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Derivable Single Valued Neutrosophic Graphs Based on $KM$-Fuzzy Metric

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ABSTRACT In this paper we consider the concept of $KM$-fuzzy metric spaces and we introduce a novel concept of $KM$-single valued neutrosophic metric graphs based on $KM$-fuzzy metric spaces. Then we investigate the finite $KM$-fuzzy metric spaces with respect to $KM$-fuzzy metrics and we construct the $KM$-fuzzy metric spaces on any given non-empty sets. We try to extend the concept of $KM$-fuzzy metric spaces to a larger class of $KM$-fuzzy metric spaces such as union and product of $KM$-fuzzy metric spaces and in this regard we investigate the class of products of $KM$-single valued neutrosophic metric graphs. In the final, we define some operations such as tensor product, Cartesian product, semi-strong product, strong product, union, semi-ring sum, suspension, and complement of $KM$-single valued neutrosophic metric graphs.

INDEX TERMS (Derivable) $KM$-single valued neutrosophic metric graph, $KM$-fuzzy metric space, triangular-norm (conorm).

I. INTRODUCTION

Classical theory is a pure concept and without quality or criteria, so it is not attractive to use in our world, that’s why we use the neutrosophic sets theory as one of a generalizations of set theory in order to deal with uncertainties, which is a key action in the contemporary world introduced by Smarandache for the first time in 1998 [22] and in 2005 [23]. This concept is a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. This theory describes an important role in modeling and controlling unsure hypersystems in nature, society and industry. In addition, fuzzy topological spaces as a generalization of topological spaces, have a fundamental role in construction of fuzzy metric spaces as an extension of the concept of metric spaces. The theory of fuzzy metric spaces works on finding the distance between two points as non-negative fuzzy numbers, which have various applications. The structure of fuzzy metric spaces is equipped with mathematical tools such as triangular norms and fuzzy subsets depending on time parameter and on other variables. This theory has been proposed by different researchers with different definitions from several points of views ([3]–[5], [12]), and that this study was applied to the notion of $KM$-fuzzy metric space introduced in 1975 [4] by Kramosil and Michalek. Fuzzy graphs, introduced by Rosenfeld, are finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with respect to different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the difference between the traditional numerical models used in engineering and sciences [19]. The generalization of the concept of a fuzzy graph is noticed by some researchers on more subjects, such as fuzzy graph based on t-norm, intuitionistic fuzzy threshold graphs, m-polar fuzzy graphs and single-valued neutrosophic graphs. Mordeson et al. [17] generalized the definition of a fuzzy graph by replacing minimum in the basic definitions with an arbitrary t-norm. They developed a measure on the susceptibility of trafficking in persons for networks by using a t-norm other than minimum [17]. Recently, F. Smarandache, introduced a new concept as a generalization of hypergraphs to n-SuperHypergraph, Plithogenic n-SuperHypergraph {with super-vertices (that are groups of vertices) and hyper-edges (defined on power-set of power-set...)} that is the most general form of graph as today), and n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra respectively, which have several properties and are connected with the real world [24]. Further materials regarding graphs, single-valued neutrosophic metric graphs, hypergraphs, intuitionistic fuzzy set, n-SuperHypergraph and Plithogenic n-SuperHypergraph, and NeutroAlgebras {Smarandache generalized the classical
algebraic structures to neutro algebraic structures (or Neutro Algebras) [whose operations and axioms are partially true, partially indeterminate, and partially false] as extensions of PartialAlgebra, and to AntiAlgebraic structures (or Anti Algebras) [whose operations and axioms are totally false], and in general, he extended any classical structure, in no matter what field of knowledge, to a Neutro structure and an Anti structure. All these are available in the literature too [2], [7]–[10], [13], [18], [20], [21], [25]–[29].

Regarding these points, we introduce the concept of KM-single valued neutrosophic metric graphs based on the concept of KM-fuzzy metrics. One of the main motivations of KM-single valued neutrosophic metric graphs is obtained from the fuzzy graphs and so we want to use this concept to model many decision making problems in uncertain environment. We need to construct the KM-single valued neutrosophic metric graphs based on finite or infinite sets, so we develop the concept of KM-fuzzy metric on any nonempty set and prove that for every given set with respect to the concept of C-graphable sets one can construct a KM-metric space. It is a natural generalization of the fuzzy graphs to the single-valued neutrosophic metric graphs, so it shows our main motivation for introducing the notion of the KM-single valued neutrosophic metric graphs. This notion is based on one of the fundamental concepts of fuzzy mathematics, which includes tools such as t-norms, t-conorms, and fuzzy subsets. We apply the notation of KM-fuzzy metric spaces to generate the finite KM-single valued neutrosophic metric graphs. We have extended some production operations on the KM-fuzzy metric spaces to the KM-single valued neutrosophic metric graphs.

II. PRELIMINARIES

In this section, we recall some definitions and results, which we use in what follows.

Definition 1 [11], [14]: Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be simple graphs, $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$, where $V_1 \times V_2$ is the vertex set of the following graphs:

(i) categorical(tensor, direct, cardinal, Kronecker) product graph $G_1 \times G_2$:

$E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2\};$

(ii) Cartesian product graph $G_1 \otimes G_2$:

$E(G_1 \otimes G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1 = y_1 \text{ and } x_2y_2 \in E_2) \text{ or } (x_1y_1 \in E_1 \text{ and } x_2 = y_2)\};$

(iii) semi-strong product graph $G_1 \cdot G_2$:

$E(G_1 \cdot G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1 = y_1 \text{ and } x_2y_2 \in E_2) \text{ or } (x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2)\};$

(iv) strong product (symmetric composition) graph $G_1 \otimes G_2$:

$E(G_1 \otimes G_2) = E(G_1 \otimes G_2) \cup E(G_1 \times G_2);$

(v) lexicographic product (composition) graph $G_1 \circ G_2$:

$E(G_1 \circ G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1y_1 \in E_1) \text{ or } (x_1 = y_1 \text{ and } x_2y_2 \in E_2)\};$

(vi) union graph $G_1 \cup G_2$:

$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$; and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2);$

(vii) join product graph $G_1 + G_2$:

$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup E'$, where $E'$ is the set of all line joining $V_1$ with $V_2$.

Definition 2 [16]: A fuzzy graph $G = (V, \sigma, \mu)$ is an algebraic structure of non-empty set $V$ together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\mu(x, y) \leq \sigma(x) \land \sigma(y)$. It is called $\sigma$ as fuzzy vertex set and $\mu$ as fuzzy edge set of $G$.

Definition 3 [1]: A single valued neutrosophic graph (SVN–G) is defined to be a form ($V, E, A, B$) where

(i) $V = \{v_1, v_2, \ldots, v_n\}$, $T_A, I_A, F_A : V \to [0, 1]$ denote the degree of membership, degree of indeterminacy and non-membership of the element $v_j \in V$; respectively, and for every $1 \leq i \leq n$, we have $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$.

(ii) $E \subseteq V \times V, T_B, I_B, F_B : E \to [0, 1]$ are called degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in E$ respectively, such that for any $1 \leq i, j \leq n$, we have $T_B(v_i, v_j) \leq \min(T_A(v_i), T_A(v_j))$, $I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j))$, $F_B(v_i, v_j) \geq \max(F_A(v_i), F_A(v_j))$ and $0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$. Also $A$ is called the single valued neutrosophic vertex set of $V$ and $B$ is called the single valued neutrosophic edge set of $E$.

Definition 4 [15]: A triplet ($X, \rho, T$) is called a KM-fuzzy metric space, if $X$ is an arbitrary non-empty set, $T$ is a left-continuous t-norm and $\rho : X^2 \times [0, \infty) \to [0, 1]$ is a fuzzy set, such that for each $x, y, z, \in X$ and $t, s \geq 0$, we have:

(i) $\rho(x, y, 0) = 0$,

(ii) $\rho(x, x, t) = 1$ for all $t > 0$,

(iii) $\rho(x, y, t) = \rho(y, x, t)$ (commutative property),

(iv) $T(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t + s)$ (triangular inequality),

(vi) $\rho(x, y, -) : [0, \infty) \to [0, 1]$ is a left-continuous map,

(vii) $\lim_{t \to \infty} \rho(x, y, t) = 1$,

(viii) $\rho(x, y, t) = 1, \forall t > 0$ implies that $x = y$.

If ($X, \rho, T$) satisfies in conditions (i)–(viii), then it is called KM-fuzzy pseudo metric space and $\rho$ is called a KM-fuzzy pseudo metric. A fuzzy version of the triangular inequality. The value $\rho(x, y, t)$ is considered as the degree of nearness from

Theorem 1 [15]: Let ($X, \rho, T$) be a KM-fuzzy metric space. Then $\rho(x, y, -) : [0, \infty) \to [0, 1]$ is a non-decreasing map.

Proof 1: See [15].

III. FINITE KM-FUZZY METRIC SPACE

In this section, we apply the concept of KM-fuzzy metric spaces and construct a new class of KM-fuzzy metric spaces under operation product and union of KM-fuzzy metric spaces. In addition, for any given non-empty set we construct
**A. FINITE KM-FUZZY METRIC SPACE BASED ON METRIC**

In this subsection, we apply the concept of finite metric for constructing of KM-fuzzy metric space on any given non-empty set.

**Definition 5:** Let X be a finite set. We say that X is a C-graphable set if G = (X, E) is a connected graph, where E ⊆ X × X and G = (X, E) is called an X-derived graph. Let GX be the set of all connected graphs which are constructed on X as set of vertices, so we have the following results.

Let G = (X, E) be a connected graph. For all x, y ∈ X, define \( d^\delta(x, y) = \min\{P_{x, y}\} \) where \( P_{x, y} \) is a path between x, y. Obviously, \( d^\delta \) is a metric on X.

**Theorem 4:** Let X be a finite set and |X| ≥ 2. Then there exists a non-discrete metric d on X such that (X, d) is a metric space.

**Proof 4:** Let |X| ≥ 2. Clearly, X is a C-graphable set and so there exists a graph G = (X, E) ∈ GX. For all x, y ∈ X, define \( d(x, y) = d^\delta(x, y) \). Clearly, (X, d^\delta) is a metric space.

**Corollary 2:** Let n ∈ \( \mathbb{N} \), X be a set and |X| = n.

(i) If G = (X, E) ∼= K_n is the complete graph, then for metric spaces (X, d^\delta) and (X, d_1), we have \( d_1 \leq d^\delta \).

(ii) If G = (X, E) ∼= C_n is the cycle graph, then for metric spaces (X, d^\delta) and (X, d_1), we have \( d_1 \leq d^\delta \leq \frac{d(|n|)}{2} \).

**Theorem 5:** Let X be a non-empty set. Then there exists a fuzzy subset \( \rho: X^2 \times \mathbb{R}^2 \rightarrow [0, 1] \), such that (X, \( \rho \), \( T_{\min} \)) is a KM-fuzzy metric space.

**Proof 5:** Let |X| ≥ 2. Then clearly, X is a C-graphable set and by Theorem 4, (X, d^\delta) is a metric space. For all x, y ∈ X and for all \( 0 \neq m, t \in \mathbb{R}^2 \), define \( \rho(x, y, 0) = 0 \) and \( \rho(x, y, t > 0) = \frac{\psi(t) + md^\delta(x, y)}{\psi(t) + md^\delta(x, y)} \), where \( \psi: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is an increasing continuous function, and for all x, y ∈ X, we have \( \psi(t) + md^\delta(x, y) \neq 0 \) and \( \psi(t) \rightarrow 0 \) when \( t \rightarrow 0 \). Now, we show that (X, \( \rho \), \( T_{\min} \)) is a KM-fuzzy metric space. We prove only the triangular inequality for all x, y ∈ X, consider the five cases: \( x = z, x = \neq z, x = \neq y, x = y \neq z \). In all cases for \( 0 \in \{ t \} \) is clear, now for \( 0 \notin \{ t \} \) we investigate it. For \( x = y \neq z \) since \( \psi(t + s) \geq \psi(s) \), we have \( \psi(t + s)\psi(s) + ma \leq \psi(s)\psi(t + s) + ma \) and so \( \frac{\psi(s)}{\psi(t + s)} \leq \frac{\psi(s) + ma}{\psi(t + s) + ma} \). If \( x \neq y \neq z \), then \( d_a(x, y) = d_a(y, z) = d_a(x, z) = \alpha \). Since \( \psi(t) \) is an increasing mapping, we get that \( \max\{\psi(t)\} \leq \max\{\psi(t) + ma\} \) and it implies that \( \psi(t)\psi(t + s) + ma \leq \psi(t + s)\psi(t + ma) \) and so \( \frac{\psi(t)}{\psi(t + s) + ma} \leq \frac{\psi(t)}{\psi(t + s) + ma} \), which means that \( \rho(x, y, t) \leq \rho(x, z, t + s) \) by a similar way, \( \rho(z, y, s) \leq \rho(x, z, t + s) \) and so \( T_{\min}(\rho(x, y, t), \rho(z, y, s)) \leq \rho(x, z, t + s) \) is a KM-fuzzy metric space.

**Corollary 3:** Let X be a non-empty set. Then there exists a fuzzy subset \( \rho: X^2 \times \mathbb{R}^2 \rightarrow [0, 1] \), such that for all left-continuous t-norm T \( \leq T_{\min} \), (X, \( \rho \), T) is a KM-fuzzy metric space.

**B. OPERATIONS ON KM-FUZZY METRIC SPACES**

In this subsection, we extend KM-fuzzy metric spaces to union and product of KM-fuzzy metric spaces. Let \( (X_1, \rho_1, T_1) \) and \( (X_2, \rho_2, T_2) \) be KM-fuzzy metric spaces, \( (x_1, y_1) \), \( (x_2, y_2) \) ∈ \( X_1 \times X_2 \) and \( t \in \mathbb{R}^2 \). For an arbitrary \( T \in T_{\mathbb{C}} \), define \( T(\rho): (X_1 \times X_2)^2 \times \mathbb{R}^2 \rightarrow [0, 1] \) by \( T(\rho)((x_1, y_1), (x_2, y_2), t) = T(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t)) \). So we have the following theorem.

**Theorem 6:** Let \( (X_1, \rho_1, T_1) \) and \( (X_2, \rho_2, T_2) \) be KM-fuzzy metric spaces. Then \( (X_1 \times X_2, T_{\min}(\rho), T) \) is a KM-fuzzy metric space.
Proof 6: Let \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2 \) and \(t, s \in \mathbb{R}^\geq 0\).

(i) Since for all \(x_1, x_2 \in X_1, y_1, y_2 \in X_2, \rho_1(x_1, x_2) = 0 \) and \(\rho_2(y_1, y_2) = 0\), have \(T_{\min}(\rho)(x_1, y_1), (x_2, y_2), 0) = 0\).

(ii) \(T_{\min}(\rho)((x_1, y_1), (x_2, y_2), t) = 1\) if and only if \(T_{\min}(\rho)(x_1, x_2, t) = 1\) and \(T_{\min}(\rho)(y_1, y_2, t) = 1\) if and only if \((x_1, y_1) = (x_2, y_2)\).

(iii) It is clear that \(T_{\min}(\rho)\) is a commutative map.

(iv) \(T(T_{\min}(\rho))(x_1, y_1, x_2, y_2, t) = T(T_{\min}(\rho)(x_1, x_2, t), T_{\min}(\rho)(y_1, y_2, t))\)

\[= T\left(T_{\min}(\rho)(x_1, x_2, t), T_{\min}(\rho)(y_1, y_2, t)\right) \leq T\left(T_{\min}(\rho)(x_1, x_2), T_{\min}(\rho)(y_1, y_2)\right) \]

\[\leq T_{\min}(\rho)(x_1, x_2, t + s), T_{\min}(\rho)(y_1, y_2, t + s)\)

\[= T_{\min}(\rho)(x_1, y_1, x_2, y_2, t + s)\).

(v) Since \(\rho_1, \rho_2\) are left-continuous maps, we get that \(\rho\) is a left-continuous map.

(vi) Clearly \(\lim_{t \to \infty} T_{\min}(\rho)(x_1, x_2, t) = T_{\min}(\rho)(x_1, x_2, t) = T_{\min}(1, 1) = 1\).

In this section, we introduce a novel concept as \(KM\)-single valued neutrosophic metric space. In doing so, the degree of truth-membership, indeterminacy-membership and falsity-membership of the element \(x \in V\), respectively. There is no restriction on the sum of \(N_T(x), T(x)\) and \(F_T(x)\), therefore \(0 \leq T(x) + I_T(x) + F_T(x) \leq 3\) for all \(x \in V\).

**IV. KM-SINGLE VALUED NEUTROSOPHIC METRIC GRAPH**

In this section, we introduce a novel concept as \(KM\)-single valued neutrosophic metric graphs and analyze some properties.

**Definition 6:** Let \((V, \rho, T)\) be a fuzzy metric space and \(G^* = (V, E)\) be a simple graph. Then \(G = (X = \{T(x), I(x), F(x)\}, T(x), T(x), S)\) is called a \(KM\)-single valued neutrosophic metric graph on \(G^*\), if there exists some time \(t \in \mathbb{R}^\geq 0\) (for \(t = 0\), we call starting time) such that for all \(xy \in E\), we have

(i) the functions \(T_V : V \to [0, 1], I_V : V \to [0, 1] \) and \(F_V : V \to [0, 1]\) represent the degree of truth-membership, indeterminacy-membership and falsity-membership of the element \(x \in V\), respectively. Since \(0 \leq T_V(x) + I_V(x) + F_V(x) \leq 3\) for all \(x \in V\).

(ii) the functions \(E_T : E \subseteq V \times V \to [0, 1], I_E : E \subseteq V \times V \to [0, 1]\) and \(F_E : E \subseteq V \times V \to [0, 1]\) are defined by \(E_T = (T(x, y), T(x, y)) \leq \rho(x, y, t)\), \(S((E_T, I(x, y), I(x, y))) = \rho(x, y, t)\) and \(S(F_E, S(F_E, F_E)) \geq \rho(x, y, t)\), where \(S\) is a triangular conorm as a dual of triangular norm \(T\), via a negation \(\eta\).

We call \(X\) as a \(KM\)-single valued neutrosophic metric vertex set of \(G\) and \(Y\) is \(KM\)-single valued neutrosophic edge set of \(G\).

In definition of \(KM\)-single valued neutrosophic metric graph, if \(t \to \infty\), then for all \(xy \in V\), \(\rho(x, y, t) \to 1\) and so it follows that \(E_T = (S(T(x, y), S(x, y))) = (E_E(x), E_E(y), E_E(y)) \leq \rho(x, y, t)\) and \(E_T = (S(T(x, y), S(x, y))) = \rho(x, y, t)\), which is any given fuzzy values. The concept of \(KM\)-single valued neutrosophic metric graph is a generalization of \(KM\)-fuzzy metric graphs, which is introduced by M. Hamidi et.al [6].

**Theorem 7:** Let \((X_1, \rho_1, T)\) and \((X_2, \rho_2, T)\) be \(KM\)-fuzzy metric spaces. Then \((X_1 \times X_2, \rho_1 \cup \rho_2, T)\) is a \(KM\)-fuzzy metric space, where \(X_1 \times X_2 = \emptyset\).

**Proof 7:** Let \(x, y, z \in X_1 \times X_2 \) and \(t, s \in \mathbb{R}^\geq 0\). We only prove the triangular inequality property and other cases are immediate. Let \(x, y \in X_1\) (for \(x, y \in X_2\), one can prove in a similar way), then \(T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) \leq (\rho_1(x, z, t + s) = (\rho_1 \cup \rho_2)(x, z, t + s).\) If \(z \in X_2\), then \(T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) \leq \rho_1(x, z, t + s) = (\rho_1 \cup \rho_2)(x, z, t + s).\) Let \(x \in X_1, y \in X_2\). Then \(T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), \rho_1(y, z, s)) \leq \rho_1(x, z, t + s) = (\rho_1 \cup \rho_2)(x, z, t + s).\) Let \(x \in X_1, y \in X_2\).
we get that $T\left( (T_E(xy), T(T_V(x), T_V(y))) \right) \leq \rho(x, y, 0)$. Hence $T(TE(xy), T(T_V(x), T_V(y))) = 0$ and so $T_V(x) = 0$ or $T_V(y) = 0$.

(ii) It is immediate by Definition.

\textbf{Example 1:} Let $V = \{1, 2, 3, 4\}$ and $x, y \in X$. Consider a fuzzy subset $\rho(x, y, 0) = 0$ and $\rho(x, y, t) > 0 = \min(x, y) + t - \max(x, y) + t$. We take the negation $\eta(m) = 1 - m(m \in [0, 1])$ and obtain a $KM$-single valued neutrosoptic metric graph $G = (V, (X = (T_V, I_V, F_V), Y = (T_E, I_E, F_E), \rho, T_{min}, S_{max}))$ on the cycle graph $C_4$ for $t = 1$. In Figure 1.

![FIGURE 1. KM-single valued neutrosoptic metric graph $G = (X, Y, \rho, T_{min}, S_{max})$.](image)

Let $(V, \rho, T)$ be a fuzzy metric space and $G = (X, Y, \rho, T, S)$ be a $KM$-single valued neutrosoptic metric graph on $G^* = (V, E)$. Define $T^a_y = \{x \in V | T_V(x) \geq a\}$, $I^x_y = \{x \in V | I_V(x) \leq \beta\}$, $F^x_y = \{x \in V | F_V(x) \leq \gamma\}$, $I^y_x = \{x \in E | T_E(x) \geq \alpha\}$, $I_{E}^{y} = \{x \in E | I_E(x) \leq \beta\}$, $F_{E}^{y} = \{x \in E | F_E(x) \leq \gamma\}$, $X^{(\alpha, \beta, \gamma)} = \{x \in V | T_V(x) \geq \alpha, I_V(x) \leq \beta, F_V(x) \leq \gamma\}$ and $Y^{(\alpha, \beta, \gamma)} = \{x \in E | T_E(x) \geq \alpha, I_E(x) \leq \beta, F_E(x) \leq \gamma\}$.

\textbf{Theorem 9:} Let $(V, \rho, T)$ be a fuzzy metric space and $G = (X, Y, \rho, T, S)$ be a $KM$-single valued neutrosoptic metric graph on $G^* = (V, E)$ and $\alpha, \beta, \gamma \in [0, 1]$. Then $X^{(\alpha, \beta, \gamma)} = T^a_y \cap I^x_y \cap F^x_y$ and $Y^{(\alpha, \beta, \gamma)} = T^a_y \cap I^x_y \cap F^x_y$.

\textbf{Proof:} Let $x \in X^{(\alpha, \beta, \gamma)}$. Then $T_V(x) \geq \alpha, I_V(x) \leq \beta$ and $F_V(x) \leq \gamma$ implies that $x \in T^a_y \cap I^x_y \cap F^x_y$ and conversely. In a similar way, one can see that $Y^{(\alpha, \beta, \gamma)} = T^a_y \cap I^x_y \cap F^x_y$.

Let $G = (X, Y, \rho, T, S)$ be a $KM$-single valued neutrosoptic metric graph on $G^* = (V, E)$. Consider $S_{min} = \bigwedge_{xy \in E} T(TE(xy), T(T_V(x), T_V(y)))$, $\beta_{max} = \bigwedge_{xy \in E} S(I_E(xy), S(I_V(xy), I_V(y)))$, $\gamma_{max} = \bigwedge_{xy \in E} S(F_E(xy), S(F_V(xy), F_V(y)))$. Then we have the following theorem.

\textbf{Theorem 10:} Let $(V, \rho, T)$ be a fuzzy metric space and $G = (X, Y, \rho, T, S)$ be a $KM$-single valued neutrosoptic metric graph on $G^* = (V, E)$. Then for any $\alpha \leq \alpha_{min}, \beta \geq \beta_{max}, \gamma \geq \gamma_{max}, G^{(\alpha, \beta, \gamma)} = (X^{(\alpha, \beta, \gamma)}, Y^{(\alpha, \beta, \gamma)})$ is a subgraph of $G^* = (X, Y)$, parameters of $R^+$.\n
\textbf{Proof 10:} Let $xy \in E$. Since $T(TE(xy), T(T_V(x), T_V(y))) \leq T_{min}(TE(xy), T(T_V(x), T_V(y)))$, we get that $T_E(xy) \geq \alpha_{min} \geq \alpha$. So for any $\alpha \leq \alpha_{min}, T^a_E \subseteq E$. Also since $S(I_E(xy), S(I_V(xy), I_V(y))) \leq S_{max}(I_E(xy), I_V(xy), I_V(y))$, we get that $I_E(xy) \leq \beta_{max} \leq \beta$. So for any $\beta \geq \beta_{max}$, $\gamma_{max} \leq \gamma$. In a similar way, one can see that $F_E(\beta) \subseteq E$. Using Theorem 9, $G^{(\alpha, \beta, \gamma)} \subseteq E$ and So $G^{(\alpha, \beta, \gamma)} = (X^{(\alpha, \beta, \gamma)}, Y^{(\alpha, \beta, \gamma)})$ is a subgraph of $G^* = (X, Y)$.

\textbf{Theorem 11:} Let $(V, \rho, T)$ be a $KM$-fuzzy metric space and $G^* = (V, E)$ be a simple graph.

(i) If $T_E \leq \rho, I_E \geq \rho$ and $F_E \geq \rho$ then $G = (X, Y, \rho, T)$ is a $KM$-single valued neutrosoptic metric graph on $G^*$.

(ii) If $G = (X, Y, \rho, T_{min}, S_{max})$ is a $KM$-single valued neutrosoptic metric graph on $G^*$ and $T_E > \rho, I_E < \rho$ and $F_E < \rho$, then $G = (X, Y)$ is not a single valued neutrosoptic graph on $G^*$.

(iii) If $G = (X, Y, \rho, T_{min}, S_{max})$ is a $KM$-single valued neutrosoptic metric graph on $G^*$ and if only if $T_E(x, y, t) \geq T_E(xy, \rho(x, y, t) \leq I_E(xy)$ and $\rho(x, y, t) \leq F_E(xy)$.

\textbf{Proof 11:} Let $x, y \in V$. Then for some $t \in R^{\geq 0}$:

(i) Since $T(TE(xy), T(T_V(x), T_V(y))) \leq T_E(xy)$, $S(I_E(xy), S(I_V(xy), I_V(y))) \geq I_E(xy)$ and $S(F_E(xy), S(F_V(xy), F_V(y))) \geq F_E(xy)$ then $T_E \leq \rho, I_E \geq \rho$ and $F_E \geq \rho$ imply that $T(TE(xy), T(T_V(x), T_V(y))) \leq T_E(xy) \leq \rho, \gamma \leq \gamma$ and $S(I_E(xy), S(I_V(xy), I_V(y))) \geq \rho(x, y, t)$ and $S(F_E(xy), S(F_V(xy), F_V(y))) \geq \rho(x, y, t)$. So $G = (X, Y, \rho, T)$ is a $KM$-single valued neutrosoptic graph metric graph on $G^*$.

(ii) Let $G = (X, Y)$ be a single valued neutrosoptic graph on $G^*$. For all $xy \in E$, since $G = (X, Y, \rho, T_{min}, S_{max})$ is a $KM$-single valued neutrosoptic metric graph on $G^*$, using $T_E(xy) \leq T_{min}(T_E(T_V(x), T_V(y)))$, $I_E(xy) \geq S_{max}(I_V(xy), I_V(y))$ and $F_E(xy) \geq S_{max}(F_V(xy), F_V(y))$. We get that $T_E(xy) = T_{min}(T_E(T_V(x), T_V(y))) \leq \rho(x, y, t)$. $I_E(xy) = S_{max}(I_V(xy), I_V(y)) \geq \rho(x, y, t)$ and $F_E(xy) = S_{max}(F_V(xy), F_V(y))$.

(iii) $G = (X, Y)$ is a single valued neutrosoptic graph on $G^*$ if and only if for all $xy \in E$, $T_E(xy) \leq T_{min}(T_E(T_V(x), T_V(y)))$, $I_E(xy) \geq S_{max}(I_V(xy), I_V(y))$ and $F_E(xy) \geq S_{max}(F_V(xy), F_V(y))$. Then $G = (X, Y)$ is a $KM$-single valued neutrosoptic graph on $G^*$ if and only if $T_E(xy) = T_{min}(T_E(T_V(x), T_V(y)))$, $I_E(xy) = S_{max}(I_V(xy), I_V(y))$ and $F_E(xy) = S_{max}(F_V(xy), F_V(y))$.

\textbf{Corollary 4:} Let $G = (X, Y, \rho, T, S)$ be a $KM$-fuzzy metric connected graph on $G^* = (V, E)$. Then for starting time $G = (X, Y)$ is not a single valued neutrosoptic graph on $G^*$.

\textbf{Theorem 12:} Let $(V, \rho, T)$ be a $KM$-fuzzy metric space, $G^* = (V, E)$ be a simple graph and $xy \in E$. Then for $T_V, I_V, F_V : V \rightarrow [0, 1]$ and $T_E, I_E, F_E : E \rightarrow [0, 1]$,
(i) If \( T_V(x) + T_V(y) \leq 1, I_V(x) + I_V(y) = 1 \) and \( F_V(x) + F_V(y) = 1 \) then \( G = (X, Y, \rho, T_{lu}, S_{lu}) \) is a \( KM \)-single valued neutrosophic graph on \( G^* \).

(ii) If \( T_E(xy) + 1 \leq T_V(xy) + T_V(x) + T_V(y) \leq 2, I_V(x) + I_V(y) = F_V(xy) + F_V(y) = 1 \), then \( G = (X, Y, \rho, T_{lu}, S_{lu}) \) is a \( KM \)-single valued neutrosophic graph on \( G^* \).

**Proof 12:** Let \( x, y \in V \). Then for some \( t \in \mathbb{R}_{\geq 0} \):

(i) We have

\[
T_{lu}(T_E(xy), T_{lu}(T_V(x), T_V(y))) = \max \left( 0, T_E(xy) + T_{lu}(T_V(x), T_V(y)) - 1 \right) = \max \left( 0, T_E(xy) + \max(0, T_V(xy) + T_V(y) - 1) - 1 \right).
\]

If \( T_V(x) + T_V(y) \leq 1 \), then \( T_{lu}(T_E(xy), T_{lu}(T_V(x), T_V(y))) = \max(0, T_E(xy) - 1) = 0 \), since for all \( x, y \in V \) we have \( T_E(xy) \leq 1 \). It concludes that for any time \( t \in \mathbb{R}_{\geq 0} \) get that \( T_{lu}(T_E(xy), T_{lu}(T_V(x), T_V(y))) \leq \rho(x, y, t) \). In addition, \( I_V(x) + I_V(y) = 1 \), implies that

\[
S_{lu}(I_E(xy), S_{lu}(I_V(x), I_V(y))) = \min \left( 1, E(xy) + S_{lu}(I_V(x), I_V(y)) \right) = \min \left( 1, E(xy) + \max(1, I_V(x) + I_V(y)) \right) = 1 \geq \rho(x, y, t).
\]

In a similar way, one can prove that \( S_{lu}(F_E(xy), S_{lu}(F_V(x), F_V(y))) \geq \rho(x, y, t) \) and so \( G = (X, Y, \rho, T_{lu}, S_{lu}) \) is a \( KM \)-single valued neutrosophic metric graph on \( G^* \).

(ii) Because \( T_E(xy) + 1 \leq T_E(xy) + T_V(x) + T_V(y) \leq 2 \), we get that \( T_V(x) + T_V(y) \geq 1 \) and by item (i), we have \( T_{lu}(T_E(xy), T_{lu}(T_V(x), T_V(y))) = T_{lu}(0, T_E(xy) + T_V(x) + T_V(y) - 2) = 0 \). Moreover, \( I_V(x) + I_V(y) = F_V(x) + F_V(y) = 1 \), implies that

\[
S_{lu}(F_E(xy), S_{lu}(F_V(x), F_V(y))) = \min \left( 1, F_E(xy) + S_{lu}(F_V(x), F_V(y)) \right) = \min \left( 1, F_E(xy) + \max(1, F_V(x) + F_V(y)) \right) = 1 \geq \rho(x, y, t).
\]

**Definition 7:** Let \( G_1, G_2 \) be \( KM \)-single valued neutrosophic metric graphs on simple graphs \( G^*_1 \) and \( G^*_2 \), respectively. Define the categorical product (tensor product) of fuzzy subsets \( X_1 \times X_2 = (T_V^{(1)} \times T_V^{(2)}, I_V^{(1)} \times I_V^{(2)}, F_V^{(1)} \times F_V^{(2)}) \), \( Y_1 \times Y_2 = (T_V^{(1)} \times T_V^{(2)}, I_V^{(1)} \times I_V^{(2)}, F_V^{(1)} \times F_V^{(2)}) \), \( Y_1 \times Y_2 = (T_V^{(1)} \times T_V^{(2)}, I_V^{(1)} \times I_V^{(2)}, F_V^{(1)} \times F_V^{(2)}) \), where

\[
(T_V^{(1)} \times T_V^{(2)})(x_1, x_2) = \min (T_V^{(1)}(x_1), T_V^{(2)}(x_2)), (I_V^{(1)} \times I_V^{(2)})(x_1, x_2) = \max (I_V^{(1)}(x_1), I_V^{(2)}(x_2)), (F_V^{(1)} \times F_V^{(2)})(x_1, x_2) = \max (F_V^{(1)}(x_1), F_V^{(2)}(x_2)),
\]

and \( T_E^{(1)} \times T_E^{(2)}(x_1, y_1) = T_{min}(T_E^{(1)}(x_1), T_E^{(2)}(y_1)), (I_E^{(1)} \times I_E^{(2)})(x_1, y_1) = T_{max}(I_E^{(1)}(x_1), I_E^{(2)}(y_1)), (F_E^{(1)} \times F_E^{(2)})(x_1, y_1) = T_{max}(F_E^{(1)}(x_1), F_E^{(2)}(y_1)) \),

and hence \( \rho(x_1, y_1, t) \) is dense in \( \mathbb{R}_{\geq 0} \).

**Theorem 13:** Let \( G_1 \) and \( G_2 \) be \( KM \)-single valued neutrosophic metric graphs on simple graphs \( G^*_1 \) and \( G^*_2 \), respectively. Then \( G_1 \times G_2 = (X_1 \times X_2, Y_1 \times Y_2, T_{min}  \rho, T, S) \) is a \( KM \)-single valued neutrosophic metric graph on \( G^*_1 \times G^*_2 \).

**Proof 13:** Firstly, by Theorem 6, \( (V_1 \times V_2, T_{min} \rho, T, S) \) is a \( KM \)-fuzzy metric space. Let \( (x_1, x_2), (y_1, y_2) \in E(G^*_1 \times G^*_2) \).

Since \( G_1 \) is a \( KM \)-single valued neutrosophic metric graph on \( G^*_1 \) and \( G_2 \) is a \( KM \)-single valued neutrosophic metric graph on \( G^*_2 \), for some \( t_1, t_2 \in \mathbb{R}_{\geq 0} \), we get that

\[
T((T_E^{(1)} \times T_E^{(2)})(x_1, x_2)(y_1, y_2)), (T^{(1)}(x_1, y_2))(y_1, y_2), T^{(1)}(x_1, x_2)) = \min (T_{min}(T_E^{(1)}(x_1), T_E^{(2)}(x_2)), (T_{min}(T_E^{(1)}(y_1), T_E^{(2)}(y_2))), (T_{min}(T_E^{(1)}(y_1), T_E^{(2)}(y_2))) \leq \min (T_E^{(1)}(x_1, y_1), T_E^{(1)}(x_1, y_1)) = \rho(x_1, x_1, t_1, t_2),
\]

\[
T((T_E^{(1)} \times T_E^{(2)})(x_1, x_2)(y_1, y_2)), (T^{(1)}(x_1, x_2))(y_1, y_2), T^{(1)}(x_1, x_2)) = \min (T_{min}(T_E^{(1)}(x_1), T_E^{(2)}(x_2)), (T_{min}(T_E^{(1)}(y_1), T_E^{(2)}(y_2))), (T_{min}(T_E^{(1)}(y_1), T_E^{(2)}(y_2))) \leq \min (T_E^{(1)}(x_1, y_1), T_E^{(1)}(x_1, y_1)) = \rho(x_2, x_2, t_1, t_2).
\]

Consider \( t = max(t_1, t_2) \), so by Theorem 1, we obtain

\[
T((T_E^{(1)} \times T_E^{(2)})(x_1, x_2)(y_1, y_2)), (T^{(1)}(x_1, x_2))(y_1, y_2), T^{(1)}(x_1, x_2)) \leq \min (T_{min}(x_1, y_1, t_1, t_2), (x_2, y_2, t_1, t_2)) \leq T_{min}(x_1, y_1, t_1, t_2).
\]

I addition,

\[
S((I_E^{(1)} \times I_E^{(2)})(x_1, x_2)(y_1, y_2)), (I^{(1)}(x_1, x_2))(y_1, y_2), I^{(1)}(x_1, x_2)) = \min (S_{min}(I_E^{(1)}(x_1, y_1), I_E^{(2)}(y_2)), (I^{(1)}(x_1, y_1), I^{(2)}(y_2))) \geq \min (S_{min}(I^{(1)}(x_1, y_1), I^{(2)}(y_2))) \geq \rho(x_1, y_1, t_1, t_2).
\]
and \(S(I_E^1 \times I_E^2)(x_1, y_1, y_2), S(I_E^1 \times I_V^2)(x_1, y_1, y_2)\), \(S(I_E^1 \times I_V^2)(y_1, y_2)\), \(S(I_E^2 \times I_V^2)(x_1, y_1, y_2)\), \(S(I_V^1 \times I_V^2)(x_1, y_1, y_2)\), \(S(I_V^1 \times I_V^2)(x_1, y_1, y_2)\)).

Consider \(t = \min\{t_1, t_2\}\), so by Theorem 1, we obtain

\[
S((I_E^1 \times I_V^2)(x_1, y_1, y_2), S(I_V^1 \times I_V^2)(x_1, y_1, y_2)).
\]

In a similar way, we can see that \(S((F_E^1 \times F_E^2)(x_1, y_1, y_2)), S((F_V^1 \times F_V^2)(x_1, y_1, y_2)) \geq S_{\max}(\rho(x_1, y_1, t_1), t_2(x_2, y_2, t_2)) \geq S_{\max}(\rho(x_1, y_1, t_1), (x_2, y_2, t_2)).
\]

Definition 8: Let \(G_1, G_2\) be \(K\)-single valued neutrosophic metric graphs on simple graphs \(G_1^*\) and \(G_2^*\) respectively. Define the Cartesian product (or product) fuzzy subsets

\[
X_1 \times X_2 = (T_{E_1} \times T_{E_2}) \times (T_{V_1} \times T_{V_2}), (F_{E_1} \times F_{E_2}) \times (F_{V_1} \times F_{V_2}), Y_1 \times Y_2 = (T_{E_1} \times T_{E_2}(x_1, x_2), (F_{E_1} \times F_{E_2}(x_1, x_2), (F_{V_1} \times F_{V_2}(x_1, x_2), (F_{V_1} \times F_{V_2}(x_1, x_2).)
\]

Thus \(G_1 \times G_2 = (X_1 \times X_2, Y_1 \times Y_2, \min(\rho, T, S)\) is a \(K\)-single valued neutrosophic metric graph on \(G_1^* \times G_2^*\).

Example 2: Consider the \(K\)-fuzzy metric spaces

\[
(V_1 = \{1, 2\}, \rho_1, \min), (V_2 = \{3, 4, 5\}, \rho_2, \min, \rho_1(1, 1, t > 0) = 1, \rho_1(2, 2, t > 0) = 1, \rho_1(1, 2, t > 0) = 1 + t, \rho_1(x, y, 0) = 0, x, y \in V_1\) and for all \(x, y \in \{3, 4, 5\}, \rho_2(x, y, 0) = \frac{\min[x, y] + t}{\max[x, y] + t} \quad \text{if} \ t > 0, \rho_2(x, y, 0) = 0 \quad \text{if} \ t = 0.
\]

We take the negation \(\eta(m) = 1 - m(n \in [0, 1])\) and obtain the \(K\)-single valued neutrosophic metric graphs

\[
G_1 = (V_1, (X = (T_{E_1} \times I_{V_1} \times F_{V_1})), Y = (T_{E_1} \times I_{V_1} \times F_{V_1})), \rho_1, \min, \max)\] in unit time \(t_1 = 1\) and \(G_2 = (V_2, (X = (T_{E_2} \times I_{V_2} \times F_{V_2})), Y = (T_{E_2} \times I_{V_2} \times F_{V_2})), \rho_2, \min, \max)\] in unit time \(t_2 = 1\) on \(G_1^*\) and \(G_2^*\) in Figure 2, where \(A = (0.6, 0.4, 0.2), B = (0.5, 0.1, 0.3), C = (0.3, 0.5, 0.7), D = (0.5, 0.6, 0.2), E = (0.1, 0.2, 0.5), AB = (0.5, 0.97, 0.95), CE = (0.1, 0.96, 0.91), ED = (0.5, 0.98, 0.99)\) and \(DC = (0.3, 0.93, 0.96)\).

Now, we obtain the \(K\)-fuzzy metric graph \(G_1 \times G_2\) in Figure 3, where \(a = (0.1, 0.4, 0.5), b = (0.1, 0.4, 0.5), c = (0.5, 0.6, 0.2), d = (0.5, 0.6, 0.3), e = (0.3, 0.5, 0.7), f = (0.3, 0.5, 0.7), ab = (0.1, 0.97, 0.95), dc = (0.5, 0.97, 0.95), ef = (0.3, 0.97, 0.95), bd = (0.5, 0.98, 0.99), df = (0.3, 0.93, 0.96), ac = (0.5, 0.98, 0.99), ce = (0.3, 0.93, 0.96), bf = (0.1, 0.96, 0.961)\) and \(ae = (0.1, 0.96, 0.91)\).

Theorem 14: Let \(G_1\) and \(G_2\) be \(K\)-single valued neutrosophic metric graphs on simple graphs \(G_1^*\) and \(G_2^*\), respectively.

(i) If \((G_1^* \times G_2^*)\), then \(T(\rho, T)\) is a \(K\)-fuzzy metric space, then \(T(\rho, T) \geq T(\rho, T)\), where \(T \in C_T\).

(ii) \(G_1 \times G_2 = (X_1 \times X_2, Y_1 \times Y_2, T(\min(\rho, T, S), T, S)\) is a \(K\)-single valued neutrosophic metric graph on \(G_1^* \times G_2^*\).

Proof 14: (i) Let \((x_1, x_2), (y_1, y_2) \in E(G_1^* \times G_2^*)\). Then \(x_1 = y_1\) and \(x_2 \neq y_2\) and \(y_1 \neq y_2\) in \(E_1^*\). Then \(x_1 = y_1\) and \(x_2 \neq y_2\) in \(E_1^*\), then \(T(\rho)(x_1, x_2), (y_1, y_2), (T(\rho)(x_1, x_2), (y_1, y_2), t) = T(\rho)(x_1, y_1, t, \rho_2(x_2, y_2, t)) = T(\rho)(x_2, y_2, t), (T(\rho)(x_1, y_1, t, \rho_2(x_2, y_2, t)) = T(\rho)(x_1, y_1, t) = (\rho_1(x_1, y_1, t), 1) = \rho_1(x_1, y_1, t).

(ii) Firstly, by Theorem 6, \((V_1 \times V_2, T(\min(\rho, T), T)\) is a \(K\)-fuzzy metric space. Let \((x_1, x_2), (y_1, y_2) \in E(G_1^* \times G_2^*)\). Since \(G_1\) is a \(K\)-single valued neutrosophic metric graph on \(G_1^*\) and \(G_2\) is a \(K\)-single valued neutrosophic metric graph on \(G_2^*\), for some \(t_1, t_2 \in \mathbb{R}^{\geq 0}\), give \(t = \max(1, t_2)\), so by item (i) and Theorem 1, we get that

\[
T((T_{E_1} \times T_{E_2})(x_1, x_2), (x_1, x_2), (T_{E_1} \times T_{E_2})(x_1, x_2), (T_{E_1} \times T_{E_2})(x_1, x_2), (T_{E_1} \times T_{E_2})(x_1, x_2), (T_{E_1} \times T_{E_2})(x_1, x_2), (T_{E_1} \times T_{E_2})(x_1, x_2).
\]
Now, give \( t = \min\{t_1, t_2\} \), so by item (i) and Theorem 1, we get that

\[
S((I_E^1 \otimes I_E^2)((x, x_2), (x, x_2))) = S(S_{\max}(I_V^1(x), I_V^2(x)), S(S_{\max}(I_V^1(x), I_V^2(x))) \\
\leq S(I_E^1(x), I_E^1(x), I_E^2(y), I_E^2(y))) \leq \rho_2(x_2, y_2, t_2) \\
\geq S_{\max}(\rho((x_2, x_2)(y), t_2) and S((I_E^1 \otimes I_E^2)((x_1, y)(y_1, y), y), (I_E^1 \otimes I_E^2)((x_1, y)(y_1, y))) \\
= S(S_{\max}(I_V^1(y), I_V^2(y)), S_{\max}(I_V^1(y), I_V^2(y))) \\
\geq S(I_E^1(x_1), I_E^1(x_1), I_E^2(y_1), I_E^2(y_1))) \geq \rho_1(x_1, y_1, t_1) \\
\geq S_{\max}(\rho((x, y)(y), y)).
\]

In a similar way, we can see that \( S((F_E^1 \otimes F_E^2)((x_1, x_2)(y_1, y_2)) \), \( S((F_V^1 \otimes F_V^2)((x_1, x_2), (y_1, y_2))) \), \( S((I_E^1 \otimes I_E^2)((x_1, y)(y_1, y), y), (I_E^1 \otimes I_E^2)((x_1, y)(y_1, y))) \), where \( T_E^1 \), \( T_E^2 \), \( T_V^1 \), \( T_V^2 \), \( T_{\max}((x_1, x_2)(y_1, y_2), y), (T_{\max}((x_1, x_2)(y_1, y_2), y))) \), respectively. Thus \( G_1 \otimes G_2 = (X_1 \otimes X_2, Y_1 \otimes Y_2, T_{\min}(\rho), T, S) \) is a KM-single valued neutrosophic metric graph on \( G_1^* \otimes G_2^* \).

**Definition 9:** Let \( G_1, G_2 \) be KM-single valued neutrosophic metric graphs on simple graphs \( G_1^* \) and \( G_2^* \), respectively.

**Definition 10:** Let \( G_1, G_2 \) be KM-single valued neutrosophic metric graphs on simple graphs \( G_1^* \) and \( G_2^* \), respectively. Define the strong product of fuzzy subsets \( X_1 \otimes X_2 = (T_E^1 \otimes T_E^2, I_E^1 \otimes I_E^2, F_E^1 \otimes F_E^2, I_V^1 \otimes I_V^2, F_V^1 \otimes F_V^2, V(G_1^* \times G_2^*), T_{\max}(\rho, T, S) \) is a KM-single valued neutrosophic metric graph on \( G_1^* \otimes G_2^* \).

**Example 3:** Consider the KM-single valued neutrosophic metric graphs \( G_1 \) and \( G_2 \) in Example 2. So we obtain the KM-fuzzy metric graph \( G_1 \cdot G_2 \) in Figure 4, where \( a = (0.1, 0.4, 0.5), b = (0.1, 0.4, 0.5), c = (0.5, 0.6, 0.2), d = (0.5, 0.6, 0.3), e = (0.3, 0.5, 0.7), f = (0.3, 0.5, 0.7), be = (0.1, 0.97, 0.95), bc = (0.5, 0.98, 0.99), af = (0.1, 0.97, 0.95), bd = (0.5, 0.98, 0.99), df = (0.3, 0.93, 0.96), ac = (0.5, 0.98, 0.99), ce = (0.3, 0.93, 0.96), bf = (0.1, 0.96, 0.961), cf = (0.3, 0.97, 0.96), de = (0.3, 0.97, 0.96), ad = (0.5, 0.98, 0.99) and ae = (0.1, 0.96, 0.91).

**Example 4:** Consider the KM-fuzzy metric spaces \( V_1 = [1, 2], \rho_1, T_{\min} \), \( V_2 = [3, 4, 5], \rho_2, T_{\min} \), where for all
Theorem 16: Let $G_1$ and $G_2$ be $KM$-single valued neutrosophic metric graphs on simple graphs $G_1^*$ and $G_2^*$, respectively. Then $G_1 \sqcup G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, T_{\min}(\rho), T, S)$ is a $KM$-single valued neutrosophic metric graph on $G_1^* \circ G_2^*$.

Proof 16: It is similar to Theorems 13 and 14.

Definition 11: Let $G_1, G_2$ be $KM$-single valued neutrosophic metric graphs on simple graphs $G_1^*$ and $G_2^*$, respectively. Define the union of fuzzy subsets $X_1 \cup X_2 = (T_{V_1}^{(1)} \cup T_{V_1}^{(2)}, I_{V_1}^{(1)} \cup I_{V_1}^{(2)}, F_{V_1}^{(1)} \cup F_{V_1}^{(2)}), Y_1 \cup Y_2 = (T_{V_2}^{(1)} \cup T_{V_2}^{(2)}, I_{V_2}^{(1)} \cup I_{V_2}^{(2)}, F_{V_2}^{(1)} \cup F_{V_2}^{(2)})$.

FIGURE 5. $KM$-single valued neutrosophic metric graphs $G_1, G_2$ for $t_1 = 2, t_2 = 1$.

$KM$-single valued neutrosophic metric graph $G_1 \circ G_2$ in Figure 6, where $a = (0.1, 0.5, 0.5), b = (0.1, 0.6, 0.5), c = (0.1, 0.5, 0.4), d = (0.2, 0.4, 0.5), e = (0.2, 0.6, 0.5), f = (0.2, 0.3, 0.3), ab = (0.1, 0.93, 0.96), bc = (0.1, 0.98, 0.99), de = (0.2, 0.93, 0.96), ef = (0.2, 0.98, 0.99), ad = (0.3, 0.97, 0.95), be = (0.4, 0.97, 0.95), cf = (0.5, 0.97, 0.95), ac = (0.3, 0.97, 0.96), bf = (0.5, 0.98, 0.99), bd = (0.3, 0.97, 0.96), ce = (0.5, 0.98, 0.99).

FIGURE 6. $KM$-single valued neutrosophic $G_1 \circ G_2$ for $t = 2$.

Example 5: Consider the $KM$-single valued neutrosophic metric graphs $G_1$ and $G_2$ in Example 4. It is easy to see that $KM$-single valued neutrosophic metric graph $G_1 \circ G_2$ with $t = 2$ in Figure 4.

Theorem 17: Let $G_1 = (X_1, Y_1, \rho_1, T, S), G_2 = (X_2, Y_2, \rho_2, T, S)$ be $KM$-single valued neutrosophic metric graphs on simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. If $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ are two simple graphs, where $V_1 \cup V_2 = \emptyset$, then $G_1 \cup G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, \rho_1 \cup \rho_2, T, S)$ is a $KM$-single valued neutrosophic metric graph on $G_1^* \cup G_2^*$.

Proof 17: Firstly, by Theorem 7, $(V_1 \cup V_2, T_{\min}(\rho), T)$ is a $KM$-fuzzy metric space. Let $xy \in E(G_1^* \cup G_2^*), T$ is a $KM$-fuzzy metric. Let $xy \in E(G_1^* \cup G_2^*), T$ is a $KM$-fuzzy metric space. Let $xy \in E(G_1^* \cup G_2^*), T$ is a $KM$-fuzzy metric space. Let $xy \in E(G_1^* \cup G_2^*), T$ is a $KM$-fuzzy metric space. Let $xy \in E(G_1^* \cup G_2^*), T$ is a $KM$-fuzzy metric space.
take \( t = \max\{t_1, t_2\} \) so by Theorems 1 and 7, if \( xy \in E_1 \setminus E_2 \), we have
\[
T((T_x^{(1)}) \cup T_y^{(1)})(x), T_y^{(1)}(x), T_y^{(1)}(y)) = T(T_x^{(1)}(x), T_y^{(1)}(x)) \leq \rho_1(x, y, t) \leq \rho_1(x, y, t).
\]

In a similar way, if \( xy \in E_2 \setminus E_1 \), one can see that \( T_{\min}(T_x^{(1)} \cup T_y^{(1)}, (x), T_y^{(1)}(x), T_y^{(1)}(y)) \leq \rho_2(x, y, t) = (\rho_1 \cup \rho_2)(x, y, t) \). Other cases are similar to the above.

Now consider \( t = \min\{t_1, t_2\} \) so by Theorems 1 and 7, if \( xy \in E_1 \setminus E_2 \), we have
\[
\begin{align*}
S((T_x^{(1)}) \cup T_y^{(1)})(x), (T_y^{(1)})(x), (T_y^{(1)})(y)) &= S(T_x^{(1)}(x), T_y^{(1)}(x)) \\
&\geq \rho_1(x, y, t) \geq (\rho_1 \cup \rho_2)(x, y, t).
\end{align*}
\]

In a similar way, if \( xy \in E_2 \setminus E_1 \), one can see that \( S((T_x^{(1)}) \cup T_y^{(1)})(x), (T_y^{(1)})(x), (T_y^{(1)})(y)) \geq \rho_2(x, y, t) = (\rho_1 \cup \rho_2)(x, y, t) \). Other cases are similar to the above and in a similar way, we can prove that \( S((T_x^{(1)}) \cup T_y^{(1)})(x), (T_y^{(1)})(x), (T_y^{(1)})(y)) \geq (\rho_1 \cup \rho_2)(x, y, t) \). Thus \( G_1 \cup G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, \rho_1 \cup \rho_2, T, S) \) is a KM-single valued neutrosophic metric graph on \( G_1^* \cup G_2^* \).

**Definition 12:** Let \( G_1, G_2 \) be KM-single valued neutrosophic metric graphs on simple graphs \( G_1^* \) and \( G_2^* \), respectively. Define the semi-ring sum of fuzzy subsets \( X \subseteq Y \) as \( (T_x^{(1)} \cup T_y^{(1)})(x), (T_y^{(1)})(x), (T_y^{(1)})(y)) \). By \( T((T_x^{(1)} \cup T_y^{(1)})(x), (T_y^{(1)})(x), (T_y^{(1)})(y)) \geq (\rho_1 \cup \rho_2)(x, y, t) \) and other cases are similar to **Theorem 17**. Since \( xy \not\in E \), we see that \( G_1 \cup G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, \rho_1 \cup \rho_2, T, S) \) is a KM-single valued neutrosophic metric graph on \( G_1^* \cup G_2^* \).

**Theorem 18:** Let \( G_1 \) and \( G_2 \) be KM-single valued neutrosophic metric graphs on simple graphs \( G_1^* \) and \( G_2^* \), respectively. If \( G_1^* \) and \( G_2^* \) are two simple graphs, where \( V_1 \cap V_2 = \emptyset \), then \( G_1 \cup G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, \rho_1 \cup \rho_2, T, S) \) is a KM-single valued neutrosophic metric graph on \( G_1^* \cup G_2^* \).

**Proof 18:** It is similar to **Theorem 17**.

**Definition 13:** Let \( G_1, G_2 \) be KM-single valued neutrosophic metric graphs on simple graphs \( G_1^* \) and \( G_2^* \), respectively. Define the join(OR suspension) of fuzzy subsets
\[
(1) \quad (T_x^{(1)} \cup T_y^{(1)})(x) = \begin{cases} T_x^{(1)}(x) & \text{if } xy \in E_1 \setminus E_2 \\ T_y^{(1)}(x) & \text{if } xy \in E_2 \setminus E_1 \\ 0 & \text{if } xy \in E_2 \cap E_1. \end{cases}
\]
\[
(2) \quad (T_x^{(1)} \cup T_y^{(1)})(x) = \begin{cases} I_x^{(1)}(x) & \text{if } xy \in E_1 \setminus E_2 \\ I_y^{(1)}(x) & \text{if } xy \in E_2 \setminus E_1 \\ 1 & \text{if } xy \in E_2 \cap E_1. \end{cases}
\]
\[
(3) \quad (T_x^{(1)} \cup T_y^{(1)})(x) = \begin{cases} F_x^{(1)}(x) & \text{if } xy \in E_1 \setminus E_2 \\ F_y^{(1)}(x) & \text{if } xy \in E_2 \setminus E_1 \\ 1 & \text{if } xy \in E_2 \cap E_1. \end{cases}
\]
Theorem 20: Let \((V, \rho, T)\) be a \(KM\)-fuzzy metric space and \(G^* = (V, E)\) be a simple graph. If \(G = (X, Y, \rho, T, S)\) is a \(KM\)-single valued neutrosophic metric graph on \(G^*\), then \(\overline{G} = (\overline{X}, \overline{Y}, \rho, T, S)\) is a \(KM\)-single valued neutrosophic metric graph.

Proof 20: Let \(x, y \in V\). Since \(G\) is a \(KM\)-single valued neutrosophic metric graph on \(G^*\), for some \(t \in \mathbb{R}^+\),
\[
T(T_E(xy), T(T_V(x), T_V(y))) = T(\rho(x, y, t) - T(T_E(xy), T(T_V(x), T_V(y))), T(T_V(x), T_V(y))) \leq \rho(x, y, t).
\]
In addition,
\[
S(T_E(xy), S(T_V(x), T_V(y))) = S(S(I_E(xy), S(I_V(x), I_V(y))), S(I_V(x), I_V(y))) \leq \rho(x, y, t).
\]
In a similar way, it is easy to see that \(S(\overline{T}_E(xy), S(\overline{T}_V(x), \overline{T}_V(y))) \geq \rho(x, y, t)\). If follows that \(\overline{G} = (\overline{X}, \overline{Y}, \rho, T, S)\) is a \(KM\)-single valued neutrosophic metric graph.

Example 6: Consider the \(KM\)-single valued neutrosophic-metric graph \(G\) in Example 1. So obtain a \(KM\)-single valued neutrosophic metric graph \(\overline{G}\) on the cycle graph \(C_4\) for \(t = 1\), in Figure 7.

V. CONCLUSION

The current paper has introduced a novel concept fuzzy algebra as \(KM\)-single valued neutrosophic metric graph and a new generalization of graphs based on \(KM\)-fuzzy metric spaces. This work extended and obtained some properties in \(KM\)-fuzzy metric spaces. Also it showed that every non empty set converted to a \(KM\)-fuzzy metric space, the product and union of \(KM\)-fuzzy metric spaces is a \(KM\)-fuzzy metric space, the extended \(KM\)-fuzzy metric spaces are constructed using the some algebraic operations on \(KM\)-fuzzy metric spaces, the concept of complement of \(KM\)-single valued neutrosophic metric graph is defined and investigated some its properties. We hope that these results are helpful for further studies in theory of graphs. In our future studies, we hope to obtain more results regarding intuitionistic metric graphs, neutrosophic metric graphs, \(KM\)-single valued neutrosophic metric hypergraphs, bipolar \(KM\)-single valued neutrosophic metric graphs, automorphism \(KM\)-single valued neutrosophic metric graphs and their applications.

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