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MBJ − Neutrosophic �� − Ideal of �� − Algebra

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MBJ – Neutrosophic $\boldsymbol{\beta}$ – Ideal of $\boldsymbol{\beta}$ – Algebra

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Abstract: This paper extends the concept of ideal of a β – algebra to MBJ – Neutrosophic β – Ideal of a β – algebra. Further discusses about the homomorphic image, inverse image, cartesian product and related results.

Keywords: Neutrosophic Set, MBJ – Neutrosophic Set, MBJ – Neutrosophic β – Ideals, Cartesian Product of MBJ – Neutrosophic β – algebra.

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1. Introduction

Zadeh [21, 22] first presented the idea of Fuzzy Set by which shown a meaningful application in many fields and this theory is welcomed to handle the uncertainty. As a generalization of fuzzy set Atanassov [7, 11] introduced Intuitionistic Fuzzy Set which assigns a pair with membership degree and non – membership degree. The Interval Valued Fuzzy Set [6, 10, 12] represents the membership degree with interval values to reflect the uncertainty in assigning membership degree. As an extension for all elements in any set, Neutrosophic Fuzzy Set provides with truth, intermediate and false membership function by Smarandache, F [16, 17, 18] and is further developed to MBJ – Neutrosophic fuzzy set [19, 20] with truth membership function, intermediate interval valued membership function and false membership function.

Neggers and Kim [18] brought a new structure of algebra called β – algebra and Jun [17] dealt some related topics on β – algebra. The fusion of fuzzy with algebra and the notion was initiated by Rosenfeld [15]. Further many researchers correlated some algebras with fuzzy sets. Ansari [5, 8] initialized the fuzzy β – subalgebra of β – algebra and also introduced fuzzy β – ideal of β – algebra. With these inspirations, this paper extends to MBJ – Neutrosophic β – ideal of β – algebra and analyzed some result.

2. Preliminaries

In this section, some definitions and examples of β – algebra and fuzzy set are discussed.

2.1 Definition: [5, 8, 14]   A non-empty set $(X, +, -, 0)$ is called a β – algebra if

   i. $x - 0 = x$

   ii. $(0 - x) + x = 0$

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iii. $(x - y) - z = x - (z + y)$ $\forall\, x, y, z\, \in X.$

2.2 Example: [9] The following Cayley's table is formed using a set $X = \{\, 0, 1, 2, 3, 4, 5\,\}$ with a constant 0 and two binary operations $+$ and $-$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	4	3	1	2	5	0
5	5	2	3	1	0	4

-	0	1	2	3	4	5
0	0	1	2	3	5	4
1	1	0	4	5	3	2
2	2	5	0	4	1	3
3	3	4	5	0	2	1
4	4	3	1	2	0	5
5	5	2	3	1	4	0

$\therefore$ The set X is a β – algebra.

2.3 Definition: [5] A non – empty subset S of a β – algebra $(\,X, +, -, 0\,)$ is known as β – subalgebra if
   i. $x - y \in S$
   ii. $x + y \in S$ $\forall\, x, y\, \in S$

2.4 Example: Let $U_1 = \{\, 0, 2\,\}$ and $U_2 = \{\, 0, 1\,\}$ be any two subset of a β – algebra $X = \{\,(\, 0, 1, 2, 3, 4, 5),\ +, -, 0\,\}$.   Here U_1 is a β – subalgebra of X where as U_2 is not a β – subalgebra of X.

2.5 Definition: [8]   A non – empty subset I of a β – algebra is said to be β – ideal of $(\,X, +, -, 0\,)$ if it has the following conditions
   i. $0 \in I$
   ii. $x + y \in I$
   iii. $x - y$ and $y \in I$ then $x \in I\ \forall\, x, y\, \in X$

2.6 Exercise: [12]   Consider a β – algebra$(\,X, +, -, 0\,)$ in the Cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

The subset $I_1 = \{\, 0, 3\,\}$ of X is a β – ideal of X.

2.7 Definition: [5] A mapping $f : X \to Y$ is said to be a β – homomorphism where X and Y are two β – algebras with constant 0 and two binary operations $+$ and $-$ if
   i. $f(x + y) = f(x) + f(y)$

ii. $f(x - y) = f(x) - f(y) \; \forall \; x, y \in X.$

2.8 Definition: [22] A Fuzzy Set in X is a mapping, $\rho : X \rightarrow [0,1]$ for each x in X, $\rho(x)$ is called the membership value of $x \in X$.

2.9 Definition: [7] A non – empty set X is said to be Intuitionistic Fuzzy Set and is defined by $A = \{ < x, \rho(x), \eta(x) > / x \in X \}$ where $\rho_A : X \rightarrow [0,1]$ is a membership function of A and $\eta_A : X \rightarrow [0,1]$ is a non – membership function of A with $0 \leq \rho_A(x) + \eta_A \leq 1$.

2.10 Definition: [6] An Interval Valued Fuzzy Set on X is represented as $A = \{ (x, \bar{\rho}_A(x)) \} x \in X$ where $\bar{\rho}_A : X \rightarrow D[0,1]$ where $D[0,1]$ is the family of all closed subintervals of $[0,1]$. Also $\bar{\rho}_A(x) = [\rho_A^L(x), \rho_A^U(x)]$ where ρ_A^L and ρ_A^U are two fuzzy sets in X such that $\rho_A^L(x) \leq \rho_A^U(x) \; \forall \; x \in X$.

Remark: Now let us illustrate refined minimum $(rmin)$ and refined maximum $(rmax)$ of two elements in $D[0,1]$. Also characterized the symbols $\leq, \geq, =$ in case of two elements in $D[0,1]$.
Let $D_1 = [a_1, b_1]$ & $D_2 = [a_2, b_2] \in D[0,1]$ then
$rmin(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)]$
$rmax(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)].$
For $D_i = [a_i, b_i] \in D[0,1]$ for $i = 1, 2, 3, \dots$
$rsup_i(D_i) = [sup_i(b_i), sup_i(b_i)]$ & $rinf_i(D_i) = [inf_i(b_i), inf_i(b_i)]$
Now $D_1 \geq D_2$ if and only if $a_1 \geq a_2$, $b_1 \geq b_2$. Likewise, for $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

2.11 Definition: [6] The representation of an Interval Valued Intuitionistic Fuzzy Set A on X is $A = \{ < x, \bar{\rho}_A(x), \bar{\eta}_A(x) > / x \in X \}$ where $\bar{\rho}_A : X \rightarrow D[0,1]$ and $\bar{\eta}_A : X \rightarrow D[0,1]$ where $\bar{\rho}_A(x) = [\rho_A^L(x), \rho_A^U(x)]$ and $\bar{\eta}_A(x) = [\eta_A^L(x), \eta_A^U(x)]$ with the condition that $0 \leq \rho_A^L(x) + \eta_A^L \leq 1$ and $0 \leq \rho_A^U(x) + \eta_A^U \leq 1$.

2.12 Definition: [16, 17] The definition of an Neutrosophic Fuzzy Set A on X is characterized by a Truth – membership function ρ_T, an indeterminacy membership function ξ_I, and a falsity – membership function η_F where ρ_T, ξ_I, η_F are subsets of $[0,1]$ that is $\rho_T, \xi_I, \eta_F : X \rightarrow [0,1]$. Thus, the Neutrosophic Set is defined as $A = \{ < x, \rho_T(x), \xi_I(x), \eta_F(x) > / x \in X \}$.

2.13 Definition: [19,20] The structure $A = \{ < x, \rho_T(x), \bar{\bar{\xi}}_I(x), \eta_F(x) > / x \in X \}$ is called MBJ – Neutrosophic Set in X where $\rho_T, \eta_F : X \rightarrow [0,1]$ and $\bar{\bar{\xi}}_I : X \rightarrow D[0,1]$ with $\rho_T(x)$ denotes the truth membership function, $\bar{\bar{\xi}}_I(x)$ denotes an intermediate interval valued membership function and $\eta_F(x)$ denotes an false membership function.

2.14 Definition: An Fuzzy set is said to have a supremum property for any subset W of X there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W} \rho_A(x)$.

2.15 Definition: An Intuitionistic Fuzzy Set A is said to have a $sup - inf$ property for any subset W of X, there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W}\rho_A(x)$ and $\eta_A(x_0) = inf_{x \in W}\eta_A(x)$.

2.16 Definition: An Interval Valued Intuitionistic Fuzzy Set A in any set X is said to have $rsup - rinf$ property if for subset W of X there exists $x_0 \in W$ such that $\bar{\rho}_A(x_0) = rsup_{x \in W}\bar{\rho}_A(x)$ and $\bar{\eta}_A(x_0) = rinf_{x \in W}\bar{\eta}_A(x)$.

2.17 Definition: [19] An MBJ – Neutrosophic Fuzzy Set A in X has $sup - rsup - inf$ property if for subset W of X there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W}\rho_A(x)$; $\bar{\xi}_A(x_0) = rsup_{x \in W} \bar{\xi}_A(x)$; $\eta_A(x_0) = inf_{x \in W} \eta_A(x)$ respectively.

2.18 Definition: [12] An Interval Valued Fuzzy Set $A = \{< x, \bar{\rho}_A(x) >/x \in X\}$ in X is said to be Interval Valued Fuzzy β – ideal of X if

    i. $\bar{\rho}_A(0) \geq \bar{\rho}_A(x)$

    ii. $\bar{\rho}_A(x + y) \geq r\min\{\bar{\rho}_A(x), \bar{\rho}_A(y)\}$

    iii. $\bar{\rho}_A(x) \geq r\min\{\bar{\rho}_A(x - y), \bar{\rho}_A(y)\} \ \forall \ x, y \in X$.

2.19 Definition: An Intuitionistic Fuzzy Set $A = \{< x, \rho(x), \eta(x) >/x \in X\}$ in X is known as Intuitionistic Fuzzy β - ideal of X if

    i. $\rho_A(0) \geq \rho_A(x)$; $\eta_A(0) \leq \eta_A(x)$

    ii. $\rho_A(x + y) \geq \min\{\rho_A(x), \rho_A(y)\}$; $\eta_A(x + y) \leq \max\{\eta_A(x), \eta_A(y)\}$

    iii. $\rho_A(x) \geq \min\{\rho_A(x - y), \rho_A(y)\}$; $\eta_A(x) \leq \max\{\eta_A(x - y), \eta_A(y)\}$

2.20 Definition: [19] Let X be a β – algebra and an MBJ Neutrosophic Set $A = \{ \rho_A, \ \bar{\xi}_A, \ \eta_A \}$ is called an MBJ – Neutrosophic β – subalgebra of X if it satisfies

    i. $\rho_A(x + y) \geq \min\{\rho_A(x), \rho_A(y)\}$; $\rho_A(x - y) \geq \min\{\rho_A(x), \rho_A(y)\}$

    ii. $\bar{\xi}_A(x + y) \geq r\min\{\bar{\xi}_A(x), \bar{\xi}_A(y)\}$; $\bar{\xi}_A(x - y) \geq r\min\{\bar{\xi}_A(x), \bar{\xi}_A(y)\}$

    iii. $\eta_A(x + y) \leq \max\{\eta_A(x), \eta_A(y)\}$; $\eta_A(x - y) \leq \max\{\eta_A(x), \eta_A(y)\}$

3 MBJ – Neutrosophic β – Ideal of β – Algebra

This part frames the structure of MBJ – Neutrosophic β – Ideal of β – Algebra and studied the related results.

3.1 Definition: Let $(X, +, -, 0)$ be a β – algebra. An MBJ – Neutrosophic Set $K = \{ \rho_K, \ \bar{\xi}_K, \ \eta_K \}$ in X is called an MBJ – Neutrosophic β – Ideal of X if it satisfies the following conditions:

    i. $\rho_K(0) \geq \rho_K(x)$

        $\rho_K(x + y) \geq \min\{ \rho_K(x), \rho_K(y)\}$

        $\rho_K(x) \geq \min\{ \rho_K(x - y), \rho_K(y)\}$

ii. $\bar{\xi}_K(0) \geq \bar{\xi}_K(x)$

$\bar{\xi}_K(x + y) \geq r\min\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}$

$\bar{\xi}_K(x) \geq r\min\{\bar{\xi}_K(x - y), \bar{\xi}_K(y)\}$

iii. $\eta_K(0) \leq \eta_K(x)$

$\eta_K(x + y) \leq \max\{\eta_K(x), \eta_K(y)\}$

$\eta_K(x) \leq \max\{\eta_K(x - y), \eta_K(y)\} \quad \forall x, y \in X$

3.2 Example : A β – algebra X in example 2.6 defines a MBJ – Neutrosophic set as $\rho_A : X \rightarrow [0,1]$; $\bar{\xi}_A : X \rightarrow D[0,1]$ and $\eta_A : X \rightarrow [0,1]$ such that

$$\rho_A(x) = \begin{cases} 0.4, & x = 0 \\ 0.2, & x = 1,3 \\ 0.3, & x = 2 \end{cases}$$

$$\bar{\xi}_{K_A}(x) = \begin{cases} [0.3, 0.7] & x = 0 \\ [0.1, 0.5] & x = 1,3 \\ [0.2, 0.6] & x = 2 \end{cases}$$

$$\eta_A(x) = \begin{cases} 0.1, & x = 0 \\ 0.4, & x = 1,3 \\ 0.5, & x = 2 \end{cases} \quad \text{is an MBJ – Neutrosophic } \beta \text{ – Ideal of } X.$$

3.3 Theorem: The intersection of any two MBJ – Neutrosophic β – Ideal of a β – algebra is also an MBJ – Neutrosophic β – Ideal.

Proof: Let $K_1 \& K_2$ be two MBJ – Neutrosophic β – Ideal of X.

Now, $(\rho_{K_1 \cap K_2})(0) \geq \min\{\rho_{K_1}(0), \rho_{K_2}(0)\}$

$= \min\{\rho_{K_1}(x), \rho_{K_2}(x)\}$

$= (\rho_{K_1 \cap K_2})(x)$

$(\rho_{K_1 \cap K_2})(x + y) \geq \min\{\rho_{K_1}(x + y), \rho_{K_2}(x + y)\}$

$= \min\{\min\{\rho_{K_1}(x), \rho_{K_1}(y)\}, \min\{\rho_{K_2}(x), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1}(x), \rho_{K_2}(x)\}, \min\{\rho_{K_1}(y), \rho_{K_2}(y)\}\}$

$= \min\{\rho_{K_1 \cap K_2}(x), \rho_{K_1 \cap K_2}(y)\}$

$\rho_{K_1 \cap K_2}(x) \geq \min\{\rho_{K_1}(x), \rho_{K_2}(x)\}$

$= \min\{\min\{\rho_{K_1}(x - y), \rho_{K_1}(y)\}, \min\{\rho_{K_2}(x - y), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1}(x - y), \rho_{K_2}(x - y)\}, \min\{\rho_{K_1}(y), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1 \cap K_2}(x - y), \rho_{K_1 \cap K_2}(y)\}\}$

$(\bar{\xi}_{K_1 \cap K_2})(0) \geq r\min\{\bar{\xi}_{K_1}(0), \bar{\xi}_{K_2}(0)\}$

$= r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}$

$= (\bar{\xi}_{K_1 \cap K_2})(x)$

$(\bar{\xi}_{K_1 \cap K_2})(x + y) \geq r\min\{\bar{\xi}_{K_1}(x + y), \bar{\xi}_{K_2}(x + y)\}$

$= r\min\{r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_1}(y)\}, r\min\{\bar{\xi}_{K_2}(x), \bar{\xi}_{K_2}(y)\}\}$

$= r\min\{r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}, r\min\{\bar{\xi}_{K_1}(y), \bar{\xi}_{K_2}(y)\}\}$

$= r\min\{\bar{\xi}_{K_1 \cap K_2}(x), \bar{\xi}_{K_1 \cap K_2}(y)\}$

$\bar{\xi}_{K_1 \cap K_2}(x) \geq r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}$

$$= \text{rmin} \{ \text{rmin} \{ \bar{\bar{\xi}}_{K_1}(x-y), \bar{\bar{\xi}}_{K_1}(y) \}, \text{rmin} \{ \bar{\bar{\xi}}_{K_2}(x-y), \bar{\bar{\xi}}_{K_2}(y) \} \}$$

$$= \text{rmin} \{ \text{rmin} \{ \bar{\bar{\xi}}_{K_1}(x-y), \bar{\bar{\xi}}_{K_2}(x-y) \}, \text{rmin} \{ \bar{\bar{\xi}}_{K_1}(y), \bar{\bar{\xi}}_{K_2}(y) \} \}$$

$$= \text{rmin} \{ \text{rmin} \{ \bar{\bar{\xi}}_{K_1 \cap K_2}(x-y), \bar{\bar{\xi}}_{K_1 \cap K_2}(y) \} \}$$

$$(\eta_{K_1 \cap K_2})(0) \leq \max\{ \eta_{K_1}(0), \eta_{K_2}(0) \}$$

$$= \max\{ \eta_{K_1}(x), \eta_{K_2}(x) \}$$

$$= (\eta_{K_1 \cap K_2})(x)$$

$$(\eta_{K_1 \cap K_2})(x+y) \leq \max\{ \eta_{K_1}(x+y), \eta_{K_2}(x+y) \}$$

$$= \max \{\max \{ \eta_{K_1}(x), \eta_{K_1}(y) \}, \max \{ \eta_{K_2}(x), \eta_{K_2}(y) \} \}$$

$$= \max \{\max \{ \eta_{K_1}(x), \eta_{K_2}(x) \}, \max \{ \eta_{K_1}(y), \eta_{K_2}(y) \} \}$$

$$= \max \{ \eta_{K_1 \cap K_2}(x), \eta_{K_1 \cap K_2}(y) \}$$

$$\eta_{K_1 \cap K_2}(x) \leq \max\{ \eta_{K_1}(x), \eta_{K_2}(x) \}$$

$$= \max \{\max \{ \eta_{K_1}(x-y), \eta_{K_1}(y) \}, \max \{ \eta_{K_2}(x-y), \eta_{K_2}(y) \} \}$$

$$= \max \{\max \{ \eta_{K_1}(x-y), \eta_{K_2}(x-y) \}, \max \{ \eta_{K_1}(y), \eta_{K_2}(y) \} \}$$

$$= \max \{\max \{ \eta_{K_1 \cap K_2}(x-y), \eta_{K_1 \cap K_2}(y) \} \}$$

Hence $K_1 \cap K_2$ is an MBJ – Neutrosophic β – Ideal of X.

3.4 Theorem: The intersection of any set of MBJ – Neutrosophic β – Ideal of a β – Algebra X is also an MBJ – Neutrosophic β – Ideal.

3.5 Theorem: Let $K = \{ \rho_K, \bar{\bar{\xi}}_K, \eta_K \}$ be an MBJ – Neutrosophic β – Ideal. If $x \leq y$ then $\rho_K(x) \geq \rho_K(y)$; $\bar{\bar{\xi}}_K(x) \geq \bar{\bar{\xi}}_K(y)$ and $\eta_K(x) \leq \eta_K(y)$.

Proof: For any $x, y \in X$, $x \leq y \Rightarrow x - y = 0$.

$$\rho_K(x) \geq \min \{ \rho_K(x-y), \rho_K(y) \}$$

$$= \min \{ \rho_K(0), \rho_K(y) \}$$

$$= \rho_K(y)$$

$$\rho_K(x) \geq \rho_K(y)$$

$$\bar{\bar{\xi}}_K(x) \geq \text{rmin} \{ \bar{\bar{\xi}}_K(x-y), \bar{\bar{\xi}}_K(y) \}$$

$$= \text{rmin} \{ \bar{\bar{\xi}}_K(0), \bar{\bar{\xi}}_K(y) \}$$

$$= \bar{\bar{\xi}}_K(y)$$

$$\bar{\bar{\xi}}_K(x) \geq \bar{\bar{\xi}}_K(y)$$

$$\eta_K(x) \leq \max \{ \eta_K(x-y), \eta_K(y) \}$$

$$= \max \{ \eta_K(0), \eta_K(y) \}$$

$$= \eta_K(y)$$

$$\eta_K(x) \leq \eta_K(y).$$

3.6 Theorem: Let K be an MBJ – Neutrosophic β – Ideal of X whenever $x \leq z + y$ then $\rho_K(x) \geq \min\{ \rho_K(z), \rho_K(y) \}$; $\bar{\bar{\xi}}_K(x) \geq \text{rmin}\{ \bar{\bar{\xi}}_K(z), \bar{\bar{\xi}}_K(y) \}$ and $\eta_K(x) \leq \max\{ \eta_K(z), \eta_K(y) \}$

Proof: For $x, y, z \in X$

$$\rho_K(x) \geq \min \{ \rho_K(x-y), \rho_K(y) \}$$

$$= \min \{ \min \{ \rho_K((x-y)-z), \rho_K(z) \}, \rho_K(y) \}$$

$$= \min\{\min\{\rho_K(x-(z+y)),\rho_K(z)\},\rho_K(y)\}$$
$$= \min\{\min\{\rho_K(0),\rho_K(z)\},\rho_K(y)\}$$
$$\geq \min\{\rho_K(z),\rho_K(y)\}$$

$$\bar{\bar{\xi}}_K(x) \geq \text{rmin}\{\bar{\bar{\xi}}_K(x-y),\bar{\bar{\xi}}_K(y)\}$$
$$= \text{rmin}\{\text{rmin}\{\bar{\bar{\xi}}_K((x-y)-z),\bar{\bar{\xi}}_K(z)\},\bar{\bar{\xi}}_K(y)\}$$
$$= \text{rmin}\{\text{rmin}\{\bar{\bar{\xi}}_K(x-(z+y)),\bar{\bar{\xi}}_K(z)\},\bar{\bar{\xi}}_K(y)\}$$
$$= \text{rmin}\{\text{rmin}\{\bar{\bar{\xi}}_K(0),\bar{\bar{\xi}}_K(z)\},\bar{\bar{\xi}}_K(y)\}$$
$$\geq \text{rmin}\{\bar{\bar{\xi}}_K(z),\bar{\bar{\xi}}_K(y)\}$$

$$\eta_K(x) \leq \max\{\eta_K(x-y),\eta_K(y)\}$$
$$= \max\{\max\{\eta_K((x-y)-z),\eta_K(z)\},\eta_K(y)\}$$
$$= \max\{\max\{\eta_K(x-(z+y)),\eta_K(z)\},\eta_K(y)\}$$
$$= \max\{\max\{\eta_K(0),\eta_K(z)\},\eta_K(y)\}$$
$$\leq \max\{\eta_K(z),\eta_K(y)\}$$

3.7 Theorem: Let $K = \{\rho_K,\bar{\bar{\xi}}_K,\eta_K\}$ be an MBJ – Neutrosophic β – Ideal of X, then sets $X_{\rho_K} = \{x \in X : \rho_K(x) = \rho_K(0)\}$; $X_{\bar{\bar{\xi}}_K} = \{x \in X : \bar{\bar{\xi}}_K(x) = \bar{\bar{\xi}}_K(0)\}$ and $X_{\eta_K} = \{x \in X : \eta_K(x) = \eta_K(0)\}$ are β – ideals of X.

Proof: Since $\rho_K(x) = \rho_K(0) \Longrightarrow 0 \in X_{\rho_K}$

If $x - y, y \in X_{\rho_K}$

$\Longrightarrow \rho_K(x-y) = \rho_K(0)$; $\rho_K(y) = \rho_K(0)$

Now, $\rho_K(x) \geq \min\{\rho_K(x-y),\rho_K(y)\}$
$$= \min\{\rho_K(0),\rho_K(0)\}$$
$$= \rho_K(0)$$

$\rho_K(x) \geq \rho_K(0)$

But $\rho_K(x) \leq \rho_K(0)$ implies $\rho_K(x) = \rho_K(0)$

$\Longrightarrow x \in X_{\rho_K}$

$x - y, y \in X_{\rho_K} \Longrightarrow x \in X_{\rho_K}$

$\therefore X_{\rho_K}$ is an β – Ideal of X

$\bar{\bar{\xi}}_K(x) = \bar{\bar{\xi}}_K(0) \Longrightarrow 0 \in X_{\bar{\bar{\xi}}_K}$

If $x - y, y \in X_{\bar{\bar{\xi}}_K}$

$\Longrightarrow \bar{\bar{\xi}}_K(x-y) = \bar{\bar{\xi}}_K(0)$; $\bar{\bar{\xi}}_K(y) = \bar{\bar{\xi}}_K(0)$

Now, $\bar{\bar{\xi}}_K(x) \geq \text{rmin}\{\bar{\bar{\xi}}_K(x-y),\bar{\bar{\xi}}_K(y)\}$
$$= \text{rmin}\{\bar{\bar{\xi}}_K(0),\bar{\bar{\xi}}_K(0)\}$$
$$= \bar{\bar{\xi}}_K(0)$$

$\bar{\bar{\xi}}_K(x) \geq \bar{\bar{\xi}}_K(0)$

But $\bar{\bar{\xi}}_K(x) \leq \bar{\bar{\xi}}_K(0)$ implies $\bar{\bar{\xi}}_K(x) = \bar{\bar{\xi}}_K(0)$

$\Longrightarrow x \in X_{\bar{\bar{\xi}}_K}$

$x - y, y \in X_{\bar{\bar{\xi}}_K} \Longrightarrow x \in X_{\bar{\bar{\xi}}_K}$

$\therefore X_{\bar{\bar{\xi}}_K}$ is an β – Ideal of X.

Similarly, X_{η_K} is also an β – Ideal of X.

3.8 Theorem: Suppose J is subset of X. An MBJ – Neutrosophic set $K = \{\, \rho_K \,, \bar{\bar{\xi}}_K \,, \eta_K \,\}$ such that

$\rho_K = \begin{cases} t, & x \in J \\ s, & x \notin J \end{cases}$; $\bar{\bar{\xi}}_K = \begin{cases} \bar{t}, & x \in J \\ \bar{s}, & x \notin J \end{cases}$ and $\eta_K = \begin{cases} \alpha, & x \in J \\ \beta, & x \notin J \end{cases}$ where $t, s, \alpha, \beta \in [\,0,1\,]$ and

$\bar{t}, \bar{s} \in D[\,0,1\,]$ with $[t_0, t_1] \geq [s_0, s_1]$. Then the MBJ – Neutrosophic set $K = \{\, \rho_K \,, \bar{\bar{\xi}}_K \,, \eta_K \,\}$ is an MBJ – Neutrosophic β – ideal of X if and only if J is an β – ideal of X.

Proof: Consider an MBJ – Neutrosophic set $K = \{\, \rho_K \,, \bar{\bar{\xi}}_K \,, \eta_K \,\}$ is an MBJ - Neutrosophic β – ideal of X

i) a) $\rho_K(0) \geq \rho_K(x) \quad \forall\, x \in X$

$\rho_K(0) = t \implies 0 \in J$

b) For $x, y \in J$

$\implies \rho_K(x) = t = \rho_K(y)$

$\therefore \rho_K(x + y) \geq \min\{\rho_K(x), \rho_K(y)\}$

$= \min\{t, t\}$

$\rho_K(x + y) = t$

$\implies x + y \in J$

c) For $x, y \in J$ if $x - y$ and $y \in J$

$\implies \rho_K(x - y) = t = \rho_K(y)$

$\therefore \rho_K(x) \geq \min\{\rho_K(x - y), \rho_K(y)\}$

$= \min\{t, t\} = t$

$\rho_K(x) = t$

$\implies x \in J$

ii) a) $\bar{\bar{\xi}}_K(0) \geq \bar{\bar{\xi}}_K(x) \quad \forall\, x \in X$

$\bar{\bar{\xi}}_K(0) = [t_0, t_1] \implies 0 \in J$

b) For $x, y \in J$

$\implies \bar{\bar{\xi}}_K(x) = [t_0, t_1] = \bar{\bar{\xi}}_K(y)$

$\therefore \bar{\bar{\xi}}_K(x + y) \geq \text{rmin}\{\bar{\bar{\xi}}_K(x), \bar{\bar{\xi}}_K(y)\}$

$= \text{rmin}\{[t_0, t_1], [t_0, t_1]\}$

$\bar{\bar{\xi}}_K(x + y) = [t_0, t_1]$

$\implies x + y \in J$

c) For $x, y \in J$ if $x - y$ and $y \in J$

$\implies \bar{\bar{\xi}}_K(x - y) = [t_0, t_1] = \bar{\bar{\xi}}_K(y)$

$\therefore \bar{\bar{\xi}}_K(x) \geq \text{rmin}\{\bar{\bar{\xi}}_K(x - y), \bar{\bar{\xi}}_K(y)\}$

$= \text{rmin}\{[t_0, t_1], [t_0, t_1]\} = [t_0, t_1]$

$\bar{\bar{\xi}}_K(x) = [t_0, t_1]$

$\implies x \in J$

iii) a) $\eta_K(0) \leq \eta_K(x) \quad \forall\, x \in X$

$\eta_K(0) = \alpha \implies 0 \in J$

b) For $x, y \in J$

$\implies \eta_K(x) = \alpha = \eta_K(y)$

$\therefore \eta_K(x + y) \leq \max\{\eta_K(x), \eta_K(y)\}$

$= \max\{\alpha, \alpha\}$

$\eta_K(x + y) = \alpha$

$$\Longrightarrow x + y \ \in J$$

c) For $x, y \in J$ if $x - y$ and $y \in J$

$$\Longrightarrow \eta_K(x - y) = \alpha = \eta_K(y)$$

$$\therefore \eta_K(x) \le \max\{\eta_K(x - y), \eta_K(y)\}$$

$$= \max\{\alpha, \alpha\} = \alpha$$

$$\eta_K(x) = \alpha$$

$$\Longrightarrow x \in J$$

$$\therefore J \text{ is an } \beta - \text{ideal of } X$$

Conversely, assuming J is an $\beta - $ideal of X. Then

i) a) If $0 \in J$

Implies $\rho_K(0) = t$

Also $\forall x \in X$, $Im(\rho_K) = [t, s]$ & $t > s$

$$\Longrightarrow \rho_K(0) \ge \rho_K(x) \ \forall x \in X$$

b) For any $x, y \in J$

$$\Longrightarrow x + y \ \in J$$

$$\Longrightarrow \rho_K(x) = \rho_K(x + y) = t = \rho_K(y)$$

$$= \min\{\rho_K(x), \rho_K(y)\}$$

$$\therefore \rho_K(x + y) \ge \min\{\rho_K(x), \rho_K(y)\}$$

c) For any $x, y \in J$

If $x - y$ and $y \in J \Longrightarrow x \in J$

$$\rho_K(x) = t = \min[t, t] = \min\{\rho_K(x - y), \rho_K(y)\}$$

ii) a) If $0 \in J$

Implies $\bar{\xi}_K(0) = \bar{t}$

Also $\forall x \in X$, $Im(\bar{\xi}_K) = [\bar{t}, \bar{s}]$ & $\bar{t} > \bar{s}$

$$\Longrightarrow \bar{\xi}_K(0) \ge \bar{\xi}_K(x) \ \forall x \in X$$

b) For any $x, y \in J$

$$\Longrightarrow x + y \ \in J$$

$$\Longrightarrow \bar{\xi}_K(x) = \bar{\xi}_K(x + y) = \bar{t} = \bar{\xi}_K(y)$$

$$= \text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}$$

$$\therefore \bar{\xi}_K(x + y) \ge \text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}$$

c) For any $x, y \in J$

If $x - y$ and $y \in J \Longrightarrow x \in J$

$$\bar{\xi}_K(x) = \bar{t} = \text{rmin}[\bar{t}, \bar{t}] = \text{rmin}\{\bar{\xi}_K(x - y), \bar{\xi}_K(y)\}$$

iii) a) If $0 \in J$

Implies $\eta_K(0) = \alpha$

Also $\forall x \in X$, $Im(\eta_K) = [\alpha, \beta]$ & $\alpha < \beta$

$$\Longrightarrow \eta_K(0) \le \eta_K(x) \ \forall x \in X$$

b) For any $x, y \in J$

$$\Longrightarrow x + y \ \in J$$

$$\Longrightarrow \eta_K(x) = \eta_K(x + y) = \alpha = \eta_K(y)$$

$$= \max\{\eta_K(x), \eta_K(y)\}$$

$$\therefore \eta_K(x + y) \le \max\{\eta_K(x), \eta_K(y)\}$$

c) For any $x, y \in J$

If $x - y$ *and* $y \in J \Rightarrow x \in J$

$\eta_K(x) = \alpha = \max[\alpha, \alpha] = \max\{\eta_K(x-y), \eta_K(y)\}$

$\therefore K$ is an MBJ $- \beta -$ Ideal of X.

3.9 Definition: Let $K = \{< x, \rho_K(x), \bar{\bar{\xi}}_K(x), \eta_K(x) >/x \in X\}$ be an MBJ- Neutrosophic Set in X and $f : X \to Y$ be a mapping then the image of K under f, $f(K)$ is defined as $f(K) = \{< x, f_{sup}(\rho_K), f_{rsup}(\bar{\bar{\xi}}_K), f_{inf}\eta_K(x) >/x \in Y\}$ where $\quad f_{sup}(\rho_K)(y) = sup_{x \in f^{-1}(y)}\rho_K(x)$; $f_{rsup}(\bar{\bar{\xi}}_K)(y) = rsup_{x \in f^{-1}(y)}\bar{\bar{\xi}}_K(x)$ and $\quad f_{inf}(\eta_K)(y) = inf_{x \in f^{-1}(y)}\eta_K(x)$

3.10 Definition: Let $f : X \to Y$ be a function and let K and L be two MBJ – Neutrosophic β – Ideal in $X \& Y$ respectively then the preimage of L under f is defined by $f^{-1}(L) = \{x, f^{-1}(\rho_K(x)), f^{-1}(\bar{\bar{\xi}}_K(x)), f^{-1}(\eta_K(x)) >/x \in X\}$ such that

$f^{-1}(\rho_K(x)) = \rho_K(f(x))$; $f^{-1}(\bar{\bar{\xi}}_K(x)) = \bar{\bar{\xi}}_K(f(x))$ and $f^{-1}(\eta_K(x)) = \eta_K(f(x))$.

3.11 Theorem: Let $f : X \to Y$ be an onto homomorphism of β - algebra. Suppose K is an MBJ – Neutrosophic β – Ideal of Y, then the preimage of $f^{-1}(K)$ is an MBJ – Neutrosophic β – Ideal of X.

Proof: Suppose K be an MBJ - Neutrosophic β - ideal of Y

i) For $x \in X$

$f^{-1}(\rho_K(0)) = \rho_K(f(0))$

$= \rho_K(0)$

$\geq \rho_K(x)$

For some $x, y \in X$

$f^{-1}(\rho_K)(x + y) = \rho_K(f(x + y))$

$= \rho_K(f(x) + f(y))$

$\geq min\{\rho_K(f(x)), \rho_K(f(y))\}$

$= min\{f^{-1}(\rho_K(x)), f^{-1}(\rho_K(y))\}$

$f^{-1}(\rho_K)(x) = \rho_K(f(x))$

$\geq min\{\rho_K(f(x) - f(y)), \rho_K(f(y))\}$

$= min\{\rho_K(f(x - y)), \rho_K(f(y))\}$

$= min\{f^{-1}(\rho_K(x - y)), f^{-1}(\rho_K(y))\}$

ii) $f^{-1}(\bar{\bar{\xi}}_K(0)) = \bar{\bar{\xi}}_K(f(0))$

$= \bar{\bar{\xi}}_K(0)$

$\geq \bar{\bar{\xi}}_K(x)$

For some $x, y \in X$

$f^{-1}(\bar{\bar{\xi}}_K)(x + y) = \bar{\bar{\xi}}_K(f(x + y))$

$= \bar{\bar{\xi}}_K(f(x) + f(y))$

$\geq rmin\{\bar{\bar{\xi}}_K(f(x)), \bar{\bar{\xi}}_K(f(y))\}$

$= rmin\{f^{-1}(\bar{\bar{\xi}}_K(x)), f^{-1}(\bar{\bar{\xi}}_K(y))\}$

$$f^{-1}(\bar{\bar{\xi}}_K)(x) = \bar{\bar{\xi}}_K(f(x))$$
$$\geq rmin\{\bar{\bar{\xi}}_K(f(x) - f(y)), \bar{\bar{\xi}}_K(f(y))\}$$
$$= rmin\{\bar{\bar{\xi}}_K(f(x - y)), \bar{\bar{\xi}}_K(f(y))\}$$
$$= rmin\{f^{-1}(\bar{\bar{\xi}}_K(x - y)), f^{-1}(\bar{\bar{\xi}}_K(y))\}$$

iii) $\quad f^{-1}(\eta_K(0)) = \eta_K(f(0))$
$$= \eta_K(0)$$
$$\leq \eta_K(x)$$

For some $x, y \in X$

$$f^{-1}(\eta_K)(x + y) = \eta_K(f(x + y))$$
$$= \eta_K(f(x) + f(y))$$
$$\leq max\{\eta_K(f(x)), \eta_K(f(y))\}$$
$$= max\{f^{-1}(\eta_K(x)), f^{-1}(\eta_K(y))\}$$
$$f^{-1}(\eta_K)(x) = \eta_K(f(x))$$
$$\leq max\{\eta_K(f(x) - f(y)), \eta_K(f(y))\}$$
$$= max\{\eta_K(f(x - y)), \eta_K(f(y))\}$$
$$= max\{f^{-1}(\eta_K(x - y)), f^{-1}(\eta_K(y))\}$$

Hence $f^{-1}(K)$ is an MBJ $- \beta -$ Ideal of X.

3.12 Theorem: Let $f : X \to X$ be an endomorphism on X. If K is an MBJ $-$ Neutrosophic $\beta -$ Ideal of X then $f(K) = \{< x, \rho_f(x) = \rho(f(x)), \bar{\bar{\xi}}_f(x) = \bar{\xi}(f(x)), \eta_f(x) = \eta(f(x)) > /x \in X\}$ is an MBJ $-$ Neutrosophic $\beta -$ Ideal of X.

Proof: Suppose K be an MBJ $-$ Neutrosophic β - ideal of X. Then,

i) $\quad \rho_f(0) = \rho(f(0))$
$$= \rho(0) \geq \rho(x) \; \forall x \in X$$
$$\rho_f(x + y) = \rho(f(x + y))$$
$$= \rho(f(x) + f(y))$$
$$= min\{\rho(f(x)) + \rho(f(y))\}$$
$$= min\{\rho_f(x), \rho_f(y)\} \; \forall x, y \in X$$

Also, $\rho_f(x) = \rho(f(x))$
$$\geq min\{\rho(f(x) - f(y)), \rho(f(y))\}$$
$$= min\{\rho(f(x - y)), \rho(f(y))\}$$
$$= min\{\rho_f(x - y), \rho_f(y)\}$$

ii) $\quad \bar{\bar{\xi}}_f(0) = \bar{\xi}(f(0))$
$$= \bar{\xi}(0) \geq \bar{\xi}(x) \; \forall x \in X$$
$$\bar{\bar{\xi}}_f(x + y) = \bar{\xi}(f(x + y))$$
$$= \bar{\xi}(f(x) + f(y))$$
$$= rmin\{\bar{\xi}(f(x)) + \bar{\xi}(f(y))\}$$
$$= rmin\{\bar{\bar{\xi}}_f(x), \bar{\bar{\xi}}_f(y)\} \; \forall x, y \in X$$

Also, $\bar{\bar{\xi}}_f(x) = \bar{\xi}(f(x))$

$$\geq \text{rmin} \ \{ \ \bar{\bar{\xi}}\left(f(x) - f(y)\right), \ \bar{\bar{\xi}}(f(y))\right) \ \}$$

$$= \text{rmin} \ \{ \ \bar{\bar{\xi}}\left(f(x - y)\right), \ \bar{\bar{\xi}}(f(y))\right) \ \}$$

$$= \text{rmin} \ \{ \ \bar{\bar{\xi}}_f(x - y), \ \bar{\bar{\xi}}_f(y) \ \}$$

iii) $\quad \eta_f(0) = \eta(f(0))$

$$= \eta(0) \ \leq \ \eta(x) \ \forall \ x \ \in X$$

$$\eta_f(x + y) = \eta(f(x + y))$$

$$= \eta(f(x) + f(y))$$

$$= \max \ \{ \ \eta\big(f(x)\big) + \eta(f(y)\big) \ \}$$

$$= \max \ \{ \eta_f(x), \ \eta_f(y) \ \} \ \forall \ x, y \ \in X$$

Also, $\quad \eta_f(x) = \eta(f(x))$

$$\leq \max \ \{ \ \eta(f(x) - f(y)), \ \eta(f(y))) \ \}$$

$$= \max \ \{ \ \eta\big(f(x - y)\big), \ \eta(f(y)) \ \}$$

$$= \max \ \{ \eta_f(x - y), \ \eta_f(y) \ \}$$

$\therefore \ f(K)$ is an MBJ $- \ \beta -$ Ideal of X.

3.13 Theorem: Let $f : X \to Y$ be a homomorphism of $\beta -$ algebra. If K is an MBJ $-$ Neutrosophic $\beta -$ Ideal of X, with sup $-$ rsup $-$ inf property and $\ker(f) \subseteq X_K$ then the image of the set K, $f(K)$ is an MBJ $-$ Neutrosophic $\beta -$ ideal of Y.

Proof: Suppose K is an MBJ $-$ Neutrosophic $\beta -$ Ideal of X, with sup $-$ rsup $-$ inf property and $\ker(f) \subseteq X_K$ then

i) $\quad f(\rho_K)(0) = sup_{x \in f^{-1}(0)}\{ \rho_K(x) \}$

$$= \rho_K(0)$$

$$\geq \rho_K(x) \ \forall \ x \ \in X$$

Hence, $f(\rho_K)(0) = sup_{x \in f^{-1}(0)}\{ \rho_K(x) \}$

$$= f(\rho_K)(y) \ \forall \ y \ \in Y$$

Let $\quad y_1, y_2 \ \in Y$

Then there exists $x_1, x_2 \ \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.

$f(\rho_K)(y_1 + y_2) = \sup \ \{ \rho_K(x_1 + x_2) : x \ \in \ f^{-1}(y_1 + y_2) \}$

$$\geq \sup \ \{ \rho_K(x_1 + x_2) : x_1 \ \in \ f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$$

$$\geq \sup\{\min\{ \rho_K(x_1), \rho_K(x_2)\}, \ x_1 \ \in \ f^{-1}(y_1), \ x_2 \in \ f^{-1}(y_2)\}$$

$$\geq \min\{\sup\{ \rho_K(x_1) : x_1 \in f^{-1}(y_1)\}, \sup\{ \rho_K(x_2) : x_2 \in \ f^{-1}(y_2)\}\}$$

$$= \min\{ \ sup_{x_1 \in f^{-1}(y_1)}\{ \rho_K(x_1) \}, sup_{x_2 \in f^{-1}(y_2)}\{ \rho_K(x_2) \}$$

$$= \min\{ f(\rho_K)(y_1), \ f(\rho_K)(y_2)\}$$

Suppose that for some $y_1, y_2 \ \in Y$

Then $f(\rho_K)(y_1) \ \leq \ \min\{ f(\rho_K)(y_1 - y_2), \ f(\rho_K)(y_2)\}$

Since f is onto $\exists \ x_1, \ x_2 \ \in X$ such that $f(x_1) = \ y_1 \ \& f(x_2) = \ y_2$

$f(\rho_K)\big(f(x_1)\big) \ < \min\{ f(\rho_K)\big(f(x_1) - f(x_2)\big), \ f(\rho_K)\big(f(x_2)\big) \}$

$$= \min\{ f(\rho_K)\big(f(x_1 - x_2)\big), \ f(\rho_K)\big(f(x_2)\big) \}$$

$$< \min\{ f^{-1}(\ f(\rho_K)) \ (x_1 - x_2), f^{-1}(f(\rho_K))(x_2)\}$$

$$\rho_K(x_1) < \min\{ \rho_K(x_1 - x_2), \ \rho_K(x_2)\}$$

ii) $\quad f(\bar{\bar{\xi}}_K)(0) = rsup_{x \in f^{-1}(0)}\{ \bar{\bar{\xi}}_K(x) \}$

$$= \bar{\bar{\xi}}_K(0)$$

$$\geq \bar{\bar{\xi}}_K(x) \ \forall \ x \in X$$

Hence, $f(\bar{\bar{\xi}}_K)(0) = rsup_{x \in f^{-1}(0)}\{ \bar{\bar{\xi}}_K(x) \}$

$$= f(\bar{\bar{\xi}}_K)(y) \ \forall \ y \in Y$$

Let $f(x_1) = y_1, f(x_2) = y_2$.

$f(\bar{\bar{\xi}}_K)(y_1 + y_2) = r \sup \{ \bar{\bar{\xi}}_K(x_1 + x_2) : x \in f^{-1}(y_1 + y_2) \}$

$\geq \mathrm{rsup} \{ \bar{\bar{\xi}}_K(x_1 + x_2) : x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$

$\geq \mathrm{rsup}\{\mathrm{rmin}\{ \bar{\bar{\xi}}_K(x_1), \bar{\bar{\xi}}_K(x_2)\}, x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$\geq \mathrm{rmin}\{ \mathrm{rsup}\{ \bar{\bar{\xi}}_K(x_1) : x_1 \in f^{-1}(y_1)\}, \mathrm{rsup}\{ \bar{\bar{\xi}}_K(x_2) : x_2 \in f^{-1}(y_2)\}\}$

$= \mathrm{rmin}\{ rsup_{x_1 \in f^{-1}(y_1)}\{ \bar{\bar{\xi}}_K(x_1) \}, rsup_{x_2 \in f^{-1}(y_2)}\{ \bar{\bar{\xi}}_K(x_2) \}\}$

$= \mathrm{rmin}\{ f(\bar{\bar{\xi}}_K)(y_1), \ f(\bar{\bar{\xi}}_K)(y_2)\}$

For $y_1, y_2 \in Y$

$f(\bar{\bar{\xi}}_K)(y_1) \leq \mathrm{rmin}\{ f(\bar{\bar{\xi}}_K)(y_1 - y_2), \ f(\bar{\bar{\xi}}_K)(y_2)\}$

$f(\bar{\bar{\xi}}_K)(f(x_1)) < \mathrm{rmin}\{ f(\bar{\bar{\xi}}_K)(f(x_1) - f(x_2)), \ f(\bar{\bar{\xi}}_K)(f(x_2)) \}$

$$= \mathrm{rmin}\{ f(\bar{\bar{\xi}}_K)(f(x_1 - x_2)), \ f(\bar{\bar{\xi}}_K)(f(x_2)) \}$$

$$< \mathrm{rmin}\{ f^{-1}(f(\bar{\bar{\xi}}_K)) (x_1 - x_2), f^{-1}(f(\bar{\bar{\xi}}_K))(x_2)\}$$

$\bar{\bar{\xi}}_K(x_1) < \mathrm{rmin}\{ \bar{\bar{\xi}}_K(x_1 - x_2), \ \bar{\bar{\xi}}_K(x_2) \}$

iii) $\quad f(\eta_K)(0) = inf_{x \in f^{-1}(0)}\{ \eta_K(x) \}$

$$= \eta_K(0)$$

$$\leq \eta(x) \ \forall \ x \in X$$

Hence, $f(\eta_K)(0) = inf_{x \in f^{-1}(0)}\{ \eta_K(x) \}$

$$= f(\eta_K)(y) \ \forall \ y \in Y$$

Let $f(x_1) = y_1, f(x_2) = y_2$.

$f(\eta_K)(y_1 + y_2) = \inf \{ \eta_K(x_1 + x_2) : x \in f^{-1}(y_1 + y_2) \}$

$\leq \inf \{ \eta_K(x_1 + x_2) : x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$

$\leq \inf \{\max\{ \eta_K(x_1), \eta_K(x_2)\}, x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$\leq \max\{ \inf\{ \eta_K(x_1) : x_1 \in f^{-1}(y_1)\}, \inf\{ \eta_K(x_2) : x_2 \in f^{-1}(y_2)\}\}$

$= \max\{ inf_{x_1 \in f^{-1}(y_1)}\{ \eta_K(x_1) \}, inf_{x_2 \in f^{-1}(y_2)}\{ \eta_K(x_2) \}$

$= \max\{ f(\eta_K)(y_1), \ f(\eta_K)(y_2)\}$

For $y_1, y_2 \in Y$

$f(\eta_K)(y_1) \leq \max\{ f(\eta_K)(y_1 - y_2), \ f(\eta_K)(y_2)\}$

$f(\eta_K)(f(x_1)) < \max\{ f(\eta_K)(f(x_1) - f(x_2)), \ f(\eta_K)(f(x_2)) \}$

$$= \max\{ f(\eta_K)(f(x_1 - x_2)), \ f(\eta_K)(f(x_2)) \}$$

$$< \max\{ f^{-1}(f(\eta)) (x_1 - x_2), f^{-1}(f(\eta_K))(x_2)\}$$

$\eta_K(x_1) < \max\{ \eta_K(x_1 - x_2), \ \eta_K(x_2)\}$

Thus, $f(K)$ is an MBJ $- \beta -$ ideal of Y.

3.14 Theorem: Let $f : X \to Y$ be an onto homomorphism of β – algebra. If K is an MBJ – Neutrosophic β – ideal of X, with $\ker(f) \subseteq X_K$ then $f^{-1}(f(K)) = K$.

Proof: To prove $f^{-1}(f(K)) = K$.

It's necessary to prove

$$f^{-1}(f(\rho_K))(x) = \rho_K(x) \; ; \; f^{-1}\left(f(\bar{\xi}_K)\right)(x) = \bar{\xi}_K(x) \text{ and } f^{-1}(f(\eta_K))(x) = \eta_K(x).$$

For $x \in X \, ; f(x) = y$

i) Now, $f^{-1}(f(\rho_K))(x) = f(\rho_K)(f(x))$

$$= f(\rho_K)(y)$$

$$= \sup_{x \in f^{-1}(y)} \{ \rho_K(x) \}$$

For $x' \in X$, $x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker f$

$x' - x \in X_{\rho_K}$

$\rho_K(x' - x) = \rho_K(0)$

$\rho_K(x') \geq \min\{\rho_K(x' - x), \rho_K(x)\}$

$$= \min\{\rho_K(0), \rho_K(x)\}$$

$$= \rho_K(x)$$

$\rho_K(x') \geq \rho_K(x)$ and similarly, $\rho_K(x) \geq \rho_K(x')$

Therefore, $\rho_K(x') = \rho_K(x)$

$f^{-1}(f(\rho_K))(x) = f(\rho_K)(f(x))$

$$= f(\rho_K)(f(x'))$$

$$= \sup_{x \in f^{-1}(y)} \{ \rho_K(x') \}$$

$$= \rho_K(x)$$

$f^{-1}(f(\rho_K))(x) = \rho_K(x)$

ii) $f^{-1}\left(f(\bar{\xi}_K)\right)(x) = f(\bar{\xi}_K)(f(x))$

$$= f(\bar{\xi}_K)(y)$$

$$= rsup_{x \in f^{-1}(y)} \{ \bar{\xi}_K(x) \}$$

For $x' \in X$, $x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker f$

$x' - x \in X_{\bar{\xi}_K}$

$\bar{\xi}_K(x' - x) = \bar{\xi}_K(0)$

$\bar{\xi}_K(x') \geq rmin\{\bar{\xi}_K(x' - x), \bar{\xi}_K(x)\}$

$$= rmin\{\bar{\xi}_K(0), \bar{\xi}_K(x)\}$$

$$= \bar{\xi}_K(x)$$

$\bar{\xi}_K(x') \geq \bar{\xi}_K(x)$ and similarly, $\bar{\xi}_K(x) \geq \bar{\xi}_K(x')$

Therefore, $\bar{\bar{\xi}}_K(x') = \bar{\bar{\xi}}_K(x)$

$$f^{-1}\left(f(\bar{\bar{\xi}}_K)\right)(x) = f(\bar{\bar{\xi}}_K)(f(x))$$

$$= f(\bar{\bar{\xi}}_K)(f(x'))$$
$$= rsup_{x \in f^{-1}(y)}\{\bar{\bar{\xi}}_K(x')\}$$
$$= \bar{\bar{\xi}}_K(x)$$

$$f^{-1}\left(f(\bar{\bar{\xi}}_K)\right)(x) = \bar{\bar{\xi}}_K(x)$$

iii)   Proceeding in the same way,

$$f^{-1}(f(\eta_K))(x) = f(\eta_K)(f(x))$$

$$= f(\eta_K)(y)$$
$$= inf_{x \in f^{-1}(y)}\{\eta_K(x)\}$$

For $x' \in X$, $x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker\ f$

$x' - x \in X_{\eta_K}$

$\eta_K(x' - x) = \eta_K(0)$

$\eta_K(x') \leq \max\{\eta_K(x' - x), \eta_K(x)\}$

$$= \max\{\eta_K(0), \eta_K(x)\}$$
$$= \eta_K(x)$$

$\eta_K(x') \geq \eta_K(x)$ and similarly, $\eta_K(x) \geq \eta_K(x')$

Therefore, $\eta_K(x') = \eta_K(x)$

$$f^{-1}(f(\eta_K))(x) = f(\eta_K)(f(x))$$

$$= f(\eta_K)(f(x'))$$
$$= inf_{x \in f^{-1}(y)}\{\eta_K(x')\}$$
$$= \eta_K(x)$$

$$f^{-1}(f(\eta_K))(x) = \eta_K(x)$$

Therefore, all these conditions are proved and hence $f^{-1}(f(K)) = K$.

4 Cartesian Product of MBJ – Neutrosophic β – Ideal

This section introduces the cartesian product of MBJ – Neutrosophic β – ideal and discusses few associated results.

4.1.Definition: Let $K = \{< x,\ \rho_K(x),\ \bar{\bar{\xi}}_K(x),\ \eta_K(x) >/x \in X\}$ and $L = \{< y,\ \rho_K(y),\ \bar{\bar{\xi}}_K(y),\ \eta_K(y) >/y \in Y\}$ be two MBJ – Neutrosophic sets $X\ \&\ Y$ respectively. The Cartesian product of $K\ and\ L$ is denoted by $K \times L$ is defined as $K \times L = \{< (x,y),\rho_{K\times L}(x,y),\ \bar{\bar{\xi}}_{K\times L}(x,y),\ \eta_{K\times L}(x,y) >/(x,y) \in X \times Y\}$ where $\rho_{K\times L}: X \times Y \to [0,1]$; $\bar{\bar{\xi}}_{K\times L}: X \times Y \to D[0,1]$ and $\eta_{K\times L}: X \times Y \to [0,1]$. $\rho_{K\times L}(x,y) =$

$\min\{\,\rho_K(x)\,,\rho_L(y)\}$; $\quad\quad\quad\quad\quad\quad\quad \bar{\bar{\xi}}_{K\times L}(x,y) = \mathrm{rmin}\{\,\bar{\bar{\xi}}_K(x)\,,\bar{\bar{\xi}}_L(y)\}$ and

$\eta_{K\times L}(x,y) = \max\{\,\eta_K(x)\,,\eta_L(y)\}$

4.2 Theorem: If K and L be two MBJ – Neutrosophic β – Ideal of $X\,\&\,Y$ respectively then $K\times L$ is an MBJ – Neutrosophic β – Ideal of $X\times Y$.

Proof: Let $K = \{\,<x,\ \rho_K(x),\ \bar{\bar{\xi}}_K(x),\ \eta_K(x)>/x\in X\,\}$ and

$L = \{\,<y,\ \rho_K(y),\ \bar{\bar{\xi}}_K(y),\ \eta_K(y)>/y\in Y\,\}$ be two MBJ – Neutrosophic sets $X\,\&\,Y$.

Take $(x,y)\in X\times Y$

i) $\quad \rho_{K\times L}(0,0) = \min\{\,\rho_K(0,0)\,,\rho_L(0,0)\}$

$\quad\quad\quad\quad\quad\geq \min\{\min\{\rho_K(0),\rho_K(0)\},\min\{\rho_L(0),\rho_L(0)\}\}$

$\quad\quad\quad\quad\quad= \min\{\min\{\rho_K(x),\rho_K(y)\},\min\{\rho_L(x),\rho_L(y)\}\}$

$\quad\quad\quad\quad\quad= \min\{\min\{\rho_K(x),\rho_L(x)\},\min\{\rho_K(y),\rho_L(y)\}\}$

$\quad\quad\quad\quad\quad= \min\{\,\rho_{K\times L}(x)\,,\rho_{K\times L}(y)\}$

$\quad\quad\quad\quad\quad\geq \rho_{K\times L}(x,y)$

$\quad\quad$ Take $(u,v))\in X\times Y$ where $u = (x_1\,,y_1), v = (x_2\,,y_2)$

$\quad\quad \rho_{K\times L}(u+v) = \rho_{K\times L}((x_1\,,y_1)+(x_2\,,y_2))$

$\quad\quad\quad\quad\quad= \rho_{K\times L}((x_1+x_2),(y_1+y_2))$

$\quad\quad\quad\quad\quad= \min\{\,\rho_K(x_1+x_2),\rho_L(y_1+y_2)\}$

$\quad\quad\quad\quad\quad\geq \min\{\min\{\,\rho_K(x_1),\rho_K(x_2)\},min\{\rho_L(y_1),\rho_L(y_2)\}\}$

$\quad\quad\quad\quad\quad= \min\{\min\{\,\rho_K(x_1),\rho_L(y_1)\},min\{(\rho_K(x_2),\rho_L(y_2)\}\}$

$\quad\quad\quad\quad\quad= \min\{\,\rho_{K\times L}(x_1\,,y_1)\,,\rho_{K\times L}((x_2\,,y_2))\}$

$\quad\quad\quad\quad\quad\geq \min\{\,\rho_{K\times L}(u)\,,\rho_{K\times L}(v)\}$

$\quad \rho_{K\times L}(u) = \rho_{K\times L}(x_1\,,y_1)$

$\quad\quad\quad\quad= \min\{\,\rho_{K\times L}(x_1\,)\,,\rho_{K\times L}(y_1)\}$

$\quad\quad\quad\quad\geq \min\{\min\{\,\rho_K(x_1-x_2),\rho_K(x_2)\},min\{\rho_L(y_1-y_2),\rho_L(y_2)\}\}$

$\quad\quad\quad\quad= \min\{\min\{\,\rho_K(x_1-x_2),\rho_L(y_1-y_2))\},min\{\rho_L(x_2),\rho_L(y_2)\}\}$

$\quad\quad\quad\quad= \min\{\,\rho_{K\times L}((x_1\,,y_1)-(x_2\,,y_2)),\rho_{K\times L}(x_2\,,y_2)\}$

$\quad\quad\quad\quad\geq \min\{\,\rho_{K\times L}(u-v),\rho_{K\times L}(v)\}$

ii) $\quad \bar{\bar{\xi}}_{K\times L}(0,0) = \mathrm{rmin}\{\,\bar{\bar{\xi}}_K(0,0)\,,\bar{\bar{\xi}}_L(0,0)\}$

$\quad\quad\quad\quad\quad\geq r\mathrm{min}\{\mathrm{rmin}\{\bar{\bar{\xi}}_K(0),\bar{\bar{\xi}}_K(0)\},\mathrm{rmin}\{\bar{\bar{\xi}}_L(0),\bar{\bar{\xi}}_L(0)\}\}$

$\quad\quad\quad\quad\quad= r\mathrm{min}\{\mathrm{rmin}\{\bar{\bar{\xi}}_K(x),\bar{\bar{\xi}}_K(y)\},\mathrm{rmin}\{\bar{\bar{\xi}}_L(x),\bar{\bar{\xi}}_L(y)\}\}$

$\quad\quad\quad\quad\quad= r\mathrm{min}\{\mathrm{rmin}\{\bar{\bar{\xi}}_K(x),\bar{\bar{\xi}}_L(x)\},\mathrm{rmin}\{\bar{\bar{\xi}}_K(y),\bar{\bar{\xi}}_L(y)\}\}$

$\quad\quad\quad\quad\quad= \mathrm{rmin}\{\,\bar{\bar{\xi}}_{K\times L}(x)\,,\bar{\bar{\xi}}_{K\times L}(y)\}$

$\quad\quad\quad\quad\quad\geq \bar{\bar{\xi}}_{K\times L}(x,y)$

$\quad \bar{\bar{\xi}}_{K\times L}(u+v) = \bar{\bar{\xi}}_{K\times L}((x_1\,,y_1)+(x_2\,,y_2))$

$\quad\quad\quad\quad\quad= \bar{\bar{\xi}}_{K\times L}((x_1+x_2),(y_1+y_2))$

$\quad\quad\quad\quad\quad= r\mathrm{min}\{\,\bar{\bar{\xi}}_K(x_1+x_2),\bar{\bar{\xi}}_L(y_1+y_2)\}$

$\quad\quad\quad\quad\quad\geq r\mathrm{min}\{\mathrm{rmin}\{\,\bar{\bar{\xi}}_K(x_1),\bar{\bar{\xi}}_K(x_2)\},rmin\{\bar{\bar{\xi}}_L(y_1),\bar{\bar{\xi}}_L(y_2)\}\}$

$\quad\quad\quad\quad\quad= r\mathrm{min}\{r\min\{\,\bar{\bar{\xi}}_K(x_1),\bar{\bar{\xi}}_L(y_1)\},rmin\{(\bar{\bar{\xi}}_K(x_2),\bar{\bar{\xi}}_L(y_2)\}\}$

$\quad\quad\quad\quad\quad= \mathrm{rmin}\{\,\bar{\bar{\xi}}_{K\times L}(x_1\,,y_1)\,,\bar{\bar{\xi}}_{K\times L}((x_2\,,y_2))\}$

$\quad\quad\quad\quad\quad\geq \mathrm{rmin}\{\,\bar{\bar{\xi}}_{K\times L}(u)\,,\bar{\bar{\xi}}_{K\times L}(v)\}$

$\quad\quad \bar{\bar{\xi}}_{K\times L}(u) = \bar{\bar{\xi}}_{K\times L}(x_1\,,y_1)$

$$= \text{rmin}\{\xi_{K \times L}(x_1), \xi_{K \times L}(y_1)\}$$

$$\geq \text{rmin}\{\text{rmin}\{\xi_K(x_1 - x_2), \xi_K(x_2)\}, \text{rmin}\{\xi_L(y_1 - y_2), \xi_L(y_2)\}\}$$

$$= \text{rmin}\{\text{rmin}\{\xi_K(x_1 - x_2), \xi_L(y_1 - y_2)\}, \text{rmin}\{\xi_L(x_2), \xi_L(y_2)\}\}$$

$$= \text{rmin}\{\xi_{K \times L}((x_1, y_1) - (x_2, y_2)), \xi_{K \times L}(x_2, y_2)\}$$

$$\geq \text{rmin}\{\xi_{K \times L}(u - v), \xi_{K \times L}(v)\}$$

i) $\quad \eta_{K \times L}(0,0) = \max\{\eta_K(0,0), \eta_L(0,0)\}$

$$\leq \max\{\max\{\eta_K(0), \eta_K(0)\}, \max\{\eta_L(0), \eta_L(0)\}\}$$

$$= \max\{\max\{\eta_K(x), \eta_K(y)\}, \max\{\eta_L(x), \eta_L(y)\}\}$$

$$= \max\{\max\{\eta_K(x), \eta_L(x)\}, \max\{\eta_K(y), \eta_L(y)\}\}$$

$$= \max\{\eta_{K \times L}(x), \eta_{K \times L}(y)\}$$

$$\leq \eta_{K \times L}(x, y)$$

$\eta_{K \times L}(u + v) = \eta_{K \times L}((x_1, y_1) + (x_2, y_2))$

$$= \eta_{K \times L}((x_1 + x_2), (y_1 + y_2))$$

$$= \max\{\eta_K(x_1 + x_2), \eta_L(y_1 + y_2)\}$$

$$\leq \max\{\max\{\eta_K(x_1), \eta_K(x_2)\}, \max\{\eta_L(y_1), \eta_L(y_2)\}\}$$

$$= \max\{\max\{\eta_K(x_1), \eta_L(y_1)\}, \max\{(\eta_K(x_2), \eta_L(y_2)\}\}$$

$$= \max\{\eta_{K \times L}(x_1, y_1), \eta_{K \times L}((x_2, y_2))\}$$

$$\leq \max\{\eta_{K \times L}(u), \eta_{K \times L}(v)\}$$

$\eta_{K \times L}(u) = \eta_{K \times L}(x_1, y_1)$

$$= \max\{\eta_{K \times L}(x_1), \eta_{K \times L}(y_1)\}$$

$$\leq \max\{\max\{\eta_K(x_1 - x_2), \eta_K(x_2)\}, \max\{\eta_L(y_1 - y_2), \eta_L(y_2)\}\}$$

$$= \max\{\max\{\eta_K(x_1 - x_2), \eta_L(y_1 - y_2)\}, \max\{\eta_L(x_2), \eta_L(y_2)\}\}$$

$$= \max\{\eta_{K \times L}((x_1, y_1) - (x_2, y_2)), \eta_{K \times L}(x_2, y_2)\}$$

$$\leq \max\{\eta_{K \times L}(u - v), \eta_{K \times L}(v)\}$$

Hence $K \times L$ is an MBJ – Neutrosophic β – Ideal of $X \times Y$.

4.3 Theorem: If $K_1, K_2, \ldots K_n$ be an MBJ – Neutrosophic β – Ideals of $X_1, X_2, \ldots X_n$ respectively, then $\prod_{i=1}^{n} K_i$ is also a MBJ – Neutrosophic β – Ideal of $\prod_{i=1}^{n} X_i$.

Proof: By induction on Theorem 4.2,

i) $\quad \prod_{i=1}^{n} \rho_{K_i}(0) \geq \prod_{i=1}^{n} \rho_{K_i}(x_i)$

$\quad \prod_{i=1}^{n} \rho_{K_i}(x_i + y_i) \geq \min\{ \prod_{i=1}^{n} \rho_{K_i}(x_i), \prod_{i=1}^{n} \rho_{K_i}(y_i) \}$

$\quad \prod_{i=1}^{n} \rho_{K_i}(x_i) \geq \min\{ \prod_{i=1}^{n} \rho_{K_i}(x_i - y_i), \prod_{i=1}^{n} \rho_{K_i}(y_i) \}$

ii) $\quad \prod_{i=1}^{n} \bar{\xi}_{K_i}(0) \geq \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i)$

$\quad \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i + y_i) \geq r\min\{ \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i), \prod_{i=1}^{n} \bar{\xi}_{K_i}(y_i) \}$

$\quad \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i) \geq r\min\{ \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i - y_i), \prod_{i=1}^{n} \bar{\xi}_{K_i}(y_i) \}$

iii) $\quad \prod_{i=1}^{n} \eta_{K_i}(0) \leq \prod_{i=1}^{n} \eta_{K_i}(x_i)$

$\quad \prod_{i=1}^{n} \eta_{K_i}(x_i + y_i) \leq \max\{ \prod_{i=1}^{n} \eta_{K_i}(x_i), \prod_{i=1}^{n} \eta_{K_i}(y_i) \}$

$\quad \prod_{i=1}^{n} \eta_{K_i}(x_i) \leq \max\{ \prod_{i=1}^{n} \eta_{K_i}(x_i - y_i), \prod_{i=1}^{n} \eta_{K_i}(y_i) \}$

Hence the proof is clear.

4.4 Theorem: For the MBJ – Neutrosophic subsets K and L of X & Y , if $K \times L$ is an MBJ – Neutrosophic β – ideal of $X \times Y$ then

   i) $\rho_K(0) \geq \rho_L(y)$ & $\rho_L(0) \geq \rho_K(x)$

   ii) $\bar{\xi}_K(0) \geq \bar{\xi}_L(y)$ & $\bar{\xi}_L(0) \geq \bar{\xi}_K(x)$

   iii) $\eta_K(0) \leq \eta_L(y)$ & $\eta_L(0) \leq \eta_K(x)$

Proof: Let K & L be MBJ – Neutrosophic subsets of X & Y with $K \times L$ is an MBJ – Neutrosophic β – ideal of $X \times Y$.

Suppose $\rho_L(y) \geq \rho_K(0)$ and $\rho_K(x) \geq \rho_L(0)$ for some $x \in X$, $y \in Y$.

$\rho_{K \times L}(x, y) = \min\{\rho_K(x), \rho_L(y)\}$

$\qquad\qquad \geq \min\{\rho_L(0), \rho_K(0)\}$

$\qquad\qquad = \rho_{K \times L}(0,0)$

which is a contradiction.

Thus, $\rho_K(0) \geq \rho_L(y)$ & $\rho_L(0) \geq \rho_K(x)$

Similarly, $\bar{\xi}_L(y) \geq \bar{\xi}_K(0)$ and $\bar{\xi}_K(x) \geq \bar{\xi}_L(0)$ for some $x \in X$, $y \in Y$.

$\bar{\xi}_{K \times L}(x, y) = r\min\{\bar{\xi}_K(x), \bar{\xi}_L(y)\}$

$\qquad\qquad \geq r\min\{\bar{\xi}_L(0), \bar{\xi}_K(0)\}$

$\qquad\qquad = \bar{\xi}_{K \times L}(0,0)$

Now, $\eta_L(y) \leq \eta_K(0)$ and $\eta_K(x) \leq \eta_L(0)$

$\eta_{K \times L}(x, y) = \max\{\eta_K(x), \eta_L(y)\}$

$\qquad\qquad \leq \max\{\eta_L(0), \eta_K(0)\}$

$\qquad\qquad = \eta_{K \times L}(0,0)$

Hence the condition is satisfied.

4.5 Theorem: Let K & L be two MBJ – Neutrosophic β – ideals of X & Y such that $K \times L$ is an MBJ – Neutrosophic β – ideals of $X \times Y$.   Then, either K is an   MBJ - β – ideals of X or L is an MBJ – Neutrosophic β – ideals of Y.

Proof: By using the above theorem

   i)   We consider $\rho_K(0) \geq \rho_L(y)$ then

          $\rho_{K \times L}(0, y) \geq \min\{\rho_K(0), \rho_L(y)\}$ …… (1)

          Given $K \times L$ is an MBJ – Neutrosophic β – ideals of $X \times Y$

          $\rho_{K \times L}\big((x_1, y_1), (x_2, y_2)\big) \geq \min\{\rho_{K \times L}((x_1, y_1) - (x_2, y_2)), \rho_{K \times L}(x_2, y_2)\}$

          $\because \rho_{K \times L}((x_1, y_1) - (x_2, y_2)) \geq \min \{ \rho_{K \times L}(x_1, y_1), \rho_{K \times L}(x_2, y_2) \}$

          $\rho_{K \times L}(x_1, y_1) \geq \min\{ \rho_{K \times L}((x_1 - x_2), (y_1 - y_2)), \rho_{K \times L}(x_2, y_2) \}$ ……. (2)

          Now,

          $\rho_{K \times L}((x_1 - x_2), (y_1 - y_2)) \geq \min\{ \rho_{K \times L}(x_1, y_1), \rho_{K \times L}(x_2, y_2) \}$ ……(3)

          Put $x_1 = x_2 = 0$ in Equation (2 & 3)

          $\rho_{K \times L}(0, y_1) \geq \min\{ \rho_{K \times L}(0, (y_1 - y_2)), \rho_{K \times L}(0, y_2) \}$ and

          $\rho_{K \times L}(0, (y_1 - y_2)) \geq \min\{ \rho_{K \times L}(0, y_1), \rho_{K \times L}(0, y_2)\}$ ……..(4)

          From (1) & (4)

          $\rho_L(y_1) \geq \min\{ \rho_L(y_1 - y_2), \rho_L(y_2) \}$ and

          $\rho_L(y_1 - y_2) \geq \min\{ \rho_L(y_1), \rho_L(y_2)\}$

ii) Consider $\bar{\bar{\xi}}_K(0) \geq \bar{\bar{\xi}}_L(y)$. Then

$\bar{\bar{\xi}}_{K \times L}(0, y) \geq r\min\{\bar{\bar{\xi}}_K(0), \bar{\bar{\xi}}_L(y)\}$ ...... (5)

$\bar{\bar{\xi}}_{K \times L}((x_1, y_1), (x_2, y_2)) \geq r\min\{\bar{\bar{\xi}}_{K \times L}((x_1, y_1) - (x_2, y_2)), \bar{\bar{\xi}}_{K \times L}(x_2, y_2)\}$

$\because \bar{\bar{\xi}}_{K \times L}((x_1, y_1) - (x_2, y_2)) \geq r\min\{\bar{\bar{\xi}}_{K \times L}(x_1, y_1), \bar{\bar{\xi}}_{K \times L}(x_2, y_2)\}$

$\bar{\bar{\xi}}_{K \times L}(x_1, y_1) \geq r\min\{\bar{\bar{\xi}}_{K \times L}((x_1 - x_2), (y_1 - y_2)), \bar{\bar{\xi}}_{K \times L}(x_2, y_2)\}$ ....(6)

Now,

$\bar{\bar{\xi}}_{K \times L}((x_1 - x_2), (y_1 - y_2)) \geq r\min\{\bar{\bar{\xi}}_{K \times L}(x_1, y_1), \bar{\bar{\xi}}_{K \times L}(x_2, y_2)\}$ ...(7)

Put $x_1 = x_2 = 0$ in Equation (6 & 7)

$\bar{\bar{\xi}}_{K \times L}(0, y_1) \geq r\min\{\bar{\bar{\xi}}_{K \times L}(0, (y_1 - y_2)), \bar{\bar{\xi}}_{K \times L}(0, y_2)\}$ and

$\bar{\bar{\xi}}_{K \times L}(0, (y_1 - y_2)) \geq r\min\{\bar{\bar{\xi}}_{K \times L}(0, y_1), \bar{\bar{\xi}}_{K \times L}(0, y_2)\}$ ....(8)

From (5) & (7)

$\bar{\bar{\xi}}_L(y_1) \geq r\min\{\bar{\bar{\xi}}_L(y_1 - y_2), \bar{\bar{\xi}}_L(y_2)\}$ and

$\bar{\bar{\xi}}_L(y_1 - y_2) \geq r\min\{\bar{\bar{\xi}}_L(y_1), \bar{\bar{\xi}}_L(y_2)\}$

iii) As in the same way if we proceed, we get

$\eta_L(y_1) \leq \max\{\eta_L(y_1 - y_2), \eta_L(y_2)\}$ and

$\eta_L(y_1 - y_2) \leq \max\{\eta_L(y_1), \eta_L(y_2)\}$

$\therefore$ B is an MBJ - β − ideals of Y.

5. Conclusion

This paper presents the characterization of MBJ − Neutrosophic β − Ideal of β − algebra. In depth, the study analysed the homomorphic image, pre − image, cartesian product and related results. The concept can be explored to other substructures of a β − algebra.

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