiplication, the expected value of the cosine product becomes

$$\begin{aligned}
 E_c &= \frac{1}{8} E\left\{\cos\left[\Gamma_{1+} + \Gamma_{2+}\right] + \cos\left[\Gamma_{1+} - \Gamma_{2+}\right] + \cos\left[\Gamma_{1+} + \Gamma_{2-}\right]\right. \\
 &+ \cos\left[\Gamma_{1+} - \Gamma_{2-}\right] + \cos\left[\Gamma_{1-} + \Gamma_{2+}\right] + \cos\left[\Gamma_{1-} - \Gamma_{2+}\right] \\
 &\cdot \\
 &+ \left.\cos\left[\Gamma_{1-} + \Gamma_{2-}\right] + \cos\left[\Gamma_{1-} - \Gamma_{2-}\right]\right\} \quad (IV.10)
 \end{aligned}$$

where the Γ are defined by

$$\Gamma_{1+} = 2\omega_c t_1 + \omega_c (\lambda t_1 - t_0) - \hat{\psi}(t_1) - \hat{\psi}(t_1 + \lambda t_1 - t_0)$$

$$\Gamma_{1-} = \omega_c (t_0 - \lambda t_1) - \hat{\psi}(t_1) + \hat{\psi}(t_1 + \lambda t_1 - t_0)$$

$$\Gamma_{2+} = 2\omega_c t_2 + \omega_c (\lambda t_2 - t_0) - \hat{\psi}(t_2) - \hat{\psi}(t_2 + \lambda t_2 - t_0)$$

$$\Gamma_{2-} = \omega_c (t_0 - \lambda t_2) - \hat{\psi}(t_2) + \hat{\psi}(t_2 + \lambda t_2 - t_0)$$

and $\hat{\psi}(t) = \psi(t) + \phi_0$. Taking the expected value of (IV.10), recalling that the expected value of a cosine which contains ϕ_0 is zero, the expected value of the cosine product becomes

$$\begin{aligned} E_c = \frac{1}{8} E \bigg\{ & \cos \left[2\omega_c (t_1 - t_2) + \omega_c \lambda (t_1 - t_2) - \psi(t_1) - \psi(t_1 + \lambda t_1 - t_0) \right. \\ & \left. + \psi(t_2) + \psi(t_2 + \lambda t_2 - t_0) \right] + \cos \left\{ \omega_c \left[2t_0 - \lambda (t_1 + t_2) \right] \right. \\ & \left. + \psi(t_1 + \lambda t_1 - t_0) - \psi(t_1) + \psi(t_2 + \lambda t_2 - t_0) - \psi(t_2) \right\} \\ & \left. + \cos \left[\omega_c \lambda (t_1 - t_2) + \psi(t_1 + \lambda t_1 - t_0) - \psi(t_1) \right. \right. \\ & \left. \left. - \psi(t_2 + \lambda t_2 - t_0) + \psi(t_2) \right] \right\} \end{aligned}$$

which may be written

$$\begin{aligned}
 E_c = \frac{1}{8} E \bigg\{ & \cos \left[2\omega_c(t_1 - t_2) + \omega_c \lambda(t_1 - t_2) \right] \cos \left[-\psi(t_1) - \psi(t_1 + \lambda t_1 - t_0) + \psi(t_2) \right. \\
 & \left. + \psi(t_2 + \lambda t_2 - t_0) \right] \\
 & + \cos \left\{ \omega_c \left[2t_0 - \lambda(t_1 + t_2) \right] \right\} \cos \left[-\psi(t_1) + \psi(t_1 + \lambda t_1 - t_0) \right. \\
 & \left. - \psi(t_2) + \psi(t_2 + \lambda t_2 - t_0) \right] \\
 & + \cos \left[\omega_c \lambda(t_1 - t_2) \right] \cos \left[-\psi(t_1) + \psi(t_1 + \lambda t_1 - t_0) \right. \\
 & \left. + \psi(t_2) - \psi(t_2 + \lambda t_2 - t_0) \right] \bigg\}. \quad (IV.11)
 \end{aligned}$$

The expected value of $\cos \left[\sum_j \psi_j \right]$ is given by

$$E \left\{ \cos \sum_j \psi_j \right\} = e^{-1/2 \sum_{i,j} u_{ij} \xi_i \xi_j}$$

where $u_{ij} = E\{\psi(t_i)\psi(t_j)\} = K_\psi(t_i - t_j)$; hence the expected values of the terms in (IV.11) are:

a. First Term

$$E_{c1} = \cos\left[2\omega_c(t_1 - t_2) + \omega_c\lambda(t_1 - t_2)\right] \exp\left(-\frac{1}{2}\left\{4\sigma^2 + 2K_\psi(t_0 - \lambda t_1) + 2K_\psi(t_0 - \lambda t_2) - 2K_\psi(t_1 - t_2) - 2K_\psi(t_1 - t_2 + t_0 - \lambda t_2) - 2K_\psi(t_2 - t_1 - \lambda t_1 + t_0) - 2K_\psi[t_1 - t_2 + \lambda(t_1 - t_2)]\right\}\right).$$

b. Second Term

$$E_{c2} = \cos\left\{\omega_c[2t_0 - \lambda(t_1 + t_2)]\right\} \exp\left(-\frac{1}{2}\left\{4\sigma^2 - 2K(t_0 - \lambda t_1) + 2K(t_1 - t_2) - 2K(t_1 - t_2 + t_0 - \lambda t_2) - 2K(t_2 - t_1 - \lambda t_1 + t_0) + 2K[t_1 - t_2 + \lambda(t_1 - t_2)] - 2K(\lambda t_2 - t_0)\right\}\right).$$

c. Third Term

$$E_{c3} = \cos[\omega_c \lambda(t_1 - t_2)] \exp \left(-\frac{1}{2} \{ 4\sigma^2 - 2K(t_0 - \lambda t_1) - 2K(t_1 - t_2) \right. \\ \left. + 2K(t_1 - t_2 + t_0 - \lambda t_2) - 2K(t_0 - \lambda t_2) \right. \\ \left. + 2K(t_2 - t_1 - \lambda t_1 + t_0) - 2K[t_1 - t_2 + \lambda(t_1 - t_2)] \} \right).$$

Substituting (IV.5) in these expressions, we obtain

$$E\{s(t)s(t+\tau)x(t)x(t+\tau)\} = \frac{1}{8} A_s^2 E\{A_r(t)A_r(t+\tau)\} \cos[2\omega_c \tau + \omega_d \tau] \\ \cdot \exp \left\{ -\frac{\sigma^2}{\tau_c} \left[-|t_0 - \lambda t| - |t_0 - \lambda(t - \tau)| + |\tau| + |\tau + t_0 - \lambda(t - \tau)| \right. \right. \\ \left. \left. + |t_0 - \lambda t - \tau| + |\tau + \lambda \tau| \right] \right\} + \cos[2\omega_c t_0 - \omega_d t - \omega_d \tau] \\ \cdot \exp \left\{ -\frac{\sigma^2}{\tau_c} \left[|t_0 - \lambda t| - |\tau| + |\tau + t_0 - \lambda t + \lambda \tau| + |t_0 - \lambda t - \tau| \right. \right. \\ \left. \left. - |\tau + \lambda \tau| + |t_0 - \lambda t + \lambda \tau| \right] \right\} + \cos \omega_d \tau \\ \cdot \exp \left\{ -\frac{\sigma^2}{\tau_c} \left[|t_0 - \lambda t| + |\tau| - |\tau + t_0 - \lambda(t - \tau)| + |t_0 - \lambda(t - \tau)| \right. \right. \\ \left. \left. - |-\tau + t_0 - \lambda t| + |\tau + \lambda \tau| \right] \right\} \quad (IV.12)$$

where $\tau = t_1 - t_2$.

We now let $t_0 - \lambda t = t'$, and consider the region $\tau \geq 0$ for the following intervals of t' :

$$(a) \quad 0 \leq t' < \tau; \quad (b) \quad \tau \leq t' \leq \tau_c; \quad (c) \quad -\tau_c \leq t' < -\tau - \lambda\tau;$$

$$(d) \quad -\tau - \lambda\tau \leq t' < -\lambda\tau; \quad (e) \quad -\lambda\tau \leq t' \leq 0.$$

[Since the correlation function is an even function of τ , we need only consider the interval $\tau \geq 0$.] Considering the equations separately in these regions allows us to remove the absolute value signs in the exponentials so that various terms can be combined. This gives the following equations for $R(t', \tau)$; $\tau \geq 0$:

$$1. \quad \underline{0 \leq t' \leq \tau}$$

$$\begin{aligned} R(t', \tau) = k_1 & \left[\cos(2\omega_c t' + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2t' + \lambda\tau)\right\} \right. \\ & + \cos(2\omega_c \tau + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (-2t' + 4\tau + \lambda\tau)\right\} \\ & \left. + \cos(\omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2t' + \lambda\tau)\right\} \right] \end{aligned} \quad (IV.13)$$

$$2. \quad \underline{\tau \leq t' \leq \tau_c}$$

$$\begin{aligned} R(t', \tau) = k_1 & \left[\cos(2\omega_c t' + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (4t' - 2\tau + \lambda\tau)\right\} \right. \\ & + \cos(2\omega_c \tau + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2\tau + \lambda\tau)\right\} \\ & \left. + \cos(\omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2\tau + \lambda\tau)\right\} \right] \end{aligned} \quad (IV.14)$$

$$3. \quad \underline{-\tau_c < t' < -\tau - \lambda\tau}$$

$$\begin{aligned} R(t', \tau) = k_1 & \left[\cos(2\omega_c t' + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (-4t' - 2\tau - 3\lambda\tau)\right\} \right. \\ & + \cos(2\omega_c \tau + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2\tau + \lambda\tau)\right\} \\ & \left. + \cos(\omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2\tau + \lambda\tau)\right\} \right] \end{aligned} \quad (\text{IV.15})$$

$$4. \quad \underline{-\tau - \lambda\tau \leq t' < -\lambda\tau}$$

$$\begin{aligned} R(t', \tau) = k_1 & \left[\cos(2\omega_c t' + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (-2t' - \lambda\tau)\right\} \right. \\ & + \cos(2\omega_c \tau + \omega_d \tau) \exp\left\{-\sigma^2/\tau_c (2t' + 4\tau + 3\lambda\tau)\right\} \\ & \left. + \cos \omega_d \tau \exp\left\{-\sigma^2/\tau_c (-2t' - \lambda\tau)\right\} \right] \end{aligned} \quad (\text{IV.16})$$

$$5. \quad \underline{-\lambda\tau \leq t' \leq 0}$$

$$\begin{aligned} R(t', \tau) = k_1 & \left[\cos(2\omega_c t' + \omega_d \tau) \exp\left\{-\frac{\sigma^2 \lambda \tau}{\tau_c}\right\} \right. \\ & + \cos(2\omega_c \tau + \omega_d \tau) \exp\left\{-\frac{\sigma^2}{\tau_c} (4\tau + \lambda\tau)\right\} \\ & \left. + \cos \omega_d \tau \exp\left\{-\frac{\sigma^2 \lambda \tau}{\tau_c}\right\} \right] \end{aligned} \quad (\text{IV.17})$$

where, for the general case, $k_1 = 1/8 A_s^2 E \{A_r(t) A_r(t + \tau)\} = 1/8 A_s^2 R_A(\tau)$.

We note from (IV.13) through (IV.17) that the correlation function of the SXS term depends on both t' and τ , implying that the SXS process out of the multiplier is inherently a nonstationary process. The theory of such processes and their applications to a practical system are discussed in Appendix I, and applied in the remaining chapters.

3. Signal-Cross-Noise Correlation Function

The total correlation function at the output of the multiplier, for the case in which $\{X(t)\}$ and $\{S(t)\}$ are correlated, is the sum of the correlated SXS correlation function plus the SXN correlation function. The SXN correlation function is given by

$$R_{SXN}(t', \tau) = R_{SXN}(\tau) = \frac{1}{2} \left(\frac{A_s A_n}{2} \right)^2 e^{-\frac{|\tau| \hat{\sigma}^2}{\tau_c}} (1 + \cos 2\omega_c \tau); \quad 0 \leq |\tau| < \tau_c$$

$$R_{SXN}(\tau) = \frac{1}{2} \left(\frac{A_s A_n}{2} \right)^2 \exp \left\{ -\sigma^2 - \frac{\sigma_n^2 |\tau|}{\tau_c} \right\} (1 + \cos 2\omega_c \tau); \quad |\tau| > \tau_c$$
(IV.18)

where $\hat{\sigma}^2 = \sigma^2 + \sigma_n^2$. [This simple equation results from the form chosen for the covariance function of the noise process, representing the case in which the signal and noise processes are of the same type. This is the case in which one might expect the system to have its poorest performance.] Then in the interval where $\{S(t)\}$ and $\{X(t)\}$ are correlated, the total correlation function at the multiplier output, for $0 \leq \tau$, is the sum of (IV.13) through (IV.17) plus (IV.18).

4. The Correlation Function in the Interval
Where $\{X(t)\}$ and $\{S(t)\}$ are Uncorrelated

In the case in which $\{X(t)\}$ and $\{S(t)\}$ are uncorrelated, the total correlation function at the multiplier output is the sum of the SXN term, and the uncorrelated SXS term. If $\{X(t)\}$ and $\{S(t)\}$ are uncorrelated,

$$R_{\mu}(\tau) = R_{ss}(\tau)R_{xx}(\tau) \quad (\text{IV.19})$$

where $R_{ss}(\tau)$ is the correlation function of the L.O. process, and $R_{xx}(\tau)$ is the correlation function of the return process. From (II.16) and (III.16),

$$R_{\mu}(\tau) = \left(\frac{A_r A_s}{2} \right)^2 \cos \omega_c \tau \cos [\omega_c (\tau + \lambda \tau)] \exp \frac{-\sigma^2}{\tau_c} |2\tau + \lambda \tau| ; \quad \left. \begin{array}{l} | \tau | \leq \tau_c \\ \\ = \left(\frac{A_r A_s}{2} \right)^2 e^{-\sigma^2} \cos \omega_c \tau \cos [\omega_c (\tau + \lambda \tau)] ; \quad | \tau | > \tau_c \end{array} \right\} \quad (\text{IV.20})$$

so that the total correlation function at the multiplier output is

$$R_{yy}(\tau) = \left(\frac{A_r A_s}{2} \right)^2 \cos (\omega_c \tau) \cos [\omega_c (\tau + \lambda \tau)] \exp \left\{ \frac{-\sigma^2}{\tau_c} |2\tau + \lambda \tau| \right\} + \left(\frac{A_s A_n}{2} \right)^2 \exp \left\{ -\frac{\hat{\sigma}^2}{\tau_c} |\tau| \right\} (1 + \cos 2\omega_c \tau) ; \quad | \tau | \leq \tau_c$$

$$R_{yy}(\tau) = \left(\frac{A_r A_s}{2} \right)^2 \cos \omega_c \tau \cos [\omega_c (\tau + \lambda \tau)] e^{-2\sigma^2} + \frac{1}{2} \left(\frac{A_s A_n}{2} \right)^2 \exp \left\{ -\sigma^2 - \frac{\sigma_n^2 |\tau|}{\tau_c} \right\} (1 + \cos 2\omega_c \tau) , \quad | \tau | > \tau_c .$$

$$(\text{IV.21})$$

CHAPTER V

THE EXPECTED VALUE AND ENERGY DENSITY SPECTRUM OF THE BANDPASS FILTER OUTPUT

In computing the expressions for the mean and variance of the decision voltage, it is necessary to compute the mean value and energy density spectrum of the output of the multiplier, and then compute the effect of the remaining signal processing circuits on these quantities. (Because of our assumption that the decision voltage has a gaussian distribution, the density function is completely specified by the mean and variance, and the variance can be obtained from a knowledge of the spectrum.) In this chapter we will derive expressions for the spectrum of the multiplier output, and the spectrum and mean value of the bandpass filter output. In deriving these expressions, simplifying approximations will be made whenever it is obvious that the error introduced by the approximations will be very small.

In the above paragraph, we have been talking about the spectra at the multiplier and bandpass filter outputs; however, for the types of processes under discussion here we need to define what we mean by the spectrum. This point is discussed in Appendix I, where it is shown that the spectrum of interest is the energy density spectrum of the processes. It is shown there that it is possible to define a time varying spectrum, and to compute the effect of various operations on this spectrum; however, it is also possible to define an average spectrum which is analogous to the power density spectrum of a stationary process. Since, for the system under consideration here, the significant terms in the spectrum of the multiplier output are due to quasi-stationary processes, and since the average spectrum is an adequate representation for this type of process, we shall use the average spectrum in this analysis.

1. The Average Covariance Function at the Multiplier Output

The equations for the average correlation function and the average energy density spectrum, as derived in Appendix I, are

$$\bar{\rho}(\tau) = \int_{-\infty}^{\infty} R(t, \tau) dt \quad (V.1)$$

$$\bar{\omega}(f) = \int_{-\infty}^{\infty} \bar{\rho}(\tau) e^{-j2\pi f\tau} d\tau \quad (V.2)$$

where $\bar{\rho}(\tau)$ is the average correlation, and $\bar{\omega}(f)$ is the average spectrum. The process out of the multiplier is not zero mean, and hence $\bar{\rho}(\tau)$ will contain the mean value; thus, it is more convenient to use the average covariance function $\bar{k}(\tau)$ given by

$$\bar{k}(\tau) = \int_{-\infty}^{\infty} K(t', \tau) dt' \quad (V.3)$$

where $K(t, \tau) = R(t, \tau) - m(t)m(t - \tau)$, and $m(t)$ is the expected value.

Let us first consider the average spectrum of the SXS component in the region where correlation exists. Equations (IV.13) through (IV.17) are the expressions for $R(t', \tau)$ for this component when $\tau \geq 0$. Since the average

correlation function is an even function of τ , we need only use this portion in computing the average spectrum. The expression for $m(t)m(t - \tau)$ is

$$m(t)m(t - \tau) = k_1 \exp\left\{-\alpha t' - \frac{\alpha\lambda\tau}{2}\right\} \left[\cos(2\omega_c t' + \omega_d \tau) + \cos \omega_d \tau\right];$$

$$0 \leq t' \leq \tau_c$$

$$= k_1 \exp\left\{\alpha\tau + \frac{\alpha\lambda\tau}{2}\right\} \left[\cos(2\omega_c t' + \omega_d \tau) + \cos \omega_d \tau\right];$$

$$-\tau_c \leq t' < -\lambda\tau$$

$$= k_1 \exp\left\{-\frac{\alpha\lambda\tau}{2}\right\} \left[\cos(2\omega_c t' + \omega_d \tau) + \cos \omega_d \tau\right];$$

$$-\lambda\tau \leq t' \leq 0 \quad (V.4)$$

where $\alpha = 2\sigma^2/\tau_c$. Then the covariance function, $K(t', \tau)$, is obtained by subtracting (V.4) from (IV.13) through (IV.17). Substituting the result of this operation in (V.3) and performing the integration, we obtain

$$\begin{aligned}
\bar{k}(\tau) = & \frac{2k_1}{\lambda\alpha} \cos(2\omega_c \tau + \omega_d \tau) \left[\exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} - \exp\left\{-2\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \right] \\
& + \frac{2k_1}{\lambda} (\tau_c - \tau) \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \cos(2\omega_c \tau + \omega_d \tau) \\
& + \frac{k_1 \lambda \tau}{\lambda} \cos(2\omega_c \tau + \omega_d \tau) \left[\exp\left\{-2\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} - \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \right] \\
& + \frac{2k_1}{\lambda} \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \left[\cos(2\omega_c \tau + \omega_d \tau) \left\{ \frac{2\alpha}{4\alpha^2 + 4\omega_c^2} - \frac{\alpha}{\alpha^2 + 4\omega_c^2} \right\} \right. \\
& \quad \left. + \sin(2\omega_c \tau + \omega_d \tau) \left\{ \frac{2\omega_c}{\alpha^2 + 4\omega_c^2} - \frac{2\omega_c}{4\alpha^2 + 4\omega_c^2} \right\} \right] \\
& + \frac{k_1 \exp\left\{-4\sigma^2 + \alpha\tau + \frac{3\alpha\lambda\tau}{2}\right\}}{\lambda(4\alpha^2 + 4\omega_c^2)} \left[2\omega_c \sin \omega_d \tau - 2\alpha \cos \omega_d \tau \right] \\
& + \frac{k_1 \exp\left\{-2\sigma^2 - \frac{\alpha\lambda\tau}{2}\right\}}{\lambda(\alpha^2 + 4\omega_c^2)} \left[\alpha \cos \omega_d \tau - 2\omega_c \sin \omega_d \tau \right] \\
& + \frac{2k_1(\tau_c - \tau)}{\lambda} \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \cos \omega_d \tau - \frac{k_1 \lambda \tau}{\lambda} \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \cos \omega_d \tau \\
& - \frac{2k_1}{\lambda\alpha} \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \cos \omega_d \tau + \frac{2k_1}{\lambda\alpha} \exp\left\{-2\sigma^2 + \frac{\alpha\lambda\tau}{2}\right\} \cos \omega_d \tau ;
\end{aligned}$$

$$\tau \geq 0. \quad (V.5)$$

Equation (V.5) is the complete expression for the average covariance function of the multiplier output, and the expression for the average energy density spectrum is the fourier transform of this equation. However, since the bandpass filter will reject those components around $2\omega_c\tau + \omega_d\tau$, we only need to consider the low-frequency terms. For a practical system, the equation for the low-frequency portion of the average covariance function can be simplified by neglecting the terms which are very small compared to the terms which are retained. Thus, since α will be much less than ω_c , and τ_c will be much greater than $1/\alpha$, while $e^{-n\sigma^2}$ will be approximately zero, the low-frequency portion of $\bar{k}(\tau)$ can be written

$$\hat{\bar{k}}(\tau) \cong \frac{2k_1\tau_c}{\lambda} \exp\left\{-\alpha\tau - \frac{\alpha\lambda\tau}{2}\right\} \cos \omega_d\tau . \quad (V.6)$$

This expression can be simplified even further by observing that λ will be many orders of magnitude less than one, so that

$$\hat{\bar{k}}(\tau) \cong \frac{2k_1\tau_c}{\lambda} \exp\{-\alpha\tau\} \cos \omega_d\tau ; \tau \geq 0 . \quad (V.7)$$

Now we must compute the average covariance function for the noise terms. The noise is made up of the SXN term, and the SXS term when $\{S(t)\}$ and $\{X(t)\}$ are uncorrelated. The covariance function for the SXN term is given by

$$E\{s(t)s(t+\tau)n(t)n(t+\tau)\} = K_{sn}(t,\tau) . \quad (V.8)$$

Since $\{S(t)\}$ and $\{N(t)\}$ are, by hypothesis, independent, we have

$$K_{sn}(t', \tau) = K_{ss}(t', \tau) K_{nn}(t', \tau) . \quad (V.9)$$

Thus, using the assumed form for $K_{nn}(t', \tau)$, and the fact that $e^{-\sigma^2} \approx 0$, we have

$$K_{sn}(t', \tau) = K_{sn}(\tau) = k'_1 \exp \left\{ -\frac{\hat{\sigma}^2 |\tau|}{\tau_c} \right\} (1 + \cos 2\omega_c \tau) \quad (V.10)$$

where $\hat{\sigma}^2 = \sigma^2 + \sigma_n^2$. Since the SXN term is a quasi-stationary process, the average covariance function is given by

$$\bar{k}(\tau) = \frac{(t_0 + \tau_c)}{\lambda} K_{sn}(\tau) = \frac{(t_0 + \tau_c)}{\lambda} k'_1 \exp \left\{ -\frac{\hat{\sigma}^2 |\tau|}{\tau_c} \right\} (1 + \cos 2\omega_c \tau) . \quad (V.11)$$

For the uncorrelated SXS term we have,

$$K_{sx}(t', \tau) = K_{ss}(t', \tau) K_{xx}(t', \tau) \quad (V.12)$$

which becomes, since $e^{-\sigma^2} \approx 0$,

$$K_{sx}(t, t + \tau) = K_{sx}(\tau) = 2k_1 \cos \omega_c \tau \cos \omega_c (\tau + \lambda \tau) \exp \left\{ -\frac{\sigma^2}{\tau_c} |2\tau + \lambda \tau| \right\} . \quad (V.13)$$

This is also a quasi-stationary process, so that the average covariance function is given by

$$\bar{k}_{sx}(\tau) = \frac{(t_o - \tau_c)}{\lambda} 2k_1 \cos \omega_c \tau \cos \omega_c (\tau + \lambda\tau) \exp \left\{ -\frac{\sigma^2}{\tau_c} |2\tau + \lambda\tau| \right\}. \quad (V.14)$$

The total average covariance function in the interval where $\{S(t)\}$ and $\{X(t)\}$ are correlated is then the sum of (V.7) and (V.10), while the average covariance function in the interval where $\{S(t)\}$ and $\{X(t)\}$ are uncorrelated is the sum of (V.10) and (V.14). The average spectrum can then be obtained by substituting these sums into (V.2), and evaluating the resulting integrals. However, since the bandpass filter will be centered in the vicinity of ω_d , those terms in the correlation function, and hence the spectrum, which are at or near $2\omega_c$, will be rejected by the filter. Thus we shall only consider the terms which are at, or near, ω_d .

2. Average Energy Density Out of the Bandpass Filter

As noted above, we need only consider the low-frequency portion of the average covariance function at the input to the bandpass filter. The equations for $\hat{k}(\tau)$ in the correlated and uncorrelated regions, respectively, are

$$\hat{k}_c(\tau) \approx \frac{2k_1 \tau_c}{\lambda} \exp \left\{ -\alpha |\tau| \right\} \cos \omega_d \tau + \frac{(t_o + \tau_c)}{\lambda} k_1' \exp \left\{ -\frac{\hat{\sigma}^2 |\tau|}{\tau_c} \right\} \quad (V.15)$$

$$\hat{k}_u(\tau) \approx \frac{k_1(t_o - \tau_c)}{\lambda} \exp \left\{ -\alpha |\tau| \right\} \cos \omega_d \tau + \frac{(t_o + \tau_c)}{\lambda} k_1' \exp \left\{ -\frac{\hat{\sigma}^2 |\tau|}{\tau_c} \right\}. \quad (V.16)$$

Substituting these in (V.3), and noting that, since $\epsilon^{-\sigma^2}$ is approximately zero, the integral from zero to τ_c is approximately equal to the integral from zero to infinity, we obtain

$$\hat{w}_c(\omega) = \frac{2k_1 \tau_c \alpha}{\lambda(\alpha^2 + \omega'^2)} + \frac{2k_1 \tau_c \alpha}{\lambda(\alpha'^2 + \omega'^2)} + \frac{2\alpha'(t_o + \tau_c) k'_1}{\lambda(\alpha'^2 + \omega^2)}; -\tau_c \leq t' \leq \tau_c \quad (V.17)$$

$$\hat{w}_u(\omega) = \frac{\alpha k_1(t_o - \tau_c)}{\lambda(\alpha^2 + \omega'^2)} + \frac{\alpha k_1(t_o - \tau_c)}{\lambda(\alpha'^2 + \omega'^2)} + \frac{2\alpha'(t_o + \tau_c) k'_1}{\lambda(\alpha'^2 + \omega^2)}; t' < -\tau_c \quad (V.18)$$

where $\omega = 2\pi f$, $\omega' = \omega - \omega_d$; $\omega'' = \omega + \omega_d$, and $\alpha' = \frac{\hat{\sigma}^2}{\tau_c}$

The impulse response of the bandpass filter is assumed to be

$$h(t) = 2b e^{-bt} \cos \omega_o t = 2h_{\ell}(t) \cos \omega_o t \quad (V.19)$$

where $h_{\ell}(t)$ is the impulse response of the equivalent low-pass filter. This form is a reasonable approximation to many practical filters. Although we realize that it is not strictly true for a filter of this type, we shall assume that the system function of the filter equals zero at $\omega = 0$, i.e., we make the narrow-band assumption. Therefore, the absolute square of the system function is given by

$$|H(f)|^2 = \left(|H_{\ell+}(\omega - \omega_o)|^2 + |H_{\ell-}(\omega + \omega_o)|^2 \right)$$

where $|H_{l+}|^2$ and $|H_{l-}|^2$ are the (nonoverlapping) positive and negative portions of the system function. Transforming (V.19), we obtain

$$|H(f)|^2 = \left(\frac{b^2}{b^2 + \tilde{\omega}'^2} + \frac{b^2}{b^2 + \tilde{\omega}'^2} \right) \quad (V.20)$$

where $\tilde{\omega}' = \omega + \omega_0$, and $\tilde{\omega}' = \omega - \omega_0$.

The average spectrum at the output of the filter is given by (see Appendix I)

$$\bar{w}_z(f) = |H(f)|^2 \hat{w}(f). \quad (V.21)$$

Substituting (V.20), (V.17), and (V.18) in (V.21), the expressions for the average density spectrum for the correlated and uncorrelated intervals are, respectively,

$$\bar{w}_{zc}(f) = \left(\frac{b^2}{b^2 + \tilde{\omega}'^2} + \frac{b^2}{b^2 + \tilde{\omega}'^2} \right) \left(\frac{2k_1 \tau_c \alpha}{\lambda(\alpha^2 + \omega'^2)} + \frac{2k_1 \tau_c \alpha}{\lambda(\alpha^2 + \omega'^2)} + \frac{2\alpha'(t_0 + \tau_c)k_1'}{\lambda(\alpha'^2 + \omega^2)} \right) \quad (V.22)$$

$$\bar{w}_{zu}(f) = \left(\frac{b^2}{b^2 + \omega'^2} + \frac{b^2}{b^2 + \tilde{\omega}'^2} \right) \left(\frac{k_1(t_0 - \tau_c)\alpha}{\lambda(\alpha^2 + \tilde{\omega}'^2)} + \frac{k_1(t_0 - \tau_c)\alpha}{\lambda(\alpha^2 + \omega'^2)} + \frac{2\alpha'(t_0 + \tau_c)k_1'}{\lambda(\alpha'^2 + \omega^2)} \right) \quad (V.23)$$

3. Expected Value at the Bandpass Filter Output

The expected value of the process $\{Z(t)\}$ out of the bandpass filter is

$$E\{Z(t)\} = \int_{-\infty}^t h(t - \tau) E\{y(\tau)\} d\tau \quad (V.24)$$

where $E\{y(\tau)\}$ is the expected value of the output of the multiplier. Thus

$$E\{y(t)\} = E\{s(t)x(t)\} + E\{s(t)n(t)\}$$

and since, by hypothesis, $\{S(t)\}$ and $\{N(t)\}$ are statistically independent,

$$E\{y(t)\} = E\{s(t)x(t)\} \quad (V.25)$$

We have two cases for (V.25): (1) The mean value for the uncorrelated interval, and (2) the mean value in the correlated interval. As has been shown previously, the respective expressions are

$$\left. \begin{aligned} E_{\mu}\{s(t)x(t)\} &= \left(\frac{A A}{r s}\right) \epsilon^{-\sigma^2} \cos \omega_c(t_o - \lambda t) \approx 0 \\ E_c\{s(t)x(t)\} &= \left(\frac{A A}{r s}\right) \exp\left\{-\frac{\sigma^2}{\tau_c} |t_o - \lambda t|\right\} \cos \omega_c(t_o - \lambda t) \end{aligned} \right\} \quad (V.26)$$

where the subscript μ denotes no correlation, and the subscript c denotes correlation.

The amplitude factor, A_r , will be a function of time because of the relative velocity between the reflecting surface and the altimeter; however, over the effective integration time of the filter, it will remain essentially constant. [There may also be variation due to random irregularities in the reflecting surface, and the same comment applies for this case, providing the correlation time of the return process is much greater than the integration time; however, we will ignore the random fluctuations here.] Then the expected value of the bandpass filter output is given by

$$E_c \{z(t)\} = \frac{A_r(t_n) A_s b}{\lambda} \int_{\frac{t_o - \tau_c}{\lambda}}^t \exp \left\{ -\frac{\sigma^2}{\tau_c} |t_o - \lambda\tau| \right\} \exp \{-bt + b\tau\} \cdot \cos [\omega_o(t - \tau)] \cos [\omega_c(t - \lambda\tau)] d\tau \quad (V.27)$$

where $A_r(t_n)$ may be defined as the average value of return in the interval $\left[\frac{t_o - \tau_c}{\lambda}, t \right]$. The expected value in the interval $\left[0, \frac{t_o - \tau_c}{\lambda} \right]$ is, of course, equal to 0. We shall assume that $\cos \omega_c t_o = 1$, and that $\omega_o = \omega_d$ [this condition maximizes the expected value]; then, ignoring the terms involving $2\omega_o \tau$, since they will be approximately = 0, (V.27) may be written

$$E_c \{z(t)\} = \frac{bA_r(t_n) A_s \cos \omega_o t}{2} \exp \left\{ -\frac{\sigma^2 t_o}{\tau_c} - bt \right\} \int_{\frac{t_o - \tau_c}{\lambda}}^t \exp \left\{ \frac{\sigma^2 \lambda \tau}{2} + b\tau \right\} d\tau; \quad t < t_o/\lambda$$

$$= E_c \{z(t_o/\lambda)\} e^{-b(t - t_o/\lambda)} + \frac{bA_r(t_n) A_s \cos \omega_o t}{2} \exp \left\{ \frac{\sigma^2 t_o}{\tau_c} - bt \right\} \int_{t_o/\lambda}^t \exp \left\{ -\frac{\lambda \tau}{2} + b\tau \right\} d\tau; \quad t_o \geq t_o/\lambda \quad (V.28)$$

where $E_c\{z(t_0/\lambda)\}$ is the expected value at $t = t_0/\lambda$. Evaluating the integrals in (V.28), using the fact that $e^{-\sigma^2} \approx 0$, we obtain,

$$E_c\{z(t)\} = \frac{b}{2b'} A_r(t_n) A_s \cos \omega_0 t \exp\left\{-\frac{\sigma^2}{\tau_c} (t_0 - \lambda t)\right\}; t < t_0/\lambda$$

$$= \frac{b}{2b'} A_r(t_n) A_s \exp\{-b(t - t_0/\lambda)\}$$

$$+ \frac{b}{2b''} A_r(t_n) A_s \cos \omega_0 t \left[\exp\left\{\frac{\sigma^2}{\tau_c} (t_0 - \lambda t)\right\} \right.$$

$$\left. - \exp\{-b(t - t_0/\lambda)\} \right]; t_0/\lambda < t \leq \frac{t_0 + \tau_c}{\tau_c} \quad (V.29)$$

$$\text{where } b' = b + \frac{\sigma^2 \lambda}{\tau_c}, \text{ and } b'' = b - \frac{\sigma^2 \lambda}{\tau_c}.$$

CHAPTER VI

THE AVERAGE SPECTRUM AND EXPECTED VALUE AT THE OUTPUT OF THE SQUARE-LAW DEVICE

In order to provide a decision voltage which is always greater than zero, the output of the bandpass filter must be subjected to a nonlinear transformation; we shall assume that this nonlinear element is a zero-memory; i.e., infinite bandwidth, square-law device.

The random process, $\{Z(t)\}$, at the input to the square-law device is the sum of a zero mean process, $\{\hat{Z}(t)\}$, and the expected value $m_z(t)$, i.e.,

$$z(t) = \hat{z}(t) + m(t) \quad (\text{VI.1})$$

and the random process, $\{U(t)\}$, at the output is the sum of a zero mean process, $\{\hat{U}(t)\}$, and the expected value $m_u(t)$

$$u(t) = \hat{u}(t) + m_u(t) \quad (\text{VI.2})$$

where

$$m_u(t) = E\{U(t)\} = E\{Z^2(t)\} = E\{\hat{Z}^2(t)\} + m_z^2(t) \quad (\text{VI.3})$$

The expression for $E\{\hat{Z}^2(t)\}$ can be obtained from (see Appendix I)

$$E\{\hat{Z}^2(t)\} = \frac{\bar{k}_{\hat{Z}}(\tau) \Big|_{\tau=0}}{2T} \quad (\text{VI.4})$$

where $2T$ is the length of the sample functions, and $\bar{k}_z(\tau) \Big|_{\tau=0}$ indicates the evaluation of the average covariance function of $\{\hat{z}(t)\}$ at $\tau = 0$. In this chapter, we shall calculate the expressions for the average spectrum and expected value of $\{\hat{u}(t)\}$.

1. Average Energy Density Spectrum and Covariance Function

In this section we shall derive expressions for the average energy density spectrum and the average covariance function at the output of the square-law device. Since the average covariance function is analogous to the covariance function of a stationary process, we can assume that the Fourier transforms involved are transforms of the sample functions of the equivalent stationary process. Furthermore, for a physical system, the Fourier transform of almost all sample functions exist, i.e., all but a set of measure zero, and they are differentiable; hence the spectral density function in the spectral representation of the equivalent stationary process is $E\{|\Phi(f)|^2\}$, where $\Phi(f)$ is the random variable obtained by transforming the sample functions (see [5], [12], and [17]).

By definition,

$$\Psi(f) = \int_{-\infty}^{\infty} \hat{u}(t) e^{-j2\pi f t} dt \quad (\text{VI.5})$$

where $\Psi(f)$ is the Fourier transform of sample functions of $\{\hat{u}(t)\}$. Since the continuous portion of the transform of $\hat{u}(t)$ is given by the convolution of the transform of $\hat{z}(t)$ with itself, we have

$$\Psi(f) = \int_{-\infty}^{\infty} \Phi(\zeta) \Phi(f - \zeta) d\zeta \quad (\text{VI.6})$$

where $\Phi(\zeta)$ is the transform of the sample functions of $\{\hat{Z}(t)\}$. Then the spectral density function of $\{\hat{U}(t)\}$ is given by

$$\begin{aligned} E\left\{|\Psi(f)|^2\right\} &= \bar{w}_{\hat{U}}(f) = E\left\{\left|\int_{-\infty}^{\infty} \Phi(\zeta)\Phi(f - \zeta)d\zeta\right|^2\right\} \\ &= \iint_{-\infty}^{\infty} E\left\{\Phi(\zeta)\Phi(f - \zeta)\tilde{\Phi}(\lambda)\tilde{\Phi}(f - \lambda)\right\}d\zeta d\lambda \end{aligned} \quad (\text{VI.7})$$

where the tilde denotes the complex conjugate. It can be shown by a straightforward, but lengthy, calculation that the random process $\Gamma_f(\zeta)$ defined as

$$\Gamma_f(\zeta) \equiv \Phi(\zeta)\Phi(f - \zeta)$$

is an orthogonal process. Therefore

$$E\left\{\Gamma_f(\zeta_1)\tilde{\Gamma}_f(\zeta_2)\right\} = 0 ; \zeta_1 \neq \zeta_2$$

so that

$$\bar{w}_{\hat{U}}(f) = \int_{-\infty}^{\infty} E\left\{|\Phi(\zeta)|^2|\Phi(f - \zeta)|^2\right\}d\zeta . \quad (\text{VI.8})$$

Similarly, the random process defined as

$$\gamma(f) = |\Phi(f)|^2 - \bar{w}_z(f)$$

is the transform of a wide sense stationary process so that

$$E\{| \Phi(\xi) |^2 | \Phi(f - \xi) |^2\} - \bar{w}_z(\xi) \bar{w}_z(f - \xi) = 0 ; f \neq 0 \quad (\text{VI.9})$$

and this implies that

$$\bar{w}_u(f) = \bar{w}_z(f) * \bar{w}_z(f) \quad (\text{VI.10})$$

where the * indicates convolution.

Since $\bar{w}_u(f)$ is the convolution of $\bar{w}_z(f)$ with itself, the covariance function of $\{\hat{u}(t)\}$ is the square of the covariance function of $\{\hat{z}(t)\}$. It is easier here to compute the covariance function of $\{\hat{z}(t)\}$, square it, and take the inverse transform, rather than perform the convolution in (VI.13) directly; this is the procedure which we shall follow.

For the correlated and uncorrelated cases, the respective expressions for $\bar{w}_2(f)$ are, from (V.22) and (V.23),

$$\left. \begin{aligned} \bar{w}_{zc}(f) &= \frac{b^2}{\lambda(b^2 + \tilde{\omega}'^2)} \left(\frac{2k_1 \tau_c \alpha}{\alpha^2 + \omega'^2} + \frac{2k_1 \tau_c \alpha}{\alpha^2 + \omega'^2} + \frac{2\alpha' k'_1 (t_o + \tau_c)}{\alpha'^2 + \omega^2} \right); \omega \geq 0 \\ \bar{w}_{zc}(f) &= \frac{b^2}{\lambda(b^2 + \tilde{\omega}'^2)} \left(\frac{2k_1 \tau_c \alpha}{\alpha^2 + \omega'^2} + \frac{2k_1 \tau_c \alpha}{\alpha^2 + \omega'^2} + \frac{2\alpha' k'_1 (t_o + \tau_c)}{\alpha'^2 + \omega^2} \right); \omega < 0 \end{aligned} \right\} \quad (VI.11)$$

$$\left. \begin{aligned} \bar{w}_{zu}(f) &= \frac{b^2}{\lambda(b^2 + \tilde{\omega}'^2)} \left(\frac{k_1 \alpha (t_o - \tau_c)}{\alpha^2 + \omega'^2} + \frac{k_1 \alpha (t_o - \tau_c)}{\alpha^2 + \omega'^2} + \frac{2\alpha' k'_1 (t_o + \tau_c)}{\alpha'^2 + \omega^2} \right); \\ &\quad \omega \geq 0 \\ \bar{w}_{zu}(f) &= \frac{b^2}{\lambda(b^2 + \tilde{\omega}'^2)} \left(\frac{k_1 \alpha (t_o - \tau_c)}{\alpha^2 + \omega'^2} + \frac{k_1 \alpha (t_o - \tau_c)}{\alpha^2 + \omega'^2} + \frac{2\alpha' k'_1 (t_o + \tau_c)}{\alpha'^2 + \omega^2} \right); \\ &\quad \omega < 0 \end{aligned} \right\} \quad (VI.12)$$

where $\omega = 2\pi f$, $\tilde{\omega}' = \omega - \omega_o$, $\tilde{\omega}' = \omega + \omega_o$; $\alpha' = \hat{\sigma}^2 / \tau_c$, and $k'_1 = \frac{1}{2} \left(\frac{A A_s}{2} \right)^2$. For most systems of practical interest, $\alpha \gg \omega$; $\alpha \gg \omega'$; $\alpha \gg \omega'$ over the frequency interval of interest, i.e., over the passband of the filter, and α' is of the order of α ; hence (VI.11) and (VI.12) may be written

$$\left. \begin{aligned} \bar{w}_{zc}(f) &\cong \frac{2b\ell \beta_1}{b^2 + \omega'^2}; \omega \geq 0 \\ \bar{w}_{zc}(f) &\cong \frac{2b\ell \beta_1}{b^2 + \omega'^2}; \omega < 0 \end{aligned} \right\} \quad (VI.13)$$

$$\left. \begin{aligned} \bar{w}_{zu}(f) &\approx \frac{2b\ell\beta_2}{b^2 + \omega'^2} ; \omega \geq 0 \\ \bar{w}_{zu}(f) &\approx \frac{2b\ell\beta_2}{b^2 + \omega'^2} ; \omega < 0 \end{aligned} \right\} \quad (\text{VI.14})$$

where

$$\ell_{\beta 1} = \frac{2bk_1\tau_c}{\lambda\alpha} + \frac{k'_1(t_o + \tau_c)b}{\lambda\alpha'}$$

and

$$\ell_{\beta 2} = \frac{bk_1(t_o - \tau_c)}{\lambda\alpha} + \frac{bk'_1(t_o + \tau_c)}{\lambda\alpha'}$$

The average covariance function of $\{\hat{Z}(t)\}$ is given by

$$\bar{k}_{\hat{Z}}(\tau) = \int_{-\infty}^{\infty} \bar{w}_{\hat{Z}}(f) e^{j2\pi f\tau} df . \quad (\text{VI.15})$$

Then substituting (VI.13) and (VI.14) in (VI.15), we obtain

$$\bar{k}_{zc}(\tau) = 2\ell_{\beta 1} e^{-b|\tau|} \cos \omega_0 \tau \quad (\text{VI.16})$$

and

$$\bar{k}_{zu}(\tau) = 2\ell_{\beta 2}^2 e^{-b|\tau|} \cos \omega_o \tau. \quad (\text{VI.17})$$

The average covariance function of the square-law device output is then

$$\bar{k}_{\hat{u}c}(\tau) = k_{\hat{z}c}^2(\tau) = 2\ell_{\beta 1}^2 e^{-2b|\tau|} (1 + \cos 2\omega_o \tau) \quad (\text{VI.18})$$

and

$$\bar{k}_{\hat{u}u}(\tau) = \bar{k}_{\hat{z}u}(\tau) = 2\ell_{\beta 2}^2 e^{-2b|\tau|} (1 + \cos 2\omega_o \tau). \quad (\text{VI.19})$$

Taking the fourier transforms of (VI.18) and (VI.19), we have

$$\bar{w}_{\hat{u}c}(f) = \frac{8b\ell_{\beta 1}^2}{4b^2 + \omega^2} + \frac{4b\ell_{\beta 1}^2}{4b^2 + \tilde{\omega}'^2} + \frac{4b\ell_{\beta 1}^2}{4b^2 + \tilde{\omega}''^2} \quad (\text{VI.20})$$

and

$$\bar{w}_{\hat{u}u}(f) = \frac{8b\ell_{\beta 2}^2}{4b^2 + \omega^2} + \frac{4b\ell_{\beta 2}^2}{4b^2 + \tilde{\omega}'^2} + \frac{4b\ell_{\beta 2}^2}{4b^2 + \tilde{\omega}''^2} \quad (\text{VI.21})$$

where $\tilde{\omega}' = 2\pi(f - 2f_o)$, and $\tilde{\omega}'' = 2\pi(f + 2f_o)$.

2. Expected Value at the Output

The expected value at the output of the square-law device is given by

$$m_u(t) = m_z^2(t) + E\{\hat{z}^2(t)\} \quad (\text{VI.22})$$

where $m_z(t)$ is given by (V.29), and $E\{\hat{z}^2(t)\}$ is given by (see Appendix I)

$$E\{\hat{z}^2(t)\} = \frac{1}{2T} \bar{k}_z(\tau) \Big|_{\tau=0} \quad (\text{VI.23})$$

where $2T$ is the length of the sample functions. From (VI.16) and (VI.17), we obtain

$$\left. \begin{aligned} E_c\{\hat{z}^2(t)\} &= \frac{2\ell_{\beta 1}}{2T} = \tilde{\ell}_{\beta 1} = \frac{2k_1 b}{\alpha} + \frac{2k'_1 b}{\alpha'} \quad (\text{a}) \\ E_u\{\hat{z}^2(t)\} &= \frac{2\ell_{\beta 2}}{2T} = \tilde{\ell}_{\beta 2} = \frac{2bk_1}{\alpha} + \frac{2bk'_1}{\alpha'} \quad (\text{b}) \end{aligned} \right\} \quad (\text{VI.24})$$

where the subscript c refers to the interval where $\{X(t)\}$ and $\{S(t)\}$ are correlated, and the subscript u refers to the interval where they are uncorrelated.

Squaring the expressions in (V.29), we obtain

$$m_z^2(t) = \left(\frac{b}{b'}\right)^2 \left(\frac{A_r(t_n)A_s}{2}\right)^2 \left\{ \exp -\alpha(t_o - \lambda t) \right\} (1 + \cos 2\omega_o t) ; \frac{t_o - \tau_c}{\lambda} \leq t < \frac{t_o}{\lambda}$$

$$m_z^2(t) = \left(\frac{b}{b'}\right)^2 \left(\frac{A_r(t_n)A_s}{2}\right)^2 \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\}$$

$$+ 2 \left(\frac{b^2}{b'b''}\right) \left(\frac{A_r(t_n)A_s}{2}\right)^2 \cos \omega_o t \left[\exp\left\{(t_o - \lambda t) \left(\frac{b}{\lambda} + \frac{\alpha}{2}\right)\right\} \right.$$

$$\left. - \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\} \right]$$

$$+ \left(\frac{A_r(t_n)A_s}{2}\right)^2 \left(\frac{b^2}{b''^2}\right) (1 + \cos 2\omega_o t) \left[\exp\left\{\alpha(t_o - \lambda t)\right\} \right.$$

$$\left. - 2 \exp\left\{(t_o - \lambda t) (\alpha/2 + b)\right\} \right]$$

$$+ \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\} \Big] ;$$

$$\frac{t_o}{\lambda} < t \leq \frac{t_o + \tau_c}{\lambda} . \quad (VI.25)$$

For the type of system under consideration here, $b \gg \frac{\sigma^2 \lambda}{\tau_c}$, so that

$b \cong b' \cong b''$; hence (VI.25) may be written

$$m_z^2(t) \cong \frac{2\tilde{k}_1}{1} \exp\{-\alpha(t_0 - \lambda t)\} (1 + \cos 2\omega_0 t) ; \frac{t_0 - \tau_c}{\lambda} \leq t \leq t_0/\lambda$$

$$m_z^2(t) \cong 2\tilde{k}_1 \exp\left\{\frac{2b}{\lambda} (t_0 - \lambda t)\right\} + 4\tilde{k}_1 \cos \omega_0 t \left[\exp\left\{(t_0 - \lambda t)(b/\lambda + \alpha/2)\right\} \right. \\ \left. - \exp\left\{\frac{2b}{\lambda} (t_0 - \lambda t)\right\} \right]$$

$$+ \frac{2\tilde{k}_1}{1} (1 + \cos 2\omega_0 t) \left[\exp\left\{\alpha(t_0 - \lambda t)\right\} - 2 \exp\left\{(t_0 - \lambda t)(\alpha/2 + b)\right\} \right]$$

$$+ \exp\left\{\frac{2b}{\lambda} (t_0 - \lambda t)\right\} \Big] ; \frac{t_0}{\lambda} < t \leq \frac{t_0 + \tau_c}{\lambda} \quad (\text{VI.26})$$

where $\tilde{k}_1 = \frac{1}{2} \left(\frac{\Lambda_r(t) \Lambda_s}{2} \right)^2$. The expected value at the square-law device output for the correlated case is given by the sum of (VI.24a) and (VI.24b), while for the uncorrelated case it is given by (VI.26).

CHAPTER VII

THE EXPECTED VALUE AND AVERAGE VARIANCE OF THE DECISION VOLTAGE

Drawing on the results of the previous chapters, we shall now derive expressions for the expected value and the variance of the decision voltage. Since we assume that the decision voltage has a gaussian distribution function, these two parameters completely specify the distribution. In deriving the expressions, we shall continue our previous practice, and make approximations which are valid for a practical system, whenever such approximations will simplify the analysis without introducing an appreciable error in the final results. The first two sections of this chapter will be devoted to the derivation of the equations for the expected value and the variance. The last section will indicate how this information can be applied to the calculation of system performance in terms of the probability of a correct decision as a function of signal-to-noise ratio (SNR) at the multiplier input.

1. The Expected Value

The expected value of the decision voltage is given by

$$E\{E_d(t)\} = m_d(t) = \int_{-\infty}^t E\{u(t)\} h_\ell(t - \tau) d\tau \quad (\text{VII.1})$$

where $h_\ell(t)$ is the impulse response of the low-pass filter, and $E\{u(t)\}$ is the expected value of the process $\{U(t)\}$ at the squaring device output. From

(VI.24) and (VI.26), the expression for $E\{u(t)\} = m_u(t)$ is

$$\begin{aligned}
 m_u(t) &= \tilde{\ell}_{\beta 2} ; t < \frac{t_o - \tau_c}{\lambda} \\
 &= \tilde{\ell}_{\beta 1} + \frac{2\tilde{k}_1}{\lambda} (1 + \cos 2\omega_o t) \exp\{-\alpha(t_o - \lambda t)\} ; \frac{t_o - \tau_c}{\lambda} \leq t \leq t_o/\lambda \\
 &= \tilde{\ell}_{\beta 1} + 2\tilde{k}_1 \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\} + 4\tilde{k}_1 \cos \omega_o t \left[\exp\left\{(t_o - \lambda t)(b/\lambda + \alpha/2)\right\} \right. \\
 &\quad \left. - \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\} \right] \\
 &\quad + \frac{2\tilde{k}_1}{\lambda} (1 + \cos 2\omega_o t) \left[\exp\left\{\alpha(t_o - \lambda t)\right\} - 2 \exp\left\{(t_o - \lambda t)\left(\frac{\alpha}{2} + b\right)\right\} \right. \\
 &\quad \left. + \exp\left\{\frac{2b}{\lambda} (t_o - \lambda t)\right\} \right] ; \frac{t_o}{\lambda} < t \leq \frac{t_o + \tau_c}{\lambda} . \quad (\text{VII.2})
 \end{aligned}$$

We shall assume that the impulse response of the filter is given by

$$h_{\ell}(t) = a e^{-at} . \quad (\text{VII.3})$$

Substituting (VII.2) and (VII.3) in (VII.1), we obtain the following

expressions for $m_d(t)$, where we have ignored the terms involving $\cos \omega_0 t$ and $\cos 2\omega_0 t$, since they will be approximately zero:

$$m_{d1}(t) = \tilde{l}_{\beta 2} ; t < \frac{t_0 - \tau_c}{\lambda}$$

$$m_{d2}(t) = \tilde{l}_{\beta 2} \exp\left\{\frac{a}{\lambda} (t_0 - \tau_c - \lambda t)\right\} + \tilde{l}_{\beta 1} \left[1 - \exp\left\{\frac{a}{\lambda} (t_0 - \tau_c - \lambda t)\right\}\right]$$

$$+ \frac{2a\tilde{k}_1}{a_1} \exp\{-at - \alpha t_0\} \left[\exp\{a_1 t\} - \exp\left\{a_1 \left(\frac{t_0 - \tau_c}{\lambda}\right)\right\} \right] ; \frac{t_0 - \tau_c}{\lambda} \leq t \leq t_0/\lambda$$

$$m_{d3}(t) = \tilde{l}_{\beta 2} \exp\left\{\frac{a}{\lambda} (t_0 - \tau_c - \lambda t)\right\} + \tilde{l}_{\beta 1} \left[1 - \exp\left\{\frac{a}{\lambda} (t_0 - \tau_c - \lambda t)\right\}\right]$$

$$+ \tilde{m}_{d2} \left(\frac{t_0}{\lambda}\right) \exp\left\{\frac{a}{\lambda} (t_0 - \lambda t)\right\} + \frac{2a\tilde{k}_1}{a_2} \exp\{\alpha t_0 - at\} \left[\exp\{a_2 t\} - \exp\left\{\frac{a_2 t_0}{\lambda}\right\} \right]$$

$$- \frac{4a\tilde{k}_1}{a_3} \exp\left\{-at + t_0 \left(\frac{\alpha}{2} + b\right)\right\} \left[\exp\{a_3 t\} - \exp\left\{\frac{a_3 t_0}{\lambda}\right\} \right]$$

$$+ \frac{2a\tilde{k}_1}{a_4} \exp\left\{-at - \frac{2bt_0}{\lambda}\right\} \left[\exp\{a_4 t\} - \exp\left\{\frac{a_4 t_0}{\lambda}\right\} \right]$$

$$+ \frac{4a\tilde{k}_1}{a_5} \exp\left\{-at + \frac{2bt_0}{\lambda}\right\} \left[\exp\{a_5 t\} - \exp\left\{\frac{a_5 t_0}{\lambda}\right\} \right] ; \frac{t_0}{\lambda} < t \leq \frac{t_0 + \tau_c}{\lambda}$$

(VII.4)

where $a_1 = a + \alpha\lambda$, $a_2 = a - \alpha\lambda$, $a_3 = a - \lambda\left(\frac{\alpha}{2} + b\right)$, $a_4 = a - 2b$, $a_5 = a - 2b$, and $\tilde{m}_{d2}(t_0/\lambda) = \frac{\tilde{a}k_1}{2a_1} \left[1 - \exp\left\{-\tau_c(a + \alpha)\right\} \right]$. Given the system parameters, the expected value of the decision voltage can be calculated at any time $t < \frac{t_0 + \tau_c}{\lambda}$ using the above equation.

2. The Average Variance

In this section we shall derive the expression for the average variance of the decision voltage. In deriving the equation, we shall again make simplifying approximations in cases where the approximations will introduce a negligible error in the final result.

The average variance of the decision voltage is given by

$$\sigma_d^2 = 2\left(\frac{1}{2T}\right)^2 \int_0^\infty |H_\ell(f)|^2 \bar{\omega}_\mu(f) df = \left(\frac{1}{2T}\right)^2 \int_{-\infty}^\infty \bar{w}_d(f) df \quad (\text{VII.5})$$

where $\bar{\omega}_\mu(f)$ is the average spectrum of the process $\{U(t)\}$, $\bar{w}_d(f)$ is the average spectrum of the decision voltage, $|H_\ell(f)|^2$ is the absolute square of the transform of $h_\ell(t)$, and $1/2T$ is the length of the sample functions. [We use the factor $(1/2T)^2$ here to account for the fact that the sample function length has been squared in the mathematical manipulations.] Transforming the expression for $h_\ell(t)$, and taking the absolute square, we obtain

$$|H_\ell(f)|^2 = \frac{a^2}{a^2 + (2\pi f)^2} \quad (\text{VII.6})$$

From (VI.20) and (VI.21), the expressions for $\bar{\omega}_\mu(f)$ are

$$\left. \begin{aligned} \bar{\omega}_{uu}(f) &= 2b\ell^2 \beta_2^2 \left[\frac{4}{4b^2 + \omega^2} + \frac{2}{4b^2 + \tilde{\omega}'^2} + \frac{2}{4b^2 + \tilde{\omega}''^2} \right]; \text{ uncorrelated case} \\ \bar{\omega}_{uc}(f) &= 2b\ell^2 \beta_1^2 \left[\frac{4}{4b^2 + \omega^2} + \frac{2}{4b^2 + \tilde{\omega}'^2} + \frac{2}{4b^2 + \tilde{\omega}''^2} \right]; \text{ correlated case} \end{aligned} \right\} \quad (\text{VII.7})$$

In systems of practical interest, the constants b and ω_o are both much greater than ω over the effective bandwidth of the filter; thus the expression for the average spectrum at the output of the low-pass filter may be written

$$\left. \begin{aligned} \bar{w}_{du}(f) &\cong \left(\frac{2\ell^2 \beta_2^2}{b} + \frac{2b\ell^2 \beta_2^2}{b^2 + \omega_o^2} \right) \frac{a^2}{a^2 + \omega^2} \\ \text{and} \\ \bar{w}_{dc}(f) &\cong \left(\frac{2\ell^2 \beta_1^2}{b} + \frac{2b\ell^2 \beta_1^2}{b^2 + \omega_o^2} \right) \frac{a^2}{a^2 + \omega^2} \end{aligned} \right\} \quad (\text{VII.8})$$

Substituting (VII.8) in (VII.5), we obtain, since $a \ll b$,

$$\left. \begin{aligned} \sigma_{du}^2 &\cong \frac{a\ell^2 \beta_2^2}{2} \left(\frac{2}{b} + \frac{2b}{b^2 + \omega_o^2} \right); \text{ uncorrelated case} \\ \text{and} \\ \sigma_{dc}^2 &\cong \frac{a\ell^2 \beta_1^2}{2} \left(\frac{2}{b} + \frac{2b}{b^2 + \omega_o^2} \right); \text{ correlated case} \end{aligned} \right\} \quad (\text{VII.9})$$

Because of the assumption that the bandpass filter is narrow band,

$$\frac{b}{(b^2 + \omega_o^2)} \ll \frac{1}{b} \text{ so that}$$

$$\left. \begin{aligned} \sigma_{du}^2 &\approx \frac{a^2 \beta_2^2}{b} \\ \sigma_{dc}^2 &\approx \frac{a^2 \beta_1^2}{b} \end{aligned} \right\} \quad \text{(VII.10)}$$

We observe from (VII.10) that as long as $a \ll b$, the average variance of the decision voltage is a function of the product of the inverse time constants of the filters, i.e., of the constants a and b .

3. Application to Calculation of Probability of a Correct Decision

In this section we shall describe the application of the results of this study to the calculation of the probability of a correct decision as a function of the SNR at the multiplier input. Values of the parameters for any particular system can then be substituted in the equations in order to obtain system performance.

Since we have assumed that the decision voltage is gaussian distributed, the expression for the d.d. is

$$f(r) = \frac{1}{\sigma_d^2 \sqrt{2\pi}} \exp \left\{ -\frac{(r - m_d)^2}{2\sigma_d^2} \right\} \quad \text{(VII.11)}$$

where m_d and σ_d are the mean and variance of the decision voltage, respectively. We assume that the system must meet some false-alarm rate, which we specify in terms of a false-alarm probability per integration interval, P_f , and we assume that a decision that the proper range has been reached will be made if $E_d(t)$ exceeds a preset threshold. Thus, we must set a threshold such that the probability that $E_d(t)$ exceeds this threshold at the end of an integration interval is $\leq P_f$. (The integration interval is determined by the vehicle velocity and the minimum acceptable accuracy.) Note that we are considering here only one point in time, i.e., the end of an integration interval. This, of course, is an approximation since we are in effect ignoring the points in time prior to this. Unfortunately, with our present state of knowledge, it is not possible to calculate the cumulative probability within an integration interval. In terms of the d.d., we require that

$$\int_{E_{th}}^{\infty} \frac{1}{\sigma_d^2 \sqrt{2\pi}} \exp \left\{ -\frac{r - m_d(t)^2}{2\sigma_d^2} \right\} dr \leq P_f,$$

or, making a change of variable,

$$\int_{\frac{E_{th} - m_d(t)}{\sigma_d}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\Gamma^2}{2}} d\Gamma \leq P_f. \quad (\text{VII.12})$$

Now, to each value of P_f there corresponds a number k_f which satisfies (VII.12), i.e., the lower limit of the integral must equal some number k_f for a given P_f ;

$$\frac{E_{th} - m_d(t)}{\sigma_d} = k_f; \quad k_f = f(P_f). \quad (\text{VII.13})$$

Using the values of $m_d(t)$ and σ_d given by (VII.10) and (VII.4), and the value of k_f determined by the required P_f (which we can look up in a table of the gaussian distribution function), we can calculate the required threshold voltage.

In computing the required threshold, we should take the time variation of A_r into account, and we could do so; however, in a practical case, it is a reasonable approximation to use an average return signal over the altitude range of interest. That is, we assume that the altimeter is turned on at some altitude h_o , and use the average return signal amplitude over the altitude interval from h_o to the preset altitude. If the reflecting surface is not smooth, we could use the same approximation for the standard deviation of the reflected signal.

Once the threshold E_{th} has been determined, we can calculate the probability of a correct decision as a function $(A_r/A_n)^2$, i.e., the SNR at multiplier input, at a given time t , since the other factors in the equations for the expected value and variance are fixed. (Actually, since we have assumed that A_n is constant, A_r is the only variable for a given instant of time.) These calculations are only valid for one instant of time; i.e., they give the probability of a correct decision regardless of what has happened previously. We cannot compute the cumulative probability of a correct decision over a time interval Δt because the probabilities are dependent within the interval. Therefore, we usually compute the probability at the preset altitude.

As an example, let us compute the SNR required to give a probability of a correct decision of 0.9 at time $t = t_o/\lambda$, for a false-alarm rate of 10^{-5} . We shall use the following system parameters for this example: $a = 10^2$; $b = 10^4$; $\alpha = \alpha' = 10^7$; $\lambda = 2 \times 10^{-6}$; $\tau_c = 10^{-8}$; $P_{fa} = 10^{-5}$. Substituting these values in (VII.4) and (VII.10), we obtain

$$E_{th} = 10^{-3} k_1' (2.85 + 2.85\rho') \quad (\text{VII.14})$$

where

$$\rho' = \left(\frac{\bar{A}_{r1}}{A_n} \right)^2$$

with $\bar{A}_{r1}$ the average value of the return signal in the time interval $\left[t_0, \frac{t_0 - \tau_c}{\lambda} \right]$. Then, evaluating (VII.4) and (VII.10) at $t = t_0/\lambda$, and substituting these results and (VII.14) in (VII.13), we obtain

$$\frac{E_{th} - m_d}{\sigma_d} = \frac{2.96(2.85 + 2.85 \rho') - (3.94\rho + 3.94 + 3.75 \times 10^3 \rho)}{\rho + 1} \quad (\text{VII.15})$$

where

$$\rho = \left(\frac{\bar{A}_{r2}}{A_n} \right)^2$$

with $\bar{A}_{r2}$ the average of the return signal in the time interval $\left[\frac{t_0 - \tau_c}{\lambda}, t_0/\lambda \right]$. For a probability of correct decision of 0.9, (VII.15) must equal -1.3; hence solving for ρ , we have

$$\rho = 10^{-3}(1.5 - 2.2\rho') \quad (\text{VII.16})$$

In most cases ρ' will be about two orders of magnitude less than ρ , so that

$$\rho \approx 1.5 \times 10^{-3} \quad (\text{VII.17})$$

Thus we have a 0.9 probability of correct decision at $t = t_0/\lambda$ with a SNR at the multiplier input of -28.2 db.

CHAPTER VIII

NUMERICAL RESULTS

A program was written for the CDC 3600 computer which simulates the system under consideration for general operating conditions; however, the amount of computer time required to perform the simulation in the general case is excessive. Therefore, in the interest of economy, we have performed the simulation only for one limiting case. However, with some extrapolation, this case illustrates the operation of an actual system. The condition simulated is the case of predetection integration, i.e., the case in which the integration is performed in the bandpass filter, so that the combination of the square-law device and the low pass is an envelope detector. From theory (see [20], [21], and [22]) we would expect that in this case, the output of the bandpass filter would have a gaussian distribution function (d.f.) and the output of the low-pass filter would have an exponential d.f. Thus, we can check the simulation against theory for this case. In addition, the results for this case can be extrapolated to the actual system, since the time constant used for the bandpass filter is approximately equal to the time constant of the low-pass filter in an actual system; hence the results for the output of the bandpass filter will apply approximately to the output of the low-pass filter in the actual system, as long as the time constant of the low-pass filter is much greater than the time constant of the bandpass filter. Hence, simulation for this case not only gives a theoretical check of the procedure used in the simulation, but it also gives us information on the distribution functions in an actual system.

The results of the simulation are presented in the form of graphs in Figures 6 through 21. (These graphs were produced by a plotter connected to the computer output, and the discrete points are connected by straight lines with no attempt made to smooth the presentation.) Figures 6 through 13 show the d.f., and the percent error between the actual distribution and a gaussian

d.f. with the same expected value and variance, for the output of the bandpass filter. The graphs are for various points within the correlation interval, in order to illustrate the effect of the delay between the transmitted and received signals on the statistics. The graphs labeled $\tau = 0$ correspond to a correlation coefficient of unity, while the graphs labeled $\tau = 0.9 \times 10^{-8}$ correspond to a correlation coefficient approximately equal to zero. The two intermediate points are useful as checks to see if any unexpected phenomena occur between the end points. Examining the figures, we observe that the output of the bandpass filter approximates a gaussian d.f. very closely over the interval from approximately -3σ to infinity. The apparent error between the actual d.f. and the gaussian d.f. for values of the random variable less than about -3σ below the expected value is probably due to the limited number of samples (1000) used in the simulation. Because of this, the number of random values obtained in the low probability region is extremely small, so that very little confidence can be placed in the results for this region. Thus, we may infer that the d.f. of the output of the bandpass filter is a very good approximation to a gaussian d.f. over the entire range of the random variable.

Figures 14 through 21 show the d.d. and d.f. at the output of the low-pass filter for the same points in the correlation interval illustrated for the bandpass filter. The d.d. is included here so that we may get an idea of the shape; however, the d.f. is a better measure of simulation because of the variation in the d.d. This variation is explained by the fact that we have only a finite number of samples, plus the fact that the simulation is performed on a discrete, rather than continuous, basis. However, we observe from the graphs that the d.d. has the general form of an exponential d.d., as we expected from theory. Thus we are confident that the simulation is a valid representation of the actual system.

In performing the simulation, the computer generated a random, gaussian distributed process $\{\Psi(t)\}$ which had the correlation function given by (II.27); the random values generated were then used in (II.1) to obtain the L.O. process.

The received process was then generated in a similar manner. In the region where $\{X(t)\}$ and $\{S(t)\}$ were uncorrelated, the values of $\psi(t)$ and $\psi(t + \lambda t - t_0)$ were picked independently, while in the region where these two processes were correlated, $\psi(t + \lambda t - t_0)$ was picked subject to the conditional density function. For purposes of simulation, it was assumed that the noise process was gaussian distributed with the covariance function

$$K_{nn}(\tau) = \frac{A_n^2}{2} e^{-\frac{\sigma_n^2 |\tau|}{\tau_c}} \cos \omega_c \tau ; 0 \leq |\tau| \leq \infty$$

The values of the noise process were picked independent of the L.O. and received processes. Using the processes so generated, the computer performed the operations indicated in the block diagram of Figure 1, and computed the statistics at the low-pass and bandpass filter outputs.

The constants used in the simulation were the following (the symbols used below are those used in the analysis):

$$\begin{array}{lll} A_s = 10^2 & \lambda = 2 \times 10^{-5} & b = 2 \times 10^{-3} \\ A_r = 1 & \tau_c = 10^{-8} & a = 1 \times 10^{-3} \\ \sigma_n^2 = 5 & \sigma = 5 & A_n = 20 \end{array}$$

The amplitude factors are, of course, not realistic; they were chosen for convenience. The other constants are representative of what we might find in a practical system when the velocity uncertainty is small. On the basis of the rapidity with the sum of independent sets of random variables converge to a gaussian d.f., we observe that the time constant of the bandpass filter could be considerably shortened without affecting the results; hence we can infer that in an actual system, the outputs of both the bandpass and low-pass filters are approximately gaussian distributed. Note that the shape of the d.f. of the bandpass and low-pass filters remain essentially constant in both the correlated and uncorrelated intervals.

mean = -1.897
Variance = 5965

Figure 6. Computed D.F. at Bandpass Filter Output; $\tau = 0$

Mean = -1.897
Variance = 5965

Figure 7. Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0$

Mean = -15.813
Variance = 11279

Figure 8. Computed D.F. at Bandpass Filter Output;
 $\tau = 0.3 \times 10^{-8}$ sec

Mean = -15.813
Variance = 11279

Figure 9. Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.3 \times 10^{-8}$ sec

Mean = 24.427
Variance = 16933

Figure 10. Computed D.F. at Bandpass Filter Output;
 $\tau = 0.6 \times 10^{-8}$ sec

Mean = 24.427
Variance = 16933

Figure 11. Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.6 \times 10^{-8}$ sec

Mean = -20.836
Variance = 11472

Figure 12. Computed D.F. at Bandpass Filter Output;
 $\tau = 0.9 \times 10^{-8}$ sec

Mean = -20.836
Variance = 11472

Figure 13. Percent Error Between Actual D.F. and Equivalent Gaussian D.F.; $\tau = 0.9 \times 10^{-8}$ sec

Mean = 10688
Variance = 8.9×10^7

Figure 14. Computed D.F. at Low-Pass Filter Output; $\tau = 0$

Figure 15. Computed D.D. at Low-Pass Filter Output; $\tau = 0$

Mean = 10375
Variance = 8.1×10^7

Figure 16. Computed D.F. at Low-Pass Filter Output;
 $\tau = 0.3 \times 10^{-8}$ sec

AVERAGE VALUE OF R.V. = 10375
VARIANCE OF R.V. = ***** 8.1×10^7

VARIANCE OF R.V. = ***** 8.1×10^7

LOWPASS FILTER

TIME= 33.5 -4

TAU= 0.3 -8

Figure 17. Computed D.D. at Low-Pass Filter Output;
 $\tau = 0.3 \times 10^{-8}$ sec

Mean = 10335
Variance = 8.1×10^7

Figure 18. Computed D.F. at Low-Pass Filter Output;
 $\tau = 0.6 \times 10^{-8}$ sec

Figure 19. Computed D.D. at Low-Pass Filter Output;
 $\tau = 0.6 \times 10^{-8}$ sec

Mean = 10224
Variance = 8×10^7

Figure 20. Computed D.F. at Low-Pass Filter Output;
 $\tau = 0.9 \times 10^{-8}$ sec

AVERAGE VALUE OF R.V.= 10224
VARIANCE OF R.V.= ***** 8×10^7

LOWPASS FILTER

TIME= 30.5 -4

TAU= 0.9 -8

Figure 21. Computed D.D. at Low-Pass Filter Output;
 $\tau = 0.9 \times 10^{-8}$ sec

CHAPTER IX

SUMMARY AND CONCLUSIONS

In this report we have presented an analysis of a particular radar altimeter which utilizes random angle modulation. We have considered the non-stationary aspect of the problem by defining an average correlation function, and an average energy density spectrum, and working with these quantities. In addition we have derived general equations for the amplitude d.d., the correlation function, and the density spectrum of the random angle modulated process; however, this is not a general analysis which answers all questions. The reasons for this, and areas requiring further study, are discussed below.

1. Limitations of the Analysis

Throughout the analysis we have emphasized the practical aspects rather than the completely theoretical aspects; thus, in deriving various equations, we have neglected terms whenever such an omission will not cause a significant error in the final result, and we have made certain simplifying assumptions about the characteristics of the elements of the system. In general, these assumptions are compatible with practical systems. In order to obtain concrete results, it was necessary to specify the form of the modulating process, the covariance function of the noise process, and the impulse responses of the bandpass and low-pass filters. However, this is no limitation, since the same techniques could be used for any specified values. The quantities used in the analysis are typical of those found in actual systems, so that the equations should be representative of what one would encounter in practice.

The most severe limitation of the analysis is the assumption that the decision voltage is gaussian distributed. While we know from published studies

and experimental data that this is a good approximation for cases where the time constant of the low-pass filter is much greater than the time constant of the bandpass filter, which is usually the case in systems of this type, the analysis is not a good approximation when this is not true. Although it is possible to derive an expression for the d.d. of the decision voltage which will hold for all physical processes, it is doubtful that even a high-speed computer could evaluate the expression in a reasonable amount of time; hence the gaussian assumption is the only practical approach to the problem at the present time. The other significant limitation of the analysis is the assumption that all system components are operating in an unsaturated mode. While this is usually the desired condition, it is not always possible to maintain this condition in practice.

2. Areas for Further Study

This analysis is a fairly rigorous study of the system. We have developed a general approach which is valid under rather broad conditions, and have developed general equations for the quantities of interest; however, detailed calculations have been carried out for only one set of conditions. In particular, we have derived equations for the statistics at the multiplier output which are valid for either a specular return or a scattered return,* but we have only indicated the extension of the analysis which is required for the case of a scattered signal. The primary reason for this is that the effect of the scattering depends on the correlation function and d.d. of the scattered return, and at present, we do not have sufficient information on these quantities to assign reasonable functional forms. Hence, a detailed study of the statistics of the return from various surfaces should be carried out, so that the effect of scattering on system performance can be evaluated.

* We define a specular return as one where the reflected signal has the same form as the transmitted signal, while for a scattered return, the reflected signal has random perturbations.

Further studies are required to determine the effect of saturation on system performance. Saturation will cause nonlinearities in the multiplier which will lead to cross-modulation products, as well as changing the d.d. of the output. Similarly, saturation in other system components will lead to modification of the calculated spectra and expected values. These studies will require the development of techniques for evaluating the perturbation of the spectra of the sum of two random processes due to limiting, and will not be easy.

An extension of the present analysis would be the consideration of the effect of various types of noise on system performance. While this would require no new ideas, it would be a valuable contribution. Similarly, the general approach developed here could be used to calculate the effect of having a decision time which is of the same order of magnitude as the effective correlation time. The most valuable extension of the present analysis would be either a computation of the error involved in using the gaussian distribution for the decision voltage, or an expression for the actual d.d. of the decision voltage which is amenable to computation. Neither of these extensions seems possible with the present state of knowledge concerning the structure of random processes. However, we entertain the hope that such computations will be possible in the future.

APPENDIX I

SPECTRAL THEORY OF A LIMITED CLASS OF NONSTATIONARY RANDOM PROCESSES

In the analysis of the performance of practical electronic systems, it is common practice to use the spectral theory of wide-sense stationary random processes. (In this section, we shall use the expression stationary synonymously with wide-sense stationary.) However, the processes involved are never truly stationary because the analysis must apply to a finite time interval. In addition, many signals of practical interest are nonstationary even over an infinite interval. In this appendix we will present the spectral theory of a class of nonstationary random processes which will cover all cases of interest for physical systems, i.e., the class of processes whose sample functions are of finite length, and for which the Fourier transform of all sample functions, except perhaps a set of measure zero, exists. Most of the material in this appendix is based on [12], [23], and [27].

1. General Discussion

In the spectral theory of stationary random processes, the spectrum of a random process tells us how the average power is distributed over frequency, and this information can then be used to compute the spectral distribution after various transformations on the random process. We would like to be able to define a similar function for a nonstationary process which would tell us how the energy of the process is distributed over frequency. (Since we are considering sample functions of finite length, we must speak of energy rather than average power.) Requirements on such a function are that it must be consistent with the theory of stationary processes, i.e., it must give the same spectrum as the theory of stationary processes when we allow the sample

functions of a stationary process to become of infinite length, and it must be related to the correlation function of the process, i.e., to $E\{X(t_1)X(t_2)\}$. It must also satisfy the basic definition of energy.

The approach used in developing a theory of the class of processes under consideration is based on a combination of intuition, ordinary Fourier analysis, and probability theory. The development includes a spectral function which is a function of time, and an average spectral function which is analogous to the spectral function of a stationary process. In addition to defining the spectral function, the theory must also consider the effects of transformations on the form of the function. The effect of filtering is considered in Section 3.

2. The Spectral Representation

We assume that almost all sample functions, i.e., all but a set of measure zero, are Fourier transformable. The transform of the i^{th} sample function is

$$i\Phi(f) = \int_{-T/2}^{T/2} i x(t) e^{-j2\pi ft} dt \quad (\text{AI.1})$$

where $i x(t)$ is the i^{th} sample function. The energy spectrum of this sample function is, from Fourier integral theory,

$$i_w(f) = |i\Phi(f)|^2 \quad (\text{AI.2})$$

hence, we would intuitively define the average energy density spectrum of the process as

$$\bar{w}(f) = E\{|^i\Phi(f)|^2\}. \quad (\text{AI.3})$$

We show this same result in a more elegant fashion below.

We describe the sample functions of the finite duration random process by

$$x_T(t) = \begin{cases} x(t) ; & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 ; & \text{V other } t \end{cases}$$

i.e., we assume that we are looking at a finite section of a process of infinite extent, subject to the restriction that $E\{x_T^2(t)\} < \infty$ for $t \in T$, where T is the interval $[-T/2, T/2]$. (We shall use the standard notation $a \in B$ to indicate that a is an element of the set B .) The correlation function of $\{x_T(t)\}$ is by definition

$$R(t_1, t_2) = \begin{cases} E\{x(t_1)x(t_2)\} ; & t_1, t_2 \in T \\ 0 ; & \text{V other } t_1, t_2 \end{cases} \quad (\text{AI.4})$$

Equation (AI.4) defines the correlation between points in the time domain. We now want to define an analogous function in the frequency domain. We proceed

as follows: It is well known that, for a broader class of processes than those considered here, there exists a spectral decomposition of the sample functions (see [5]), given by

$$x_T(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} dZ(f) \quad (\text{AI.5})$$

where $Z(f)$ is a complex random function, and the integral is a Stieltjes integral. Then the time correlation function of the process is

$$\begin{aligned} R_T(t_1, t_2) &= E\{x_T(t_1)x_T(t_2)\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{j2\pi(f_1 t_1 - f_2 t_2)\} E\{dZ(f_1)d\bar{Z}(f_2)\} \end{aligned} \quad (\text{AI.6})$$

where the overbar denotes the complex conjugate. For the class of processes considered here,

$$E\{dZ(f_1)dZ(f_2)\} = \Gamma(f_1, f_2)df_1df_2$$

i.e., we assume that $E\{dZ(f_1)dZ(f_2)\}$ is differentiable. (This is no restriction for a practical situation.) But, for this class of processes, the ordinary Fourier transform of almost all sample functions exists, so that

$$i_{Z(f)} = i_{\Phi(f)}$$

and

$$\Gamma(f_1, f_2) = E | \dot{\Phi}(f_1) \dot{\bar{\Phi}}(f_2) | . \quad (\text{AI.7})$$

Thus we can write

$$R_T(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{ j2\pi(f_1 t_1 - f_2 t_2) \} \Gamma(f_1, f_2) df_1 df_2 \quad (\text{AI.8})$$

and, conversely,

$$\Gamma(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_T(t_1, t_2) \exp \{ -j2\pi(f_1 t_1 - f_2 t_2) \} dt_1 dt_2 . \quad (\text{AI.9})$$

Thus $\Gamma(f_1, f_2)$ is the function which specifies the correlation relation in the frequency domain. Now if $t_1 = t_2 = t$, we have, from (AI.8),

$$E \{ x_T^2(t) \} = R_T(t, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{ j2\pi t(f_1 - f_2) \} \Gamma(f_1, f_2) df_1 df_2 .$$

Letting $f_1 - f_2 = \zeta$, and integrating over f_2 , we obtain

$$E \{ x_T^2(t) \} = R_T(t, t) = \int_{-\infty}^{\infty} \bar{\Gamma}(\zeta) e^{j2\pi \zeta t} d\zeta \quad (\text{AI.10})$$

where

$$\bar{\Gamma}(\zeta) = \int_{-\infty}^{\infty} \Gamma(\zeta + f_2, f_2) df_2.$$

Writing $\Gamma(f_1, f_2)$ as the double transform of $R_T(t, t - \tau)$, and letting $f_1 = f_2 = f$, we can similarly show that

$$\Gamma(f, f) = \int_{-\infty}^{\infty} \bar{R}(\tau) e^{-j2\tau f\pi} d\tau \quad (\text{AI.11})$$

where

$$\bar{R}(\tau) = \int_{-\infty}^{\infty} R(t, t - \tau) dt.$$

Thus, we define the average correlation function as $\bar{R}(\tau)$, and we define the transform of $\bar{R}(\tau)$ as the average energy spectrum* $\bar{\omega}(f)$; hence, we observe from (AI.11) that $\bar{\omega}(f) = \Gamma(f, f)$. We also note that $\bar{\Gamma}(0)$ equals the average energy, and that

$$\bar{\Gamma}(0) = \int_{-\infty}^{\infty} \Gamma(f, f) df = \int_{-\infty}^{\infty} \bar{\omega}(f) df.$$

*

Since

$$\bar{R}(0) = \int_{-\infty}^{\infty} R(t, t) dt = \int_{-\infty}^{\infty} E\{x^2(t)\} dt = \text{the average energy} = \int_{-\infty}^{\infty} \Gamma(f, f) df,$$

we define $\Gamma(f, f) \equiv \bar{\omega}(f)$ as the average energy density spectrum. This is analogous to the power density for stationary processes.

From (AI.7), $\Gamma(f, f) = E\{|\dot{\Phi}(f)|^2\}$, which bears out our previous intuitive definition.

It is important to note that the variance in the case of a nonstationary process is not given by $\bar{R}(0) - m^2(t)$ as it is in the stationary case. The variance of a zero mean process as a function of time is given by $R_T(t, t)$, which, from (AI.10), is

$$R_T(t, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\xi + f, f) e^{j2\pi\xi t} df d\xi.$$

This equation can be written as

$$R_T(t, t) = \int_{-\infty}^{\infty} W(f, t) df \quad (\text{AI.12})$$

where $W(f, t)$ is the transform of $\Gamma(\xi + f, f)$ with respect to ξ . However, for purposes of this analysis, we shall define an average variance by

$$\sigma^2 = \frac{\bar{R}(\tau)}{T}$$

where T is the length of the sample function. When the process $\{X_T(t)\}$ is a section of a stationary process, with correlation function $R(\tau)$, and when $T \ll \tau_c$, where τ_c is the correlation time,

$$\bar{R}(\tau) \approx TR(\tau)$$

hence, our definition of the average variance of the process coincides closely with the variance of the stationary process, as it should. When the sample functions of the process are of nearly the same length as the correlation time, (AI.12) must be used. We shall not have occasion to use this equation in the present analysis.

E. The Effect of a Linear Filter on Nonstationary Processes

Let the nonstationary random process $\{X(t)\}$ with correlation function $R_{xx}(t_1, t_2)$ be the input to a linear filter, and let the impulse response of the filter be $h(t)$. Then the output process $\{Y(t)\}$ is defined by

$$y(t) = \int_{-\infty}^t h(t - \tau)x(\tau)d\tau \quad (\text{AI.13})$$

hence,

$$E\{y(t)\} = \int_{-\infty}^t h(t - \tau)E\{x(\tau)\}d\tau. \quad (\text{AI.14})$$

In order to compute the correlation function of $\{Y(t)\}$, it is convenient to first compute the cross correlation of the input and output.

Write

$$x(t_1)y(t_2) = \int_{-\infty}^{t_2} x(t_1)x(t_2 - \tau)h(\tau)d\tau$$

then

$$R_{xy}(t_1, t_2) = E\{x(t_1)y(t_2)\} = \int_{-\infty}^{t_2} R_{xx}(t_1, t_2 - \tau)h(\tau) d\tau \quad (AI.15)$$

which we can write as

$$R_{xy}(t_1, t_2) = R_{xx}(t_1, t_2) * h(t_2) \quad (AI.16)$$

where the * indicates convolution. We now write

$$y(t_1)y(t_2) = \int_{-\infty}^{t_1} x(t_1 - \tau)y(t_2)h(\tau) d\tau$$

so that

$$\begin{aligned} E\{y(t_1)y(t_2)\} &= R_{yy}(t_1, t_2) = \int_{-\infty}^t R_{xy}(t_1 - \tau, t_2)h(\tau) d\tau \\ &= R_{xy}(t_1, t_2) * h(t_1) . \end{aligned} \quad (AI.17)$$

Thus, from (AI.16) and (AI.17),

$$R_{yy}(t_1, t_2) = R_{xx}(t_1, t_2) * h(t_2) * h(t_1) . \quad (\text{AI.18})$$

Equation (AI.18) means that the convolution of $R_{xx}(t_1, t_2)$ with $h(t_2)$, for a constant t_1 , followed by a convolution of this function with $h(t_1)$ gives the output correlation function.

In order to compute the average energy density spectrum at the filter output we proceed as follows: From the above discussion we have

$$\Gamma_{yy}(f_1, f_2) = \Gamma_{xx}(f_1, f_2) H(f_1) \overline{H}(f_2)$$

so that

$$\Gamma_{yy}(f, f) = \Gamma_{xx}(f, f) |H(f)|^2$$

thus, since $\Gamma(f, f) = \overline{\omega}(f)$,

$$\overline{\omega}_y(f) = \overline{\omega}_x(f) |H(f)|^2 . \quad (\text{AI.19})$$

Note that this is the analog of the equation for the stationary case.

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