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Air Pollution Model using Neutrosophic Cubic Einstein Averaging Operators

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Abstract: The neutrosophic cubic averaging and Einstein averaging aggregation operators are presented and applied to the air pollution model of the city of Peshawar, Pakistan. Neutrosophic cubic set (NCS) is a more generalized version of the neutrosophic set (NS) and an interval neutrosophic set (INS). It is in a better position to express consistent, indeterminant and incomplete information, thus it is able to be applied to aggregate the air pollution model. Aggregation operators have a key role in science and engineering problems. Firstly, the neutrosophic cubic weighted averaging (NCWA) operator, neutrosophic cubic ordered weighted averaging (NCOWA) operator, neutrosophic cubic hybrid aggregation (NCHA) operator, neutrosophic cubic Einstine weighted averaging (NCEWA) operator, neutrosophic cubic Einstine ordered weighted averaging (NCEOWA) operator and neutrosophic cubic Einstine hybrid aggregation (NCEHA) operator are defined. Secondly, these operators are applied to the air pollution model of particulate matter with the size of less than 10 micron (PM10) in Peshawar. Subsequently, the results are compared with the World Health Organization (WHO) standards using score/accuracy function. The pollution of PM10 is found to be very much higher than WHO standards. Hence, strong measures are required to control air pollution.

Keywords: Air pollution; neutrosophic cubic weighted averaging; neutrosophic cubic hybrid averaging; neutrosophic cubic Einstein weighted averaging; neutrosophic cubic Einstein hybrid averaging.

1. Introduction

The uncertainty is a complex phenomenon that occurs in the real world. Since uncertainty is inevitably involved in problems, it occurs in different areas of life such that conventional methods have failed to cope with such problems. The big task is to deal with uncertain information. Many models have been introduced to incorporate uncertainty into the description of the system. The fuzzy set was initiated by Zadeh [1]. Henceforth, it is applied in different fields of sciences like artificial intelligence, information sciences, medical sciences, decision making theory and much more. Due to its applicability in sciences and daily life problem, fuzzy set has been extended into interval valued fuzzy sets (IVFS) [2,3], intuitionistic fuzzy set (IFS) [4], interval valued fuzzy set (IVIFS) [5] and cubic set [6] among others, besides Q-fuzzy [7-11] and vague soft set [12]. IFS consists of two components, membership and non-membership whereas the hesitant component is considered under the condition that sum of these components is one. Smarandache presented the idea of neutrosophic sets (NS) [13], which provides a more general form to extend the ideas of classic theory and fuzzy set theory. The NS expresses three components namely truth, indeterminacy and falsity and all these components are independent, which makes NS more general than IFS such that NS can be seen as a generalization of IFS [14]. For sciences and engineering problems Wang *et al.* [15] presented a single valued neutrosophic set (SVNS), while Wang *et al*. [16] introduced the interval neutrosophic set (INS). Jun *et al.* [17] combined INS and NS to form a neutrosophic cubic set (NCS) which enables us to choose both interval value and single value membership, indeterminacy and falsehood components, hence presenting a more general form for uncertain and vague data.

Aggregation operators are an imperative part of decision making. The lack of data or knowledge makes it difficult for decision maker to give the exact decision. This uncertain situation can be minimized due to the vague nature of NS and its extensions. Researchers [18-27] introduced different aggregation operators and multicriteria decision making methods in NS and INS. Khan *et al.* [28] presented neutrosophic cubic Einstein geometric aggregation operators. Zhan *et al.* [29] worked on multi criteria decision making on neutrosophic cubic sets. Banerjee *et al.* [30] used grey rational analysis (GRA) techniques to neutrosophic cubic sets. Lu and Ye [31] defined a cosine measure to neutrosophic cubic set. Pramanik *et al.* [32] used similarity measure to neutrosophic cubic set. Shi and Ji [33] defined Dombi aggregation operators on neutrosophic cubic sets. Ye [34] defined aggregation operators over the neutrosophic cubic numbers. Alhazaymeh *et al*. [35] presented a hybrid geometric aggregation operator with application to multiple attribute decision making method on neutrosophic cubic sets.

According to WHO, air pollution causes millions of premature deaths every year globally. 90% of these deaths are caused by air pollution in middle and low income countries, mainly in Africa and Asia. Indeed it is a great threat to the environment. Inhaling polluted air may cause different types of diseases like lung cancer, respiratory diseases etc. In the last few years Pakistan witnessed a significant increased in cancer, asthma and chronic lung disease. The particulate matter (PM) is one of the major factors that cause such types of diseases. The data extracted from Alam *et al.* [36] consists of particulate matter with the size of less than 10 micron (PM10) in Peshawar, Pakistan.

The collection of accurate data has always been a tough job which may cause some uncertain results. That is why the need was felt to analyze the data using vague set. The neutrosophic cubic set is one of the better choices to deal with vague and inconsistent data. For this purpose, firstly the neutrosophic cubic averaging and Einstein averaging operators are defined. Then these operators are used to analyze the air pollution of PM10 model for the city of Peshawar, Pakistan with WHO standards. In this paper, the NCWA, NCOWA, NCHA, NCEWA, NCEOWA and NCEHA are defined. Both algebraic and Einstein operators are applied to an air pollution model [36] and compared. The goal of this work is to analyze the PM10 in the city of Peshawar and compare it with WHO standards.

The methodology to measure the aggregate value of neutrosophic cubic values is as follows. Firstly, the data is extracted from [36] and converted to neutrosophic cubic values so that the aggregated value can be measured. Secondly, the data is analyzed using the WHO standard. It is to be noted that the neutrosophic cubic set is the combination of both interval neutrosophic and neutrosophic set, which enable us to deal with both interval-valued neutrosophic and neutrosophic set at the same time.

This paper is structured as follows. In section 2, some preliminaries are reviewed. In section 3, the air pollution data is formed. In section 4, neutrosophic cubic averaging operators are defined and applied over the data of section 3. In section 5, neutrosophic cubic Einstein averaging operators are defined and applied over the data of section 3. The analyzing results are concluded by both numerically and graphically. We hope to expand the study further to numerical analysis [37-42], construction management [43-45], Q-neutrosophic soft environment [46], geometric programming [47] and binomial factorial problem [48].

2. Preliminaries

This section consists of some definitions and results which provide the foundation of the work.

Definition 2.1 [13] A structure $N=\left\{\left(T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is neutrosophic set (NS), where $\left\{T_N(u), I_N(u), F_N(u) \in \left]0^-, 1^+\right[\right\}$ *and* $T_N(u), I_N(u), F_N(u)$ are truth, indeterminancy and falsity function respectively.

Definition 2.2 [15] A structure $N=\left\{\left(T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is single value neutrosophic set (SVNS), where $\left\{T_N(u), I_N(u), F_N(u) \in [0,1]\right\}$ respectively called truth, indeterminancy and falsity functions.,simply denoted by $N=\left(T_N, I_N, F_N\right)$.

Definition 2.3 [16] An interval neutrosophic set (INS) in U is a structure

$$N=\left\{\left(\tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u)\right) \mid u \in U\right\} \text{ where}$$

$\left\{\tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u) \in D[0,1]\right\}$ are respectively called truth, indeterminacy an falsity function in U.

Simply denoted by $N=\left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N\right)$ for convenience being actually

$$N=\left(\tilde{T}_N=\left[T_N^L, T_N^U\right], \tilde{I}_N=\left[I_N^L, I_N^U\right], \tilde{F}_N=\left[F_N^L, F_N^U\right]\right).$$

Definition 2.4 [17] A structure $N=\left\{\left(u, \tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u), T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is neutrosophic cubic set (NCS) in U in which $\left(\tilde{T}_N=\left[T_N^L, T_N^U\right], \tilde{I}_N=\left[I_N^L, I_N^U\right], \tilde{F}_N=\left[F_N^L, F_N^U\right]\right)$ is an interval neutrosophic set and $\left(T_N, I_N, F_N\right)$ is neutrosophic set in U, where

$$N=\left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N, T_N, I_N, F_N\right),\ [0,0] \leq \tilde{T}_N+\tilde{I}_N+\tilde{F}_N \leq [3,3] \text{ and } 0 \leq T_N+I_N+F_N \leq 3$$

such that N^U denotes the collection of neutrosophic cubic sets in U.

Definition 2.5 [22] The t-operators are basically union and intersection in the fuzzy sets which are denoted by t-conorm $\left(\Gamma^*\right)$ and t-norm (Γ). The role of t-operators is very important in fuzzy theory and its applications.

Definition 2.6 [22] $\Gamma^*:[0,1]\times[0,1]\rightarrow[0,1]$ is t-conorm if the following axioms hold.

Axiom 1 $\Gamma^*(1,u)=1$ and $\Gamma^*(0,u)=0$

Axiom 2 $\Gamma^*(u,v)=\Gamma^*(v,u)$ for all u and v.

Axiom 3 $\Gamma^*(u,\Gamma^*(v,w))=\Gamma^*(\Gamma^*(u,v),w)$ for all u, v and w.

Axiom 4 If $u\leq u'$ and $v\leq v'$, then $\Gamma^*(u,v)\leq\Gamma^*(u',v')$

Definition 2.7 [22] $\Gamma:[0,1]\times[0,1]\rightarrow[0,1]$ is t-norm if it the following axioms hold.

Axiom 1 $\Gamma(1,u)=u$ and $\Gamma(0,u)=0$

Axiom 2 $\Gamma(u,v)=\Gamma(v,u)$ for all u and v.

Axiom 3 $\Gamma(u,\Gamma(v,w))=\Gamma(\Gamma(u,v),w)$ for all u, v and w.

Axiom 4 If $u\leq u'$ and $v\leq v'$, then $\Gamma(u,v)\leq\Gamma(u',v')$

The t-conorms and t-norms families have a vast range, which correspond to unions and intersections, among these Einstein sum and Einstein product are good choices since they give the smooth approximation like algebraic sum and algebraic product, respectively. Einstein sums $\oplus_E$ and Einstein products $\otimes_E$ are the examples of t-conorm and t-norm respectively:

$$\Gamma_E^*(u,v)=\frac{u+v}{1+uv},\quad \Gamma_E(u,v)=\frac{uv}{1+(1-u)(1-v)}$$

Definition 2.8 [28] The sum of two neutrosophic cubic sets (NCS),

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and } B=(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B), \text{where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U] \text{ and } \tilde{T}_B=[T_B^L,T_B^U],\tilde{I}_B=[I_B^L,I_B^U],\tilde{F}_B=[F_B^L,F_B^U]$$

is defined as

$$A\oplus B=\left([T_A^L+T_B^L-T_A^LT_B^L,T_A^U+T_B^U-T_A^UT_B^U],[I_A^L+I_B^L-I_A^LI_B^L,I_A^U+I_B^U-I_A^UI_B^U],[F_A^LF_B^L,F_A^UF_B^U],T_AT_B,I_AI_B,F_A+F_B-F_AF_B\right).$$

Definition 2.9 [28] The product between two neutrosophic cubic sets (NCS),

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and } B=(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B), \text{ where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U] \text{ and } \tilde{T}_B=[T_B^L,T_B^U],\tilde{I}_B=[I_B^L,I_B^U],\tilde{F}_B=[F_B^L,F_B^U]$$

is defined as

$$A\otimes B=\left([T_A^LT_B^L,T_A^UT_B^U],[I_A^LI_B^L,I_A^UI_B^U],[F_A^L+F_B^L-F_A^LF_B^L,F_A^U+F_B^U-F_A^UF_B^U],T_A+T_B-T_AT_B,I_A+I_B-I_AI_B,F_AF_B\right)$$

Definition 2.10 [28] The scalar multiplication on a neutrosophic cubic set (NCS)

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and a scalar } k \text{ where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U]$$

is defined as

$$kA = \left(\left[1-(1-T_A^L)^k, 1-(1-T_A^U)^k\right], \left[1-(1-I_A^L)^k, 1-(1-I_A^U)^k\right], \left[\left(F_A^L\right)^k, \left(F_A^U\right)^k\right], (T_A)^k, (I_A)^k, 1-(1-F_A)^k\right)$$

Definition 2.11 [18] The Einstein sum between two neutrosophic cubic sets (NCS),

$A = \left(\tilde{T}_A, \tilde{I}_A, \tilde{F}_A, T_A, I_A, F_A\right)$ and $B = \left(\tilde{T}_B, \tilde{I}_B, \tilde{F}_B, T_B, I_B, F_B\right)$, where

$\tilde{T}_A = \left[T_A^L, T_A^U\right], \tilde{I}_A = \left[I_A^L, I_A^U\right], \tilde{F}_A = \left[F_A^L, F_A^U\right]$ and $\tilde{T}_B = \left[T_B^L, T_B^U\right], \tilde{I}_B = \left[I_B^L, I_B^U\right], \tilde{F}_B = \left[F_B^L, F_B^U\right]$

is defined as

$$A \oplus_E B = \left(\left[\frac{T_A^L + T_B^L}{1+T_A^L T_B^L}, \frac{T_A^U + T_B^U}{1+T_A^U T_B^U}\right], \left[\frac{I_A^L + I_B^L}{1+I_A^L I_B^L}, \frac{I_A^U + I_B^U}{1+I_A^U I_B^U}\right], \left[\frac{F_A^L F_B^L}{1+(1-F_A^L)\left(1-F_B^L\right)}, \frac{F_A^U F_B^U}{1+(1-F_A^U)\left(1-F_B^U\right)}\right], \frac{T_A T_B}{1+(1-T_A)(1-T_B)}, \frac{I_A I_B}{1+(1-I_A)(1-I_B)}, \frac{F_A + F_B}{1+F_A F_B}\right)$$

Definition 2.12 [28] The Einstein product between two neutrosophic cubic sets (NCS),

$A = \left(\tilde{T}_A, \tilde{I}_A, \tilde{F}_A, T_A, I_A, F_A\right)$ and $B = \left(\tilde{T}_B, \tilde{I}_B, \tilde{F}_B, T_B, I_B, F_B\right)$, where

$\tilde{T}_A = \left[T_A^L, T_A^U\right], \tilde{I}_A = \left[I_A^L, I_A^U\right], \tilde{F}_A = \left[F_A^L, F_A^U\right]$ and $\tilde{T}_B = \left[T_B^L, T_B^U\right], \tilde{I}_B = \left[I_B^L, I_B^U\right], \tilde{F}_B = \left[F_B^L, F_B^U\right]$

is defined as

$$A \otimes_E B = \left(\left[\frac{T_A^L T_B^L}{1+(1-T_A^L)\left(1-T_B^L\right)}, \frac{T_A^U T_B^U}{1+(1-T_A^U)\left(1-T_B^U\right)}\right], \left[\frac{I_A^L I_B^L}{1+(1-I_A^L)\left(1-I_B^L\right)}, \frac{I_A^U I_B^U}{1+(1-I_A^U)\left(1-I_B^U\right)}\right], \left[\frac{F_A^L + F_B^L}{1+F_A^L F_B^L}, \frac{F_A^U + F_B^U}{1+F_A^U F_B^U}\right], \frac{T_A + T_B}{1+T_A T_B}, \frac{I_A + I_B}{1+I_A I_B}, \frac{F_A F_B}{1+(1-F_A)(1-F_B)}\right)$$

Definition 2.13 [28] The Einstein scalar multiplication on a neutrosophic cubic set (NCS),

$A = \left(\tilde{T}_A, \tilde{I}_A, \tilde{F}_A, T_A, I_A, F_A\right)$, and a scalar k where

$\tilde{T}_A = \left[T_A^L, T_A^U\right], \tilde{I}_A = \left[I_A^L, I_A^U\right], \tilde{F}_A = \left[F_A^L, F_A^U\right]$

is defined as

$$k_E A =$$

$$\left(\left[\frac{(1+T_A^L)^k-(1-T_A^L)^k}{(1+T_A^L)^k+(1-T_A^L)^k}, \frac{(1+T_A^U)^k-(1-T_A^U)^k}{(1+T_A^U)^k+(1-T_A^U)^k}\right], \left[\frac{(1+I_A^L)^k-(1-I_A^L)^k}{(1+I_A^L)^k+(1-I_A^L)^k}, \frac{(1+I_A^U)^k-(1-I_A^U)^k}{(1+I_A^U)^k+(1-I_A^U)^k}\right], \left[\frac{2\left(F_A^L\right)^k}{\left(2-F_A^L\right)^k+\left(F_A^L\right)^k}, \frac{2\left(F_A^U\right)^k}{\left(2-F_A^U\right)^k+\left(F_A^U\right)^k}\right], \frac{2\left(T_A\right)^k}{\left(2-T_A\right)^k+\left(T_A\right)^k}, \frac{2\left(I_A\right)^k}{\left(2-I_A\right)^k+\left(I_A\right)^k}, \frac{(1+F_A)^k-(1-F_A)^k}{(1+F_A)^k+(1-F_A)^k}\right)$$

Definition 2.14 [28] Let $N = \left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N, T_N, I_N, F_N\right)$, where $\tilde{T}_N = \left[T_N^L, T_N^U\right], \tilde{I}_N = \left[I_N^L, I_N^U\right], \tilde{F}_N = \left[F_N^L, F_N^U\right]$ be a neutrosophic cubic value. The score function is defined as

$$Scr\left(N\right) = \left[T_N^L - F_N^L + T_N^U - F_N^U + T_N - F_N\right] \quad (1)$$

If the score function of two values are equal, the accuracy function is used to compare the neutrosophic cubic values.

Definition 2.15 [28] Let $N = \left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N, T_N, I_N, F_N\right)$, where $\tilde{T}_N = \left[T_N^L, T_N^U\right], \tilde{I}_N = \left[I_N^L, I_N^U\right], \tilde{F}_N = \left[F_N^L, F_N^U\right]$ be a neutrosophic cubic vlaue. The accuracy function is defined as

$$Acu(u) = \frac{1}{9}\left\{T_N^L + I_N^L + F_N^L + T_N^U + I_N^U + F_N^U + T_N + I_N + F_N\right\} \quad (2)$$

The following definition describes the comparison relation between two neutrosophic cubic values.

Definition 2.16 [28] Let N_1, N_2 be two neutrosophic cubic values, with score functions $Scr(N_1), Scr(N_2)$ and accuracy functions $Acu(N_1), Acu(N_2)$. Then

1). $Scr(N_1) > Scr(N_2) \Rightarrow N_1 > N_2$

2). If $Scr(N_1) = Scr(N_2)$, then

(i). $Acu(N_1) > Acu(N_2) \Rightarrow N_1 > N_2$, (ii). $Acu(N_1) = Acu(N_2) \Rightarrow N_1 = N_2$

3. Model Formulation of Air Pollution

Air pollution is a great threat to the environment. It causes different diseases to the human being. Inhaling polluted air may cause different types of disease like lung cancer and other respiratory diseases. According to WHO, air pollution causes 7 million premature deaths globally in 2016. Ambient air pollution alone caused 4.2 million deaths, while the atmospheric contamination of households from the kitchen with fuels and contaminating technologies led to an estimated 3.8 million deaths in the same year. More than 90% of deaths related to air pollution occur in middle- and low-income countries, mainly in Africa and Asia. In the last few years Pakistan witnessed a significant increase in cancer, asthma and chronic lung diseases. The particulate matter (PM) cause such type of diseases. The PM size is categorized as PM25, PM10 and PM2.5. The recommendation of the WHO for air quality call the countries to reduce their annual air pollution to the annual mean value of 20ug/m^3 for PM10. In this model, the data for PM10 was considered.

The collection of data is a hard task to do since most of the time we are unable to collect the correct and appropriate data. The problems may arise due to unskilled data collectors, inappropriate methods of collecting data and others. These obstacles can be minimized by using neutrosophic cubic sets which provide a vast variety to choose and decide. In this paper, a problem regarding PM10 in Peshawar, Pakistan is considered and their values aggregated using a neutrosophic cubic environment. Data is taken from [36] and converted to neutrosophic cubic form. To consider overall values, data aggregation operators are being proposed so that its value can be compared with WHO standards. According to WHO recommendation, the neutrosophic cubic value for PM10 is calculated as

$$N_{WHO} = ([0.15, 0.30], [0.10, 0.30], [0.70, 0.85], 0.20, 0.40, 0.75) \quad (3)$$

The neutrosophic cubic data for 1st, 5th, 10th, 15th and 20th April 2014 are respectively shown as follows.

$$N_A = ([0.82, 0.92], [0.39, 0.66], [0.18, 0.38], 0.88, 0.7, 0.42),\ N_B = ([0.59, 0.78], [0.68, 0.73], [0.22, 0.41], 0.68, 0.78, 0.32),$$

$$N_C = ([0.86, 0.96], [0.8, 0.85], [0.24, 0.36], 0.17, 0.8, 0.4),\ N_D = ([0.8, 0.93], [0.11, 0.41], [0.5, 0.9], 0.9, 0.5, 0.4)$$

and $N_E = \left([0.61, 0.79], [0.41, 0.53], [0.39, 0.56], 0.45, 0.30, 0.45\right)$.

The accuracy function is used to rank the air pollution in their relevant dates, in which day the air is most polluted by PM10.

$$Acu(N_A) = 0.5944, Acu(N_B) = 0.5766, Acu(N_C) = 0.6044, Acu(N_D) = 0.6055, \text{ and } Acu(N_E) = 0.4988.$$

We observe that

$$Acu\left(N_D\right) > Acu\left(N_C\right) > Acu\left(N_A\right) > Acu\left(N_B\right) > Acu\left(N_E\right).$$

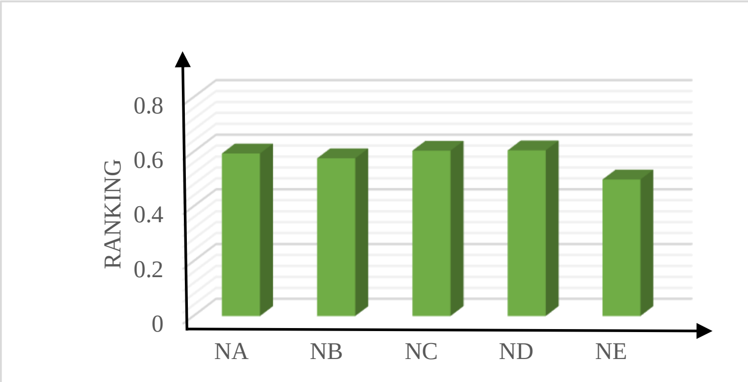


Figure 1. Pollution Graph in Peshawar City in April 2014

The graphical analysis can be seen in Figure 1. To analyze the overall pollution of PM10, the aggregation operators are needed. To fulfill this desire, the notion of neutrosophic cubic aggregation operators and neutrosophic cubic Einstein aggregation operators are proposed.

4. Neutrosophic Cubic Weighted Averaging Aggregation Operator

This section consist of some fundamental definitions of neutrosophic cubic weighted averaging (NCWA), neutrosophic cubic ordered weighted averaging (NCOWA) and neutrosophic cubic Einstein hybrid avregaing (NCEHA) aggregation operator, which are defined as follows.

Definition 4.1 The neutrosophic cubic weighted averaging is a function, $NCWA : R^n \to R$ defined by

$$NCWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} w_k N_k, \text{ where} \tag{4}$$

$W = (w_1, w_2, ..., w_n)^T$ of $N_k (k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

Note that in NCWA, the neutrosophic values are weighted first and then aggregated.

Definition 4.2 The neutrosophic cubic ordered weighted averaging is a function, $NCOWA : R^n \to R$ defined by

$$NCOWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} w_k S_k, \text{ where} \tag{5}$$

s_k denotes the ordered position of neutrosophic cubic (NC) values whereby the NC values are ordered in descending order, $W = (w_1, w_2, ..., w_n)^T$ of $N_k\,(k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

Note that in NCOWA, the neutrosophic cubic values are first ordered and then aggregated. The basic concept of NCOWA is to rearrange the neutrosophic cubic values in descending order and then aggregate them.

Theorem 4.3 Let $N_k = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T^L_{N_k}, T^U_{N_k}\right], \tilde{I}_{N_k} = \left[I^L_{N_k}, I^U_{N_k}\right], \tilde{F}_{N_k} = \left[F^L_{N_k}, F^U_{N_k}\right]$ $(k = 1, 2, ..., n)$ be a collection of neutrosophic cubic values, then the neutrosophic cubic weighted average operator (NCWA) operator of N_k is also a neutrosophic cubic value and

$$NCWA(N_k) = \left(\left[1-\prod_{k=1}^{n}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{n}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{n}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{n}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{n}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{n}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{n}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{n}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{n}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

where $W = (w_1, w_2, ..., w_n)^T$ of $N_k\,(k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

Proof: By mathematical induction for $n = 2$,

$$w_1 N_1 \oplus w_2 N_2 = \left(\left[1-\prod_{k=1}^{2}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{2}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{2}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{2}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{2}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{2}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{2}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

Assume that, the result holds for $n = m$. That is

$$\sum_{k=1}^{m} w_k N_k = \left(\left[1-\prod_{k=1}^{m}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{m}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{m}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

Consider $n = m+1$, the following result will be proven.

$$\sum_{k=1}^{m} w_k N_k \oplus w_{k+1} N_{k+1} = \left\{\begin{array}{l} \left(\left[1-\prod_{k=1}^{m+1}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{m+1}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m+1}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right) \\ \oplus\left(\left[1-(1-T^L_{N_{k+1}})^{w_{k+1}}, 1-(1-T^U_{N_{k+1}})^{w_{k+1}}\right], \left[1-(1-I^L_{N_{k+1}})^{w_{k+1}}, 1-(1-I^U_{N_{k+1}})^{w_{k+1}}\right], \left[\left(F^L_{N_{k+1}}\right)^{w_{k+1}}, \left(F^U_{N_{k+1}}\right)^{w_{k+1}}\right], \left(T_{N_{k+1}}\right)^{w_{k+1}}, \left(I_{N_{k+1}}\right)^{w_{k+1}}, 1-\left(1-\left(F_{N_{k+1}}\right)\right)^{w_{k+1}}\right) \end{array}\right\}$$

$$= \left(\left[1-\prod_{k=1}^{m+1}\left(1-T^L_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-T^U_{N_k}\right)^{w_k}\right], \left[1-\prod_{k=1}^{m+1}\left(1-I^L_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-I^U_{N_k}\right)^{w_k}\right], \left[\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m+1}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-F_{N_k}\right)^{w_k}\right)$$

Hence proved.

Example 4.4 The NCWA operator is applied on the data as stated in section 3 with corresponding weight, $w=(0.21,0.14,0.25,0.29,0.11)^T$. This weight calculated by Xu and Yager [27] is an essential part of aggregation operators and will be used throughout this paper.

The value of $NCWA=\left(\left[0.7871,0.9171\right],\left[0.5311,0.7292\right],\left[0.2912,0.5075\right],0.5216,0.6071,0.3995\right)$.

Theorem 4.5 Let $N_k=\left(\tilde{T}_{N_k},\tilde{I}_{N_k},\tilde{F}_{N_k},T_{N_k},I_{N_k},F_{N_k}\right)$, where $\tilde{T}_{N_k}=\left[T^L_{N_k},T^U_{N_k}\right],\tilde{I}_{N_k}=\left[I^L_{N_k},I^U_{N_k}\right],\tilde{F}_{N_k}=\left[F^L_{N_k},F^U_{N_k}\right]$, $(k=1,2,...,n)$ be collection of neutrosophic cubic values with weight $W=(w_1,w_2,...,w_n)^T$ of $N_k\,(k=1,2,3,...,n)$, such that $w_k\in[0,1]$ and $\sum_{k=1}^{n}w_k=1$. The following properties are true.

1. Idempotence: If for all $N_k=\left(\tilde{T}_{N_k},\tilde{I}_{N_k},\tilde{F}_{N_k},T_{N_k},I_{N_k},F_{N_k}\right)$, *where* $\tilde{T}_{N_k}=\left[T^L_{N_k},T^U_{N_k}\right],\tilde{I}_{N_k}=\left[I^L_{N_k},I^U_{N_k}\right],\tilde{F}_{N_k}=\left[F^L_{N_k},F^U_{N_k}\right]$

   $(k=1,2,...,n)$ are equal, i.e. $N_k=N$ for all k, then $\text{NCWA}_w(N_1,N_2,...,N_n)=N$

2. Monotonicity**:** Let $B_k=\left(\tilde{T}_{B_k},\tilde{I}_{B_k},\tilde{F}_{B_k},T_{B_k},I_{B_k},F_{B_k}\right)$, *where* $\tilde{T}_{B_k}=\left[T^L_{B_k},T^U_{B_k}\right],\tilde{I}_{B_k}=\left[I^L_{B_k},I^U_{B_k}\right],\tilde{F}_{B_k}=\left[F^L_{B_k},F^U_{B_k}\right]$

   be the collection of neutrosophic cubic values. If $S_B(u)\geq S_N(u)$ *and* $B_k(u)\geq N_k(u)$, where $u\in U$, then

$$\text{NCWA}_w(N_1,N_2,...,N_n)\leq \text{NCWA}_w(B_1,B_2,...,B_n)$$

3. Boundary: $N^-\leq NCWA_w\left\{(N_1),(N_2),...,(N_n)\right\}\leq N^+$, where

$$N^-=\left\{\min_k T^L_{N_k},\min_k I^L_{N_k},1-\max_k F^L_{N_k},\min_k T_{N_k},\min_k I_{N_k},1-\max_k F^L_{N_k},\min_k T_{N_k},\min_k I_{N_k},1-\max_k F_{N_k}\right\}$$

$$N^+=\left\{\max_k T^L_{N_k},\max_k I^L_{N_k},1-\min_k F^L_{N_k},\max_k T_{N_k},\max_k I_{N_k},1-\min_k F^L_{N_k},\max_k T_{N_k},\max_k I_{N_k},1-\min_k F_{N_k}\right\}$$

Proof.

1. Idempotence: Since $N_k=N$ so

$$NCWA\left(N_k\right)=NCWA(N)$$

$$\left(\left[1-\prod_{k=1}^{n}(1-T^L_N)^{w_k},1-\prod_{k=1}^{n}(1-T^U_N)^{w_k}\right],\left[1-\prod_{k=1}^{n}(1-I^L_N)^{w_k},1-\prod_{k=1}^{n}(1-I^U_N)^{w_k}\right],\left[\prod_{k=1}^{n}\left(F^L_N\right)^{w_k},\prod_{k=1}^{n}\left(F^U_N\right)^{w_k}\right],\prod_{k=1}^{n}(T_N)^{w_k},\prod_{k=1}^{n}(I_N)^{w_k},1-\prod_{k=1}^{n}\left(1-(F_N)\right)^{w_k}\right)$$

$$=\left(\tilde{T}_N,\tilde{I}_N,\tilde{F}_N,T_N,I_N,F_N\right)$$

2. Monotonicity: Since neutrosophic cubic ordered weighted average operator (NCOWA) is strictly monotone function, hence the proof is trivial.

3. Boundary: Let $u = \min N^{-}$ and $y = \max N^{+}$, then by the idempotent law we have

$$u \le NCOWA(N_k) \le y \Rightarrow N^{-} \le NCOWA(N_k) \le N^{+}$$

Theorem 4.6 Let $N_k = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T_{N_k}^L, T_{N_k}^U\right], \tilde{I}_{N_k} = \left[I_{N_k}^L, I_{N_k}^U\right], \tilde{F}_{N_k} = \left[F_{N_k}^L, F_{N_k}^U\right]$, $(k = 1, 2, ..., n)$ be the collection of neutrosophic cubic values and $W = (w_1, w_2, ..., w_n)^T$ is weight of the NCOWA with $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$. The following properties will hold, where N_k is the largest kth of $(N_1, N_2, ..., N_n)$.

1. If $W = (1, 0, ..., 0)^T$, then NCOWA $(N_1, N_2, ..., N_n) = \max N_k$
2. If $W = (0, 0, ..., 1)^T$, then NCOWA $(N_1, N_2, ..., N_n) = \min N_k$
3. If $w_k = 1, w_l = 0$, and $k \ne l$, then NCOWA $(N_1, N_2, ..., N_n) = N_k$.

Proof: Since in NCOWA, the neutrosophic values are ordered in descending order, hence NCWA operator aggregates the weighted values. On the other hand NCOWA weights only the ordering positions.

The idea of neutrosophic cubic hybrid aggregation operators (NCHA) is developed to not only weigh the values but also weigh their ordering position as well.

Definition 4.7 $NCHA : \Omega^n \to \Omega$ is a mapping of n-dimension, which has associated weight

$$W = (w_1, w_2, ..., w_n)^T, \text{ where } w_k \in [0,1] \text{ and } \sum_{k=1}^{n} w_k = 1, \text{ such that}$$

$$NCHA_w(N_1, N_2, ..., N_n) = w_1 N_{\sigma(1)}^{\varpi} \oplus w_2 N_{\sigma(2)}^{\varpi} \oplus ... \oplus w_n N_{\sigma(n)}^{\varpi} \text{ where}$$

$N_k^{\square}$ is the largest kth of the weighted neutrosophic cubic values N_k^{ϖ}. The N_k^{ϖ} can be calculated by the following formula $N_k^{\varpi} = nw_k N_k, k = 1, 2, 3, ,, n$, $W = (w_1, w_2, ..., w_n)^T, w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$, where n is the balancing coefficient.

Theorem 4.8 Let $N_{\sigma(k)}^{\varpi} = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T_{N_k}^L, T_{N_k}^U\right], \tilde{I}_{N_k} = \left[I_{N_k}^L, I_{N_k}^U\right], \tilde{F}_{N_k} = \left[F_{N_k}^L, F_{N_k}^U\right]$ $(k = 1, 2, n..$ be a collection of neutrosophic cubic values. Then the aggregated value by NCHA is also a neutrosophic cubic value and

$NCHA_w(N_k)$

$$=\left(\left[1-\prod_{k=1}^{n}\left(1-\left(T_{\sigma(k)}^{\varpi L}\right)_T\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(T_{\sigma(k)}^{\varpi U}\right)_T\right)^{w_k}\right],\left[1-\prod_{k=1}^{n}\left(1-\left(I_{\sigma(k)}^{\varpi L}\right)_I\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(I_{\sigma(k)}^{\varpi U}\right)_I\right)^{w_k}\right],\left[\prod_{k=1}^{n}\left(F_{\sigma(k)}^{\varpi L}\right)_F^{w_k},\prod_{k=1}^{n}\left(F_{\sigma(k)}^{\varpi U}\right)_F^{w_k}\right],\prod_{k=1}^{n}\left(T_{\sigma(k)}^{\varpi}\right)^{w_k},\prod_{k=1}^{n}\left(I_{\sigma(k)}^{\varpi}\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(F_{\sigma(k)}^{\varpi}\right)\right)^{w_k}\right)$$

where the weight $W = (w_1, w_2, ..., w_n)^T$ is such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Proof: The proof is directly concluded by Theorem 4.3.

Theorem 4.9 The NCWA operator is a special case of NCHA operator when all the components of w are equal, i.e. $w_1 = w_2 = ... = w_n$.

Proof. Let $W = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$.

Then NCHA $_w w(N_1, N_2, ..., N_n)$

$$= w_1 N_{\sigma(1)}^{\varpi} \oplus w_2 N_{\sigma(2)}^{\varpi} \oplus ... \oplus w_n N_{\sigma(n)}^{\varpi}$$

$$= \frac{1}{n}(N_{\sigma(1)}^{\varpi} \oplus N_{\sigma(2)}^{\varpi} \oplus ... \oplus N_{\sigma(n)}^{\varpi}) = \frac{1}{n}(N_1, N_2, ..., N_n)$$

$$= w_1 N_1, w_2 N_2, ..., w_n N_n = NCWA(N_1, N_2, ..., N_n).$$

Theorem 4.10 The NCOWA is a special case of NCHA when all the components of w are equal, i.e. $w_1 = w_2 = ... = w_n$.

Proof. Let $W = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$.

Then $NCHA_w w(N_1, N_2, ..., N_n)$

$$= w_1 N_{\sigma(1)}^{\varpi} \oplus w_2 N_{\sigma(2)}^{\varpi} \oplus ... \oplus w_n N_{\sigma(n)}^{\varpi}$$

$$= w_1 N_{\sigma(1)} \oplus w_2 N_{\sigma(2)} \oplus ... \oplus w_n N_{\sigma(n)} = NCOWA(N_1, N_2, ., N_n$$

Example 4.11 The NCHA is applied to the data as stated in section 3 with corresponding weight $w = (0.21, 0.14, 0.25, 0.29, 0.11)^T$ of Xu and Yager [27].

Solution The weighted values are

$$N_A^{\varpi} = ([0.8384, 0.9294], [0.4049, 0.6778], [0.1652, 0.3620], 0.8744, 0.6876, 0.4355)$$

$$N_B^{\varpi} = ([0.4643, 0.6535], [0.5496, 0.6001], [0.3465, 0.5357], 0.7635, 0.8404, 0.3170)$$

$$N_C^{\varpi} = \left(\left[0.9143, 0.9821\right], \left[0.8662, 0.9066\right], \left[0.1679, 0.2788\right], 0.1091, 0.7566, 0.4719\right)$$

$$N_D^{\varpi} = \left(\left[0.9030, 0.9788\right], \left[0.1555, 0.5347\right], \left[0.3660, 0.8583\right], 0.8583, 0.3660, 0.5232\right)$$

$$N_E^{\varpi} = \left(\left[0.4042, 0.5761\right], \left[0.2519, 0.3398\right], \left[0.5958, 0.7269\right], 0.6446, 0.5157, 0.2802\right)$$

$$Scr\left(N_A\right) = 1.6759,\ Scr\left(N_B\right) = 0.6821,\ Scr\left(N_C\right) = 1.0856,\ Scr\left(N_D\right) = 0.9926,\ Scr\left(N_E\right) = 0.0220.,$$

Here, $Scr\left(N_A\right) > Scr\left(N_C\right) > Scr\left(N_D\right) > Scr\left(N_B\right) > Scr\left(N_E\right)$.

According to their ranking, the values are

$$N_{\sigma(1)}^{\varpi} = \left(\left[0.8384, 0.9294\right], \left[0.4049, 0.6778\right], \left[0.1652, 0.3620\right], 0.8744, 0.6876, 0.4355\right)$$

$$N_{\sigma(2)}^{\varpi} = \left(\left[0.9143, 0.9821\right], \left[0.8662, 0.9066\right], \left[0.1679, 0.2788\right], 0.1091, 0.7566, 0.4719\right)$$

$$N_{\sigma(3)}^{\varpi} = \left(\left[0.9030, 0.9788\right], \left[0.1555, 0.5347\right], \left[0.3660, 0.8583\right], 0.8583, 0.3660, 0.5232\right)$$

$$N_{\sigma(4)}^{\varpi} = \left(\left[0.4643, 0.6535\right], \left[0.5496, 0.6001\right], \left[0.3465, 0.5357\right], 0.7635, 0.8404, 0.3170\right)$$

$$N_{\sigma(5)}^{\varpi} = \left(\left[0.4042, 0.5761\right], \left[0.2519, 0.3398\right], \left[0.5958, 0.7269\right], 0.6446, 0.5157, 0.2802\right)$$

The new associated weight is derived by the normal distribution method [19]. Here the associated weight $W = (0.110, 0.237, 0.303, 0.235, 0.115)^T$ is the weighting of the NCHA operator.

$$NCHA_W(N_{\sigma(1)}^{\varpi}, N_{\sigma(2)}^{\varpi}, N_{\sigma(3)}^{\varpi}, N_{\sigma(4)}^{\varpi}, N_{\sigma(5)}^{\varpi})$$

$$= \left(\left[1-\prod_{i=1}^{5}\left(1-\left(T_{\sigma(i)}^{\varpi L}\right)_T\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(T_{\sigma(i)}^{\varpi U}\right)_T\right)^{w_i}\right], \left[1-\prod_{i=1}^{5}\left(1-\left(I_{\sigma(i)}^{\varpi L}\right)_I\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(I_{\sigma(i)}^{\varpi U}\right)_I\right)^{w_i}\right], \left[\prod_{i=1}^{5}\left(F_{\sigma(i)}^{\varpi L}\right)_F^{w_i}, \prod_{i=1}^{5}\left(F_{\sigma(i)}^{\varpi U}\right)_F^{w_i}\right], \prod_{i=1}^{5}\left(T_{\sigma(i)}^{\varpi}\right)^{w_i}, \prod_{i=1}^{5}\left(I_{\sigma(i)}^{\varpi}\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(F_{\sigma(i)}^{\varpi}\right)\right)^{w_i}\right)$$

$$= \left(\left[0.8165, 0.9367\right], \left[0.5533, 0.6932\right], \left[0.2911, 0.5251\right], 0.4966, 0.5893, 0.4322\right)$$

In order to analyze these results with WHO standard, the scores of NCWA and NCHA operators are calculated and indicated as follows.

$Ac(N_{NCWA}) = 0.5879$, $Ac(N_{NCHA}) = 0.5927$ and $Ac(N_{WHO}) = 0.4166$

The graphical analysis is illustrated in Figure 2. It is observed that in both scores that Peshawar has highly polluted air.

Figure 2. Comparison of aggregations with WHO standard

5. Neutrosophic Cubic Einstein Aggregation Operators

This section consist of some fundamental definitions of neutrosophic cubic Einstein weighted averaging (NCEWA), neutrosophic cubic Einstein ordered weighted averaging (NCEOWA) and neutrosophic cubic Einstein hybrid avregaing (NCEHA) aggregation operator, which are defined as follows. These are defined using the Einstein addition, Einstein multiplication and Einstein scalar multiplication.

Definition 5.1 The neutrosophic cubic Einstein weighted averaging is a function, NCEWA $: R^n \to R$ defined by

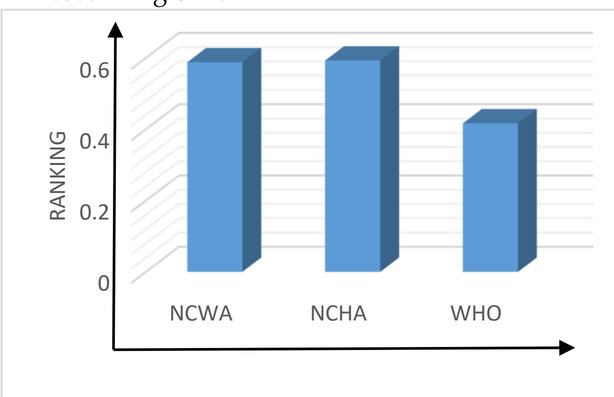
$$NCEWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} \left(w_k N_k\right)_E \tag{6}$$

where $W = (w_1, w_2, ..., w_n)^T$ is the weight of $N_k (k = 1, 2, 3, ..., n), w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

This implies that the neutrosophic cubic values are weighted and then aggregated using Einstein operations.

Definition 5.2 Order neutrosophic cubic Einstein weighted average operator (NCEOWA) is defined as NCEOWA $: R^n \to R$ by $NCEOWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} \left(w_k B_k\right)_E$ where, B_k denotes the ordered position of neutrosophic cubic (NC) values in descending order, $W = (w_1, w_2, ..., w_n)^T$ is the weight of $N_k\ (k = 1, 2, 3, ..., n),$ be such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

Note that, NCEOWA values are ordered and then weighted. Thereafter, the ordering values are aggregated using Einstein operations. The basic concept of ordered weighted operator is to rearrange the values in descending order.



Theorem 5.3 Let $N_k = \left(T_{N_k}, \tilde{I}_{N_k}, F_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $T_{N_k} = \left[T_{N_k}^L, T_{N_k}^U\right], \tilde{I}_{N_k} = \left[I_{N_k}^L, I_{N_k}^U\right], F_{N_k} = \left[F_{N_k}^L, F_{N_k}^U\right]$, $(k = 1, 2, ..., n)$ be the collection of neutrosophic cubic values. Then their NCEWA operator is also a neutrosophic cubic value where $W = (w_1, w_2, ..., w_n)^T$ is the weight vector of $N_k (k = 1, 2, 3, ..., n)$, such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Proof. By mathematical induction for $n = 2$, using Einstein's addition and ascalar multiplication, we will have the following.

$$\left(w_1 N_1\right)_E \oplus \left(w_2 N_2\right)_E = \left\{ \begin{array}{l} \left(\begin{array}{l} \left[\frac{\left(1+T_{N_1}^L\right)^{w_1} - \left(1-T_{N_1}^L\right)^{w_1}}{\left(1+T_{N_1}^L\right)^{w_1} + \left(1-T_{N_1}^L\right)^{w_1}}, \frac{\left(1+T_{N_1}^U\right)^{w_1} - \left(1-T_{N_1}^U\right)^{w_1}}{\left(1+T_{N_1}^U\right)^{w_1} + \left(1-T_{N_1}^U\right)^{w_1}} \right], \left[\frac{\left(1+I_{N_1}^L\right)^{w_1} - \left(1-I_{N_1}^L\right)^{w_1}}{\left(1+I_{N_1}^L\right)^{w_1} + \left(1-I_{N_1}^L\right)^{w_1}}, \frac{\left(1+I_{N_1}^U\right)^{w_1} - \left(1-I_{N_1}^U\right)^{w_1}}{\left(1+I_{N_1}^U\right)^{w_1} + \left(1-I_{N_1}^U\right)^{w_1}} \right], \\ \left[\frac{2\left(F_{N_1}^L\right)^{w_1}}{\left(2-F_{N_1}^L\right)^{w_1} + \left(F_{N_1}^L\right)^{w_1}}, \frac{2\left(F_{N_1}^U\right)^{w_1}}{\left(2-F_{N_1}^U\right)^{w_1} + \left(F_{N_1}^U\right)^{w_1}} \right], \frac{2\left(T_{N_1}\right)^{w_1}}{\left(2-T_{N_1}\right)^{w_1} + \left(T_{N_1}\right)^{w_1}}, \frac{2\left(I_{N_1}\right)^{w_1}}{\left(2-I_{N_1}\right)^{w_1} + \left(I_{N_1}\right)^{w_1}}, \frac{\left(1+F_{N_1}\right)^{w_1} - \left(1-F_{N_1}\right)^{w_1}}{\left(1+F_{N_1}\right)^{w_1} + \left(1-F_{N_1}\right)^{w_1}} \end{array} \right) \\ \oplus \left(\begin{array}{l} \left[\frac{\left(1+T_{N_2}^L\right)^{w_2} - \left(1-T_{N_1}^L\right)^{w_2}}{\left(1+T_{N_2}^L\right)^{w_2} + \left(1-T_{N_2}^L\right)^{w_2}}, \frac{\left(1+T_{N_2}^U\right)^{w_2} - \left(1-T_{N_2}^U\right)^{w_2}}{\left(1+T_{N_2}^U\right)^{w_2} + \left(1-T_{N_2}^U\right)^{w_2}} \right], \left[\frac{\left(1+I_{N_2}^L\right)^{w_2} - \left(1-I_{N_1}^L\right)^{w_2}}{\left(1+I_{N_1}^L\right)^{w_2} + \left(1-I_{N_2}^L\right)^{w_1}}, \frac{\left(1+I_{N_2}^U\right)^{w_2} - \left(1-I_{N_2}^U\right)^{w_2}}{\left(1+I_{N_2}^U\right)^{w_2} + \left(1-I_{N_2}^U\right)^{w_2}} \right], \\ \left[\frac{2\left(F_{N_2}^L\right)^{w_2}}{\left(2-F_{N_2}^L\right)^{w_2} + \left(F_{N_2}^L\right)^{w_2}}, \frac{2\left(F_{N_1}^U\right)^{w_2}}{\left(2-F_{N_1}^U\right)^{w_2} + \left(F_{N_2}^U\right)^{w_2}} \right], \frac{2\left(T_{N_2}\right)^{w_2}}{\left(2-T_{N_2}\right)^{w_1} + \left(T_{N_2}\right)^{w_2}}, \frac{2\left(I_{N_2}\right)^{w_2}}{\left(2-I_{N_2}\right)^{w_2} + \left(I_{N_2}\right)^{w_2}}, \frac{\left(1+F_{N_2}\right)^{w_2} - \left(1-F_{N_2}\right)^{w_2}}{\left(1+F_{N_2}\right)^{w_2} + \left(1-F_{N_2}\right)^{w_2}} \end{array} \right) \end{array} \right\}$$

$$\sum_{k=1}^{2}\left(w_k N_k\right)_E = \left(\begin{array}{l} \left[\frac{\prod_{k=1}^{2}\left(1+T_{N_k}^L\right)^{w_k} - \prod_{k=1}^{2}\left(1-T_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(1+T_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(1-T_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+T_{N_k}^U\right)^{w_k} - \prod_{k=1}^{2}\left(1-T_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(1+T_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(1-T_{N_k}^U\right)^{w_k}} \right], \left[\frac{\prod_{k=1}^{2}\left(1+I_{N_k}^L\right)^{w_k} - \prod_{k=1}^{2}\left(1-I_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(1+I_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(1-I_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+I_{N_k}^U\right)^{w_k} - \prod_{k=1}^{2}\left(1-I_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(1+I_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(1-I_{N_k}^U\right)^{w_k}} \right], \\ \left[\frac{2\prod_{k=1}^{2}\left(F_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(2-F_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(F_{N_k}^L\right)^{w_k}}, \frac{2\prod_{k=1}^{2}\left(F_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(2-F_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(F_{N_k}^U\right)^{w_k}} \right], \frac{2\prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-T_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}}, \frac{2\prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-I_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+F_{N_k}\right)^{w_k} - \prod_{k=1}^{2}\left(1-F_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(1+F_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(1-F_{N_k}\right)^{w_k}} \end{array} \right)$$

Assuming that for $n = m$ the result holds true, that is

$$\sum_{k=1}^{m}\left(w_k N_k\right)_E = \left(\begin{array}{l} \left[\frac{\prod_{k=1}^{m}\left(1+T_{N_k}^L\right)^{w_k} - \prod_{k=1}^{m}\left(1-T_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(1-T_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{m}\left(1+T_{N_k}^U\right)^{w_k} - \prod_{k=1}^{m}\left(1-T_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(1-T_{N_k}^U\right)^{w_k}} \right], \left[\frac{\prod_{k=1}^{m}\left(1+I_{N_k}^L\right)^{w_k} - \prod_{k=1}^{m}\left(1-I_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(1-I_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{m}\left(1+I_{N_k}^U\right)^{w_k} - \prod_{k=1}^{m}\left(1-I_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(1-I_{N_k}^U\right)^{w_k}} \right] \\ \left[\frac{2\prod_{k=1}^{m}\left(F_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(F_{N_k}^L\right)^{w_k}}, \frac{2\prod_{k=1}^{m}\left(F_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(F_{N_k}^U\right)^{w_k}} \right], \frac{2\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-T_{N_k}\right)^{w_k} + \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}, \frac{2\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-I_{N_k}\right)^{w_k} + \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}, \frac{\prod_{k=1}^{m}(1+F_{N_k})^{w_k} - \prod_{k=1}^{m}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m}(1+F_{N_k})^{w_k} + \prod_{k=1}^{m}(1-F_{N_k})^{w_k}} \end{array} \right)$$

The result is proven for $n = m+1$, since

$$w_{m+1}N_{m+1}=\left(\begin{array}{l}\left[\dfrac{\left(1+T^{L}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^{L}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^{L}_{N_{m+1}}\right)^{w_{m+1}}},\dfrac{\left(1+T^{U}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^{U}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^{U}_{N_{m+1}}\right)^{w_{m+1}}}\right],\left[\dfrac{\left(1+I^{L}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^{L}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^{L}_{N_{m+1}}\right)^{w_{m+1}}},\dfrac{\left(1+I^{U}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^{U}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^{U}_{N_{m+1}}\right)^{w_{m+1}}}\right]\\ \left[\dfrac{2\left(F^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^{L}_{N_{m+1}}\right)^{w_{m+1}}+F^{L}_{N_{m+1}}},\dfrac{2\left(F^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^{U}_{N_{m+1}}\right)^{w_{m+1}}+F^{U}_{N_{m+1}}}\right],\dfrac{2\left(T_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-T_{N_{m+1}}\right)^{w_{m+1}}+T_{N_{m+1}}},\dfrac{2\left(I_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-I_{N_{m+1}}\right)^{w_{m+1}}+I_{N_{m+1}}},\dfrac{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}-\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}+\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}\end{array}\right)$$

$$\sum_{k=1}^{m}\left(w_kN_k\right)_E\oplus\left(w_{m+1}N_{m+1}\right)_E=$$

$$\left\{\begin{array}{l}\left(\begin{array}{l}\left[\dfrac{\prod_{k=1}^{m}\left(1+T^{L}_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-T^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-T^{L}_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m}\left(1+T^{U}_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-T^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-T^{U}_{N_k}\right)^{w_k}}\right],\left[\dfrac{\prod_{k=1}^{m}\left(1+I^{L}_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-I^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-I^{L}_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m}\left(1+I^{U}_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-I^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-I^{U}_{N_k}\right)^{w_k}}\right]\\ \left[\dfrac{2\prod_{k=1}^{m}\left(F^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(F^{L}_{N_k}\right)^{w_k}},\dfrac{2\prod_{k=1}^{m}\left(F^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(F^{U}_{N_k}\right)^{w_k}}\right],\dfrac{2\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-T_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}},\dfrac{2\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-I_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m}(1+F_{N_k})^{w_k}-\prod_{k=1}^{m}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m}(1+F_{N_k})^{w_k}+\prod_{k=1}^{m}(1-F_{N_k})^{w_k}}\end{array}\right)\\ \oplus\left(\begin{array}{l}\left[\dfrac{\left(1+T^{L}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^{L}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^{L}_{N_{m+1}}\right)^{w_{m+1}}},\dfrac{\left(1+T^{U}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^{U}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^{U}_{N_{m+1}}\right)^{w_{m+1}}}\right],\left[\dfrac{\left(1+I^{L}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^{L}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^{L}_{N_{m+1}}\right)^{w_{m+1}}},\dfrac{\left(1+I^{U}_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^{U}_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^{U}_{N_{m+1}}\right)^{w_{m+1}}}\right]\\ \left[\dfrac{2\left(F^{L}_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^{L}_{N_{m+1}}\right)^{w_{m+1}}+F^{L}_{N_{m+1}}},\dfrac{2\left(F^{U}_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^{U}_{N_{m+1}}\right)^{w_{m+1}}+F^{U}_{N_{m+1}}}\right],\dfrac{2\left(T_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-T_{N_{m+1}}\right)^{w_{m+1}}+T_{N_{m+1}}},\dfrac{2\left(I_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-I_{N_{m+1}}\right)^{w_{m+1}}+I_{N_{m+1}}},\dfrac{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}-\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}+\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}\end{array}\right)\end{array}\right\}$$

$$\sum_{k=1}^{m+1}\left(w_kN_k\right)_E=$$

$$\left(\begin{array}{l}\left[\dfrac{\prod_{k=1}^{m+1}\left(1+T^{L}_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-T^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+T^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-T^{L}_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m+1}\left(1+T^{U}_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-T^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+T^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-T^{U}_{N_k}\right)^{w_k}}\right],\left[\dfrac{\prod_{k=1}^{m+1}\left(1+I^{L}_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-I^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+I^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-I^{L}_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m+1}\left(1+I^{U}_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-I^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+I^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-I^{U}_{N_k}\right)^{w_k}}\right]\\ \left[\dfrac{2\prod_{k=1}^{m+1}\left(F^{L}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-F^{L}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(F^{L}_{N_k}\right)^{w_k}},\dfrac{2\prod_{k=1}^{m+1}\left(F^{U}_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-F^{U}_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(F^{U}_{N_k}\right)^{w_k}}\right],\dfrac{2\prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-T_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}},\dfrac{2\prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-I_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}},\dfrac{\prod_{k=1}^{m+1}(1+F_{N_k})^{w_k}-\prod_{k=1}^{m+1}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m+1}(1+F_{N_k})^{w_k}+\prod_{k=1}^{m+1}(1-F_{N_k})^{w_k}}\end{array}\right)$$

Hence proved.

Example 5.4 The NCEWA is applied to data in section 3 with corresponding weight of Xu and Yager [27], $w=(0.21,0.14,0.25,0.29,0.11)^{T}$.

Then $NCEWA=\left(\left[0.7848,0.9163\right],\left[0.5058,0.6650\right],\left[0.2957,0.5241\right],0.5652,0.6187,0.3990\right)$

Note that the NCEWA operator aggregates the weighted value whereas the NCEOWA operator weight the ordering position and then aggregates the values. The idea of NCEHA is developed to overcome to not only weight the neutrosophic cubic values but their order positioning as well.

In order to analyze these results with WHO standard, the score of NCEWA and NCEHA operators are compared in terms of the accuracy functions as follows as illustrated in Figure 3.

$$Ac(N_{NCEWA}) = 0.5861,\ ,\quad Ac(N_{NCEHA}) = 0.6099 \quad \text{and} \quad Ac(N_{WHO}) = 0.4166.$$

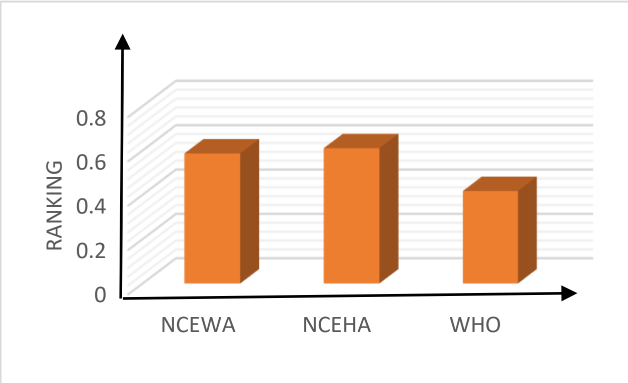


Figure 3. The comparison of Einstein aggregation with WHO

Observe that using both operators NCEWA and NCEHA, Peshawar has highly polluted air as shown in Figure 3. The overall graphical presentation is illustrated in Figure 4. Hence we conclude that serious measures by the relevant government agencies are needed to overcome the situation.

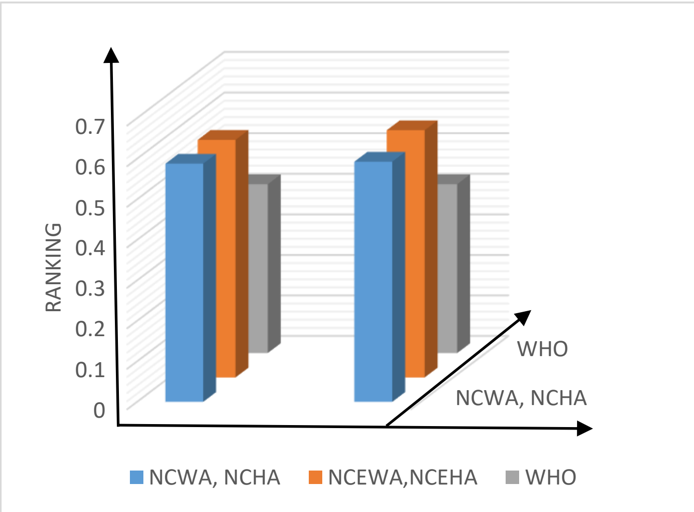


Figure 4. Comparison of neutrosophic cubic aggregation operators with WHO

6. Conclusions

In this research, NCWA, NCHA, NCEWA, NCEHA operators are compared with WHO standards. The aggregation operators are applied to the numerical data of PM10. These aggregation operators enabled us to analyze the air pollution model in the city of Peshawar, Pakistan. We computed the accuracy functions of all of these aggregation operators and WHO standard. The analysis is then presented graphically to illustrate the comparison. It is observed that in the month of April 2014, the pollution of PM10 is very much higher than WHO standards. Strong measures are thus required to control air pollution. Our future research will be to apply further the NCWA, NCHA, NCEWA, NCEHA operators to construction management, geometric programming, binomial factorial problem, and numerical convergence of polynomial roots [49-50].

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