A GENERAL MODEL OF NEUTROSOPHIC IDEALS IN BCK/BCI-ALGEBRAS BASED ON NEUTROSOPHIC POINTS

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A GENERAL MODEL OF
NEUTROSOPHIC IDEALS IN
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Abstract

More general form of \((\in, \in \lor q)\)-neutrosophic ideal is introduced, and their properties are investigated. Relations between \((\in, \in)\)-neutrosophic ideal and \((\in, \in \lor q(k_T, k_I, k_F))\)-neutrosophic ideal are discussed. Characterizations of \((\in, \in \lor q(k_T, k_I, k_F))\)-neutrosophic ideal are discussed, and conditions for a neutrosophic set to be an \((\in, \in \lor q(k_T, k_I, k_F))\)-neutrosophic ideal are displayed.

Keywords: Ideal, neutrosophic \(\in\)-subset, neutrosophic \(q_k\)-subset, neutrosophic \(\in \lor q_k\)-subset, \((\in, \in \lor q(k_T, k_I, k_F))\)-neutrosophic ideal.

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1. Introduction

Smarandache [23, 24] introduced the concept of neutrosophic sets which is a more general platform to extend the notions of the classical set and
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Song et al. [25] introduced the notion of commutative $N$-ideal in $BCK$-algebras and investigated several properties. Bordbar, Jun and et al. [21] and [17] introduced the notion of $(q, \in \lor q)$-neutrosophic ideal, and $(\epsilon, \in \lor q)$-neutrosophic ideal in $BCK/BCI$-algebras, and investigated related properties. Also in [7, 26], they discussed the notion of $BMBJ$-neutrosophic sets, subalgebra and ideals, as a generalisation of neutrosophic set, and investigated it’s application and related properties to $BCI/BCK$-algebras.

For more information about the mentioned topics, please refer to [3, 4, 8, 12, 16, 18, 19, 20].

In this paper, we introduce a more general form of $(\epsilon, \in \lor q)$-neutrosophic ideal, and investigate their properties. We discuss relations between $(\epsilon, \in)$-neutrosophic ideal and $(\epsilon, \in \lor q(k_T,k_1,k_F))$-neutrosophic ideal. We consider characterizations of $(\epsilon, \in \lor q(k_T,k_1,k_F))$-neutrosophic ideal. We investigate conditions for a neutrosophic set to be an $(\epsilon, \in \lor q(k_T,k_1,k_F))$-neutrosophic ideal. We find conditions for an $(\epsilon, \in \lor q(k_T,k_1,k_F))$-neutrosophic ideal to be an $(\epsilon, \in)$-neutrosophic ideal.

2. Preliminaries

By a $BCI$-algebra we mean a set $X$ with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0,$

(a2) $(x * (x * y)) * y = 0,$

(a3) $x * x = 0,$
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(a4) $x * y = y * x = 0 \Rightarrow x = y,$

for all $x, y, z \in X.$ If a BCI-algebra $X$ satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X,$

then we say that $X$ is a BCK-algebra. A subset $I$ of a BCK/BCI-algebra $X$ is called an ideal of $X$ (see [9, 15]) if it satisfies:

\begin{align}
0 & \in I, \\
(\forall x, y \in X) \ (x * y \in I, y \in I) & \Rightarrow x \in I.
\end{align}

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X).$

We refer the reader to the books [9] and [15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee \{a_i \mid i \in \Lambda\} = \sup \{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge \{a_i \mid i \in \Lambda\} = \inf \{a_i \mid i \in \Lambda\}.$$ 

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee \{a_i \mid i \in \{1, 2\}\}$ and $\bigwedge \{a_i \mid i \in \{1, 2\}\}$, respectively.

Let $X$ be a non-empty set. A neutrosophic set (NS) in $X$ (see [23]) is a structure of the form:

$$A := \{(x; A_T(x), A_I(x), A_F(x)) \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{(x; A_T(x), A_I(x), A_F(x)) \mid x \in X\}.$$ 

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [10]):

$$T_\varepsilon(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$$
In what follows, let \( k \). Generalizations of neutrosophic ideals based on neutrosophic points

We say that a neutrosophic set \( A \) is a neutrosophic \( k \)-point if \( k \) is denoted by \( k_T \), i.e., \( k = k_T = k_I = k_F \).

Given a neutrosophic set \( A = (A_T, A_I, A_F) \) in a set \( X \), \( \alpha, \beta, \gamma \in (0, 1] \), we consider the following sets:

\[
T_{\psi}(A; \alpha) := \{ x \in X | A_T(x) + \alpha + k_T > 1 \},
I_{\psi}(A; \beta) := \{ x \in X | A_I(x) + \beta + k_I > 1 \},
F_{\psi}(A; \gamma) := \{ x \in X | A_F(x) + \gamma + k_F < 1 \},
\]

\[
I_{\psi Treatment}(A; \beta) := \{ x \in X | A_I(x) + \beta + k_I > 1 \},
F_{\psi Treatment}(A; \gamma) := \{ x \in X | A_F(x) + \gamma + k_F < 1 \}.\]

We say that \( T_{\psi Treatment}(A; \alpha) \), \( I_{\psi Treatment}(A; \beta) \) and \( F_{\psi Treatment}(A; \gamma) \) are neutrosophic \( qk \)-subsets; and \( T_{\psi Treatment}(A; \alpha) \), \( I_{\psi Treatment}(A; \beta) \) and \( F_{\psi Treatment}(A; \gamma) \) are neutrosophic \( \psi \)-subsets. For \( \psi \in \{ \varepsilon, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in \psi, q, q_k, \in \psi k, \in \psi q, \in \psi T, \in \psi I, \in \psi F \} \), the element of \( T_{\psi}(A; \alpha) \) (resp., \( I_{\psi}(A; \beta) \) and \( F_{\psi}(A; \gamma) \)) is called a neutrosophic \( T_{\psi \text{-point}} \) (resp., neutrosophic \( I_{\psi \text{-point}} \) and neutrosophic \( F_{\psi \text{-point}} \)) with value \( \alpha \) (resp., \( \beta \) and \( \gamma \)).

It is clear that

\[
T_{\psi Treatment}(A; \alpha) = T_{\varepsilon}(A; \alpha) \cup T_{qT}(A; \alpha),
I_{\psi Treatment}(A; \beta) = I_{\varepsilon}(A; \beta) \cup I_{qI}(A; \beta),
F_{\psi Treatment}(A; \gamma) = F_{\varepsilon}(A; \gamma) \cup F_{qF}(A; \gamma).
\]

**Theorem 3.1.** Given a neutrosophic set \( A = (A_T, A_I, A_F) \) in \( X \in B(X) \), the following assertions are equivalent.
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(1) The nonempty neutrosophic ε-subsets $T_ε(A; α), I_ε(A; β)$ and $F_ε(A; γ)$ are ideals of $X$ for all $α ∈ (\frac{1-κ_T}{2}, 1], β ∈ (\frac{1-κ_I}{2}, 1]$ and $γ ∈ [0, \frac{1-κ_F}{2})$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$
(∀x ∈ X) \begin{cases}
A_T(x) ≤ A_T(0) \lor \frac{1-κ_T}{2} \\
A_I(x) ≤ A_I(0) \lor \frac{1-κ_I}{2} \\
A_F(x) ≥ A_F(0) \land \frac{1-κ_F}{2}
\end{cases}
$$

and

$$
(∀x, y ∈ X) \begin{cases}
A_T(x) \lor \frac{1-κ_T}{2} ≥ A_T(x \ast y) \land A_T(y) \\
A_I(x) \lor \frac{1-κ_I}{2} ≥ A_I(x \ast y) \land A_I(y) \\
A_F(x) \land \frac{1-κ_F}{2} ≤ A_F(x \ast y) \lor A_F(y)
\end{cases}
$$

PROOF: Assume that the nonempty neutrosophic ε-subsets $T_ε(A; α), I_ε(A; β)$ and $F_ε(A; γ)$ are ideals of $X$ for all $α ∈ (\frac{1-κ_T}{2}, 1], β ∈ (\frac{1-κ_I}{2}, 1]$ and $γ ∈ [0, \frac{1-κ_F}{2})$. If there are $a, b ∈ X$ such that $A_T(a) > A_T(0) \lor \frac{1-κ_T}{2}$, then $a ∈ T_ε(A; α_a)$ and $0 \notin T_ε(A; α_a)$ for $α_a := A_T(a) ∈ (\frac{1-κ_T}{2}, 1]$. This is a contradiction, and so $A_T(x) ≤ A_T(0) \lor \frac{1-κ_T}{2}$ for all $x ∈ X$. We also know that $A_I(x) ≤ A_I(0) \lor \frac{1-κ_I}{2}$ for all $x ∈ X$ by the similar way. Now, let $x ∈ X$ be such that $A_F(x) < A_F(0) \lor \frac{1-κ_F}{2}$. If we take $γ_x := A_F(x)$, then $γ_x ∈ [0, \frac{1-κ_F}{2})$ and $0 \notin F_ε(A; γ_x)$ since $F_ε(A; γ_x)$ is an ideal of $X$. Hence $A_F(0) ≤ γ_x = A_F(0)$, which is a contradiction. Hence $A_F(x) ≥ A_F(0) \lor \frac{1-κ_F}{2}$ for all $x ∈ X$. Suppose that $A_I(x) \lor \frac{1-κ_I}{2} < A_I(x \ast y) \land A_I(y)$ for some $x, y ∈ X$ and take $β := A_I(x \ast y) \land A_I(y)$. Then $β ∈ (\frac{1-κ_I}{2}, 1]$ and $x \ast y, y ∈ I_ε(A; β)$. But $x \notin I_ε(A; β)$ which is a contradiction. Thus $A_I(x) \lor \frac{1-κ_I}{2} ≥ A_I(x \ast y) \land A_I(y)$ for all $x, y ∈ X$. Similarly, we have $A_T(x) \lor \frac{1-κ_T}{2} ≥ A_T(x \ast y) \land A_T(y)$ for all $x, y ∈ X$. Suppose that there exist $x, y ∈ X$ such that $A_F(x) \land \frac{1-κ_F}{2} > A_F(x \ast y) \lor A_F(y)$. Taking $γ := A_F(x \ast y) \lor A_F(y)$ implies that $γ ∈ [0, \frac{1-κ_F}{2})$, $x \ast y ∈ F_ε(A; γ)$ and $y ∈ F_ε(A; γ)$, but $x \notin F_ε(A; γ)$. This is a contradiction, and so $A_F(x) \land \frac{1-κ_F}{2} ≤ A_F(x \ast y) \lor A_F(y)$ for all $x, y ∈ X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.4) and (3.5). Let $α ∈ (\frac{1-κ_T}{2}, 1], β ∈ (\frac{1-κ_I}{2}, 1]$ and $γ ∈ [0, \frac{1-κ_F}{2})$ be such that $T_ε(A; α), I_ε(A; β)$ and $F_ε(A; γ)$ are nonempty. For any $x ∈ T_ε(A; α), y ∈ I_ε(A; β)$ and $z ∈ F_ε(A; γ)$, we get
and so $A_T(0) \geq \alpha, A_I(0) \geq \beta$ and $A_F(0) \leq \gamma$. Hence $0 \in T_{\varepsilon}(A;\alpha)$, $0 \in I_{\varepsilon}(A;\beta)$ and $0 \in F_{\varepsilon}(A;\gamma)$. Let $a,b,x,y,u,v \in X$ be such that $a \ast b \in T_{\varepsilon}(A;\alpha)$, $b \in T_{\varepsilon}(A;\alpha)$, $x \ast y \in I_{\varepsilon}(A;\beta)$, $y \in I_{\varepsilon}(A;\beta)$, $u \ast v \in F_{\varepsilon}(A;\gamma)$, and $v \in F_{\varepsilon}(A;\gamma)$. It follows from (3.5) that

$$A_T(a) \lor \frac{1-k_T}{2} \geq A_T(a \ast b) \land A_T(b) \geq \alpha > \frac{1-k_T}{2},$$

$$A_I(x) \lor \frac{1-k_I}{2} \geq A_I(x \ast y) \land A_I(y) \geq \beta > \frac{1-k_I}{2},$$

$$A_F(u) \land \frac{1-k_F}{2} \leq A_F(u \ast v) \lor A_F(v) \leq \gamma < \frac{1-k_F}{2}. $$

Hence $A_T(a) \geq \alpha, A_I(x) \geq \beta$ and $A_F(u) \leq \gamma$, that is, $a \in T_{\varepsilon}(A;\alpha)$, $x \in I_{\varepsilon}(A;\beta)$ and $u \in F_{\varepsilon}(A;\gamma)$. Therefore $T_{\varepsilon}(A;\alpha)$, $I_{\varepsilon}(A;\beta)$ and $F_{\varepsilon}(A;\gamma)$ are ideals of $X$ for all $\alpha \in \left(\frac{1-k_T}{2}, 1\right]$, $\beta \in \left(\frac{1-k_I}{2}, 1\right]$ and $\gamma \in [0, \frac{1-k_F}{2}).$ \qed

**Corollary 3.2 ([21]).** Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.

(1) The nonempty neutrosophic $\varepsilon$-subsets $T_{\varepsilon}(A;\alpha)$, $I_{\varepsilon}(A;\beta)$ and $F_{\varepsilon}(A;\gamma)$ are ideals of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0, 0.5)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$\left(\forall x \in X\right) \begin{cases} A_T(x) \leq A_T(0) \lor 0.5 \\ A_I(x) \leq A_I(0) \lor 0.5 \\ A_F(x) \geq A_F(0) \land 0.5 \end{cases}$$

and

$$\left(\forall x, y \in X\right) \begin{cases} A_T(x) \lor 0.5 \geq A_T(x \ast y) \land A_T(y) \\ A_I(x) \lor 0.5 \geq A_I(x \ast y) \land A_I(y) \\ A_F(x) \land 0.5 \leq A_F(x \ast y) \lor A_F(y) \end{cases}$$

**Definition 3.3.** A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\varepsilon, \varepsilon \lor q_{(k_T, k_I, k_F)})$-neutrosophic ideal of $X$ if the following assertions are valid.
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\[(\forall x \in X) \begin{cases} x \in T_e(A; \alpha_x) &\Rightarrow 0 \in T_{e \vee q_{x,y}}(A; \alpha_x) \\ x \in I_e(A; \beta_x) &\Rightarrow 0 \in I_{e \vee q_{x,y}}(A; \beta_x) \\ x \in F_e(A; \gamma_x) &\Rightarrow 0 \in F_{e \vee q_{x,y}}(A; \gamma_x) \end{cases}, \quad (3.6)\]

\[(\forall x, y \in X) \begin{cases} x \ast y \in T_e(A; \alpha_x), y \in T_e(A; \alpha_y) &\Rightarrow x \in T_{e \vee q_{x,y}}(A; \alpha_x \land \alpha_y) \\ x \ast y \in I_e(A; \beta_x), y \in I_e(A; \beta_y) &\Rightarrow x \in I_{e \vee q_{x,y}}(A; \beta_x \land \beta_y) \\ x \ast y \in F_e(A; \gamma_x), y \in F_e(A; \gamma_y) &\Rightarrow x \in F_{e \vee q_{x,y}}(A; \gamma_x \lor \gamma_y) \end{cases}, \quad (3.7)\]

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

**Example 3.4.** Let $X = \{0,1,2,3,4\}$ be a set with the binary operation $\ast$ which is given in Table 1.

**Table 1:** Cayley table for the binary operation “$\ast$”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X, \ast, 0)$ is a $BCK$-algebra (see [15]). Consider a neutrosophic set $A = (A_T, A_I, A_F)$ in $X$ which is given by Table 2.

**Table 2:** Tabular representation of $A = (A_T, A_I, A_F)$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$A_T(x)$</th>
<th>$A_I(x)$</th>
<th>$A_F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6</td>
<td>0.5</td>
<td>0.45</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.3</td>
<td>0.93</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.7</td>
<td>0.67</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>0.3</td>
<td>0.93</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
<td>0.2</td>
<td>0.74</td>
</tr>
</tbody>
</table>
Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \in \cup q(k_T, k_I, k_F))$-neutrosophic ideal of $X$ for $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$.

**Theorem 3.5.** A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in B(X)$ is an $(\varepsilon, \in \cup q(k_T, k_I, k_F))$-neutrosophic ideal of $X \in B(X)$ if and only if $A = (A_T, A_I, A_F)$ satisfies the following assertions.

\[
(\forall x \in X) \begin{cases} 
A_T(0) \geq A_T(x) \land \frac{1-k_T}{2} \\
A_I(0) \geq A_I(x) \land \frac{1-k_I}{2} \\
A_F(0) \leq A_F(x) \lor \frac{1-k_F}{2}
\end{cases}, \quad (3.8)
\]

\[
(\forall x, y \in X) \begin{cases} 
A_T(x) \geq \bigwedge \{A_T(x+y), A_I(y), \frac{1-k_T}{2}\} \\
A_I(x) \geq \bigwedge \{A_I(x+y), A_I(y), \frac{1-k_I}{2}\} \\
A_F(x) \leq \bigvee \{A_F(x+y), A_F(y), \frac{1-k_F}{2}\}
\end{cases}. \quad (3.9)
\]

**Proof:** Assume that $A = (A_T, A_I, A_F)$ in $X \in B(X)$ is an $(\varepsilon, \in \cup q(k_T, k_I, k_F))$-neutrosophic ideal of $X \in B(X)$. If $A_T(0) < A_T(a) \land \frac{1-k_T}{2}$ for some $a \in X$, then there exists $\alpha_a \in (0,1]$ such that $A_T(0) < \alpha_a \leq A_T(a) \land \frac{1-k_T}{2}$. It follows that $\alpha_a \in (0, \frac{1-k_T}{2}]$, $a \in T_\varepsilon(A; \alpha_a)$ and $0 \notin T_\varepsilon(A; \alpha_a)$. Also, $A_T(0) + \alpha_a + k_T < 2\alpha_a + k_T \leq 1$, i.e., $0 \notin T_{q_F}(A; \alpha_a)$. Hence $0 \notin T_{\varepsilon \cup q_F}(A; \alpha_a)$, a contradiction. Thus $A_T(0) \geq A_T(x) \land \frac{1-k_T}{2}$ for all $x \in X$. Similarly, we have $A_I(0) \geq A_I(x) \land \frac{1-k_I}{2}$ for all $x \in X$. Suppose that $A_F(0) > A_F(z) \lor \frac{1-k_F}{2}$ for some $z \in X$ and take $\gamma_z := A_F(z) \lor \frac{1-k_F}{2}$. Then $\gamma_z \geq \frac{1-k_F}{2}$, $z \in F_\varepsilon(A; \gamma_z)$ and $0 \notin F_\varepsilon(A; \gamma_z)$. Also $A_F(0) + \gamma_z + k_F \geq 1$, that is, $0 \notin F_{q_F}(A; \gamma_z)$. This is a contradiction, and thus $A_F(0) \leq A_F(x) \lor \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(a) < \bigwedge \{A_I(a \ast b), A_I(b), \frac{1-k_I}{2}\}$ for some $a, b \in X$ and take $\beta := \bigwedge \{A_I(a \ast b), A_I(b), \frac{1-k_I}{2}\}$. Then $\beta \leq \frac{1-k_I}{2}$, $a \ast b \in I_\varepsilon(A; \beta)$, $b \in I_\varepsilon(A; \beta)$ and $a \notin I_\varepsilon(A; \beta)$. Also, we have $A_I(a) + \beta + k_I \leq 1$, i.e., $a \notin I_{q_F}(A; \beta)$. This is impossible, and therefore $A_I(x) \geq \bigwedge \{A_I(x \ast y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. By the similar way, we can verify that $A_T(x) \geq \bigwedge \{A_T(x \ast y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now assume that $A_F(a) > \bigvee \{A_F(a \ast b), A_F(b), \frac{1-k_F}{2}\}$ for some $a, b \in X$. Then there exists $\gamma \in [0,1]$ such that $A_F(a) > \gamma \geq \bigvee \{A_F(a \ast b), A_F(b), \frac{1-k_F}{2}\}$. Then $\gamma \geq \frac{1-k_F}{2}$, $a \ast b \in F_\varepsilon(A; \gamma)$, $b \in F_\varepsilon(A; \gamma)$ and $a \notin F_\varepsilon(A; \gamma)$. Also, $A_F(a) + \gamma + k_F \geq 1$, i.e., $a \notin F_{q_F}(A; \gamma)$. Thus $a \notin F_{\varepsilon \cup q_F}(A; \gamma)$, which is a contradiction. Hence $A_F(x) \leq \bigvee \{A_F(x \ast y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. 

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Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.8) and (3.9). For any $x, y, z \in X$, let $\alpha_x, \beta_y, \gamma_z \in (0, 1]$ be such that $x \in T_{\alpha_x}(A; \beta_y), \ y \in I_{\beta_y}(A; \gamma_z)$ and $z \in F_{\gamma_z}(A; \alpha_x)$. Then $A_T(x) \geq \alpha_x, \ A_I(y) \geq \beta_y$ and $A_F(z) \leq \gamma_z$. Assume that $A_T(0) < \alpha_x, \ A_I(0) < \beta_y$ and $A_F(0) > \gamma_z$. If $A_T(x) < \frac{1-k_T}{2}$, then

$$A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2} = A_T(x) \geq \alpha_x,$$

a contradiction. Hence $A_T(x) \geq \frac{1-k_T}{2}$, and so

$$A_T(0) + \alpha_x + k_T > 2A_T(0) + k_T \geq 2 \left( A_T(x) \wedge \frac{1-k_T}{2} \right) + k_T = 1.$$

Hence $0 \in T_{\alpha_x}(A; A_T(x)) \subseteq T_{\alpha_x}(A; \alpha_x)$. Similarly, we get $0 \in I_{\beta_y}(A; \beta_y) \subseteq I_{\beta_y}(A; \beta_y)$. If $A_F(z) > \frac{1-k_F}{2}$, then $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2} = A_F(z) \leq \gamma_z$ which is a contradiction. Hence $A_F(z) \leq \frac{1-k_F}{2}$, and thus

$$A_F(0) + \gamma_z + k_F < 2A_F(0) + k_F \leq 2 \left( A_F(z) \vee \frac{1-k_F}{2} \right) + k_F = 1.$$

Hence $0 \in F_{\alpha_x}(A; \gamma_z) \subseteq F_{\alpha_x}(A; \gamma_z)$. For any $a, b, p, q, x, y \in X$, let $\alpha_a, \alpha_b, \beta_p, \beta_q \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $a \ast b \in T_{\alpha_a}(A; \alpha_a), \ b \in T_{\alpha_b}(A; \alpha_b), \ p \ast q \in I_{\beta_p}(A; \beta_p), \ q \in I_{\beta_q}(A; \beta_q), \ x \ast y \in F_{\gamma_x}(A; \gamma_x)$, and $y \in F_{\gamma_y}(A; \gamma_y)$. Then $A_T(a \ast b) \geq \alpha_a, \ A_T(b) \geq \alpha_b, \ A_I(p \ast q) \geq \beta_p, \ A_I(q) \geq \beta_q, \ A_F(x \ast y) \leq \gamma_x$, and $A_F(y) \leq \gamma_y$. Suppose that $a \notin T_{\alpha_a}(A; \alpha_a \land \alpha_b)$. Then $A_T(a) < \alpha_a \land \alpha_b$. If $A_T(a \ast b) \land A_T(b) < \frac{1-k_T}{2}$, then

$$A_T(a) \geq \bigwedge \{ A_T(a \ast b), A_T(b), \frac{1-k_T}{2} \} = A_T(a \ast b) \land A_T(b) \geq \alpha_a \land \alpha_b.$$

This is a contradiction, and so $A_T(a \ast b) \land A_T(b) \geq \frac{1-k_T}{2}$. Thus

$$A_T(a) + (\alpha_a \land \alpha_b) + k_T > 2A_T(a) + k_T \geq 2 \left( \bigwedge \{ A_T(a \ast b), A_T(b), \frac{1-k_T}{2} \} \right) + k_T = 1,$$

which induces $a \in T_{\alpha_a}(A; \alpha_a \land \alpha_b) \subseteq T_{\alpha_a}(A; \alpha_a \land \alpha_b)$. By the similarly way, we get $p \in I_{\beta_p}(A; \beta_p \land \beta_q)$, and thus $A_I(p \ast q) \geq \beta_p \land \beta_q$. Suppose that $x \notin F_{\gamma_x}(A; \gamma_x \lor \gamma_y)$, that is, $A_F(x) > \gamma_x \lor \gamma_y$. If $A_F(x \ast y) \lor A_F(y) > \frac{1-k_F}{2}$, then

$$A_F(x) \leq \bigvee \{ A_F(x \ast y), A_F(y), \frac{1-k_F}{2} \} = A_F(x \ast y) \lor A_F(y) \leq \gamma_x \lor \gamma_y,$$

which is impossible. Thus $A_F(x \ast y) \lor A_F(y) \leq \frac{1-k_F}{2}$, and so
A_F(x) + (\gamma_x \lor \gamma_y) + k_F < 2A_F(x) \\
\leq 2 \left( \bigvee \{A_F(x \ast y), A_F(y), \frac{1-k_F}{2}\} \right) + k_F = 1.

This implies that \( x \in F_{q_{k_F}}(A; \gamma_x \lor \gamma_y) \subseteq F_{\lor q_{k_F}}(A; \gamma_x \lor \gamma_y) \). Consequently, \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q)\)-neutrosophic ideal of \( X \in \mathcal{B}(X) \).

**Corollary 3.6 ([21]).** For a neutrosophic set \( A = (A_T, A_I, A_F) \) in \( X \in \mathcal{B}(X) \), the following are equivalent.

1. \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q)\)-neutrosophic ideal of \( X \in \mathcal{B}(X) \).
2. \( A = (A_T, A_I, A_F) \) satisfies the following assertions.

\[
(\forall x \in X) \begin{pmatrix}
A_T(0) \geq A_T(x) \land 0.5 \\
A_I(0) \geq A_I(x) \land 0.5 \\
A_F(0) \leq A_F(x) \lor 0.5
\end{pmatrix},
\]

\[
(\forall x, y \in X) \begin{pmatrix}
A_T(x) \geq \bigwedge \{A_T(x \ast y), A_T(y), 0.5\} \\
A_I(x) \geq \bigwedge \{A_I(x \ast y), A_I(y), 0.5\} \\
A_F(x) \leq \bigvee \{A_F(x \ast y), A_F(y), 0.5\}
\end{pmatrix}.
\]

**Theorem 3.7.** A neutrosophic set \( A = (A_T, A_I, A_F) \) in \( X \in \mathcal{B}(X) \) is an \((\in, \in \lor q_{(k_T,k_I,k_F)})\)-neutrosophic ideal of \( X \in \mathcal{B}(X) \) if and only if the nonempty neutrosophic \( \in \)-subsets \( T_\in(A; \alpha), I_\in(A; \beta) \) and \( F_\in(A; \gamma) \) are ideals of \( X \) for all \( \alpha \in (0, \frac{1-k_F}{2}], \beta \in (0, \frac{1-k_I}{2}] \) and \( \gamma \in [\frac{1-k_F}{2}, 1) \).

**Proof:** Suppose that \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q_{(k_T,k_I,k_F)})\)-neutrosophic ideal of \( X \in \mathcal{B}(X) \) and let \( \alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}] \) and \( \gamma \in [\frac{1-k_F}{2}, 1) \) be such that \( T_\in(A; \alpha), I_\in(A; \beta) \) and \( F_\in(A; \gamma) \) are nonempty. Using (3.4), we get \( A_T(0) \geq A_T(x) \land \frac{1-k_T}{2}, A_I(0) \geq A_I(y) \land \frac{1-k_I}{2}, \) and \( A_F(0) \leq A_F(z) \lor \frac{1-k_F}{2} \) for all \( x \in T_\in(A; \alpha), y \in I_\in(A; \beta) \) and \( z \in F_\in(A; \gamma) \). It follows that \( A_T(0) \geq \alpha \land \frac{1-k_T}{2} = \alpha, A_I(0) \geq \beta \land \frac{1-k_I}{2} = \beta, \) and \( A_F(0) \leq \gamma \lor \frac{1-k_F}{2} = \gamma, \) that is, \( 0 \in T_\in(A; \alpha), 0 \in I_\in(A; \beta) \) and \( 0 \in F_\in(A; \gamma) \).

Let \( x, y, a, b, u, v \in X \) be such that \( x \ast y \in T_\in(A; \alpha), y \in T_\in(A; \alpha), a \ast b \in I_\in(A; \beta), b \in I_\in(A; \beta), u \ast v \in F_\in(A; \gamma), \) and \( v \in F_\in(A; \gamma) \) for \( \alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}] \) and \( \gamma \in [\frac{1-k_F}{2}, 1) \). Then \( A_T(x \ast y) \geq \alpha, A_T(y) \geq \alpha, A_I(a \ast b) \geq \beta, A_I(b) \geq \beta, A_F(u \ast v) \leq \gamma, \) and \( A_F(v) \leq \gamma \). It follows from (3.5) that
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$A_T(x) \geq \bigwedge \{A_T(x \ast y), A_T(y), \frac{1-k_T}{2}\} \geq \alpha \wedge \frac{1-k_T}{2} = \alpha,$

$A_I(a) \geq \bigwedge \{A_I(a \ast b), A_I(b), \frac{1-k_I}{2}\} \geq \beta \wedge \frac{1-k_I}{2} = \beta,$

$A_F(u) \leq \bigvee \{A_F(u \ast v), A_F(v), \frac{1-k_F}{2}\} \leq \gamma \vee \frac{1-k_F}{2} = \gamma$

and so that $x \in T_\varepsilon(A; \alpha), a \in I_\varepsilon(A; \beta)$ and $u \in F_\varepsilon(A; \gamma)$. Therefore $T_\varepsilon(A; \alpha), I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in \{\frac{1-k_F}{2}, 1\}$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in B(X)$ such that the nonempty neutrosophic $\varepsilon$-subsets $T_\varepsilon(A; \alpha), I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in \{\frac{1-k_F}{2}, 1\}$.

Therefore $T_\varepsilon(A; \alpha), I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in \{\frac{1-k_F}{2}, 1\}$.

Corollary 3.8 ([21]). A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in B(X)$ is an $(\varepsilon, \in, \exists q)$-neutrosophic ideal of $X \in B(X)$ if and only if the nonempty neutrosophic $\varepsilon$-subsets $T_\varepsilon(A; \alpha), I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}], \beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in \{\frac{1-k_F}{2}, 1\}$.

It is clear that every $(\varepsilon, \in, \exists q)$-neutrosophic ideal is an $(\varepsilon, \in, \exists q)$-neutrosophic ideal. But the converse is not true in general. For example, the $(\varepsilon, \in, \exists q)$-neutrosophic ideal $A = (A_T, A_I, A_F)$ with $k_T = 0.24, k_I = 0.08$ and $k_F = 0.16$ in Example 3.4 is not an $(\varepsilon, \in, \exists q)$-neutrosophic ideal since $2 \in I_\varepsilon(A; 0.56)$ and $0 \notin I_\varepsilon(A; 0.56)$. 
We now consider conditions for an \((\in, \in \vee q_{kT, k_i, k_F})\)-neutrosophic ideal to be an \((\in, \in)\)-neutrosophic ideal.

**Theorem 3.9.** Let \(A = (A_T, A_I, A_F)\) be an \((\in, \in \vee q_{kT, k_i, k_F})\)-neutrosophic ideal of \(X \in B(X)\) such that
\[
(\forall x \in X) \left( A_T(x) < \frac{1-k_T}{2}, A_I(x) < \frac{1-k_i}{2}, A_F(x) > \frac{1-k_F}{2} \right).
\]
Then \(A = (A_T, A_I, A_F)\) is an \((\in, \in)\)-neutrosophic ideal of \(X \in B(X)\).

**Proof:** Let \(x, y, z \in X\), \(\alpha, \beta \in [0,1]\) and \(\gamma \in [0,1]\) be such that \(x \in T_{\in} (A; \alpha), y \in I_{\in} (A; \beta)\) and \(z \in F_{\in} (A; \gamma)\). Then \(A_T(x) \geq \alpha, A_I(y) \geq \beta\) and \(A_F(z) \geq \gamma\). It follows from (3.9) that
\[
A_T(0) \geq A_T(x) \land \frac{1-k_T}{2} = A_T(x) \geq \alpha,
\]
\[
A_I(0) \geq A_I(y) \land \frac{1-k_i}{2} = A_I(y) \geq \beta,
\]
\[
A_F(0) \leq A_F(z) \lor \frac{1-k_F}{2} = A_F(z) \leq \gamma.
\]
Hence \(0 \in T_{\in} (A; \alpha), 0 \in I_{\in} (A; \beta)\) and \(0 \in F_{\in} (A; \gamma)\). For any \(x, y, a, b, u, v \in X\), let \(x_{\in}, \alpha, \beta, \gamma \in [0,1]\) such that \(x \in T_{\in} (A; \alpha), y \in I_{\in} (A; \beta), a \in I_{\in} (A; \beta), b \in I_{\in} (A; \beta), u \in I_{\in} (A; \beta), v \in I_{\in} (A; \beta)\). Then \(A_T(x \land y) \geq \alpha, A_I(y) \geq \beta, A_I(a \land b) \geq \beta, A_T(0) \leq \gamma, A_F(0) \leq \gamma\). It follows from (3.9) that
\[
A_T(x) \geq \land \{A_T(x \land y), A_T(y), \frac{1-k_T}{2}\} = A_T(x \land y) \land A_I(y) \geq \alpha \land \alpha,
\]
\[
A_I(a) \geq \land \{A_I(a \land b), A_I(b), \frac{1-k_i}{2}\} = A_I(a \land b) \land A_I(b) \geq \beta \land \beta,
\]
\[
A_T(u) \leq \lor \{A_T(u \land v), A_T(v), \frac{1-k_F}{2}\} = A_T(u \land v) \lor A_T(v) \leq \gamma \lor \gamma.
\]
Thus \(x \in T_{\in} (A; \alpha \land \alpha), a \in I_{\in} (A; \beta \land \beta)\) and \(u \in I_{\in} (A; \gamma \lor \gamma)\). Therefore \(A = (A_T, A_I, A_F)\) is an \((\in, \in)\)-neutrosophic ideal of \(X \in B(X)\).

**Corollary 3.10 ([21]).** Let \(A = (A_T, A_I, A_F)\) be an \((\in, \in \vee q)\)-neutrosophic ideal of \(X \in B(X)\) such that
\[
(\forall x \in X) \left( A_T(x) < 0.5, A_I(x) < 0.5, A_F(x) > 0.5 \right).
\]
Then \(A = (A_T, A_I, A_F)\) is an \((\in, \in)\)-neutrosophic ideal of \(X \in B(X)\).

**Theorem 3.11.** Given a neutrosophic set \(A = (A_T, A_I, A_F)\) in \(X \in B(X)\), if the nonempty neutrosophic \(v \in \vee q_k\)-subsets \(T_{\in \vee q_k} (A; \alpha), I_{\in \vee q_k} (A; \beta)\)
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and $F_{E\vee q\vee}(A;\gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_T}{2}, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X$.

**PROOF:** Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in B(X)$ such that the nonempty neutrosophic $\in \vee q_k$-subsets $T_{E\vee q\vee}(A;\alpha)$, $I_{E\vee q\vee}(A;\beta)$ and $F_{E\vee q\vee}(A;\gamma)$ are ideals of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_T}{2}, 1)$. If $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2} := \alpha_x$, $A_T(0) < A_T(y) \wedge \frac{1-k_T}{2} := \beta_y$ and $A_F(0) > A_F(z) \vee \frac{1-k_T}{2} := \gamma_z$ for some $x, y, z \in X$, then $x \in T_{\alpha}(A;\alpha_x) \subseteq T_{E\vee q\vee}(A;\alpha_x)$, $y \in I_{\beta}(A;\beta_y) \subseteq I_{E\vee q\vee}(A;\beta_y)$, $z \in F_{\gamma}(A;\gamma_z) \subseteq F_{E\vee q\vee}(A;\gamma_z)$, $0 \notin T_{\alpha}(A;\alpha_x)$, $0 \notin I_{\beta}(A;\beta_y)$, and $0 \notin F_{\gamma}(A;\gamma_z)$. Also, since $A_T(0) + \alpha_x + k_T < 2\alpha_x + k_T \leq 1$, i.e., $0 \notin T_{\alpha}(A;\alpha_x)$, $A_T(0) + \beta_y + k_I < 2\beta_y + k_I \leq 1$, i.e., $0 \notin I_{\beta}(A;\beta_y)$. $A_F(0) + \gamma_z + k_F > 2\gamma_z + k_F \geq 1$, i.e., $0 \notin F_{\gamma}(A;\gamma_z)$. This is a contradiction, and thus (3.8) is valid. Suppose that there exist $a, b \in X$ such that $A_I(a) \in \{A_I(a * b), A_I(b), \frac{1-k_T}{2}\}$. Taking $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_T}{2}\}$ implies that $a \ast b \in I_{E\vee q\vee}(A;\beta)$, $b \in I_{E\vee q\vee}(A;\beta)$, and so that $a \in I_{q}(A;\beta)$, i.e., $A_I(a) + \beta + k_I > 1$, since $a \notin I_{E\vee q\vee}(A;\beta)$. But $A_I(a) + \beta + k_I < 2\beta + k_I \leq 1$, a contradiction. Hence $A_I(x) \geq \bigvee\{A_I(x * y), A_I(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Similarly, we can verify that $A_T(x) \geq \bigvee\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Assume that $A_F(a) \in \bigvee\{A_F(a * b), A_F(b), \frac{1-k_T}{2}\} := \gamma$ for some $a, b \in X$. Then $a \notin F_{E\vee q\vee}(A;\gamma)$, $a \ast b \in F_{E\vee q\vee}(A;\gamma)$, $b \in F_{E\vee q\vee}(A;\gamma) \subseteq F_{E\vee q\vee}(A;\gamma)$. Since $F_{E\vee q\vee}(A;\gamma)$ is an ideal of $X$, we have $a \in F_{E\vee q\vee}(A;\gamma)$. On the other hand, $A_F(a) + \gamma + k_F > 2\gamma + k_F \geq 1$, that is, $a \notin F_{E\vee q\vee}(A;\gamma)$. Hence $a \notin y \in X$. Therefore (3.9) is valid, and consequently $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X$ by Theorem 3.5.

Corollary 3.12 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in B(X)$, if the nonempty neutrosophic $\in \vee q$-subsets $T_{E\vee q\vee}(A;\alpha)$, $I_{E\vee q\vee}(A;\beta)$ and $F_{E\vee q\vee}(A;\gamma)$ are ideals of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic ideal of $X$. \qed
4. Conclusions

More general form of \((\in, \in \lor q)\)-neutrosophic ideal was introduced, and their properties were investigated. Relations between \((\in, \in)\)-neutrosophic ideal and \((\in, \in \lor q(k_T,k_I,k_F))\)-neutrosophic ideal were discussed. Characterizations of \((\in, \in \lor q(k_T,k_I,k_F))\)-neutrosophic ideal were discussed, and conditions for a neutrosophic set to be an \((\in, \in \lor q(k_T,k_I,k_F))\)-neutrosophic ideal were displayed.

These results can be applied to characterize the neutrosophic ideals in a \(BCK/BCI\)-algebra. In our future research, we will focus on some properties of ideal such as intersections, unions, maximality, primeness and height, and try to find the relations between these properties of ideals and the results of this paper. For instance, how we can define the prime and maximal neutrosophic ideals? What is the meaning of height of these types of ideals? For information about the maximality, primeness and height of ideals, please refer to [1, 2, 6, 5].

References


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