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R. Narmada Devi

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A Novel of neutrosophic τ-Structure Ring $ExtB$ and $ExtV$ Spaces

R. Narmada Devi
Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R & D Institute of Science and Technology, Chennai, India.
E-mail: narmadadevi23@gmail.com

Abstract: In this paper, the concepts of a neutrosophic τ-structure ring spaces, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces are introduced. Some interesting functions that preserve neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces in the context of image and preimage are derived with the necessary examples.

Keywords: neutrosophic τ-structure ring space, neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, neutrosophic τ-structure ring $ExtB$ space and neutrosophic τ-structure ring $ExtV$ space.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [16]. consequent to the introduction of fuzzy sets, fuzzy logic has been applied in many real life situations to handle uncertainty. Chang [7] introduced the concept of fuzzy topological spaces. There are several kinds of fuzzy set extensions such as intuitionistic fuzzy set, interval-valued fuzzy sets, etc. After the introduction of intuitionistic fuzzy sets and its topological spaces by Atanassov [6] and Coker [8], the concept of imprecise data called neutrosophic sets was introduced by Smarandache [9]. The concept of neutrosophic topological space was introduced by Salama [15]. Later R.Narmada Devi [10,11,12,13,14] introduced the concepts of intuitionistic fuzzy G_δ sets, intuitionistic fuzzy exterior spaces and neutrosophic complex topological spaces. Moreover, the neutrosophic theory plays a viral role in all fields of branches like medial diagnosis [1,2,5], multiple criteria group decision making [3,4], etc. In this paper, the concepts of neutrosophic τ-structure ring spaces, neutrosophic G_δ rings, neutrosophic first category rings, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces are introduced. Further, neutrosophic τ-structure ring continuous (resp. open, hardly open) functions and somewhat neutrosophic τ-structure ring continuous functions are presented. Some interesting properties among of functions along with the spaces are discussed and necessary examples are provided.

2 Preliminiaries

We need the following basic definitions for our study.

Definition 2.1. [9] Let X be a nonempty set. A neutrosophic set A in X is defined as an object of the form $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$ such that $T_A, I_A, F_A : X \to [0,1]$. and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2. [9] Let $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$ and $B = \langle x, T_B(x), I_B(x), F_B(x)\rangle$ be any two neutrosophic sets in X. Then

(i) $A \cup B = \langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle$ where $T_{A\cup B}(x) = T_A(x) \vee T_B(x), I_{A\cup B}(x) = I_A(x) \vee I_B(x)$ and $F_{A\cup B}(x) = F_A(x) \wedge F_B(x)$.

(ii) $A \cap B = \langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle$ where $T_{A\cap B}(x) = T_A(x) \wedge T_B(x), I_{A\cap B}(x) = I_A(x) \wedge I_B(x)$ and $F_{A\cap B}(x) = F_A(x) \vee F_B(x)$.

(iii) $A \subseteq B$ if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x),$ for all $x \in X$.

(iv) the complement of A is defined as $C(A) = \langle x, T_{C(A)}(x), I_{C(A)}(x), F_{C(A)}(x)\rangle$ where $T_{C(A)}(x) = 1 - T_A(x), I_{C(A)}(x) = 1 - I_A(x)$ and$F_{C(A)}(x) = 1 - F_A(x)$.

(v) $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

Definition 2.3. [10,11] Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space (X, T). Then A is said be an intuitionistic fuzzy G_δ set if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, \mu_{A_i}, \gamma_{A_i}\rangle$ is an intuitionistic fuzzy open set in an intuitionistic fuzzy topological space (X, T). The complement of an intuitionistic fuzzy G_δ set is said to be an intuitionistic fuzzy F_σ set.

Definition 2.4. [12,13] Let $A = \langle \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space (X, τ). An intuitionistic fuzzy exterior of A is defines as follows: if $IFExt(A) = IFint(\overline{A})$

Definition 2.5. [12,13] Let R be a ring. An intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A\rangle$ in R is called an intuitionistic fuzzy ring on R if it satisfies the following conditions on the membership and nonmembership values:

(i) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$,

(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$,

(iii) $\gamma_A(x + y) \leq \gamma_A(x) \vee \gamma_A(y)$,

(iv) $\gamma_A(xy) \leq \mu_A(x) \vee \gamma_A(y)$,

for all $x, y \in R$.

3 Properties of neutrosophic τ-Structure Ring Exterior B Spaces

Definition 3.1. Let R be a ring. A neutrosophic set $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$ in R is called a neutrosophic ring on R if it satisfies the following conditions:

(i) $T_A(x + y) \geq T_A(x) \wedge T_A(y)$ and $T_A(xy) \geq T_A(x) \wedge T_A(y)$

(ii) $I_A(x + y) \geq I_A(x) \wedge I_A(y)$ and $I_A(xy) \geq I_A(x) \wedge I_A(y)$

(iii) $F_A(x+y) \leq F_A(x) \vee F_A(y)$ and $F_A(xy) \leq F_A(x) \vee F_A(y)$, for all $x, y \in R$.

Definition 3.2. Let R be a ring. A family $\mathscr{S}$ of a neutrosophic rings in R is said to be neutrosophic τ-structure ring on R if it satisfies the following conditions:

(i) $0_N, 1_N \in \mathscr{S}$.

(ii) $G_1 \cap G_2 \in \mathscr{S}$ for any $G_1, G_2 \in \mathscr{S}$.

(iii) $\cup G_i \in \mathscr{S}$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \mathscr{S}$.

The ordered pair $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring space. Every member of $\mathscr{S}$ is called a neutrosophic τ-open ring in $(R, \mathscr{S})$. The complement $C(A)$ of a neutrosophic τ-open ring A is a neutrosophic τ-closed ring in $(R, \mathscr{S})$.

Example 3.1. Let $R = \{0, 1\}$ be a set of integers module 2 with two binary operations '+' and '.' are specified by the following tables:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings B and D on R as follows: $T_B(0) = 0.5, T_B(1) = 0.7, I_B(0) = 0.5, I_B(1) = 0.7, F_B(0) = 0.3, F_B(1) = 0.2, T_D(0) = 0.3, T_D(1) = 0.4, I_D(0) = 0.3, I_D(1) = 0.4, F_D(0) = 0.5, F_D(1) = 0.6$. Then $\mathscr{S} = \{0_N, B, D, 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ- structure ring space.

Notation 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Then $NO(R)$ (resp. $NC(R)$) denotes the family of all neutrosophic τ-open(resp. closed) rings of $(R, \mathscr{S})$.

Definition 3.3. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be a neutrosophic ring in R. Then the neutrosophic ring interior and neutrosophic ring closure A are defined and denoted as $NF_Rint(A) = \cup\{B \mid B \in NO(R)$ and $B \subseteq A\}$ and $NF_Rcl(A) = \cap\{B \mid B \in NC(R)$ and $A \subseteq B$ respectively.

Remark 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then the following statements hold:

(i) $NF_Rcl(A) = A$ if and only if A is a neutrosophic τ-closed ring.

(ii) $NF_Rint(A) = A$ if and only if A is a neutrosophic τ-open ring.

(iii) $NF_Rint(A) \subseteq A \subseteq NF_Rcl(A)$.

(iv) $NF_Rint(1_N) = 1_N$ and $NF_Rint(0_N) = 0_N$.

(v) $NF_Rcl(1_N) = 1_N$ and $NF_Rcl(0_N) = 0_N$.

(vi) $NF_Rcl(C(A)) = C(NF_Rint(A))$ and $NF_Rint(C(A)) = C(NF_Rcl(A))$.

(vii) $\cup_{i=1}^{\infty} NF_Rcl(A_i) \subseteq NF_Rcl(\cup_{i=1}^{\infty} A_i)$.

(viii) $\cap_{i=1}^{n} NF_Rcl(A_i) = NF_Rcl(\cup_{i=1}^{n} A_i)$.

(ix) $\cap_{i=1}^{\infty} NF_R cl(A_i) \subseteq NF_R cl(\cup_{i=1}^{\infty} A_i)$.

(x) $\cup_{i=1}^{\infty} NF_R int(A_i) \subseteq NF_R int(\cup_{i=1}^{\infty} A_i)$.

Definition 3.4. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be a neutrosophic ring in R. Then $NF_R int(C(A))$ is called a neutrosophic ring exterior of A and is denoted by $NF_R Ext(A)$.

Proposition 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A and B be any two neutrosophic rings in R. Then the following statements hold:

(i) $NF_R Ext(A) \subseteq C(A)$.

(ii) $NF_R Ext(A) = C(NF_R cl(A))$.

(iii) $NF_R Ext(NF_R Ext(A)) = NF_R int(NF_R cl(A))$.

(iv) If $A \subseteq B$ then $NF_R Ext(A) \supseteq NF_R Ext(B)$.

(v) $NF_R Ext(1_N) = 0_N$ and $NF_R Ext(0_N) = 1_N$.

(vi) $NF_R Ext(A \cup B) = NF_R Ext(A) \cap NF_R Ext(B)$.

Definition 3.5. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said be to a neutrosophic G_δ ring in $(R, \mathscr{S})$ if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, T_{A_i}, I_{A_i}, F_{A_i} \rangle$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. The complement of a neutrosophic G_δ ring is a neutrosophic F_σ ring in $(R, \mathscr{S})$.

Definition 3.6. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said be to a

(i) neutrosophic dense ring if there exists no neutrosophic τ-closed ring B in $(R, \mathscr{S})$ such that $A \subset B \subset 1_N$.

(ii) neutrosophic nowhere dense ring if there exists no neutrosophic τ-open ring B in $(R, \mathscr{S})$ such that $B \subset NF_R cl(A)$. That is, $NF_R int(NF_R cl(A)) = 0_N$.

Definition 3.7. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be any neutrosophic fuzzy ring in R. Then A is said be to a neutrosophic first category ring in $(R, \mathscr{S})$ if $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. The complement of a neutrosophic first category ring is a neutrosophic residual ring in $(R, \mathscr{S})$.

Proposition 3.2. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. If A is a neutrosophic G_δ ring and the neutrosophic ring exterior of $C(A)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$.

Proof:

A being a neutrosophic G_δ ring in $(R, \mathscr{S})$, $A = \cap_{i=1}^{\infty} A_i$ where A_i's are neutrosophic τ-open rings. Since the neutrosophic ring exterior of $C(A)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, $NF_R cl(NF_R Ext(C(A))) = 1_N$. Because $NF_R Ext(C(A)) \subseteq A \subseteq NF_R cl(A)$, one has $NF_R Ext(C(A)) \subseteq NF_R cl(A)$.
This implies that $NF_R cl(NF_R Ext(C(A))) \subseteq NF_R cl(A)$, that is, $1_N \subseteq NF_R cl(A)$. Therefore, $NF_R cl(A) = 1_N$. That is, $NF_R cl(A) = NF_R cl(\cap_{i=1}^{\infty} A_i) = 1_N$. However, $IF_R cl(\cap_{i=1}^{\infty} A_i) \subseteq \cap_{i=1}^{\infty} NF_R cl(A_i)$. Hence,

$1_N \subseteq \cap_{i=1}^{\infty} NF_R cl(A_i)$. That is, $\cap_{i=1}^{\infty} NF_R cl(A_i) = 1_N$. This implies that $NF_R cl(A_i) = 1_N$, for each $A_i \in \mathscr{S}$. Hence $NF_R cl(NF_R int(A_i)) = 1_N$. Now, $NF_R int(NF_R cl(C(A_i))) = NF_R int(C(NF_R int(A_i))) = C(NF_R cl(NF_R int(A_i))) = 0_N$. Therefore, $C(A_i)$ is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$. Now, $C(A) = C(\cap_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} C(A_i)$. Hence, $C(A) = \cup_{i=1}^{\infty} C(A_i)$ where $C(A_i)$'s are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Consequently, $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$.

Proposition 3.3. If A is a neutrosophic first category ring in a neutrosophic τ-structure ring space $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then A is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$.

Proof:

Let A be a neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now $C(NF_R cl(A_i))$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} C(NF_R cl(A_i))$. Then B is non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Now, $B = \cap_{i=1}^{\infty} C(NF_R cl(A_i)) = C(\cup_{i=1}^{\infty} NF_R cl(A_i)) \subseteq C(\cup_{i=1}^{\infty} A_i) = C(A)$. Hence $B \subseteq C(A)$. Then $A \subseteq C(B)$. Now,

$$\begin{aligned} NF_R int(NF_R cl((A)) &\subseteq NF_R int(NF_R cl((C(B))) \\ &= NF_R int(C(NF_R int(B))) \\ &= C(NF_R cl(NF_R int(B))) \\ &= C(NF_R cl(NF_R Ext(C(B))) \end{aligned}$$

Since $NF_R Ext(C(B))$ is a neutrosophic dense ring in $(R, \mathscr{S})$, $NF_R cl(Ext(C(B)))$ $= 1_N$. Therefore, $NF_R int(NF_R cl(A)) \subseteq 0_N$. Then, $NF_R int(NF_R cl(A)) = 0_N$. Hence A is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$.

Definition 3.8. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said to be a neutrosophic τ-regular closed ring in $(R, \mathscr{S})$ if $NF_R cl(NF_R int(A)) = A$. The complement of a neutrosophic τ-regular closed ring in $(R.\mathscr{S})$ is a neutrosophic τ-regular open ring in $(R.\mathscr{S})$.

Remark 3.2. Every neutrosophic τ-regular closed ring is a neutrosophic τ-closed ring.

Definition 3.9. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space if every non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$.

Proposition 3.4. If the neutrosophic τ-structure ring space $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space and if A is a neutrosophic first category ring in $(R, \mathscr{S})$, then A is not a neutrosophic dense ring in $(R, \mathscr{S})$.

Proof:

Assume the contrary. Suppose that A is a neutrosophic first category ring in $(R, \mathscr{S})$ such that A is a neutrosophic dense ring in $(R, \mathscr{S})$, that is, $NF_R cl(A) = 1_N$. Then, $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now, $C(NF_R cl(A_i))$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} C(NF_R cl(A_i))$. Then, B is non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Now, $B = \cap_{i=1}^{\infty} C(NF_R cl(A_i)) = C(\cup_{i=1}^{\infty} NF_R cl(A_i)) \subseteq C(\cup_{i=1}^{\infty} A_i) = C(A)$. Hence $B \subseteq C(A)$. Then, $NF_R int(B) \subseteq NF_R int(C(A)) \subseteq C(NF_R cl(A)) = 0_N$. That is, $NF_R int(B) = 0_N$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, $B = NF_R int(B)$, which implies that $B = 0_N$. This is a contradiction. Hence A is not a neutrosophic dense ring in $(R, \mathscr{S})$.

Proposition 3.5. If $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, then $NF_R Ext(\cup_{i=1}^{\infty} C(A_i)) = \cap_{i=1}^{\infty} A_i$.
Proof:
Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space. Assume that A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$. Then, the A_i's are neutrosophic τ-closed rings in $(R, \mathscr{S})$, which implies that $C(A_i)$'s are neutrosophic τ-open rings in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} A_i$. Then B is a non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Since $(R, \mathscr{S})$ is a neutrosophic τ-ring $G_\delta T_{1/2}$ space, $B = NF_R int(B)$ is a neutrosophic τ-open ring, which implies that $NF_R int(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} A_i$. Now, $NF_R Ext(\cup_{i=1}^{\infty} C(A_i)) = NF_R int(C(\cup_{i=1}^{\infty} C(A_i))) = NF_R int(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} A_i$. Hence the proof.

Definition 3.10. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring exterior B (in short, $ExtB$) space if $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$.

Example 3.2. Let $R = \{0, 1\}$ be a set of integers of module 2 with two binary operations provided by the following tables:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B, M, D, E, F and G on R as follows: $T_A(0) = 0.5, T_A(1) = 0.7, I_A(0) = 0.5, I_A(1) = 0.7, F_A(0) = 0.3, F_A(1) = 0.3, T_B(0) = 0.5, T_B(1) = 0.7, I_B(0) = 0.5, I_B(1) = 0.7, F_B(0) = 0.3, F_B(1) = 0.2, T_M(0) = 0.3, T_M(1) = 0.4, I_M(0) = 0.3, I_M(1) = 0.4, F_M(0) = 0.5, F_M(1) = 0.6, T_D(0) = 0.4, T_D(1) = 0.5, I_D(0) = 0.4, I_D(1) = 0.5, F_D(0) = 0.3, F_D(1) = 0.5, T_E(0) = 0.3, T_E(1) = 0.2, I_E(0) = 0.3, I_E(1) = 0.2, F_E(0) = 0.5, F_E(1) = 0.7, T_F(0) = 0.3, T_F(1) = 0.2, I_F(0) = 0.3, I_F(1) = 0.2, F_F(0) = 0.5, F_F(1) = 0.8, T_G(0) = 0.3, T_G(1) = 0.2, I_G(0) = 0.3, I_G(1) = 0.2, F_G(0) = 0.6, F_G(1) = 0.7, T_H(0) = 0.3, T_H(1) = 0.2, I_H(0) = 0.3, I_H(1) = 0.2, F_H(0) = 0.6, F_H(1) = 0.8$. Then $\mathscr{S} = \{0_N, A, B, M, D, 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ-structure ring space. Let $\{E, F, G, H\}$ be neutrosophic nowhere dense rings in $(R, \mathscr{S})$.

Then $NF_R Ext(\cap\{C(E), C(F), C(G), C(H)\}) = NF_R Ext(C(E)) = NF_R int(E) = 0_N$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.6. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then the following statements are equivalent:

(i) $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

(ii) $NF_R int(A) = 0_N$, for every neutrosophic first category ring A in $(R, \mathscr{S})$.

(iii) $NF_R cl(A) = 1_N$, for every neutrosophic residual ring A in $(R, \mathscr{S})$.

Proof:
(i)⇒(ii)
Let A be any neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now, $NF_R int(A) = NF_R int(\cup_{i=1}^{\infty} A_i) = NF_R int(C(\cap_{i=1}^{\infty} C(A_i))) = NF_R Ext(\cap_{i=1}^{\infty} C(A_i))$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$. Therefore, $NF_R int(A) = 0_N$. Hence (i) ⇒ (ii).
(ii)⇒(iii)

Let A be any neutrosophic residual ring in $(R, \mathscr{S})$. Then $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$. By (ii), $NF_R int(C(A)) = 0_N$. That is, $NF_R int(C(A)) = 0_N = C(NF_R cl(A))$. Therefore, $NF_R cl(A) = 1_N$. Hence (ii) ⇒ (iii).

(iii)⇒(i)

Let A be any neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Since A is a neutrosophic first category ring, $C(A)$ is a neutrosophic residual ring in $(R, \mathscr{S})$. Then by (iii), $NF_R cl(C(A)) = 1_N$. Now, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = NF_R int(C(\cap_{i=1}^{\infty} C(A_i))) = NF_R int(\cup_{i=1}^{\infty} A_i) = NF_R int(A) = C(NF_R cl(C(A))) = 0_N$. Hence, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.7. If A is a neutrosophic first category ring in a neutrosophic τ-structure ring space $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proof:

Let A be any neutrosophic first category ring in $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is aneutrosophic dense ring in $(R, \mathscr{S})$. Then by Proposition 3.3., A is a neutrosophic nowhere dense ring $(R, \mathscr{S})$, that is, $NF_R int(NF_R cl(A)) = 0_N$. Then, $NF_R int(A) \subseteq NF_R int(NF_R cl(A))$ implies that $NF_R int(A) = 0_N$. By Proposition 3.6., $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.8. If $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space and if $\cup_{i=1}^{\infty} A_i = 1_N$ where A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$, then $NF_R cl(\cup_{i=1}^{\infty} NF_R Ext(C(A_i))) = 1_N$.

Proof:

Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring $ExtB$ space. Assume that A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$. Suppose that $NF_R int(A_i) = 0_N$, for each $i \in J$. Since A_i is a neutrosophic τ- regular closed ring in $(R, \mathscr{S})$, A_i is a neutrosophic τ-closed ring in $(R, \mathscr{S})$. Also, $NF_R int(A_i) = 0_N$ implies that $NF_R int(NF_R cl(A_i)) = 0_N$. Therefore, A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Since $\cup_{i=1}^{\infty} A_i = 1_N$, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = NF_R Ext(C(\cup_{i=1}^{\infty} A_i)) = NF_R int(\cup_{i=1}^{\infty} A_i) = NF_R int(1_N) = 1_N$. Hence, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 1_N$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$, which is a contradiction. Hence $NF_R int(A_i) \neq 0_N$, for atleast one $i \in J$. Therefore, $\cup_{i=1}^{\infty} NF_R int(A_i) \neq 0_N$. Since A_i is a neutrosophic τ-regular closed rings in $(R, \mathscr{S})$ and $\cup_{i=1}^{\infty} NF_R cl(A_i) \subseteq NF_R cl(\cup_{i=1}^{\infty} A_i)$,

$$\begin{aligned}
&\Rightarrow \cup_{i=1}^{\infty} NF_R cl(NF_R int(A_i)) \subseteq NF_R cl(\cup_{i=1}^{\infty} NF_R int(A_i))\\
&\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq NF_R cl(\cup_{i=1}^{\infty} NF_R int(A_i))\\
&\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq NF_R cl(\cup_{i=1}^{\infty} NF_R Ext(C(A_i)))\\
&\Rightarrow 1_N \subseteq NF_R cl(\cup_{i=1}^{\infty} NF_R Ext(C(A_i))).
\end{aligned}$$

But $1_N \supseteq NF_R cl(\cup_{i=1}^{\infty} NF_R Ext(C(A_i)))$. Hence, $NF_R cl(\cup_{i=1}^{\infty} NF_R Ext(C(A_i))) = 1_N$.

4 On neutrosophic τ-Structure Ring Exterior V Spaces

Definition 4.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring exterior V (in short, $ExtV$)space if $NF_R cl(\cap_{i=1}^{n} A_i) = 1_N$ where A_i's are neutrosophic G_δ

rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.

Example 4.1. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B and D on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_D(0) = 0.7, T_D(1) = 0.4, T_D(2) = 1, I_D(0) = 0.7, I_D(1) = 0.4, I_D(2) = 1, F_D(0) = 0.3, F_D(1) = 0.6, F_D(2) = 0$.
Then $\mathscr{S} = \{0_N, A, B, D, A \cap B, A \cup B, A \cap D, A \cup D, B \cap D, B \cup D, D \cap (A \cup B), A \cup (B \cap D), B \cup (A \cap D), 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ-structure ring space.

Now, $A \cap D = \cap\{B \cup (A \cap D), D \cap (A \cup B), D, A\}$ and $D \cap (A \cup B) = \cap\{A \cup B, D \cap (A \cup B), A \cup D\}$ are neutrosophic G_δ rings in $(R, \mathscr{S})$. Also, the neutrosophic ring exterior of $C(A \cap D)$ and $C(D \cap (A \cup B))$ are neutrosophic dense rings in $(R, \mathscr{S})$. Now, $NF_Rcl(\cap\{A \cap D, D \cap (A \cup B)\}) = NF_Rcl(A \cap D) = 1_N$.Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.1. Let $(R, \mathscr{S})$ be a neutrosophic structure ring space. Then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space iff $NF_Rint(\cup_{i=1}^{n} C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.

Proof:

Let $(R, \mathscr{S})$ be a neutrosophic ring $ExtV$ space. Assume that A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^{n} A_i) = 1_N$. Now, $NF_Rint(\cup_{i=1}^{n} C(A_i)) = NF_Rint(C(\cap_{i=1}^{n} A_i)) = C(NF_Rcl(\cap_{i=1}^{n} A_i)) = 0_N$. Therefore, $NF_Rint(\cup_{i=1}^{n} C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.

Conversely, let $NF_Rint(\cup_{i=1}^{n} C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$. Now, $NF_Rcl(\cap_{i=1}^{n} A_i) = NF_Rcl(C(\cup_{i=1}^{n} A_i)) = C(NF_Rint(\cup_{i=1}^{n} A_i)) = 1_N$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.2. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. If every neutrosophic first category ring in $(R, \mathscr{S})$ is formed from the neutrosophic G_δ rings and the neutrosophic ring exterior of its complements are neutrosophic dense rings in a neutrosophic τ-structure ring $ExtV$ space $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proof:

Assume that A_i's are neutrosophic G_δ rings in $(R, \mathscr{S})$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space and by Proposition 4.1., $NF_Rint(\cup_{i=1}^{n} C(A_i)) = 0_N$. But $\cup_{i=1}^{n} NF_Rint(C(A_i)) \subseteq NF_Rint(\cup_{i=1}^{n} C(A_i))$, which implies that $\cup_{i=1}^{n} NF_Rint(C(A_i)) = 0_N$. Then $NF_Rint(C(A_i)) = 0_\sim$. Since A_i's are neutrosophic G_δ rings in $(R, \mathscr{S})$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. By Proposition 3.2., $C(A_i)$'s are neutrosophic first category rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Therefore, $NF_Rint(C(A_i)) = 0_N$, for every $C(A_i)$ is a neutrosophic first category rings in $(R, \mathscr{S})$. By Proposition 3.6., $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Definition 4.2. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then f is said to be a

(i) neutrosophic τ-structure ring continuous function if $f^{-1}(A)$ is a neutrosophic τ-open ring in $(R_1, \mathscr{S}_1)$, for every neutrosophic τ-open ring A in $(R_2, \mathscr{S}_2)$.

(ii) somewhat neutrosophic τ-structure ring continuous function if $A \in \mathscr{S}_2$ and $f^{-1}(A) \neq 0_\sim$ implies that there exists a neutrosophic τ-open ring B in $(R_1, \mathscr{S}_1)$ such that $B \neq 0_N$ and $B \subseteq f^{-1}(A)$.

(iii) neutrosophic τ-structure ring hardly open function if for each neutrosophic dense ring A in $(R_2, \mathscr{S}_2)$ such that $A \subseteq B \subset 1_N$ for some neutrosophic τ-open ring B in $(R_2, \mathscr{S}_2)$, $f^{-1}(A)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$.

(iv) neutrosophic τ-structure ring open function if $f(A)$ is a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$, for every neutrosophic τ-open ring A in $(R_1, \mathscr{S}_1)$.

Proposition 4.3. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then the following statements are equivalent:

(i) f is a neutrosophic τ-structure ring continuous function.

(ii) $f^{-1}(B)$ is a neutrosophic τ-closed ring in $(R_1, \mathscr{S}_1)$, for every neutrosophic τ-closed ring B in $(R_2, \mathscr{S}_2)$.

(iii) $NF_Rcl(f^{-1}(A)) \subseteq f^{-1}(NF_Rcl(A))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.

(iv) $f^{-1}(NF_Rint(A)) \subseteq NF_Rint(f^{-1}(A))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.

Remark 4.1. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring continuous function, then $f^{-1}(NF_RExt(C(A)) \subseteq NF_RExt(C(f^{-1}(A)))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.
Proof: The proof follows from the Definition 3.4 and Proposition 4.3..

Proposition 4.4. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ into another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is neutrosophic τ-structure ring continuous, 1-1 and if A is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.
Proof:

Suppose that $f(A)$ is not a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then there exists a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f(A) \subset D \subset 1_N$. Then, $f^{-1}(f(A)) \subset f^{-1}(D) \subset f^{-1}(1_N)$. Since f is 1-1, $f^{-1}(f(A)) = A$. Hence $A \subset f^{-1}(D) \subset 1_N$. Since f is a neutrosophic τ-structure ring continuous function and D is a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(D)$ is a neutrosophic τ-closed ring in $(R_1, \mathscr{S}_1)$. Then $NF_Rcl(A) \neq 1_N$, which is a contradiction. Therefore $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.

Remark 4.2. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Then

(i) the neutrosophic τ-structure ring continuous image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

(ii) the neutrosophic τ-structure ring open image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proof: It is clear from the following Examples.

Example 4.2. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings $A, B, V, D, E,$ and F on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_V(0) = 0.7, T_V(1) = 0.4, T_V(2) = 1, I_V(0) = 0.7, I_V(1) = 0.4, I_V(2) = 1F_V(0) = 0.3, F_V(1) = 0.6, F_V(2) = 0, T_D(0) = 0.9, T_D(1) = 1, T_D(2) = 0.2, I_D(0) = 0.9, I_D(1) = 1, I_D(2) = 0.2, F_D(0) = 0.1, F_D(1) = 0, F_D(2) = 0.8, T_E(0) = 0.2, T_E(1) = 0.2, T_E(2) = 1, I_E(0) = 0.2, I_E(1) = 0.2, I_E(2) = 1, F_E(0) = 0.8, F_E(1) = 0.8, F_E(2) = 0, T_F(0) = 1, T_F(1) = 0.7, T_F(2) = 0.4, I_F(0) = 1, I_F(1) = 0.7, I_F(2) = 0.4, F_F(0) = 0, F_F(1) = 0.3, F_F(2) = 0.6$.

Then $\mathscr{S}_1 = \{0_N, A, B, V, A\cap B, A\cup B, A\cap V, A\cup V, B\cap V, B\cup V, V\cap(A\cup B), A\cup(B\cap V), B\cup(A\cap V), 1_N\}$ and $\mathscr{S}_2 = \{0_N, D, E, F, D\cap E, D\cup E, D\cap F, D\cup F, E\cap F, E\cup F, F\cap(D\cup E), D\cup(E\cap F), E\cup(D\cap F), 1_N\}$ are two neutrosophic τ-structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are neutrosophic τ-structure ring spaces. Now, $A\cap V = \cap\{B\cup(A\cap V), V\cap(A\cup B), V, A\}$ and $V\cap(A\cup B) = \cap\{A\cup B, V\cap(A\cup B), A\cup V\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_1)$. Also, the neutrosophic ring exterior of $C(A\cap V)$ and $C(V\cap(A\cup B))$ are neutrosophic dense rings in $(R, \mathscr{S}_1)$. Now, $NF_Rcl(\cap\{A\cap V, V\cap(A\cup B)\}) = NF_Rcl(A\cap V) = 1_N$. Therefore, $(R, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space. Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 1, f(1) = 2$ and $f(2) = 0$. Clearly, f is a neutrosophic τ-structure ring continuous function. Also, $f(A) = D, f(B) = E$ and $f(V) = F$. Now, $D = \cap\{D, D\cup E, D\cup(E\cap F)\}$, $D\cap F = \cap\{F, D\cup F, D\cap F, F\cap(D\cup E)\}$ and $E = \cap\{E, E\cup F, E\cup(D\cap F)\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. Also, the neutrosophic ring exterior of $C(D)$, $C(F)$ and $C(D\cap F)$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. But, $NF_Rcl(\cap\{D, E, D\cap F\}) = C(E\cap F) \neq 1_N$. Therefore, $(R, \mathscr{S}_2)$ is not a neutrosophic τ-structure ring $ExtV$ space. Therefore the neutrosophic τ-structure ring continuous image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Example 4.3. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B, V and D on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_V(0) = 0.7, T_V(1) = 0.4, T_V(2) = 1, I_V(0) = 0.7, I_V(1) = 0.4, I_V(2) = 1, F_V(0) = 0.3, F_V(1) = 0.6, F_V(2) = 0, T_D(0) = 0.5, T_D(1) = 0.6, T_D(2) = 0.4, I_D(0) = 0.5, I_D(1) = 0.6, I_D(2) = 0.4, F_D(0) = 0.5, F_D(1) = 0.4, F_D(2) = 0.6$.

Then $\mathscr{S}_1 = \{0_N, A, B, V, A\cap B, A\cup B, A\cap V, A\cup V, B\cap V, B\cup V, V\cap(A\cup B), A\cup(B\cap V), B\cup(A\cap V), 1_N\}$ and $\mathscr{S}_2 = \{0_N, A, B, V, D, A\cup B, A\cup V, A\cup D, B\cup V, B\cup D, V\cup D, A\cap B, A\cap V, A\cap D, B\cap V, B\cap$

$D, V \cap D, D \cup (A \cap V), V \cap (A \cup B), A \cup (B \cap V), B \cup (A \cap V), 1_N\}$ are two neutrosophic τ-structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are neutrosophic τ-structure ring spaces. Now, $A \cap V = \cap\{B \cup (A \cap V), V \cap (A \cup B), V, A\}$ and $V \cap (A \cup B) = \cap\{A \cup B, V \cap (A \cup B), A \cup V\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_1)$. Also, the neutrosophic ring exterior of $C(A \cap V)$ and $C(V \cap (A \cup B))$ are neutrosophic dense rings in $(R, \mathscr{S}_1)$. Now, $NF_Rcl(\cap\{A \cap V, V \cap (A \cup B)\}) = NF_Rcl(A \cap V) = 1_V$. Therefore, $(R, \mathscr{S}_1)$ is a neutrosophic ring $ExtV$ space. Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 0, f(1) = 1$ and $f(2) = 2$. Clearly, f is a neutrosophic τ-structure ring open function. Also, $f(A) = A, f(B) = B, f(V) = V$ and $f(D) = D$. Now, $A = \cap\{A, A \cup B, A \cup V, A \cup (B \cap V)\}$, $D \cup (A \cap V) = \cap\{V, V \cup D, A \cap V, D \cup (A \cap V), V \cap (A \cup B)\}$ and $B = \cap\{B, B \cup V, B \cup D, B \cup (A \cap V)\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. Also, the neutrosophic ring exterior of $C(A), C(B)$ and $C(D \cup (A \cap V))$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. But, $NF_Rcl(\cap\{A, B, D \cup (A \cap V)\}) = C(B \cap V) \neq 1_N$. Therefore, $(R, \mathscr{S}_2)$ is not a neutrosophic τ-ring $ExtV$ space. Therefore the neutrosophic τ-structure ring open image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proposition 4.5. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is onto function, then the following statements are equivalent:

(i) f is a neutrosophic τ-structure ring hardly open function.

(ii) $NF_Rint(f(A)) \neq 0_N$, for all neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and there exists a neutrosophic τ-closed ring $B \neq 0_N$ in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$.

(iii) $NF_Rint(f(A)) \neq 0_N$, for all neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and there exists a neutrosophic τ-closed ring $B \neq 0_N$ in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(B) \subseteq A$.

Proof:

(i)⇒(ii)

Assume that (i) is true. Let A be any neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and $B \neq 0_N$ be a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$. Suppose that $NF_Rint(A) = 0_N$. This implies that $NF_Rcl(C(f(A))) = 1_N$. Thus, $C(f(A))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$ and $C(f(A)) \subseteq C(B)$. By assumption, $f^{-1}(C(f(A)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. That is, $NF_Rcl(f^{-1}(C(f(A)))) = 1_N$. Now, $NF_Rint(A) = NF_Rint(f^{-1}(f(A))) = C(NF_Rcl(C(f^{-1}(f(A))))) = C(NF_Rcl(f^{-1}(C(f(A))))) = 0_N$. This is a contradiction. Hence (i)⇒(ii).

(ii)⇒(iii)

Assume that (ii) is true. Since f is onto function and by assumption, $B \subseteq f(A)$. This implies that $f^{-1}(B) \subseteq f^{-1}(f(A))$, that is, $f^{-1}(B) \subseteq A$. Hence (ii)⇒(iii).

(iii)⇒(i)

Let $V \subseteq C(D)$ where C is a neutrosophic dense ring and D is non-zero neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$. Let $A = f^{-1}(C(V))$ and $B = C(D)$. Now, $f^{-1}(B) = f^{-1}(C(D)) \subseteq f^{-1}(C(V)) = A$.

Consider, $NF_Rint(f(A)) = NF_Rint(f(f^{-1}(C(V))) = NF_Rint(C(V)) = C(NF_Rint(V)) = 0_N$. Therefore, $NF_Rint(A) = 0_N$, which implies that $NF_Rint(f^{-1}(C(V))) = NF_Rint(C(f^{-1}(V))) = 0_N$. Therefore, $C(NF_Rcl(f^{-1}(V))) = 0_N$. Thus, $NF_Rcl(f^{-1}(V)) = 1_N$. Therefore, $f^{-1}(V)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. This implies that f is a neutrosophic τ-structure ring hardly open function. Hence (iii)⇒(i). This completes the proof.

Proposition 4.6. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ onto another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is neutrosophic τ-structure ring continuous, 1-1 and neutrosophic τ-structure ring hardly open function and if $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space, then $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Proof:

Let $(R_1, \mathscr{S}_1)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^{\infty} B_{ij}) = \cap_{j=1}^{\infty} f^{-1}(B_{ij}) \tag{4.1}$$

Since f is a neutrosophic τ-structure ring continuous function and B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^{\infty} f^{-1}(B_{ij})$ is an neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring hardly open function and $NF_RExt(C(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Now,

$$\begin{aligned} f^{-1}(NF_RExt(C(A_i))) &= f^{-1}(NF_Rint(A_i)) \\ &\subseteq NF_Rint(f^{-1}(A_i)) \\ &= NF_RExt(C(f^{-1}(A_i))). \end{aligned}$$

Therefore $1_N = NF_Rcl(f^{-1}(NF_RExt(C(A_i)))) \subseteq NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$, which implies that $1_N = NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$. Hence $NF_RExt(C(f^{-1}(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Since $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-strucuture ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) = 1_N$ where $f^{-1}(A_i)$'s are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(f^{-1}(A_i))$'s are neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Thus, $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) = 1_N = NF_Rcl(f^{-1}(\cap_{i=1}^{n} A_i))$. Therefore, $f^{-1}(\cap_{i=1}^{n} A_i)$ is a neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring continuous, 1-1 and by Proposition 3.4., $f(f^{-1}(\cap_{i=1}^{n} A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Hence $NF_Rcl(f(f^{-1}(\cap_{i=1}^{n} A_i))) = 1_N$. Since f is 1-1, $f(f^{-1}(\cap_{i=1}^{n} A_i)) = \cap_{i=1}^{n} A_i$. Then, $NF_Rcl(\cap_{i=1}^{n} A_i) = 1_N$. Therefore, $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_2, \mathscr{S}_2)$.

Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^{\infty} B_{ij}) = \cap_{j=1}^{\infty} f^{-1}(B_{ij}) \tag{4.2}$$

Since f is a neutrosophic τ-structure ring continuous function and B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^{\infty} f^{-1}(B_{ij})$ is a neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring hardly open function and $NF_RExt(C(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. By Remark 4.2., $f^{-1}(NF_RExt(C(A_i))) \subseteq NF_RExt(C(f^{-1}(A_i)))$.

Thus, $NF_Rcl(f^{-1}(NF_RExt(C(A_i)))) = 1_N \subseteq NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$. Hence, $NF_RExt(C(f^{-1}(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Suppose that $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) \neq 1_N$. This implies that

$$\begin{gathered} \overline{NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i))} \neq 0_N \\ \Rightarrow NF_Rint(\cup_{i=1}^{n} C(f^{-1}(A_i))) \neq 0_N \\ \Rightarrow NF_Rint(\cup_{i=1}^{n} f^{-1}(C(A_i))) \neq 0_N. \end{gathered}$$

Then, there is a non-zero neutrosophic τ-open ring E_i in $(R_1, \mathscr{S}_1)$ such that $E_i \subseteq \cup_{i=1}^n f^{-1}(C(A_i))$. Now,

$$\begin{aligned} f(E_i) &\subseteq f(\cup_{i=1}^n f^{-1}(C(A_i))) \\ &\subseteq \cup_{i=1}^n f(f^{-1}(C(A_i))) \\ &\subseteq \cup_{i=1}^n C(A_i) \\ &= C(\cap_{i=1}^n A_i). \end{aligned}$$

$$\text{Then, } NF_R int(f(E_i)) \subseteq NF_R int(C(\cap_{i=1}^n A_i)) = C(NF_R cl(\cap_{i=1}^n A_i)). \tag{4.3}$$

Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_R cl(\cap_{i=1}^n A_i) = 1_N$. Hence from (4.3), $NF_R int(f(E_i)) \subseteq 0_N$. This implies that $NF_R int(f(E_i)) = 0_N$, which is a contradiction. Hence $NF_R cl(\cap_{i=1}^n f^{-1}(A_i)) = 1_N$. Therefore, $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.7. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any bijective function. Then the following statements are equivalent:

(i) f is somewhat neutrosophic τ-structure ring continuous function.

(ii) If A is a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$, then there exists a neutrosophic τ-closed ring $0_N \neq E \neq 1_N$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subset E$.

(iii) If A is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.

Proof:

(i)⇒(ii)

Assume that (i) is true. Let A be a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$. Then $C(A)$ is a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$ such that $C(f^{-1}(A)) = f^{-1}(C(A)) \neq 0_N$. Since f is somewhat neutrosophic τ-structure ring continuous, there exists a neutrosophic τ-open ring E in $(R_1, \mathscr{S}_1)$ such that $E \subseteq f^{-1}(C(A))$. Then there exists a neutrosophic τ-closed ring $C(E) \neq 0_N$ in $(R_1, \mathscr{S}_1)$ such that $C(E) \subset f^{-1}(A)$. Hence (i)⇒(ii).

(ii)⇒(iii)

Assume that (ii) is true. Let A be a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$ such that $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then, there exists a neutrosophic τ-closed ring C in $(R_2, \mathscr{S}_2)$ such that

$$f(A) \subset E \subset 1_N.$$

This implies that $f^{-1}(E) \neq 1_N$. Then by (ii), there exists a neutrosophic τ-closed ring $0_N \neq D \neq 1_N$ such that $A \subset f^{-1}(E) \subset D \subset 1_N$. This is a contradiction. Hence (ii)⇒(iii).

(iii)⇒(ii)

Assume that (iii) is true. Suppose (ii) is not true. Then there exists a neutrosophic τ-closed ring A in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$. But there is no neutrosophic τ-closed ring $0_N \neq E \neq 1_N$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subseteq E$. This implies that $f^{-1}(A)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. But from hypothesis $f(f^{-1}(A)) = A$ must be neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, which is a contradiction. Hence (iii)⇒(ii).

(ii)⇒(i)

Let A be a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$ and $f^{-1}(A) \neq 0_N$. Then, $f^{-1}(C(A)) = C(f^{-1}(A)) = 0_N$. Then by (ii), there exists a neutrosophic τ-closed ring $0_N \neq B \neq 1_N$ such that $f^{-1}(C(A)) \subset B$. This implies that $C(B) \subset f^{-1}(A)$ and $C(B) \neq 0_N$ is a neutrosophic τ-open ring in $(R_1, \mathscr{S}_1)$. Hence (ii)⇒(i). Hence the proof.

Proposition 4.8. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ onto another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is somewhat neutrosophic τ-structure ring continuous, 1-1 and neutrosophic τ-structure ring open function and if $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space, then $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.
Proof:

Let $(R_1, \mathscr{S}_1)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Then, $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring open function, $f(B_{ij})$'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Now, $\cap_{j=1}^{\infty} f(B_{ij})$ is a neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{j=1}^{\infty} f(B_{ij})) = \cap_{j=1}^{\infty} f^{-}(f(B_{ij})) = \cap_{j=1}^{\infty} B_{ij} = A_i \tag{4.4}$$

$$\text{Since } f \text{ is onto, } f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}) \tag{4.5}$$

Therefore, $f(A_i)$ is a neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is somewhat neutrosophic τ-structure ring continuous function, $NF_RExt(C(A_i`))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$ and by Proposition 4.7., $f(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, which implies that $NF_RExt((f(A_i)))$. Now we claim that $NF_Rcl(\cap_{i=1}^{\infty} f(A_i)) = 1_N$. Suppose that $NF_Rcl(\cap_{i=1}^{n} f(A_i)) \neq 1_N$. This implies that

$$\begin{aligned}
& C(NF_Rcl(\cap_{i=1}^{n} f(A_i))) \neq 0_N \\
& \Rightarrow NF_Rint(\cup_{i=1}^{n} C(f(A_i))) \neq 0_N \\
& \Rightarrow NF_Rint(\cup_{i=1}^{n} f(C(A_i))) \neq 0_N.
\end{aligned}$$

Therefore there is an non-zero neutrosophic τ-open ring E_i in $(R_2, \mathscr{S}_2)$ such that $E_i \subseteq \cup_{i=1}^{n} f(C(A_i))$. Then $f^{-1}(E_i) \subseteq f^{-1}(\cup_{i=1}^{n} f(C(A_i)))$. Since f is somewhat neutrosophic τ-structure ring continuous function and $E_i \in \mathscr{S}_2$, $NF_Rint(f^{-1}(E_i)) \neq 0_N$ implies that $NF_Rint(f^{-1}(\cup_{i=1}^{n} f(C(A_i)))) \neq 0_N$.
Then $NF_Rint(\cup_{i=1}^{n} f^{-1}(f(C(A_i)))) \neq 0_N$. Since f is a bijective function, $NF_Rint(\cap_{i=1}^{n} C(A_i)) \neq 0_N$, which implies that $C(NF_Rcl(\cap_{i=1}^{n} A_i)) \neq 0_N$. That is, $NF_Rcl(\cap_{i=1}^{n} A_i) \neq 1_N$. This is a contradiction. Hence $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Since f is somewhat neutrosophic τ-structure ring continuous function, $NF_RExt(C(A_i))$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$ and By Proposition 4.7., $f(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. That is, $NF_Rcl(NF_RExt(C(A_i))) = 1_N$. Since f is a neutrosophic τ-structure ring open function and B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$, $f(B_{ij})$'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence $\cap_{j=1}^{\infty} f(B_{ij})$ is a neutrosophic G_δ ring in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{i=1}^{n} f(B_{ij})) = \cap_{i=1}^{n}(f^{-1}(f(B_{ij})) = \cap_{i=1}^{n} B_{ij}. \tag{4.6}$$

Since f is onto,

$$f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}). \tag{4.7}$$

Hence $f(A_i)$ is a neutrosophic G_δ ring in $(R_2, \mathscr{S}_2)$. Now,

$$\begin{aligned} NF_Rcl(NF_RExt(C(f(A_i))) &= NF_Rcl(NF_RExt(f(C(A_i))) \\ &= NF_Rcl(NF_Rint(f(A_i)) \\ &\supseteq NF_Rcl(f(NF_Rint(A_i)) \\ &\supseteq f(NF_Rcl(NF_Rint(A_i))) \\ &= f(1_N) = 1_N. \end{aligned}$$

This implies that $NF_RExt(C(f(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Hence the neutrosophic ring exterior of $C(f(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$. Now we claim that $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$ where A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Suppose that $NF_Rcl(\cap_{i=1}^n A_i) \neq 1_N$. This implies that

$$\begin{aligned} &C(NF_Rcl(\cap_{i=1}^n A_i)) \neq 0_N \\ &\Rightarrow NF_Rint(C(\cap_{i=1}^n A_i)) \neq 0_N \\ &\Rightarrow NF_Rint(\cup_{i=1}^n C(A_i)) \neq 0_N. \end{aligned}$$

Then there is a non-zero neutrosophic τ-open ring E_i in $(R_1, \mathscr{S}_1)$ such that $E_i \subseteq \cup_{i=1}^n C(A_i)$. Now,

$$\begin{aligned} f(E_i) &\subseteq f(\cup_{i=1}^n C(A_i)) \\ &\subseteq \cup_{i=1}^n f(C(A_i)) \\ &\subseteq \cup_{i=1}^n C(f(A_i)) \\ &= C(\cap_{i=1}^n f(A_i)). \end{aligned}$$

$$\text{Then, } NF_Rint(f(E_i)) \subseteq NF_Rint(C(\cap_{i=1}^n f(A_i))) \subseteq C(NF_Rcl(\cap_{i=1}^n f(A_i))) \tag{4.8}$$

Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$. Hence from (4.8), $NF_Rint(f(E_i)) \subseteq 0_N$, which implies that $NF_Rint(f(E_i)) = 0_N$, which is a contradiction. Hence $NF_Rcl(\cap_{i=1}^n A_i) = 1_N$. Therefore $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space.

5 Conclusion

A neutrosophic set model provides a mechanism for solving the modeling problems which involve indeterminacy, and inconsistent information in which human knowledge is necessary and human evaluation is needed. It deals more flexibility and compatibility to the system as compared to the classical theory, fuzzy theory and intuitionistic fuzzy models. In this paper, a new idea of a neutrosophic τ-structure ring spaces, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces have been introduced. Further, neutrosophic τ-structure ring continuous (resp. open,hardly open)functions, somewhat neutrosophic τ-structure ring continuous functions are studied. Their characterization are derived and illustrated with examples.

References

[1] M. Abdel-Basset, M. El-hoseny,A. Gamal & F. Smarandache, A Novel Model for Evaluation Hospital Medical Care Systems Based on Plithogenic Sets, *Artificial Intelligence in Medicine* , 101710(2019).

[2] M. Abdel-Basset, G. Manogaran, A. Gamal & V. Chang, A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT, *IEEE Internet of Things Journal* , (2019).

[3] M. Abdel-Basset, R. Mohamed, A. E. N. H.Zaied & F. Smarandache, A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, 11(7)(2019),903.

[4] M. Abdel-Basset, A. Atef & F.Smarandache, A hybrid Neutrosophic multiple criteria group decision making approach for project selection, *Cognitive Systems Research*, 57(2019), 216–227.

[5] M. Abdel-Basset, A. Gamal, G. Manogaran & H. V. Long, A novel group decision making model based on neutrosophic sets for heart disease diagnosis, *Multimedia Tools and Applications* , (2019), 1–26.

[6] K.T.Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20(1986), 87–96.

[7] C.L.Chang, Fuzzy Topological Spaces,J.Math.Anal. Appl.,24(1968),182–190.

[8] D.Coker, An Introduction to Intuitionistic Fuzzy Topological Spaces,*Fuzzy Sets and Systems,* 88(1997), No.1, 81–89.

[9] Florentin Smarandache, Neutrosophy and Neutrosophic Logic, *First International Conference on Neutrosophy, Neutrosophic Logic set, Probability and Statistics*, University of New Mexico, Gallup, NM 87301, USA(2002).

[10] R.Narmada Devi, E. Roja and M.K.Uma, On Some Applications of Intuitionistic Fuzzy G_δ-α-locally closed sets,*The Journal of Fuzzy Mathematics,* 21(2013), No. 2, 85–98.

[11] R.Narmada Devi, E. Roja and M.K.Uma, Intuitionistic Fuzzy G_{delta}-α-locally Continuous Functions,*Annals of Fuzzy Mathematics and Informatics*, 5(2013), No. 2, 399–416.

[12] R.Narmada Devi, E.Roja and M.K.Uma, Intuitionistic Fuzzy Exterior Spaces Via Rings, *Annals of Fuzzzy mathematics and Informatics*,6(2014),2,554–559.

[13] R.Narmada Devi, E.Roja and M.K.Uma,Basic Compactness and Extremal Compactness in Intuitionistic Fuzzy Structure Ring Spaces,*The Journal of fuzzy Mathematics*,23(2015),6,643–660.

[14] R.Narmada Devi, Neutrosophic Complex $\mathscr{N}$-continuity,*Annals of Fuzzy Mathematics nad Informatics*,13(2017),1,109–122.

[15] A. A. Salama and S. A. Alblowi, Neutrosophic Set ans Neutrosophic Topological Spaces,*ISOR Journal of Mathematics*,3(2012), No.6,31–35.

[16] L.A.Zadeh, Fuzzy Sets,*Information and Control*, 9(1965), 338–353.

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