[Neutrosophic Sets and Systems](https://digitalrepository.unm.edu/nss_journal)

[Volume 33](https://digitalrepository.unm.edu/nss_journal/vol33) [Article 1](https://digitalrepository.unm.edu/nss_journal/vol33/iss1/1)

5-1-2020

Eight Kinds of Graphs of BCK-algebras Based on Ideal and Dual Ideal

Atena Tahmasbpour Meikola

Neutrosophic Sets and Systems, Vol. 33, 2020
University of New Mexico

Eight Kinds of Graphs of BCK-algebras Based on Ideal and Dual Ideal

Atena Tahmasbpour Meikola [1,*]

[1] Department of mathematics. Islamic azad university, central Tehran branch, Iran;

* Correspondence: atena.tahmasbpour63@gmail.com.

Abstract: In this paper, at first we introduce the concepts of ideal-annihilator, dual ideal- annihilator, right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left- dual ideal-annihilator. Then by using of these concepts, we constructed six new types of graphs in a bounded BCK-algebra $(X,*,0)$ based on ideal and dual ideal which are denoted by $\Phi_I(X), \Phi_{I^N}(X), \Delta_I(X), \Sigma_I(X), \Delta_{I^N}(X)$, and $\Sigma_{I^N}(X)$, respectively. Then basic properties of graph theory such as connectivity, regularity, and planarity on the structure of these graphs are investigated. Finally, by utilizing of binary operations $\wedge$ and $\vee$, we construct graphs $\Upsilon_I(X)$ and $\Upsilon_{I^N}(X)$, respectively, some their interesting properties are presented.

Keywords: BCK- algebra; Diameter; Chromatic number; Euler graph.

1. Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. The story goes back to a paper of Beck [4] in 1998, where he introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma(R)$ to be the graph whose vertices are elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. Recently, Halas and Jukl in [7] introduced the zero divisor graphs of posets. The study of the zero-divisor graphs of posets was then continued by Xue and Liu in [23], Maimani in [12]. More recently, a different method of associating a zero-divisor graph to a poset P was proposed by Lu and Wu in [11]. In this paper, we deal with zero-divisor graphs of BCI/BCK-algebras based on ideal and dual ideal. Imai and Iseki [8] in 1966 introduced the notion of BCK- algebra. In the same year, Iseki [9] introduced BCI-algebra as a super class of the class of BCK- algebras. Jun and Lee [10] defined the concept of associated graph of BCK- algebra and verified some properties of this graph. Zahiri and Borzooei [24] associated a new graph to a BCI-algebra X which is denoted by $G(X)$, this definition is based on branches of X, Tahmasbpour in [16, 17] studied chordality of graph defined by Zahiri and Borzooei and introduced four types of graphs of BCK- algebras which are constructed by equivalence classes determined by ideal I and dual ideal $I^{\vee}$. Also, Tahmasbpour in [18, 21] introduced two new graphs of lattice implication algebras based on LI-ideal. Further, Tahmasbpour in [19, 20] introduced two new graphs of BCK-algebras based on fuzzy ideal μ_I and fuzzy dual ideal $\mu_{I^{\vee}}$, two new graphs of lattice implication algebras based on fuzzy filter μ_F and fuzzy LI- ideal μ_A. Futhermore, Tahmasbpour in [22] introduced twelve kinds of graphs of lattice implication algebras based on filter and LI- ideal. This paper is divided into six parts.

In Section 2, we recall some concepts of graph theory such as connected graph, planar graph, outerplanar graph, Eulerian graph, and chromatic number, among others.

Section 3, is an introduction to a general theory of BCK- algebras. We will first give the notions of BCI/BCK- algebras, and investigate their elementary and fundamental properties , and then deal with a number of basic concepts, such as ideal, and dual ideal, among others.

In Section 4, inspired by ideas from Behzadi et al. [5], we study the graphs of BCK-algebras which are constructed from ideal-annihilator and dual ideal-annihilator, denoted by $\Phi_I(X), \Phi_{I^{\vee}}(X)$, respectively.

In Sction 5, inspired by ideas from Behzadi et al. [5], we study the graphs of BCK- algebras which are constructed from right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left-dual ideal- annihilator, denoted by $\Delta_I(X), \Sigma_I(X), \Delta_{I^{\vee}}(X)$, $\Sigma_{I^{\vee}}(X)$, respectively.

In Section 6, inspired by ideas from Alizadeh et al. [3], we introduce the associated graphs $\Upsilon_I(X)$ and $\Upsilon_{I^{\vee}}(X)$ which are constructed from binary operations $\wedge$ and $\vee$, respectively.

2. Preliminaries of graph theory

In this section, for convenience of the reader, we recall some definitions and notations concerning graphs and posets for later use.

Definition 2.1. ([3, 6]) For a graph G, we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. The greatest induced complete subgraph denotes a clique. If graph G contains a clique with n elements, and every clique has at most n elements, we say that the clique number of G is n and write $\omega(G)=n$. Also, a graph G is said to be connected if there is a path between any given pairs of vertices, otherwise the graph is disconnected. For distinct vertices x and y of G, let $d(x,y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x,y)\coloneqq\infty$. The diameter of G is $diam(G)\coloneqq\sup\{d(x,y);x,y\in V(G)\}$. Also, the girth of a graph G, is denoted by $gr(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, we get $gr(G)\coloneqq\infty$. The neighborhood of a vertex x is the set $N_G(x)=\{y\in V(G);xy\in E(G)\}$. Graph H is called a subgraph of G if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. A graph G is called regular of degree k when every vertex has precisely k neighbors. A cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3- regular graph. Moreover, for distinct vertices x and y, we use the notation $x-y$ to show that is x connected to y. Let $P=(V,\leq)$ be a poset. If $x\leq y$ but $x\neq y$, then we write $x<y$. If x and y are in V, then y covers x in P if $x<y$ and there is no $z\in V$, with $x<z<y$. Two sets $\{x\in P; x\ covers\ 0\}$ and $\{x\in P; 1\ covers x\}$, denoted by $Atom(P)$ and $Coatom(P)$, respectively. Let $L\subseteq P$, we say L is a chain if for all $x,y\in L, x\leq y$ or $y\leq x$. Chain L is maximal if for all chain $L', L\subseteq L'$ implies that $L=L'$.

Definition 2.2. ([4]) If K is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, we say that the chromatic number of G is K and write $\chi(G)=K$. Moreover, we have $\chi(G)\geq\omega(G)$.

Definition 2.3. ([6]) A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected. Euler's theorem says that the connected graph G is Eulerian if and only if all vertices of G are of even degree.

Definition 2.4. ([2]) A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Graph G is planar if it can be drawn in a plane without the edges having to cross. Proving that a graph is planar amounts to redrawing the edges in such a way that no edges will cross. One may need to move the vertices around and the edges may have to be drawn in a very indirect fashion. Kuratowski's theorem says that a finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. The clique number of any planar graph is less than or equal to four.

Definition 2.5. ([15]) Let G be a plane graph. A face is a region bounded by edges. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Definition 2.6. ([14]) The number g is called the genus of the surface if it is homeomorphic to a sphere with g handles or equivalently holes. Also, the genus g of a graph G is the smallest genus of all surfaces in such a way that the graph G can be drawn on it without any edge-crossing. The graphs of genus zero are precisely the planar graphs since the genus of a plane is zero. The graphs that can be drawn on a torus without edge- crossing are called toroidal. They have a genus of one since the genus of a torus is one. The notation $\gamma(G)$ stands for the genus of a graph G.

Theorem 2.7. ([1]) For the positive integers m and n, we have:

(i) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$ if $n \geq 3$,

(ii) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$ if $m, n \geq 2$.

3. Introduction of BCI/BCK- algebras

In this section, we submit some concepts related to BCI/BCK-algebra, which are necessary for our discussion.

Definition 3.1. ([13]) A BCI- algebra $(X,*,0)$ is an algebra of type $(2,0)$ satisfying in the following conditions:

$(BCI\ 1)$ $\big((x*y)*(x*z)\big)*(z*y)=0,$

$(BCI\ 2)$ $x*0=0,$

$(BCI\ 3)$ $x*y=0, y*x=0$ imply $y=x$.

If X satisfies in the following identity:

$$(\forall\, x\in X)\ (0*x=0),$$

Therefore X is called a BCK-algebra. Any BCI/BCK- algebra X satisfies in the following conditions:

$(i)\big(x*(x*y)\big)*y=0,$

$(ii)x*x=0,$

$(iii)(x*y)*z=(x*z)*y,$

$(iv)x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$, for any $z\in X$.

Moreover, the relation $\leq$ was defined by $x\leq y\leftrightarrow x*y=0$, for any $x,y\in X$, which is a partial order on X. $(X,*,0)$ is said to be commutative if it satisfies for all $x,y\in X,$

$$x*(x*y)=y*(y*x)$$

Definition 3.2. ([13]) A subset I is called an ideal of X if it satisfies the following conditions:

$(i)\ 0\in I,$

$(ii)\ (\forall x,y\in X), (x*y\in I, y\in I\rightarrow x\in I).$

An ideal P of X is prime if $x*(x*y)\in P$ implies $x\in P$ or $y\in P$.

Note: A BCK- algebra X is said to be bounded if there exists $e \in X$ in such a way that $x \leq e$ for any $x \in X$, and the element e is said to be the unit of X. In a bounded BCK-algebra, we denote $e * x$ by $N(X)$.

Definition 3.3. ([13]) A nonempty subset $I^{\vee}$ of a bounded BCK-algebra X is said to be a dual ideal of X if

$(i)\, 1 \in I^{\vee}$.

$(ii)\; N(Nx * Ny) \in I^{\vee}$ and $y \in I^{\vee}$ imply $x \in I^{\vee}$, for any $x, y \in X$.

A dual ideal $P^{\vee}$ of X is prime if $N(Nx * (Nx * Ny)) \in P^{\vee}$ implies $x \in P^{\vee}$ or $y \in P^{\vee}$.

Theorem 3.4. ([13]) Let X be a bounded BCK-algebra with the greatest element 1. Then, the following statements hold for any $x, y \in X$:

$(i)\, N1 = 0$ and $N0 = 1$.

$(ii)\, Nx * Ny \leq y * x$.

$(iii)\, y \leq x$ implies $Nx \leq Ny$.

Theorem 3.7. ([13]) Let X be a bounded BCK-algebra. Then X is commutative if and only if $x \wedge y = x * (x * y), x \vee y = N(Nx \wedge Ny)$.

4. Graphs of BCK-algebras based on ideal and dual ideal by the concepts of ideal- annihilator, dual ideal- annihilator

Definition 4.1. Let A be a nonempty subset of X, I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the set of all zero-divisors of A by I and $I^{\vee}$ are defined as follows:

(i) $Ann_I A = \{x \in X; x * a \in I \text{ or } a * x \in I, \forall\, a \in A\}$.

(ii) $Ann_{I^\vee} A = \{x \in X; N(Nx * Na) \in I^\vee \text{ or } N(Na * Nx) \in I^\vee, \forall\, a \in A\}$.

Proposition 4.2. Let A and B be nonempty subsets of X, I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $I \cup \{1\} \subseteq Ann_I A$, $I^\vee \cup \{0\} \subseteq Ann_{I^\vee} A$.

(ii) If $A \subseteq B$, then $Ann_I B \subseteq Ann_I A$ and $Ann_{I^\vee} B \subseteq Ann_{I^\vee} A$.

(iii) If $0 \in A$, then $Ann_I A = Ann_I(A - \{0\})$ and $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{0\})$.

(iv) If $1 \in A$, then $Ann_I A = Ann_I(A - \{1\})$ and $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{1\})$.

(v) $Ann_I I = X$ and $Ann_{I^\vee} I^\vee = X$.

(vi) If $I = \{0\}, I^\vee = \{1\}$, then we have

$Ann_I A = \{y; y \text{ is comparable to any element in } A\}$,

$Ann_{I^\vee} A = \{y; y \text{ is comparable to any element in } A\}$.

Proof. (i) Let $x \in I$, then by Definition 3.1 (iii), we have $x * a \in I, \forall\, a \in A$. Also, $x * 1 = 0, \forall x \in X$, So $I \cup \{1\} \subseteq Ann_I A$. Similarly, we can prove $I^\vee \cup \{0\} \subseteq Ann_{I^\vee} A$.

(ii) Suppose that $x \in Ann_I B$, then $x * b \in I$ or $b * x \in I$, $\forall\, b \in B$, but $A \subseteq B$, therefore $x * b \in I$ or $b * x \in I, \forall b \in A$. i.e $x \in Ann_I A$, hence $Ann_I B \subseteq Ann_I A$.

(iii) According to Definition 4.1 (i), we have $Ann_I A = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{0\} = X$. Then, $Ann_I A = Ann_I(A - \{0\})$. Similarly, we can prove $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{0\})$.

(iv) According to Definition 4.1 (i), we have $Ann_I A = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{1\} = X$. Then, $Ann_I A = Ann_I(A - \{1\})$. Similarly, we can prove $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{1\})$.

(v) Let $x \in X$, we know by Definition 3.1 (iii), $a * x \in I, \forall\, a \in I$, then $x \in Ann_I I$, hence $Ann_I I = X$. Similarly we can prove $Ann_{I^\vee} I^\vee = X$.

(vi) The proof is easy.

Definition 4.3. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, we have:

(i) $\Phi_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_I\{x, y\} = I \cup \{1\}$.

(ii) $\Phi_{I^\vee}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_{I^\vee}\{x, y\} = I^\vee \cup \{0\}$.

Example 4.4. Let $X = \{0, a, b, c, 1\}$ and the operation $*$ be defined by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	0
1	1	1	1	1	0

TABLE 1

Therefore, $(X, *, 0)$ is a bounded BCK-algebra. Also, we have

$$E\left(\Phi_{\{0\}}(X)\right) = E\left(\Phi_{\{1\}}(X)\right) = \{ab, bc, ac\}.$$

Theorem 4.5. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

(i) $N_G(\{0\}) = N_G(\{1\}) = \emptyset$, where $G = \Phi_I(X)$.

(ii) $N_G(\{0\}) = N_G(\{1\}) = \emptyset$, where $G = \Phi_{I^\vee}(X)$.

Proof. (i) We know $Ann_I\{0\} = X$ and $Ann_I\{1\} = X$, for all $x \in X, x \neq 0, 1,$ we have, $I \cup \{x, 1\} \subseteq Ann_I\{x\}$. Then $I \cup \{x, 1\} \subseteq Ann_I\{0, x\}$ and $I \cup \{x, 1\} \subseteq Ann_I\{x, 1\}$, for all $x \in X, x \neq 0, 1$. So, by Definition 4.3 (i) of graph $\Phi_I(X)$, for all $x \in X, x \neq 0, 1, x$ is connected to elements $0, 1$ if and only if $x \in I$, if $x \in I$, then by proposition 4.2 (v), $Ann_I\{x\} = X$. So, $0, 1$ are not connected to x, for all $x \in X$.

(ii) We know $Ann_{I^\vee}\{0\} = X$ and $Ann_{I^\vee}\{1\} = X$, for all $x \in X, x \neq 0, 1$, we have, $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{x\}$. Then $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{0, x\}$ and $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{x, 1\}$, for all $x \in X, x \neq 0, 1$. So, by Definition 4.3 (ii) of graph $\Phi_{I^\vee}(X)$, for all $x \in X, x \neq 0, 1, x$ is connected to elements $0, 1$ if and only if $x \in I^\vee$, then by Proposition 4.2 (v), $Ann_{I^\vee}\{x\} = X$. So, $0, 1$ are not connected to x, for all $x \in X$.

Theorem 4.6. Let $X = \{0, 1\} \cup Atom(X), I = \{0\}$ and $I^\vee = \{1\}$ be an ideal, a dual ideal of X, respectively. Then, $E(\Phi_I(X)) = E(\Phi_{I^\vee}(X)) = \{xy; x, y \in Atom(X)\}$.

Proof. We know $Ann_{\{0\}}\{0\} = X$ and $Ann_{\{0\}}\{1\} = X$, by proposition 4.2 (vi), since $X = Atom(X) \cup \{0, 1\}$, we have, for all $x \in Atom(X)$, $Ann_{\{0\}}\{x\} = \{0, x, 1\}$. On the other hand we know $Ann_{\{0\}}\{x, y\} = Ann_{\{0\}}\{x\} \cap Ann_{\{0\}}\{y\}$. Then by Definition 4.3$(i)$ of graph $\Phi_{\{0\}}(X)$, x and y are adjacent if and only if $x, y \in Atom(X)$. Similarly, we have $Ann_{\{1\}}\{0\} = X$ and $Ann_{\{1\}}\{1\} = X$, for all $x \in Atom(X)$, $Ann_{\{1\}}\{x\} = \{0, x, 1\}$. Then by Definition 4.3$(ii)$ of graph $\Phi_{\{1\}}(X)$, x and y are adjacent if and only if $x, y \in Atom(X)$.

Theorem 4.7. Let $X = \{0, 1\} \cup Atom(X)$. Then, the following statements hold:

(i) $\omega\left(\Phi_{\{0\}}(X)\right) = |Atom(X)|$.

(ii) $\omega\left(\Phi_{\{1\}}(X)\right) = |Atom(X)|$.

Proof. (i) Straightforward by Theorem 4.6(i).

(ii) Straightforward by Theorem 4.6(ii).

Theorem 4.8. Let $I = \{0\}$ and $I^\vee = \{1\}$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

$(i)\, N_G(\{x\}) = \{y; y \text{ is not comparable to } x\}$, where $G = \Phi_I(X), x \neq 0, 1$.

$(ii)\, N_G(\{x\}) = \{y; y \text{ is not comparable to } x\}$, where $G = \Phi_{I^\vee}(X), x \neq 0, 1$.

Proof. (i) We have, for all $x \in X, x \neq 0, 1$, $Ann_{\{0\}}\{x\} = \{y; y \text{ is comparable to } x\}$. On the other hand we know $Ann_{\{0\}}\{x, y\} = Ann_{\{0\}}\{x\} \cap Ann_{\{0\}}\{y\}$. Then by Definition 4.3 (i) of graph $\Phi_{\{0\}}(X)$, x and y are adjacent if and only if x and y are not comparable to each other.

(ii) We have, for all $x \in X, x \neq 0, 1$, $Ann_{\{1\}}\{x\} = \{y; y \text{ is comparable to } x\}$. On the other hand we know $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{x\} \cap Ann_{\{1\}}\{y\}$. Then by Definition 4.3 (ii) of graph $\Phi_{\{1\}}(X)$, x and y are adjacent if and only if x and y are not comparable to each other.

Theorem 4.9. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

$(i)\, \alpha(\Phi_I(X)) \geq |I|$.

$(ii)\, \alpha(\Phi_{I^\vee}(X)) \geq |I^\vee|$.

Proof. (i) We suppose that $x, y \in I$. Then by Proposition 4.2 (v), we have, $Ann_I\{x\} = X$, $Ann_I\{y\} = X$. Therefore, by Definition 4.3 (i) of graph $\Phi_I(X)$, $xy \notin E(\Phi_I(x))$. Therefore, by Definition 2.1 of independent set, we have $\alpha(\Phi_I(X)) \geq |I|$.

(ii) We suppose that $x, y \in I^\vee$. Then by Proposition 4.2 (v), we have, $Ann_{I^\vee}\{x\} = X$, $Ann_{I^\vee}\{y\} = X$. Therefore, by Definition 4.3 (ii) of graph $\Phi_{I^\vee}(X)$, $xy \notin E(\Phi_{I^\vee}(X))$. Therefore, by Definition 2.1 of independent set, we have $\alpha(\Phi_{I^\vee}(X)) \geq |I^\vee|$.

Theorem 4.10. Let $|X| > 2$ and I be a prime ideal, $I^\vee$ be a prime dual ideal of X. Then the following statements hold:

(i) $\Phi_I(X)$ is an empty graph.

(i) $\Phi_{I^\vee}(X)$ is an empty graph.

Proof. (i) We suppose, on the contrary, that $\Phi_I(X)$ is not an empty graph. Then there exist $x,y \in X$, such that $xy \in E(\Phi_I(X))$. So, by Definition 4.3 (i) of graph $\Phi_I(X)$, we have, $Ann_I\{x,y\} = I \cup \{1\}$. On the other hand, since $|X - I| > 1$, we can choose $z \in X, z \notin I, z \neq 1$. Since I is a prime ideal, then $z * x \in I$ or $* z \in I$, and $z * y \in I$ or $y * z \in I$, hence $z \in Ann_I\{x,y\}$ that is contradiction, complete proof.

(ii) We suppose, on the contrary, that $\Phi_{I^\vee}(X)$ is not an empty graph. Then there exist $x,y \in X$, such that $xy \in E(\Phi_{I^\vee}(X))$. So, by Definition 4.3 (ii) of graph $\Phi_{I^\vee}(X)$, we have, $Ann_{I^\vee}\{x,y\} = I^\vee \cup \{0\}$. On the other hand, since $|X - I^\vee| > 1$, we can choose $z \in X, z \notin I^\vee, z \neq 0$. Since $I^\vee$ is a prime dual ideal, then $N(Nz * Nx) \in I^\vee$ or $N(Nx * Nz) \in I^\vee$ and $N(Nz * Ny) \in I^\vee$ or $N(Ny * Nz) \in I^\vee$, hence $z \in Ann_{I^\vee}\{x,y\}$ that is contradiction, complete proof.

5. Graphs of BCK- algebras based on ideal and dual ideal by the concepts of right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, and left- dual ideal- annihilator.

Definition 5.1. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Denote $Ann_I^R\{x\} = \{y \in X; x * y \in I\}, Ann_I^L\{x\} = \{y \in X; y * x \in I\}$, $Ann_{I^\vee}^R\{x\} = \{y \in X; N(Nx * Ny) \in I^\vee\}$, $Ann_{I^\vee}^L\{x\} = \{y \in X; N(Ny * Nx) \in I^\vee\}$

, which are called right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left- dual ideal- annihilator, respectively.

Definition 5.2. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, we have:

(i) $\Delta_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_I^R\{x\} \subseteq Ann_I^R\{y\}$ or $Ann_I^R\{y\} \subseteq Ann_I^R\{x\}$, there is an edge between x and y in the graph $\Sigma_I(X)$ if and only if $Ann_I^L\{x\} \subseteq Ann_I^L\{y\}$ or $Ann_I^L\{y\} \subseteq Ann_I^L\{x\}$.

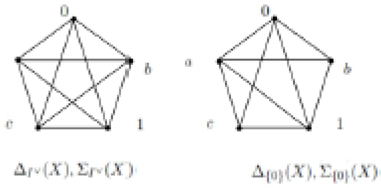
(ii) $\Delta_{I^{\vee}}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann^{R}_{I^{\vee}}\{x\} \subseteq Ann^{R}_{I^{\vee}}\{y\}$ or $Ann^{R}_{I^{\vee}}\{y\} \subseteq Ann^{R}_{I^{\vee}}\{x\}$, there is an edge between x and y in the graph $\Sigma_{I^{\vee}}(X)$ if and only if $Ann^{L}_{I^{\vee}}\{x\} \subseteq Ann^{L}_{I^{\vee}}\{y\}$ or $Ann^{L}_{I^{\vee}}\{y\} \subseteq Ann^{L}_{I^{\vee}}\{x\}$.

Example 5.3. Let $X = \{0, a, b, c, 1\}$ and the operation $*$ is given by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
1	1	c	c	a	0

TABLE 2

Therefore, $(X, *, 0)$ is a bounded BCK- algebra, $I^{\vee} = \{c, 1\}$ is a dual ideal of X. Also, in the Figure 2, we can see the graphs $\Delta_{\{0\}}(X), \Sigma_{\{0\}}(X), \Delta_{I^{\vee}}(X)$, and $\Sigma_{I^{\vee}}(X)$.



Proposition 5.4. Let I and $I^{\vee}$ be an ideal , a dual ideal of X, respectively. Then, the following statements hold:

$(i)\, \omega(\Delta_I(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(ii)\, \omega(\Sigma_I(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(iii)\, \omega(\Delta_{I^{\vee}}(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(iv)\, \omega(\Sigma_{I^{\vee}}(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

Proof. (i) According to Definition 3.1 (iv), if $x \leq y$ then, $x * z \leq y * z$. On the other hand now we let $x \leq y, z \in Ann^{R}_{I}\{y\}$. Then, by Definition 5.1, $y * z \in I$. So, by Definition 3.2 of ideal, $x * z \in I$. So, $z \in Ann^{R}_{I}\{x\}$. Then, $Ann^{R}_{I}\{y\} \subseteq Ann^{R}_{I}\{x\}, xy \in E(\Delta_I(X))$, complete proof.

(ii) According to Definition 3.1 (iv), if $x \leq y$ then, $z * y \leq z * x$. On the other hand now we let $x \leq y, z \in Ann_I^L\{x\}$. Then, by Definition 5.1, $z * x \in I$. So, by Definition 3.2 of ideal, $z * y \in I$. So, $z \in Ann_I^L\{y\}$. Then, $Ann_I^L\{x\} \subseteq Ann_I^L\{y\}, xy \in E\big(\Sigma_I(X)\big)$, complete proof.

(iii) According to Definition 3.1 (iv), Theorem 3.4 (iii), if $x \leq y$ then $N(Nx * Nz) \leq N(Ny * Nz)$. On the other hand now we let $x \leq y, z \in Ann_{I^\vee}^R\{x\}$ then, by Definition 5.1 $N(Nx * Nz) \in I^\vee$. So, by Definition 3.3 of dual ideal, $N(Ny * Nz) \in I^\vee$. So, $z \in Ann_{I^\vee}^R\{y\}$ then,

$Ann_{I^\vee}^R\{x\} \subseteq Ann_{I^\vee}^R\{y\}, xy \in E\big(\Delta_{I^\vee}(X)\big)$, complete proof.

(iv) According to Definition 3.1 (iv), Theorem 3.4 (iii), if $x \leq y$ then $N(Nz * Ny) \leq N(Nz * Nx)$. On the other hand now we let $x \leq y, z \in Ann_{I^\vee}^L\{y\}$ then, by Definition 5.1 $N(Nz * Ny) \in I^\vee$. So, by **Definition 3.3** of dual ideal, $N(Nz * Nx) \in I^\vee$. So, $z \in Ann_{I^\vee}^L\{x\}$ then,

$Ann_{I^\vee}^L\{y\} \subseteq Ann_{I^\vee}^L\{x\}, xy \in E\big(\Sigma_{I^\vee}(X)\big)$, complete proof.

Theorem 5.5. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\ \Delta_I(X)$ is connected, $diam\big(\Delta_I(X)\big) \leq 2, gr\big(\Delta_I(X)\big) = 3$.

$(ii)\ \Sigma_I(X)$ is connected, $diam\big(\Sigma_I(X)\big) \leq 2$, $gr\big(\Sigma_I(X)\big) = 3$.

$(iii)\ \Delta_{I^\vee}(X)$ is connected, $diam\big(\Delta_{I^\vee}(X)\big) \leq 2, gr\big(\Delta_{I^\vee}(X)\big) = 3$.

$(iv)\ \Sigma_{I^\vee}(X)$ is connected, $diam\big(\Sigma_{I^\vee}(X)\big) \leq 2, gr\big(\Sigma_{I^\vee}(X)\big) = 3$.

Proof. (i) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (i), $0, 1$ are connected to any element in X. So, $\Delta_I(X)$ is connected, $diam\big(\Delta_I(X)\big) \leq 2, gr\big(\Delta_I(X)\big) = 3$.

(ii) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (ii), $0, 1$ are connected to any element in X. So, $\Sigma_I(X)$ is connected, $diam(\Sigma_I(X)) \leq 2, gr(\Sigma_I(X)) = 3$.

(iii) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (iii), $0, 1$ are connected to any element in X. So, $\Delta_{I^\vee}(X)$ is connected, $diam(\Delta_{I^\vee}(X)) \leq 2, gr(\Delta_{I^\vee}(X)) = 3$.

(iv) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (iv), $0, 1$ are connected to any element in X. So, $\Sigma_{I^\vee}(X)$ is connected, $diam(\Sigma_{I^\vee}(X)) \leq 2, gr(\Sigma_{I^\vee}(X)) = 3$.

Theorem 5.6. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\Delta_I(X)$ is regular if and only if it is complete.

(ii) $\Sigma_I(X)$ is regular if and only if it is complete.

(iii) $\Delta_{I^\vee}(X)$ is regular if and only if it is complete.

(iv) $\Sigma_{I^\vee}(X)$ is regular if and only if it is complete.

Proof. (i) Suppose that $\Delta_I(X)$ is regular. By Theorem 5.5(i), $\deg(0) = |X| - 1$. Since $\Delta_I(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Delta_I(X)$ is complete. Conversely, a complete graph is regular.

(ii) Suppose that $\Sigma_I(X)$ is regular. By Theorem 5.5(ii), $\deg(0) = |X| - 1$. Since $\Sigma_I(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Sigma_I(X)$ is complete. Conversely, a complete graph is regular.

(iii) Suppose that $\Delta_{I^\vee}(X)$ is regular. By Theorem 5.5(iii), $\deg(0) = |X| - 1$. Since $\Delta_{I^\vee}(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Delta_{I^\vee}(X)$ is complete. Conversely, a complete graph is regular.

(iv) Suppose that $\Sigma_{I^\vee}(X)$ is regular. By Theorem 5.5(iv), $\deg(0) = |X| - 1$. Since $\Sigma_{I^\vee}(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Sigma_{I^\vee}(X)$ is complete. Conversely, a complete graph is regular.

Proposition 5.7. Let X be a chain, I and $I^\vee$ be an ideal, a dual of X, respectively. Then, the following statements hold:

$(i)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are planar graphs if and only if $|X| \leq 4$.

$(ii)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are outerplanar graphs if and only if $|X| \leq 3$.

$(iii)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are toroidal graphs if and only if $|X| \leq 7$.

Proof. (i) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 5$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_5, respectively, then by Kuratowski's theorem $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not planar, respectively. Conversely, we knowK_5 has five vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not planar, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least five vertices, respectively, which is contrary to $|X| \leq 4$.

(ii) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 4$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_4, respectively, then by Definition 2.5 $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not outerplanar, respectively. Conversely, we knowK_4 has four vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not outerplanar, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least four vertices, respectively, which is contrary to $|X| \leq 3$.

(iii) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 8$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_8, respectively, then by Theorem 2.7 $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not toroidal, respectively. Conversely, we know K_8 has eight vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not toroidal, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least eight vertices, respectively, which is contrary to $|X| \leq 7$.

6. Graphs of BCK- algebras based on ideal and dual ideal by the binary operations ∧ and ∨.

From now on, X is a bounded commutative BCK-algebra.

Definition 6.1. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, we have:

$(i)\, \Upsilon_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y are adjacent if and only if $x \wedge y \in I$.

$(ii)\, \Upsilon_{I^\vee}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y are adjacent if and only if $x \vee y \in I^\vee$.

Example 6.2. Let $X = \{0, a, b, c, d, 1\}$ and the operation $*$ be defined by the table:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	a	0	0
b	b	b	0	0	0	0
c	c	c	b	0	b	0
d	d	b	a	a	0	0
1	1	c	d	a	b	0

TABLE 3

Therefore, $(X, *, 0)$ is a bounded commutative BCK-algebra. It is easy to verify that $I = \{0, a\}$ is an ideal of X. Also, we let $I^\vee = \{1\}$ be a dual ideal of X, then in the Figure 3 , we can see the graphs $\Upsilon_I(X)$ and $\Upsilon_{I^\vee}(X)$.

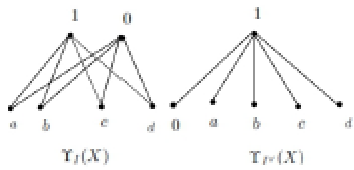


FIGURE 3

Lemma 6.3. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\deg(x) = |X| - 1$, in the graph $\Upsilon_I(X)$, where $x \in I$.

(ii) $\deg(x) = |X| - 1$, in the graph $\Upsilon_{I^{\vee}}(X)$, where $x \in I^{\vee}$.

Proof. (i) Let $x \in I, y$ be an arbitrary element in X, then $y * (y * x) \in I$. Since $y * (y * x) \leq x, I$ is an ideal of X. So, $xy \in E(\Upsilon_I(X))$, complete proof.

(ii) Let $x \in I^{\vee}, y$ be an arbitrary element in X, then $N(Ny * (Ny * Nx)) \in I^{\vee}$. Since $N(Ny * (Ny * Nx)) \geq x, I^{\vee}$ is a dual ideal of X. So, $xy \in E(\Upsilon_{I^{\vee}}(X))$, complete proof.

Theorem 6.4. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\Upsilon_I(X)$ is regular if and only if it is complete.

$(ii)\Upsilon_{I^{\vee}}(X)$ is regular if and only if it is complete.

Proof. (i) Let $\Upsilon_I(X)$ be a regular graph. By Lemma 6.3 (i), we have $\deg(0) = |X| - 1$. Now, since $\Upsilon_I(X)$ is regular, then for any $x \in X, \deg(x) = |X| - 1$. This means that $\Upsilon_I(X)$ is a complete graph. Conversely, a complete graph is regular.

(ii) Let $\Upsilon_{I^{\vee}}(X)$ be a regular graph. By Lemma 6.3 (ii), we have $\deg(1) = |X| - 1$. Now, since $\Upsilon_{I^{\vee}}(X)$ is regular, then for any $x \in X, \deg(x) = |X| - 1$. This means that $\Upsilon_{I^{\vee}}(X)$ is a complete graph. Conversely, a complete graph is regular.

Proposition 6.5. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\,\omega(\Upsilon_I(X)) \geq |I|$.

$(ii)\ \omega(\Upsilon_{I^\vee}(X)) \geq |I^\vee|$.

Proof. (i) Straightforward by Lemma 6.3 (i).

(ii) Straightforward by Lemma 6.3 (ii).

Theorem 6.6. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\Upsilon_I(X)$ is connected, $diam(\Upsilon_I(X)) \leq 2$.

(ii) $\Upsilon_{I^\vee}(X)$ is connected, $diam(\Upsilon_{I^\vee}(X)) \leq 2$.

Proof. (i) Straightforward by Lemma 6.3 (i).

(ii) Straightforward by Lemma 6.3 (ii).

Theorem 6.7. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\ gr(\Upsilon_I(X)) = 3$.

$(ii)\ gr(\Upsilon_{I^\vee}(X)) = 3$.

Proof. (i) Let $a \neq 0$ be an element in I, x be an arbitrary element in X, then $0 - a - x - 0$ is a cycle of length 3 in $\Upsilon_I(X)$, complete proof.

(ii) Let $a \neq 1$ be an element in $I^\vee, x$ be an arbitrary element in X, then $1 - a - x - 1$ is a cycle of length 3 in $\Upsilon_{I^\vee}(X)$, complete proof.

Proposition 6.8. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $\Upsilon_I(X)$ is planar, then $|I| \leq 4$.

(ii) If $\Upsilon_I(X)$ is outerplanar, then $|I| \leq 3$.

(iii) If $\Upsilon_I(X)$ is toroidal, then $|I| \leq 7$.

(iv) If $\Upsilon_I(X)$ is planar, then $|I^\vee| \leq 4$.

(v) If $Y_I(X)$ is outerplanar, then $|I^{\vee}| \leq 3$.

(vi) If $Y_I(X)$ is toroidal, then $|I^{\vee}| \leq 7$.

Proof. (i) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 5$ then $Y_I(X)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $Y_I(X)$ is not planar.

(ii) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 4$ then $Y_I(X)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $Y_I(X)$ is not outerplanar.

(iii) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 7$ then $Y_I(X)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $Y_I(X)$ is not toroidal.

(iv) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 5$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $Y_{I^{\vee}}(X)$ is not planar.

(v) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 4$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $Y_{I^{\vee}}(X)$ is not outerplanar.

(vi) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 7$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $Y_{I^{\vee}}(X)$ is not toroidal.

Theorem 6.9. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $Y_I(X)$ is an Euler graph then $|X|$ is odd.

(ii) If $Y_{I^{\vee}}(X)$ is an Euler graph then $|X|$ is odd.

Proof. (i) According to Lemma 6.3 (i), for all $x \in I$, $\deg(x) = |X| - 1$. Now, if $Y_I(X)$ is an Euler graph then degree of every vertex in I is even. So, $|X|$ is odd, complete proof.

(ii) According to Lemma 6.3 (ii), for all $x \in I$, $\deg(x) = |X| - 1$. Now, if $\Upsilon_{I^\vee}(X)$ is an Euler graph then degree of every vertex in $I^\vee$ is even. So, $|X|$ is odd, complete proof.

Theorem 6.10. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $I = \cap_{1 \le i \le n} P_i$ and, for each $1 \le j \le n, I \ne \cap_{1 \le i \le n, i \ne j} P_i$, where P_i are prime ideals of X. Then $\omega(\Upsilon_I(X)) = n = \chi(\Upsilon_I(X))$.

(ii) If $I^\vee = \cap_{1 \le i \le n} P_i^\vee$ and, for each $1 \le j \le n, I^\vee \ne \cap_{1 \le i \le n, i \ne j} P_i^\vee$, where $P_i^\vee$ are prime dual ideals of X. Then $\omega(\Upsilon_{I^\vee}(X)) = n = \chi(\Upsilon_{I^\vee}(X))$.

Proof. (i) For each j with $1 \le j \le n$, consider an element x_j in $\left(\cap_{1 \le i \le n, i \ne j} P_i\right) - P_j$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_I(X)$. Hence $\omega(\Upsilon_I(X)) \ge n$. Now, we prove that $\chi(\Upsilon_I(X)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k$ and $x \wedge y \in I$. Since P_k is prime, $y \in P_k$, and so $f(y) \ne k$. Now, since $\omega(\Upsilon_I(X)) \le \chi(\Upsilon_I(X))$, the result hold.

(ii) For each j with $1 \le j \le n$, consider an element x_j in $\left(\cap_{1 \le i \le n, i \ne j} P_i^\vee\right) - P_j^\vee$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_{I^\vee}(X)$. Hence $\omega(\Upsilon_{I^\vee}(X)) \ge n$. Now, we prove that $\chi(\Upsilon_{I^\vee}(X)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i^\vee\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k^\vee$ and $x \vee y \in I^\vee$. Since $P_k^\vee$ is prime, $y \in P_k^\vee$, and so $f(y) \ne k$. Now, since $\omega(\Upsilon_{I^\vee}(X)) \le \chi(\Upsilon_{I^\vee}(X))$, the result hold.

Theorem 6.11. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $I = \cap_{j \in J} P_j$, where P_j are prime ideals of X, J is an infinite set and, for each $i \in J$, $I \neq \cap_{j \neq i} P_j$. Then $\omega(\Upsilon_I(X)) = \infty = \chi(\Upsilon_I(X))$.

(ii) If $I = \cap_{j \in J} P_j^{\vee}$ where $P_j^{\vee}$ are prime dual ideals of X, J is an infinite set and, for each $i \in J$, $I^{\vee} \neq \cap_{j \neq i} P_j^{\vee}$. Then $\omega(\Upsilon_{I^{\vee}}(X)) = \infty = \chi(\Upsilon_{I^{\vee}}(X))$.

Proof. (i) For each $i \in J$, there exists $x_i \in (\cap_{j \neq i} P_j - P_i)$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_I(X)$. Since $\omega(\Upsilon_I(X)) \leq \chi(\Upsilon_I(X))$, the assertion holds.

(ii) For each $i \in J$, there exists $x_i \in (\cap_{j \neq i} P_j^{\vee} - P_i^{\vee})$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_{I^{\vee}}(X)$. Since $\omega(\Upsilon_{I^{\vee}}(X)) \leq \chi(\Upsilon_{I^{\vee}}(X))$, the assertion holds.

Funding: This study was not funded by any profit or nonprofit organization.
Acknowledgments: The author is grateful to the editors and anonymous reviewers for many suggestions which improved the presentation of the paper.

References

1. Afkhami, M.; Ahadjavaheri, K. H.; Khashyarmanesh, K. On the comaximal graphs associated to a lattice of genus one, Proceeding of the 7th Algebraic Combinatorics Conference, Iran, 16-17 October 2014; Ferdowsi university of Mashhad, Iran, 2014.
2. Afkhami, M.; Barati, Z.; Khashyarmanesh, K. A graph associated to lattice. Springer **2014**, 63, 67-78.
3. Alizadeh, M.; Maimani, H. R.; Pournaki, M. R.; Yassemi, S. An ideal theoretic approach to complete partite zero-divisor graphs of posets. Math-IPM **2013**, 37, 161-180.
4. Beck, I. coloring of commutative rings, J. Algebra **1998**, 116, 208-226.
5. Behzadi, M.; Torkzadeh, Z.; Ahadpanah. A graph on residuated lattices, Proceeding of the 1th Algebraic structures Conference, Iran, 15-16 November 2012; Hakim sabzevar university, Sabzevar, Iran, 2012.
6. Diestel, R. Graph theory , Springer: New York, NY, USA, 1997.
7. Halas, R; Jukl, M. On Beck's coloring of posets, Discrete Mathematics **2009**, 309, 4584-4589.
8. Imai, Y.; Iseki, K. On axiom systems of propositional calculi. XIV, Proc. JPn. Acad **1966**, 42, 19-22.
9. Iseki, K. An algebra related with a propositional calculus, Proc. Jpn. Acad **1966**, 42, 26-29.
10. Jun, Y. B.; Lee, K. J. Graphs based on BCI/BCK-algebras, Int. J. Math. Math. Sci. **2011**, 2011, 1-8.
11. Lu. D.; Wu. T. The zero-divisor graphs of posets and application to semigroup, Graphs, Combin **2016**, 26, 793-804.

12. Maimani, H. R.; Pournaki, M. R.; Yassemi, S. Zero-divisor graphs with respect to an ideal, Comm Algebra **2006**, 34, 923-929.
13. Meng, J.; Jun, Y. B. BCK-algebras, Kyung Moon So Co: seoul, south korea, 1994.
14. Mohammadian, A.; Erfanian, A.; Farrokhi, M. Planar, toroidal and projective generalized Peterson graphs, Proceeding of the 7th Algebraic Combinatorics Conference, Iran, 16-17 october 2014; Ferdowsi university of Mashhad, Mashhad, Iran, 2014.
15. Shahsavar, F.; Afkhami, M.; Khashyarmanesh, K. On End-Regular , Planar and outerplanar of zero-divisor graphs of posets, Proceeding of the 6th Algebraic Conference, Iran, 30-31 October 2013; University of Mazandaran, Babolsar, Iran, 2013.
16. Tahmasbpour Meikola, A.; Graphs of BCI/BCK-algebras, Turk. J. Math **2018**, 42, 1272-1293.
17. Tahmasbpour Meikola, A.; Graphs of BCK-algebras based on dual ideal, Proceeding of the 9th seminar on algebraic hyperstructures and fuzzy mathematics, Iran, 30 Apr- 02 May 2019; University of Mazandaran, Babolsar, Iran, 2019.
18. Tahmasbpour Meikola, A.; Graphs of lattice implication algebras based on LI-ideal, Proceeding of the 3th International conference on soft computing, Iran, 20-21 November 2019; University of guilan, Iran, 2019.
19. Tahmasbpour Meikola, A.; Graphs of BCK- algebras based on fuzzy ideal and fuzzy dual ideal, Proceeding of the 1th International conference on physics, mathematics and development of basic sciences, Iran, 23 January 2020, Tehran, Iran, 2020.
20. Tahmasbpour Meikola, A.; Graphs of lattice implication algebras based on fuzzy filter and fuzzy LI- ideal, Proceeding of the 1th International conference on physics, mathematics and development of basic sciences, Iran, 23 January 2020, Tehran, Iran, 2020.
21. Tahmasbpour Meikola, A.; Graphs of lattice implication algebras based on LI- ideal, Elites. J. Sci. Eng, 4, (2020), 169-179.
22. Tahmasbpour Meikola, A.; Twelve kinds of graphs of lattice implication algebras based on filter and LI-ideal, Cankaya. Univ. J. Sci. Eng, to appear.
23. Xue, Z.; Liu, S. Zero-divisor graphs of partially ordered sets, Appl. Math. Lett **2010**, 23, 449-452.
24. Zahiri, O.; Borzooei, R. A. Graph of BCI-algebra, Int. J. Math. Math. Sci **2012**, 1-16.

Received: Nov 20, 2019. Accepted: May 02, 2020