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Abstract. A significant area of module theory is the concept of free modules, projective modules and injective modules. The goal of this study is to characterize the projective G-modules under a single-valued neutrosophic set. So we define neutrosophic G-submodule as a generic version of projective G-submodule. It also describes and derives fundamental algebraic properties including quotient space and direct sum of neutrosophic projective G-submodules

Keywords: Neutrosophic set; Neutrosophic G-module;Direct sum; Projective G-module; Neutrosophic projective G-module

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1. **Introduction**

The projective G-module in the abstract algebra plays a pivotal role to analyze the algebraic structure G-module and its characteristics. Cartan and Eilemberge [16] introduced the concept of projective modules that offer significant ideas through the theoretical approach to module theory. The algebraic structure G-module widely used to study the representation of finite groups developed by Frobenius G and Burnside [11] in the 19th century. Several researchers have studied the algebraic structure in pure mathematics associated with uncertainty. Since Zadeh [35] introduced fuzzy sets, fuzzification of algebraic structures was an important milestone in classical algebraic studies. The notion of a fuzzy submodule was introduced by Negoita and Ralescu [25] and further developed by Mashinchi and Zahedi [24]. This basic notion has been generalized in several ways after Zadeh's implementation of fuzzy sets [4, 5]. In 1986 Atanassov [6] put forward intuitionistic fuzzy set theory in which each element coincides with membership grades and non-membership grades. Biswas [9] applied the idea of the intuitionistic fuzzy set to the algebraic structure group and K. Hur et.al. [21] additionally studied it. In

2011 P. Isaac, P.P.John [22] studied about algebraic nature of intuitionistic fuzzy submodule of a classical module.

The theory of neutrosophy first appeared in philosophy [30] and then evolved neutrosophic set as a mathematical tool. In 1995, Smarandache [31] outlined the neutrosophicset as a combination of tri valued logic with non-standard analysis in which three different types of membership values represent each element of a set.The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Neutrosophic set theory gives a thorough scientific and mathematical model knowledge in which speculative and uncertain hypothetical phenomena can be managed by hierarchal membership of the components " truth / indeterminacy / falsehood " [2,3,32]. Neutrosophic set generalizes a classical set, fuzzy set, interval-valued fuzzy set and intuitionistic fuzzy set that can be used to make a mathematical model for the real problems of science and engineering. From a scientific and engineering perspective, Wang et.al. [20] specified the definition of a neutrosophic set, which is called a single-valued neutrosophic set. Several scientists dealt with the neutrosophic set notion as a new evolving instrument for uncertain information processing and a general framework for uncertainty analysis in data set [1, 7, 17, 28].

The consolidation of the neutrosophic set hypothesis with algebraic structures is a growing trend in mathematical research. Among the various branches of applied and pure mathematics, abstract algebra was one of the first few topics where the research was carried out using the neutrosophic set concept. W. B. Vasantha Kandasamy and Florentin Smarandache [23] initially presented basic algebraic neutrosophic structures and their application to advanced neutrosophic models. Vidan Cetkin [12, 13] consolidated the neutrosophic set theory and algebraic structures, creating neutrosophic subgroups and neutrosophic submodules. F. Sherry [18, 19] introduced the concept of fuzzy G-modules in which the concept of fuzzy sets was combined with G-module and the theory of group representation. One of the key developments in the neutrosophic set theory is the hybridization of the neutrosophic set with the algebraic structure G-module. The above fact leads to inspiration for conducting an exploratory study in the field of abstract algebra, especially in the theory of G-modules in conjunction with neutrosophic set. In this paper we described neutrosophic projective G-submodule as the general case of projective G-module and derived its algebraic properties.

The reminder of this work is structured as follows. Section 2 briefs about necessary preliminary definitions and results which are basic for a better and clear cognizance of next sections. Section 3 defines neutrosophic projective G-modules, algebraic extension of projective G-submodules and derive the theorems related to quotient space and direct sum of neutrosophic G-submodules. A comprehensive overview, relevance and future study of this work is defined at the end of the paper in Section 4.

2. Preliminaries

In this section, we recall some of the preliminary definitions and results which are essential for a better and clear comprehension of the upcoming sections.

Definition 2.1. [14] Let $(G, *)$ be a group. A vector space M over the field K is called a *G-module*, denoted as G_M, if for every $g \in G$ and $m \in M$; $\exists$ a product (called the action of G on M), $g \cdot m \in M$ satisfies the following axioms

(1) $1_G \cdot m = m; \quad \forall m \in M \quad (1_G$ being the identity element of $G)$
(2) $(g * h) \cdot m = g \cdot (h \cdot m); \quad \forall m \in M$ and $g, h \in G$
(3) $g \cdot (k_1 m_1 + k_2 m_2) = k_1(g \cdot m_1) + k_2(g \cdot m_2); \forall\ k_1, k_2 \in K; m_1, m_2 \in M$ ".

Example 2.1. [18] Let $G = \{1, -1, i, -i\}$ and $M = \mathbb{C}^n$; $(n \geq 1)$. Then M is a vector space over $\mathbb{C}$ and under the usual addition and multiplication of complex numbers we can show that M is a G-module.

Definition 2.2. [15] Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G.

Definition 2.3. [15] Let M and M^* be G-modules. A mapping $f : M \to M^*$ is called a G module homomorphism $(Hom_G(M, M^*))$ if $\forall\ k_1, k_2 \in K, m_1, m_2 \in M, g \in G$ satisfies the following conditions

(1) $f(k_1 m_1 + k_2 m_2) = k_1 f(m_1) + k_2 f(m_2)$
(2) $f(gm) = gf(m)$

Definition 2.4. [10, 29] A G-module M is projective if for any G-module M^* and any G-submodule N^* of M^*, every homomorphism $\varphi : M \to M^*/N^*$ can be lifted to a homomorphism $\psi : M \to M^*$ or $\pi \circ \psi = \varphi$ where $\pi : M^* \to M^*/N^*$.

Remark 2.1. A G-module M is projective if and only if M is M^* projective for every G-module M^*

Theorem 2.2. [29] Let M and M^* be G-modules such that M is M^* projective. Let N^* be any G-submodule of M^*. Then M is N^* projective and M is M^*/N^* projective.

Proposition 2.1. [29] Let M *and* M_i be G-modules.Then M is $\oplus_{i=1}^{n} M_i$-projective if and only if M is M_i-projective $\forall\ i$

Definition 2.5. [32, 34] A neutrosophic set P of the universal set X is defined as $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$ where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components t_P, i_P *and* f_P represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^{-}0, 1^{+})$ [27].

Remark 2.3. [20, 32]

(1) If $t_P, i_P, f_P : X \to [0,1]$, then P is known as single valued neutrosophic set(SVNS).
(2) In this paper, we discuss about the algebraic structure R-module with underlying set as SVNS. For simplicity SVNS will be called neutrosophic set.
(3) U^X denotes the set of all neutrosophic subset of X or neutrosophic power set of X.

Definition 2.6. [26, 32] Let $P, Q \in U^X$. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(\eta) \leqslant Q(\eta)\ \forall \eta \in X$, this means that $t_P(\eta) \leq t_Q(\eta), i_P(\eta) \leq i_Q(\eta), f_P(\eta) \geq f_Q(\eta),\ \forall\ \eta \in X$.

Definition 2.7. [26, 33]For any neutrosophic subset $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$, the support P^* of the neutrosophic set P can be defined as $P^* = \{\eta \in X, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\}$.

Definition 2.8. [8] Let $(G, *)$ be a group and M be a G module over a field K. A neutrosphic G-submodule is a neutrosophic set $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in M\}$ in G_M such that the following conditions are satisfied;

(1) $t_P(\varrho\eta + \tau\theta) \geq t_P(\eta) \wedge t_P(\theta)$
$i_P(\varrho\eta + \tau\theta) \geq i_P(\eta) \wedge i_P(\theta)$
$f_P(\varrho\eta + \tau\theta) \leq f_P(\eta) \vee f_P(\theta)$,
$\forall\ \eta, \theta \in M, \varrho, \tau \in K$
(2) $t_P(\xi\eta) \geq t_P(\eta)$
$i_P(\xi\eta) \geq i_P(\eta)$
$f_P(\xi\eta) \leq f_P(\eta)\ \forall\ \xi \in G, \eta \in M$

Remark 2.4. We denote neutrosophic G-submodules using single valued neutrosophic set by $U(G_M)$.

Example 2.2. Consider the example 2.1 for G-module M. Define a neutrosophic set

$$P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\}$$

of M where

$$t_P(\eta) = \begin{cases} 1 & if\ \eta = 0 \\ 0.5 & if\ \eta \neq 0 \end{cases},\ i_P(\eta) = \begin{cases} 1 & if\ \eta = 0 \\ 0.5 & if\ \eta \neq 0 \end{cases},\ f_P(\eta) = \begin{cases} 0 & if\ \eta = 0 \\ 0.25 & if\ \eta \neq 0 \end{cases}$$

Then P is a neutrosophic G-submodule of M.

Definition 2.9. [8] Let $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\} \in U(^{(G_M)})$. The support P^* of the neutrosophic G-submodule P can be defined as $P^* = \{x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1, \forall x \in G_M\}$.

Proposition 2.2. If $P \in U(G_M)$, then the support $P^* \in G_M$.

Definition 2.10. [8] Let $P \in U(G_M)$ and N be a G-submodule of M. Then the restriction of P to N is denoted by $P|_N$ and it is a neutrosophic set of N defined as follows $P|_N(\eta) = (\eta, t_{P|_N}(\eta), i_{P|_N}(\eta), f_{P|_N}(\eta))$ where

$$t_{P|_N}(\eta) = t_P(\eta),\ i_{P|_N}(\eta) = i_P(\eta),\ f_{P|_N}(\eta) = f_P(\eta),\ \forall \eta \in N.$$

Proposition 2.3. [8] Let $P \in U(G_M)$ and $N \subseteq M$ then $P|_N \in U(G_N)$.

Definition 2.11. [8] Let $M \in G_M$ and N be a G-submodule of M. Then the neutrosophic set P_N of M/Ndefined as $P_N(\eta + N) = \{\eta + N, t_{P_N}(\eta + N), i_{P_N}(\eta + N), f_{P_N}(\eta + N)\}$,where

$$t_{P_N}(\eta + N) = \vee t_P(\eta + n) : n \in N$$

$$i_{P_N}(\eta + N) = \vee i_P(\eta + n) : n \in N$$

$$f_{P_N}(\eta + N) = \wedge f_P(\eta + n) : n \in N, \forall \eta \in M$$

Proposition 2.4. [8] Let $M \in G_M$. Let N be a G-submodule of M. Then $P_N \in U(G_{M/N})$.

Proposition 2.5. [8] Let $P \in U(G_M)$ and $Q \in U(G_{M^*})$ where M and M^* are G-modules over the field K. Let $r \in [0, 1]$, the neutrosophic set $Q_r = \{\eta, t_{Q_r}(\eta), i_{Q_r}(\eta), f_{Q_r}(\eta) : \eta \in M^*\}$ defined by $t_{Q_r}(\eta) = t_Q(\eta) \wedge r,\ i_{Q_r}(\eta) = i_Q(\eta) \wedge r,\ f_{Q_r}(\eta) = f_Q(\eta) \vee (1 - r)\ \forall\ \eta \in M^\star$ be a neutrosophic G-submodule.

Definition 2.12. [8] Let M and M^* be G-modules over K and a mapping $\Upsilon : M \to M^*$ is a G-module homomorphism. Also $P \in U(G_M)$ *and* $Q \in U(G_{M^*})$. A homomorphism Υ of M on to M^* is called weak neutrosophic G-submodule homomorphism of P into Q if $\Upsilon(P) \subseteq Q$. If Υ is a **weak neutrosophic G-module homomorphism** of P into Q, then P is weakly homomorphic to Q and we write $P \sim Q$.

A homomorphism Υ of M on to M^* is called a **neutrosophic G-module homomorphism** of P onto Q if $\Upsilon(P) = Q$ and we represent it as $P \approx Q$.

3. **Neutrosophic Projective G module**

In this section we discuss the generalized notion of projective G-modules, called neutrosophic projective G-modules, and study several characteristics of projective G-modules in the neutrosophic domain.

Definition 3.1. Let M and M^* be G-modules. Let $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\}$ be neutrosophic G submodule of M and $Q = \{\eta, t_Q(\eta), i_Q(\eta), f_Q(\eta) : \eta \in M^*\}$ be neutrosophic G-submodule of M^*. Then P is said to be Q projective, if the following conditions are satisfied;

(1) M is M^* projective
(2) $t_P(\eta) \leq t_Q(\psi(\eta))$
(3) $i_P(\eta) \leq i_Q(\psi(\eta))$
(4) $f_P(\eta) \geq f_Q(\psi(\eta)),\ \forall\ \psi \in Hom(M, M^*),\ \eta \in M$

Theorem 3.1. Let P and Q be neutrosophic G-submodules of finite dimensional G-modules of M and M^* respectively and M is M^* projective. Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be a basis for M^*. If

(1) $t_P(\eta) \leq min\{t_Q(\beta_j); j = 1, 2, ..., n\}$
(2) $i_P(\eta) \leq min\{i_Q(\beta_j); j = 1, 2, ..., n\}$
(3) $f_P(\eta) \geq max\{f_Q(\beta_j); j = 1, 2, ..., n\}, \forall\ \eta \in M$

Then P is Q-projective.

Proof. Let $Q = \{\eta, t_B(\eta), i_B(\eta), f_B(\eta) : \eta \in M^*\}$ be a neutrosophic G submodule of M^*. Then $\forall\ \eta_1, \eta_2 \in M^*; \varrho, \tau \in K$;

(1) $t_Q(\varrho\eta_1 + \tau\eta_2) \geq t_Q(\eta_1) \wedge t_Q(\eta_2)$
(2) $i_Q(\varrho\eta_1 + \tau\eta_2) \geq i_Q(\eta_1) \wedge i_Q(\eta_2)$
(3) $f_Q(\varrho\eta_1 + \tau\eta_2) \leq f_Q(\eta_1) \vee f_Q(\eta_2)$
(4) $t_Q(\xi\eta) \geq t_P(\eta),\ i_Q(\xi\eta) \geq i_Q(\eta),\ f_Q(\xi\eta) \leq f_Q(\eta)\ \forall\ \eta \in M^*,\ \xi \in G$

Also P is a neutrosophic G-submodule of M and M is M^* projective G-module and $\psi \in Hom(M, M^*)$ be any G-module homomorphism. For any $\eta \in M,\ \psi(\eta) \in M^*$.

$\therefore \psi(\eta) = \alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n,\ \alpha_i \in K, \beta_i \in M^*,\ i = 1, 2, ..., n$

$$\begin{aligned} t_Q(\psi(\eta)) &= t_Q(\alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n) \\ &\geq t_Q(\beta_1) \wedge t_Q(\beta_2) \wedge ... \\ &\quad \wedge t_Q(\beta_n) \\ &= mini\{t_Q(\beta_1),\ t_Q(\beta_2),\ ..., \\ &\quad t_Q(\beta_n)\} \\ &\geq t_P(\eta) \end{aligned}$$

Similarly $i_Q(\psi(\eta)) \geq i_P(\eta)$

$$\begin{aligned} f_Q(\psi(\eta)) &= f_Q(\alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n) \\ &\leq f_Q(\beta_1) \wedge t_Q(\beta_2) \wedge ... \wedge t_Q(\beta_n) \\ &= max\{f_Q(\beta_1),\ f_Q(\beta_2), \\ &\quad ...\ ,\ f_Q(\beta_n)\} \\ &\leq f_P(\eta) \end{aligned}$$

$\therefore$ P is Q projective. □

Theorem 3.2. Let $P \in U(G_M),\ Q \in U(G_{M^*})$ and P is Q projective. If N^* is a G-submodule of M^* and $C \in U(G_{N^*})$, then P is C-Projective if $Q|_{N^*} \subseteq C$

Proof. Given P is Q projective, then

(1) M is M^* projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$
$f_P(\eta) \geq f_Q(\psi(\eta))$

$\forall\ \psi \in Hom_G(M, M^*),\ \eta \in M$. Since N^* is a G-submodule of M^*, by a theorem 2.2, M is N^* projective. Let $\varphi \in Hom_G(M, N^*)$ and $\theta : N^* \to M^*$ be the inclusion homomorphism. Then $\theta \circ \varphi = \psi$

$\therefore$ from the condition 2

$t_P(\eta) \leq t_Q(\psi(\eta)) = t_Q(\theta \circ \varphi)(\eta)$

$= t_Q(\theta(\varphi(\eta))) = t_Q(\varphi(\eta))$.

Similarly $i_P(\eta) \leq i_Q(\varphi(\eta))$ *and* $f_P(\eta) \geq f_Q(\varphi(\eta)\ \forall\ \eta \in M,\ \varphi \in Hom_G(M, N^*)$.

Given $C \in U(G_{N^*})$, $\varphi(\eta) \in N^*$ and $Q|_{N^*} \subseteq C$

$$t_{Q|_{N^*}}(\varphi(\eta)) = t_Q(\varphi(\eta)) \leq t_C(\varphi(\eta)$$

$\Rightarrow$ $t_P(\eta) \leq t_C(\varphi(\eta))$. Similarly, $i_P(\eta) \leq i_C(\varphi(\eta))$ and $f_P(\eta) \geq f_C(\varphi(\eta))$. Hence P is C-Projective. □

Theorem 3.3. Let M and M^* be G-modules where P and Q are neutrosophic G-submodules of M and M^* respectively. Let $r \in [0, 1]$, the neutrosophic set $Q_r = \{\eta, t_{Q_r}(\eta), i_{Q_r}(\eta), f_{Q_r}(\eta) : \eta \in M^*\}$ defined by $t_{Q_r}(\eta) = t_Q(\eta) \wedge r,\ i_{Q_r}(\eta) = i_Q(\eta) \wedge r,\ f_{Q_r}(\eta) = f_Q(\eta) \vee (1 - r)\ \forall\ \eta \in M^\star$ be a neutrosophic G- submodule. If P is Q_r projective, then P is Q projective.

Proof. Consider P as Q_r projective where $r \in [0, 1]$. Then

(1) M is M^* projective

(2) $t_P(\eta) \leq t_{Q_r}(\psi(\eta))$,
$i_P(\eta) \leq i_{Q_r}(\psi(\eta))$,
$f_P(\eta) \geq f_{Q_r}(\psi(\eta))$,
$\psi \in Hom_G(M, M^*)$ *and* $\eta \in M$

Since $Q_r \subseteq Q$, $\Rightarrow t_{Q_r}(\psi(\eta)) \leq t_Q(\psi(\eta)$,
$i_{Q_r}(\psi(\eta)) \leq i_Q(\psi(\eta))$ *and*
$f_{Q_r}(\psi(\eta)) \geq f_Q(\psi(\eta)), \forall\ \psi(\eta) \in M^*$.
$\Rightarrow t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$ *and*
$f_P(\eta) \geq f_Q(\psi(\eta))\ \forall\ \eta \in M$.
$\therefore P$ is Q projective. □

Proposition 3.1. Let $M = \oplus_{i=1}^n M_i$ be a G-module where $M_i's$ are G-submodules of M. If $P_i \in U(G_{M_i})$ $(1 \leq i \leq n)$, then the neutrosophic set P of M defined by $t_P(\eta) = \wedge\{t_{p_i}(\eta_i) : i = 1, 2, ..., n\}$,$i_P(\eta) = \wedge\{i_{p_i}(\eta_i) : i = 1, 2, ..., n\}$ and $f_P(\eta) = \vee\{f_{P_i}(\eta_i) : i = 1, 2, ..., n\}$ where $\eta = \sum_{i=1}^{i=n}(\eta_i),\ \eta_i \in M_i$, is a neutrosophic G-submodule of M.

Proof. Let $\eta, \nu \in M$ where $\eta = \sum_{i=1}^{i=n}\eta_i$ and $\nu = \sum_{i=1}^{i=n}\nu_i$. Each $\eta_i,\ \nu_i \in M_i$ and $\varrho, b \in K$. Then by definition, $\varrho\eta + \tau\nu = \sum_{i=1}^{i=n}[\varrho\eta_i + \tau\nu_i]$ where $\varrho\eta_i + \tau\nu_i \in M_i$ $(1 \leq i \leq n)$. Now

$$\begin{aligned} t_P(\varrho\eta + \tau\nu) &= \wedge\ t_{P_i}(\varrho\eta_i + \tau\nu_i) \\ &\geq \wedge\ \{t_{P_i}(\eta_i), t_{P_i}(\nu_i)\} \\ &= \{\wedge\ t_{P_i}(\eta_i)\} \wedge \{\wedge\ t_{P_i}(\nu_i)\} \\ &= t_P(\eta) \wedge t_P(\nu) \end{aligned}$$

Similarly $i_P(\varrho\eta + \tau\nu) \geq i_P(\eta) \wedge i_P(\nu)$
Now consider

$$\begin{aligned} f_P(\varrho\eta + \tau\nu) &= \vee\ f_{P_i}(\varrho\eta_i + \tau\nu_i) \\ &\leq \vee\ \{f_{P_i}(\eta_i), f_{P_i}(\nu_i)\} \\ &= \{\vee\ f_{A_i}(\eta_i)\} \vee \{\vee\ f_{P_i}(\nu_i)\} \\ &= f_P(\eta) \vee f_P(\nu) \end{aligned}$$

Now, for $g \in G, \eta \in M$

$$\begin{aligned} t_P(g\eta) &= \wedge\ t_{P_i}(g\eta_i) \\ &\geq \wedge\ \{t_{P_i}(\eta_i)\} \\ &= t_P(\eta) \end{aligned}$$

Similarly $i_P(g\eta) \geq i_P(\eta)$, $f_P(g\eta) \leq f_P(\eta)$ $\therefore P \in U(G_M)$. $\square$

Definition 3.2. Let $M = \oplus_{i=1}^{n} M_i$ be a G module where $M_i's$ are G-submodules of M. If $P_i \in U(G_{M_i})$ $(1 \leq i \leq n)$ and $P \in U(G_{M=\oplus_{i=1}^{n} M_i})$ with $t_P(0) = t_{P_i}(0), i_P(0) = i_{P_i}(0)$ *and* $f_P(0) = f_{P_i}(0)$ $\forall$ i Then P is called the direct sum of P_i and it is denoted as $P = \oplus_{i=1}^{n} P_i$.

Theorem 3.4. Let $M = \oplus_{i=1}^{n} M_i$ be G module where $M_i's$ are G submodules of M. Let $P \in U(G_M)$ and $Q_i \in U(G_{M_i})$ such that $Q = \oplus_{i=1}^{n} Q_i$. Then P is Q projective if and only if P is Q_i projective $\forall i$.

Proof. Assume that P is Q-projective, then

(1) M is M projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta)$
$f_P(\eta) \geq f_Q(\psi(\eta)$
$\psi \in Hom_G(M, M); \eta \in M$

To prove that P is Q_i projective where $i = 1, 2, ..., n$, it is enough to prove the following conditions.

(1) M is M_i -projective

(2) $t_P(\eta) \leq t_{Q_i}(\varphi(\eta))$,
$i_P(\eta) \leq i_{Q_i}(\varphi(\eta))$
$f_P(\eta) \geq f_{Q_i}(\varphi(\eta))$
where $\forall$ $\varphi \in Hom_G(M, M_i), \eta \in M$.

Here M is $M = \oplus_{i=1}^{n} M_i$-projective and by the the proposition 2.2, M is M_i projective $\forall$ $i = 1, 2, ..., n$. Let $\varphi \in Hom_G(M, M_i)$ and $\theta : M_i \to M \in Hom_G(M_i, M)$ (inclusion) such that $\psi = \theta \circ \varphi$. Then $\forall$ $\varphi \in Hom_G(M, M_i)$

$$\begin{aligned} t_P(\eta) &\leq t_Q(\psi(\eta) \\ &= t_Q((\theta \circ \varphi)(\eta)) \\ &= t_Q(\theta(\varphi(\eta))) \\ &= t_Q(\varphi(\eta)) \end{aligned}$$

Similarly $i_P(\eta) \leq i_Q(\varphi(\eta))$ and

$$\begin{aligned} f_P(\eta) &\geq f_Q(\psi(\eta)) \\ &= f_Q((\theta \circ \varphi)(\eta)) \\ &= f_Q(\theta(\varphi(\eta))) \\ &= f_Q(\varphi(\eta)) \end{aligned}$$

Now $\varphi(\eta) \in M_i \subseteq M$ and $\eta \in M$ and consider

$\varphi(\eta) = 0 + 0 + ... + \varphi(\eta) + ... + 0$

Then

$$\begin{aligned} t_Q(\varphi(\eta)) &= t_Q(0 + 0 + ... + \varphi(\eta) + ... + 0) \\ &= t_{Q_1}(0) \wedge t_{Q_2}(0) \wedge ... \wedge t_{Q_i}\varphi(\eta) \wedge ... \wedge t_{Q_n}(0) \\ &= t_{Q_i}(\varphi(\eta)) \end{aligned}$$

Similarly $i_Q(\varphi(\eta)) = i_{Q_i}(\varphi(\eta))$ and

$f_Q(\varphi(\eta)) = f_{Q_i}(\varphi(\eta)) \;\forall\; i$

$\Rightarrow\; t_P(\eta) \leq t_Q(\varphi(\eta)) = t_{Q_i}(\varphi(\eta))$.

Also $i_P(\eta) \leq i_Q(\varphi(\eta)) = i_{Q_i}(\varphi(\eta)$ and

$f_P(\eta) \geq f_Q(\varphi(\eta)) = f_{Q_i}(\varphi(\eta), \forall\; \eta \in M, \varphi \in Hom_G(M, M_i)$.

Then P is Q_i projective.

Conversely Assume that P is Q_i projective where $i = 1, 2, ..., n$.Then

(1) M is M_i-projective

(2) $t_P(m) \leq t_{Q_i}(\varphi_i(m))$,
$i_P(m) \leq i_{Q_i}(\varphi_i(m)\; and$
$f_P(m) \geq f_{Q_i}(\varphi_i(m)$,
$\varphi_i \in Hom_G(M, M_i); m \in M$

To prove P is Q projective, it is enough to prove the following conditions

(1) M is M projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$
$f_P(\eta) \geq f_Q(\psi(\eta)),\; \psi \in Hom_G(M, M); \eta \in M$

1. :- Since P is Q_i projective and proposition 2.1, M is M-Projective where $M = \oplus_{i=1}^{n} M_i$.

2. :- Let $\psi \in Hom_G(M, M)\; where\; M = \oplus_{i=1}^{n} M_i$ such that $\forall\; \eta \in M$,

$\psi(\eta) \in M,\; i.e.\; \psi(\eta) = \eta_1 + \eta_2 + ... + \eta_n, \forall\; \eta_i \in M_i, 1 \leq i \leq n$ and $\pi_i : M \to M_i$ be the projection map where $i = 1, 2, ..., n$ such that $\pi_i(\psi(\eta)) = \eta_i,\ \forall\ i.$, then

$$\begin{aligned}
\psi(\eta) &= \eta_1 + \eta_2 + ... + \eta_n, \\
&\quad \forall\ \eta_i \in M_i, 1 \leq i \leq n \\
&= \pi_1(\psi(\eta)) + \pi_2(\psi(\eta)) + ... \\
&\quad ... + \pi_n(\psi(\eta)) \\
&= (\pi_1 \circ \psi)(\eta) + (\pi_2 \circ \psi)(\eta) + ... + \\
&\quad (\pi_n \circ \psi)(\eta) \\
&= \varphi_1(\eta) + \varphi_2(\eta) + ... + \varphi_n(\eta)
\end{aligned}$$

Also

$$\begin{aligned}
t_Q(\psi(\eta)) &= t_Q(\varphi_1(\eta)) + t_Q(\varphi_2(\eta)) + ... + \\
&\quad t_Q(\varphi_n(\eta)) \\
&= \wedge\{t_{Q_i}(\varphi_i(\eta)) : 0 \leq i \leq n\} \\
&\quad [\mathit{by\ the\ proposition}\ 3.1] \\
&\geq t_P(\eta)
\end{aligned}$$

Similarly $i_Q(\psi(\eta)) \geq i_P(\eta)$ and

$$\begin{aligned}
f_Q(\psi(\eta)) &= f_Q(\varphi_1(\eta)) + f_Q(\varphi_2(\eta)) + \\
&\quad ... + f_Q(\varphi_n(\eta)) \\
&\leq \vee\{f_{Q_i}(\varphi_i(\eta)) : 0 \leq i \leq n\} \\
&\leq f_P(m)
\end{aligned}$$

$\therefore$ A is Q projective. □

4. **Conclusion**

The study of G-module in a neutrosophic set domain using a single-valued neutrosophic set provides a new step in the algebra sector and helps to analyze group action in application level on a vector space. Projective G-modules expand the free G-modules class by maintaining a portion of the free module's primary properties. Neutrosophic projective G-module is one of the most generalizations of classical projective G-module. This paper has developed, the notion of projectivity of neutrosophic G-modules and its quotient and direct sum properties of M projectivity. This analysis leads to the extension of the quasi projective module, neutrosophic injective & projective modules and its features in neutrosophic domain.

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