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# Neutrosophic Fixed Point Theorems and Cone Metric Spaces

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**Abstract**. The intention of this paper is to give the general definition of cone metric space in the context of the neutrosophic theory. In this relation, we obtain some fundamental results concerting fixed points for weakly compatible mapping.

**Keywords:** neutrosophic theory, neutrosophic Fixed Point, neutrosophic topology, neutrosophic cone metric space, neutrosophic metric space.

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## 1. **Introduction**

Zadeh [13] introduced the notion of fuzzy sets. After that there have been a number of generalizations of this fundamental concept. The study of fuzzy topological spaces was first initiated by Chang [6] in the year 1968. Atanassov [12] introduced the notion of intuitionistic fuzzy sets. This notion was extended to intuitionistic $L$-fuzzy setting by Atanassov and Stoeva [20], which currently holds the name "intuitionistic $L$-topological spaces". Using the notion of intuitionistic fuzzy sets, Coker [7] introduced the notion of intuitionistic fuzzy topological space. The concept of generalized fuzzy closed set was introduced by G. Balasubramanian and P. Sundaram [11]. In various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). F. Smarandache and A. Al Shumrani also defined the notion of neutrosophic topology on the non-standard interval [2, 9, 14, 16]. Also, ( [8, 15, 17]) introduced the metric topology and neutrosophic geometric and studied various properties. Recently, Wadei Al-Omeri and Smarandache [18, 19] introduce and study the concepts of neutrosophic open sets and its complements in neutrosophic topological space, continuity in neutrosophic topology, and obtain some characterizations concerning neutrosophic connectedness and neutrosophic mapping.

This paper is arranged as follows. In Section 2, we will recall some notions which will be used throughout this paper. In Section 3, neutrosophic Cone Metric Space and investigate its basic properties. In Section 4, we study the neutrosophic Fixed Point Theorems and study some of their properties. Finally, Banach contraction theorem and some fixed point results on neutrosophic cone metric space are stated and proved.

## 2. Preliminaries

**Definition 2.1.** [4] Let $\Sigma$ be a non-empty fixed set. A neutrosophic set (briefly $NS$) $B$ is an object having the form $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$, where $\xi_B(r), \varrho_B(r)$, and $\eta_B(r)$ which represent the degree of membership function (namely $\xi_B(r)$), the degree of indeterminacy (namely $\varrho_B(r)$), and the degree of non-membership (namely $\eta_B(r)$ ) respectively, of each element $r \in \Sigma$ to the set $B$.

A neutrosophic set $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$ can be identified to an ordered triple $\langle \xi_B(r), \varrho_B(r)$
$, \eta_B(r)\rangle$ in $\rfloor 0^-, 1^+\lfloor$ on $\Sigma$.

*Remark* 2.1. [4] For the sake of simplicity, we shall use the symbol $B = \{r, \xi_B(r),$
$\varrho_B(r), \eta_B(r)\}$ for the NS $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$.

**Definition 2.2.** [5] Let $B = \langle \xi_B(r), \varrho_B(r), \eta_B(r)\rangle$ be an $NS$ on $\Sigma$. The complement of $B$(briefly$C(B)$), are defined as three types of complements

(1) $C(B) = \{\langle r, \eta_B(r), 1 - \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$ ,
(2) $C(B) = \{\langle r, 1 - \xi_B(r), 1 - \eta_B(r)\rangle : r \in \Sigma\}$
(3) $C(B) = \{\langle r, \eta_B(r), \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$

We have the following NSs (see [4]) which will be used in the sequel:

(1) $0_N = \{\langle r, 0, 0, 1\rangle : r \in \Sigma\}$ or
(2) $0_N = \{\langle r, 0, 1, 1\rangle : r \in \Sigma\}$ or
(3) $0_N = \{\langle r, 0, 0, 0\rangle : r \in \Sigma\}$ or
(4) $0_N = \{\langle r, 0, 1, 0\rangle : r \in \Sigma\}$

2- $1_N$ may be defined as four types:

(1) $1_N = \{\langle r, 1, 1, 1\rangle : r \in \Sigma\}$ or
(2) $1_N = \{\langle r, 1, 0, 0\rangle : r \in \Sigma\}$ or
(3) $1_N = \{\langle r, 1, 1, 0\rangle : r \in \Sigma\}$ or

(4) $1_N = \{\langle r, 1, 0, 1\rangle : r \in \Sigma\}$

**Definition 2.3.** [4] Let $x \neq \emptyset$, and generalized neutrosophic sets ($GNSs$) $B$ and $\Gamma$ be in the form $B = \{r, \xi_B(r), \varrho_B(r), \eta_B(r)\}$, $\Gamma = \{r, \xi_\Gamma(r), \varrho_\Gamma(r), \eta_\Gamma(r)\}$. We think of two possible definitions $A \subseteq \Gamma$.

(1) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \leq \eta_\Gamma(r)$

(2) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \geq \eta_\Gamma(r)$.

**Definition 2.4.** [4] Let $\{B_j : j \in J\}$ be an arbitrary family of an $NSs$ in $\Sigma$. Then

(1) $\cap B_j$ defined as two types:
- $\cap B_j = \langle r, \underset{j\in J}{\wedge}\xi_{Bj}(r), \underset{j\in J}{\wedge}\varrho_{Bj}(r), \underset{j\in J}{\vee}\eta_{Bj}(r)\rangle$ < Type 1 >
- $\cap B_j = \langle r, \underset{j\in J}{\wedge}\xi_{Bj}(r), \underset{j\in J}{\vee}\varrho_{Bj}(r), \underset{j\in J}{\vee}\eta_{Bj}(r)\rangle$ < Type 2 >.

(2) $\cup B_j$ defined as two types:
- $\cup B_j = \langle r, \underset{j\in J}{\vee}\xi_{Bj}(r), \underset{j\in J}{\vee}\varrho_{Bj}(r), \underset{j\in J}{\wedge}\eta_{Bj}(r)\rangle$ < Type 1 >
- $\cup B_j = \langle r, \underset{j\in J}{\vee}\xi_{Bj}(r), \underset{j\in J}{\wedge}\varrho_{Bj}(r), \underset{j\in J}{\wedge}\eta_{Bj}(r)\rangle$ < Type 2 >

**Definition 2.5.** [3] A neutrosophic topology (briefly $NT$) and a non empty set $\Sigma$ is a family $\Upsilon$ of neutrosophic subsets of $\Sigma$ satisfying the following axioms

(1) $0_N, 1_N \in \Upsilon$

(2) $S_1 \cap S_2 \in \Upsilon$ for any $S_1, S_2 \in \Upsilon$

(3) $\cup S_i \in \Upsilon, \forall \{S_i | i \in I\} \subseteq \Upsilon$.

The pair $(\Sigma, \Upsilon)$ is called a neutrosophic topological space (briefly $NTS$ ) and any neutrosophic set in $\Upsilon$ is defined as neutrosophic open set ( $NOS$ for short) in $\Sigma$. The elements of $\Upsilon$ are called open neutrosophic sets. A neutrosophic set $S$ is closed if f its $C(S)$ is neutrosophic open. For any $NTS$ $A$ in $(\Sigma, \Upsilon)$ ( [21]), we have $Int(A^c) = [Cl(A)]^c$ and $Cl(A^c) = [Int(A)]^c$.

**Definition 2.6.** A subset $\omega$ of $\Omega$ is called a cone if

(1) For non-empty $\omega$ is closed, and $\omega \neq 0$,

(2) If both $u \in \omega$ and $-u \in \omega$ then $u = 0$,

(3) If $u,\, v \in S,\, u,\, v \geq 0$ and $x,\, y \in \omega$ then $ux + vy \in \omega$.

Throughout this paper, we assume that all cones have non-empty interior. For any cone, $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(\omega)$. a partial ordering $\preccurlyeq$ on $\Omega$ via $\omega$ is defined by $x \preccurlyeq y$ iff $y - x \in \omega$.

**Definition 2.7.** A cone metric space (briefly $CMS$) an ordered $(\Sigma, d)$, where $\Sigma$ is any set and $d : \Sigma \times \Sigma \longmapsto \Omega$ is a mapping satisfying:

(1) $d(s_1, s_2) = d(s_2, s_1)$ for all $s_1,\, s_2 \in \Sigma$,

(2) $d(s_1, s_2) = 0$ iff $s_1 = s_2$,

(3) $0 \preccurlyeq d(s_1, s_2)$ for all $s_1,\ s_2 \in \Sigma$,

(4) $d(s_1, s_3) \preccurlyeq d(s_1, s_2) + d(s_2, s_3)$ for all $s_1,\ s_2,\ s_3 \in \Sigma$.

**Definition 2.8.** Let $(\Sigma, d)$ be a $CMS$. Then, for each $c_1 \gg 0$ and $c_2 \gg 0$, $c_1,\ c_2 \in \Omega$, there exists $c \gg 0$, $c \in \Omega$ such that $c \ll c_1$ and $c \ll c_2$.

**Definition 2.9.** A binary operation $\bigotimes : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if $\bigotimes$ satisfies the following conditions:

(1) $\bigotimes$ is continuous,

(2) $\bigotimes$ is commutative and associative,

(3) $m_1 \bigotimes m_2 \leq m_3 \bigotimes m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1,\ m_2,\ m_3,\ m_4 \in [0, 1]$,

(4) $m_1 \bigotimes 1 = m_1$ $\forall m_1 \in [0, 1]$.

**Definition 2.10.** A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:

(1) $\diamond$ is continuous,

(2) $\diamond$ is commutative and associative,

(3) $m_1 \diamond m_2 \leq m_3 \diamond m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1,\ m_2,\ m_3,\ m_4 \in [0, 1]$,

(4) $m_1 \diamond 1 = m_1$ $\forall m_1 \in [0, 1]$.

**Definition 2.11.** Let $\Sigma$ be a non-empty set. The mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ are called commutative if $\mathcal{H}(\mathcal{G}(x, y)) = \mathcal{G}(\mathcal{H}(x),\ \mathcal{H}(y))$ $\forall x,\ y \in \Sigma$.

**Definition 2.12.** Let $\Sigma \neq \emptyset$. An element $x \in \Sigma$ is called a common fixed point of mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ if $x = \mathcal{H}(x) = \mathcal{G}(x, x)$.

**Definition 2.13.** If $U$ and $V$ are two maps then, a pair of maps is called weakly compatible (briefly WCP) pair if they commute at (CP).

**Definition 2.14.** Let $\Sigma$ be a set, $\mathcal{G}$, $\mathcal{H}$ self maps of $\Sigma$. A point $x$ in $\Sigma$ is called a coincidence point (briefly CP) of $\mathcal{G}$ and $\mathcal{H}$ if and only if $\mathcal{G}(x) = \mathcal{H}(x)$. We call $w = \mathcal{G}(x) = \mathcal{H}(x)$ a point of coincidence of $\mathcal{G}$ and $\mathcal{H}$.

**Definition 2.15.** Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set $\Sigma$ are sporadically weakly compatible of $\Sigma$. If $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence, $z = \mathcal{G}(u) = \mathcal{H}(v)$, then $z$ is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$.

**Lemma 2.2.** *Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set $\Sigma$ are sporadically weakly compatible of $\Sigma$. then $z$ is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$, if $z = \mathcal{G}(u) = \mathcal{H}(u)$ $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence.*

**Definition 2.16.** A pair of maps $\mathcal{G}$ and $\mathcal{H}$ which $\mathcal{G}$ and $\mathcal{H}$ commute of a set $\Sigma$ are sporadically weakly compatible iff there is a point $x$ in $\Sigma$ which is a coincidence point of $\mathcal{G}$ and $\mathcal{H}$.

## 3. neutrosophic Cone Metric Space

**Definition 3.1.** A 3-tuple $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be a neutrosophic $CMS$ if $\omega$ is a neutrosophic cone metric (briefly NCMS) of $\Omega$, $\Sigma$ is an arbitrary set, $\diamond$ is a neutrosophic continuous t-conorm , $\bigotimes$ is a neutrosophic continuous t-norm, $\forall \epsilon_1, \epsilon_2, \epsilon_3 \in \Sigma$ and $m$, $n \in Int(\omega)$ (that is $n \gg 0_\Theta$, $s \gg 0_\Theta$), and $\Xi$, $\Theta$ are neutrosophic set on $\Sigma^2 \times Int(\omega)$ satisfying the following conditions:

(1) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) + \Theta(\epsilon_1, \epsilon_2, \epsilon_3) \leq 1_\Theta$;
(2) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) > 0_\Theta$;
(3) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = 1$ iff $\epsilon_1 = \epsilon_2$;
(4) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = \Xi(\epsilon_2, \epsilon_1, m)$;
(5) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) \bigotimes \Xi(\epsilon_2, \epsilon_3, n) \leq \Xi(\epsilon_1, \epsilon_3, m + n)$;
(6) $\Xi(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous;
(7) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) < 0_\Theta$;
(8) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = 0_\Theta$ if and only if $\epsilon_1 = \epsilon_2$;
(9) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = \Theta(\epsilon_2, \epsilon_3, r)$;
(10) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) \diamond \Theta(\epsilon_2, \epsilon_3, n) \geq \Theta(\epsilon_1, \epsilon_3, m + n)$;
(11) $\Theta(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous.

Then $(\Xi, \Theta)$ is called a neutrosophic cone metric on $\Sigma$. The functions $\Theta(\epsilon_1, \epsilon_2, m)$ and $\Xi(\epsilon_1, \epsilon_2, m)$ denote the degree of non-nearness and the degree of nearness between $\epsilon_1$ and $\epsilon_2$ with respect to $n$, respectively.

**Example 3.2.** Let $\Omega = R$, $\omega = [0, \infty)$ and $a \diamond b = max\{a, b\}$, $a \bigotimes b = min\{a, b\}$, then every neutrosophic metric space $(\Sigma, \Xi, \Theta)$ becomes a $NCMS$.

**Example 3.3.** If we take $\omega$ be an any cone, $a \bigotimes b = min\{a, b\}$, $\Sigma = \Theta$, $\Xi, \Theta : \Sigma^2 \times Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ defined by

$$\Xi(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

$$\Theta(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_2 - \epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$. Then $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$.

**Definition 3.4.** Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\{\epsilon_{1n}\}$ be a sequence in $\Sigma$ and $\epsilon_1 \in \Sigma$. Then $\{\epsilon_{1n}\}$ is said to converge to $\epsilon_1$ if for any $s \in (0,1)$ and any $m \gg 0_\Theta$ $\exists$ a natural number $n_0$ such that $\Xi(\epsilon_{1n}, x, m) > 1 - s, \Theta(\epsilon_{1n}, \epsilon_1, m) \leq s$ for all $n \geq n_0$. We denote this by $\lim_{\epsilon_{1n} \to \infty} = \epsilon_1$ or $\epsilon_{1n} \to \epsilon_1$ as $\to \infty$.

**Definition 3.5.** Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. For $m \gg 0_\Theta$, the open ball $\Gamma(x, s, m)$ with radius $s \in (0,1)$ and center $\epsilon_1$ is defined by $\Gamma(\epsilon_1, s, m) = \{\epsilon_2 \in \Sigma : \Xi(\epsilon_1, \epsilon_2, m) > 1 - s, \Theta(\epsilon_1, \epsilon_2, m) < s\}$.

**Definition 3.6.** The neutrosophic cone metric $CMS$ $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is called complete neutrosophic $CMS$ if every Cauchy sequence in $NCMS$ $(\Sigma, \Xi, \Theta)$ is convergent.

**Definition 3.7.** Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. A subset $P$ of $\Sigma$ is said to be $\mathcal{FC}$-bounded if $\exists$ $s \in (0,1)$ and $m \gg \theta$ such that $\Xi(\epsilon_1, \epsilon_2, t) > 1 - m, \Theta(\epsilon_1, \epsilon_2, m) < s$ for all $\epsilon_1, \epsilon_2 \in P$.

**Definition 3.8.** Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic $CMS$ and $h : \Sigma \to \Sigma$ is a self mapping. Then $h$ is said to be neutrosophic cone contractive if there exists $c \in (0,1)$ such that
$\frac{1}{\Xi(h(\epsilon_1), h(\epsilon_2), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_2, m)} - 1)$
$\Theta(h(\epsilon_1), h(\epsilon_2), m) \leq c\Theta(\epsilon_1, \epsilon_2, m)$
for each $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$. The constant $c$ is called the contractive constant of $h$.

**Lemma 3.9.** *If for two points $\epsilon_1, \epsilon_2 \in \Sigma$ and $c \in (0,1)$ such that $\Xi(\epsilon_1, \epsilon_2, cm) \geq \Xi(\epsilon_1, \epsilon_2, m)$, $\Theta(\epsilon_1, \epsilon_2, cm) \geq \Theta(\epsilon_1, \epsilon_2, m)$ then $\epsilon_1 = \epsilon_2$.*

**Theorem 3.10.** *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. Define $\mathcal{T} = \{K \subseteq \Sigma : \epsilon_1 \in K \text{iff there exists} s \in (0,1) \text{and} m \gg 0_\Theta \text{such that} L(\epsilon_1, s, m) \subseteq K\}$ , then $\mathcal{T}$ is a neutrosophic topology on $\Sigma$.*

*Proof.* If $\epsilon_1$ is empty, then $\emptyset = L(\epsilon_1, s, m) \subseteq \emptyset$. Hence the empty set belong to $\mathcal{T}$ Since for any $\epsilon_1 \in \Sigma$, any $s \in (0,1)$ and any $m \gg 0_\Theta$, $L(\epsilon_1, s, m) \subseteq \Sigma$, then $\Sigma \in \mathcal{T}$.
Let $K, L \in \mathcal{T}$ and $\epsilon_1 \in K \cap L$. Then $\epsilon_1 \in K$ and $\epsilon_1 \in L$, so there exist $m_1 \gg 0_\Theta$; $m_2 \gg 0_\Theta$ and $m_1, m_2 \in (0,1)$ such that $L(\epsilon_1, s_1, m_1) \subseteq K$ and $L(\epsilon_1, s_2, m_2) \subseteq L$.
By Proposition 2.8, for $m_1 \gg 0$; $m_2 \gg 0$, there exists $m \gg 0_\Theta$ such that $m \gg m_1$; $r \gg m_2$ and take $s = min\{m_1, m_2\}$. Then $L(\epsilon_1, s, m) \subseteq \Sigma$ $L(\epsilon_1, s_1, m_1) \cap L(\epsilon_1, s_2, m_2) \subseteq K \cap L$. Thus $K \cap L \in \mathcal{T}$. Let $K_i \in \mathcal{T}$ for each $i \in I$ and $\epsilon_1 \in \cup_{i \in I} K_i$. Then there exists $i_0 \in I$ such that $\epsilon_1 \in K_{i0}$. So, there exist $r \gg 0_\Theta$ and $s \in (0,1)$ such that $L(\epsilon_1, s, m) \subseteq K_{i_0}$. Since$K_{i_0} \subseteq \cup_{i \in I} K_i$, $L(\epsilon_1, s, m) \subseteq \cup_{i \in I} K_i$. Thus $\cup_{i \in I} K_i \in \mathcal{T}$. Hence, $\mathcal{T}$ is a neutrosophic topology on $\Sigma$. □

**Theorem 3.11.** *If $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$, then the neutrosophic topology $(\Sigma, \mathcal{T})$ is Hausdorff.*

*Proof.* Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic $CMS$. Let $\epsilon_1, \epsilon_2$ be two distinct points of $\Sigma$. Then $0 < \Xi(\epsilon_1, \epsilon_2, m) < 1_\Theta$ and $0 < \Theta(\epsilon_1, \epsilon_2, m) < 1_\Theta$. Let $\Xi(\epsilon_1, \epsilon_2, m) = s_1$, $\Theta(\epsilon_1, \epsilon_2, m) = s_2$ and $s = max\{s_1, s_2\}$. Then for each $s_0 \in (s, 1)$, there exists $s_3$ and $s_4$ such that $s_3 \bigotimes s_3 \geq s_0$ and $(1_\Theta - s_4) \diamond (1_\Theta - s_4) \leq (1_\Theta - s_0)$. Put $s_4 = max\{s_3, s_4\}$ and consider the open balls $L(\epsilon_1, 1_\Theta - s_5, m/2)$ and $L(\epsilon_2, 1_\Theta - s_5, m/2)$.
Then clearly $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) = \emptyset$
. Suppose that $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) \neq \emptyset$. Then there exists $\epsilon_3 \in L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1_\Theta - s_5, m/2)$.

$$
\begin{aligned}
s_1 =& \Xi(\epsilon_1, \epsilon_2, m) \\
\geq& \Xi(\epsilon_1, \epsilon_3, m/2) \bigotimes \Xi(\epsilon_3, \epsilon_2, m/2) \\
\geq& s_5 \bigotimes s_5 \\
\geq& s_3 \bigotimes s_3 \\
\geq& s_0 > s_1
\end{aligned}
$$

and

$$
\begin{aligned}
s_2 =& n(\epsilon_1, \epsilon_2, m) \\
\geq& n(\epsilon_1, \epsilon_3, m/2) \bigotimes n(\epsilon_3, \epsilon_2, m/2) \\
\geq& (1_\Theta - s_5) \diamond (1_\Theta - s_5) \\
\geq& (1_\Theta - s_4) \diamond (1_\Theta - s_4) \\
\leq& 1_\Theta - s_0 < s_2
\end{aligned}
$$

This is a contradiction. Hence $((\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is Hausdorff. □

**Theorem 3.12.** *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\epsilon_1 \in \Sigma$ and $(\epsilon_{1n})$ a sequence in $\Sigma$. Then $(\epsilon_{1n})$ converges to $\epsilon_1$ if and only if $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1_\Theta$, for each $m \gg 0_\Theta$.*

*Proof.* Let $(\epsilon_{1n}) \to \epsilon_1$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number $n_0$ such that $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$, $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \gg n_0$. We have $1 - \Xi(\epsilon_{1n}, \epsilon_1, m) < m$ and $\Xi(\epsilon_{1n}, \epsilon_1, m) < m$. Hence $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1$. Conversely, Suppose that $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1_\Theta$ as $n \to 1_\Theta$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number $n_0$ such that $1_\Theta - \Xi(\epsilon_{1n}, \epsilon_1, m) < s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \geq n_0$. In that case, $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ Hence $(\epsilon_{1n}) \to \epsilon_1$ as $n \to 1_\Theta$. □

## 4. **Neutrosophic Fixed Point Theorems**

**Theorem 4.1.** *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete NCMS in which neutrosophic cone contractive sequences are Cauchy. Let $\mathcal{H}$ a neutrosophic cone contractive mapping. Then $\mathcal{H}$ has a unique fixed point. Where $\mathcal{H} : \Sigma \to \Sigma$ with $c$ as the contractive constant.*

*Proof.* Let $\epsilon_1 \in \Sigma$ and fix $\epsilon_{1n} = \mathcal{H}^n(x), n \in \Theta$ For $m \gg 0_\Theta$, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m)} - 1_\Theta \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{11}, m)} - 1_\Theta),$$

$$\Theta(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m) \leq c\Theta(\epsilon_1, \epsilon_{11}, m).$$

And by induction

$$\frac{1}{\Xi(\epsilon_{1n+1}, \epsilon_{1n+2}, m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{1n+1}, m)} - 1)$$

,

$$\Theta(\epsilon_{1n+1}, \epsilon_{1n+2}, m) \leq c\Theta(\epsilon_1, \epsilon_{1n+1}, m) \text{ for all } n \in \Theta.$$

Then $(\epsilon_{1n})$ is a neutrosophic contractive sequence, by assumptions $(\epsilon_{1n})$ converges to $\epsilon_2$ and it is a Cauchy sequence, for some $\epsilon_2 \in \Sigma$. By Theorem 3.12, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_2, \epsilon_{1n}, m)} - 1) \to 0$$

$$\Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) \leq c\Theta(\epsilon_2, \epsilon_{1n}, m) \to o$$

as $n \to 1$. Then for each $m \gg 0_\Theta$,

$$\lim_{n\to\infty} \Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 1, \lim_{n\to\infty} \Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 0_\Theta,$$

and hence $\lim_{n\to\infty}\mathcal{H}(\epsilon_{1n}) = \mathcal{H}(\epsilon_2)$, i.e.,$\lim_{n\to\infty}\epsilon_{1n+1} = \mathcal{H}(\epsilon_2)$ and $\mathcal{H}(\epsilon_2) = \epsilon_2$. To show uniqueness. Let $\mathcal{H}(kkk) = \epsilon_3$ for some $\epsilon_3 \in W$. For $m \gg 0_\Theta$, we have

$$\begin{aligned}\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1 =& \frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1\\ \leq & \quad c(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\ =& c(\frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1)\\ \leq & c^2(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\ \leq & ... \leq c^n(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1) \to 0 \,\text{as}\, n \to \infty.\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Theta(\epsilon_2,\epsilon_3,m) =& \Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\ \leq & c(\Theta(\epsilon_2,\epsilon_3,m)\\ =& c\Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\ \leq & c^2\Theta(\epsilon_2,\epsilon_3,m)\\ \leq & ... \leq c^n\Theta(\epsilon_2,\epsilon_3,m) \to 0 \,\text{as}\, n \to \infty.\end{aligned} \tag{4.2}$$

Hence $\Xi(\epsilon_2,\epsilon_3,m) = 1_\Theta$ and $\Theta(\epsilon_2,\epsilon_3,m) = 0_\Theta$ and $\epsilon_2 = \epsilon_3$. □

**Theorem 4.2.** *Let $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ be a complete $NCMS$, for $\mathcal{G}$ be self mappings of $\Sigma$ and let $K, L, G$. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. If there exists $c \in (0, 1)$ such that*

$$\begin{aligned}\Xi(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \geq min\{&\Xi(G(\epsilon_1),\mathcal{G}(\epsilon_2),m), \Xi(G(\epsilon_1),K(\epsilon_1),m)\\ &\Xi(L(\epsilon_2),\mathcal{G}(\epsilon_2),m), \Xi(K(\epsilon_1),\mathcal{G}(\epsilon_2),m), \Xi(L(\epsilon_2),G(\epsilon_1),m)\}.\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Theta(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \leq max\{&\Theta(G(\epsilon_1),\mathcal{G}(\epsilon_2),r), \Theta(G(\epsilon_1),K(\epsilon_1),m)\\ &\Theta(L(\epsilon_2),\mathcal{G}(\epsilon_2),m), \Theta(K(\epsilon_1),\mathcal{G}(\epsilon_2),r), \Theta(L(\epsilon_2),G(\epsilon_1),m)\}.\end{aligned} \tag{4.4}$$

*for all $\epsilon_1, \epsilon_2 \in \Sigma$ and for all $r \gg 0_\Theta$, there exists a unique point $z \in \Sigma$ such that $K(z) = G(z) = z$ and a unique point $y \in \Sigma$ such that $L(y) = \mathcal{G}(y) = y$. Moreover $y = z$, so that there is a unique common fixed point of $K, L, G$ and $\mathcal{G}$.*

*Proof.* Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible, so there are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequality 4.3,

$$\begin{aligned}\Xi(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \geq & min\{\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}\\ = & min\{\Xi(K(\epsilon_1), L(\epsilon_2), r), \Xi(K(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m)\}\\ = & \Xi(K_{\epsilon_1}, L_{\epsilon_2}, m). \end{aligned} \tag{4.5}$$

$$\begin{aligned}\Theta(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \leq & max\{\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}\\ = & max\{\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(L(\epsilon_2), K(\epsilon_1), m)\}\\ = & \Theta(K_{\epsilon_1}, L_{\epsilon_2}, m). \end{aligned} \tag{4.6}$$

By Lemma 3.9, $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = L(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is another point $y$ such that $K(y) = G(y)$ and by 4.3, we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of $K$ and $G$. By Lemma 2.2, $z$ is the unique common fixed point of $K$ and $G$. Similarly there is a only point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Assume that $z \neq y$, we have

$$\begin{aligned}\Xi(z, y, c(m)) = & \Xi(K(z), L(y), c(m))\\ \geq & min\{\Xi(G(z), \mathcal{G}(y), r), \Xi(G(z), K(y), m), \Xi(L(y), \mathcal{G}(y), m)\\ & \Xi(K(z), \mathcal{G}(y), m), \Xi(L(y), G(z), m)\}\\ = & min\{\Xi(z, y, m), \Xi(z, y, m), \Xi(y, y, m), \Xi(z, y, m), \Xi(y, z, m)\}\\ = & \Xi(z, y, m). \end{aligned} \tag{4.7}$$

$$\begin{aligned}\Theta(z, y, c(r)) = & \Theta(K(z), L(y), c(m))\\ \geq & min\{\Theta(G(z), \mathcal{G}(y), m), \Theta(G(z), K(y), m), \Theta(L(y), \mathcal{G}(y), m)\\ & \Theta(K(z), \mathcal{G}(y), r), \Theta(L(y), G(z), m)\}\\ = & min\{\Theta(z, y, m), \Theta(z, y, m), \Theta(y, y, m), \Theta(z, y, m), \Theta(y, z, m)\}\\ = & \Theta(z, y, m). \end{aligned} \tag{4.8}$$

by Lemma 2.2 and $y$ is a common fixed point of $K, L, G$ and $\mathcal{G}$. Then we have $y = z$. The uniqueness of the fixed point come from 4.6. □

**Theorem 4.3.** *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete* $NCMS$ *and* $K, L, G$ *and* $\mathcal{G}$ *be self-mappings of* $\Sigma$*. Let the pairs* $\{K, G\}$ *and* $\{L, \mathcal{G}\}$ *be sporadically weakly compatible. If there exists* $c \in (0, 1)$ *such that*

$$\begin{aligned} \Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi[min\{&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m) \\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}]. \end{aligned} \tag{4.9}$$

$$\begin{aligned} \Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta[max\{&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m) \\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}]. \end{aligned} \tag{4.10}$$

*for all* $\epsilon_1, \epsilon_2 \in \Sigma$ *and* $\phi, \zeta : \rfloor 0^-, 1^+ \lfloor \to \rfloor 0^-, 1^+ \lfloor$ *such that* $\zeta(m) < m$*,* $\phi(m) > m$*, for all* $0_\Theta \ll r < 1_\Theta$*, thus there is a unique common fixed point of* $K, L, G$ *and* $\mathcal{G}$*.*

*Proof.* The proof follows from Theorem 4.4 □

**Theorem 4.4.** *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete* $NCMS$ *and* $K, L, G$ *and* $\mathcal{G}$ *be self-mappings of* $\Sigma$*. Let* $\{K, G\}$ *and* $\{L, \mathcal{G}\}$ *are pairs be sporadically weakly compatible. If* $\exists c \in (0, 1)$ *such that*

$$\begin{aligned} \Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi(&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m) \\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m) \\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)). \end{aligned} \tag{4.12}$$

*for all* $\epsilon_1, \epsilon_2 \in \Sigma$ *and* $\phi, \zeta : \rfloor 0^-, 1^{+^5} \lfloor \to \rfloor 0^-, 1^+ \lfloor$ *such that* $\phi(r, 1_\Theta, 1_\Theta, m, m) > m$*,* $\zeta(m, 0_\Theta, 0_\Theta, m, m) < m$ *for all* $0 \ll m < 1$ *then there exists a unique common fixed point of* $K, L, G$ *and* $\mathcal{G}$*.*

*Proof.* Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. There are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.11) and (4.12), we have

$$\begin{aligned} \Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi(&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m), \\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), mr), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)) \\ =&\phi(\Xi(K(\epsilon_1), L(\epsilon_2), m), \Xi(K(\epsilon_1), K(\epsilon_1), m), \\ &\Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m)) \\ =&\phi((\Xi(K(\epsilon_1), L(\epsilon_2), m), 1_\Theta, 1_\Theta, \Xi(K(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m)) \\ >&\Xi(K(\epsilon_1), L(\epsilon_2), m). \end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m))\\ =&\zeta(\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), L(L(\epsilon_2), K(\epsilon_1), m))\\ =&\zeta((\Theta(K(\epsilon_1), L(\epsilon_2), m), 0_\Theta, 0_\Theta, \Theta(K(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m))\\ <&\Theta(K(\epsilon_1), L(\epsilon_2), m).\end{aligned}$$

a contradiction, therefore $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point $y$ such that $K(y) = G(y)$. Then by 4.11 we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$, so $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = \mathcal{G}(\epsilon_1)$ is the unique point of coincidence. $z$ is a unique common fixed point of $K$ and $G$, by Lemma 2.2. Similarly, for $K$ and $G$ there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus for $K, L, G$, $y$ is a common fixed point and $\mathcal{G}$. For the uniqueness fixed point holds from (4.11). □

**Theorem 4.5.** *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete NCMS and $K, L, G$ and $\mathcal{G}$ be self-mappings of $\Sigma$. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible. If there exists $c \in (0, 1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Xi\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.14}$$

*then there exists a unique common fixed point of $K, L, G$ and $\mathcal{G}$.*

*Proof.* Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ are sporadicallyweakly compatible, there are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.13) and (4.14), we have

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(G(\epsilon_1), L(\epsilon_2), m) &\bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ &\bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ =&\Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), K(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m)\\ &\bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ \geq&\Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes 1_\Theta \bigotimes 1_\Theta \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ \geq&\Xi(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ =& \Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), K(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ \leq& \Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond 0_\Theta \diamond 0_\Theta \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ \leq& \Theta(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

By Lemma 3.9, we have $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point $y$ such that $K(y) = G(y)$. Then by (4.13, 4.14), we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of $K$ and $G$. Then there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus $z$ is a common fixed point of $K, L, G$ and $\mathcal{G}$. □

**Theorem 4.6.** *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete neutrosophic CMS and $\mathcal{G}$ and $K, L, G$ be self-mappings of $\Sigma$. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are the pairs be sporadically weakly compatible. If $\exists c \in (0,1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), r) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.16}$$

*then for $K, L, G$ and $\mathcal{G}$ there exists a unique common fixed point.*

*Proof.* We have,

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ =& \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \bigotimes \Xi(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ \geq& \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \diamond \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ =& \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \diamond \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \diamond \Theta(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ \leq& \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

and therefore by Theorem 4.5, $K, L, G$ and $\mathcal{G}$ have a common fixed point. □

**Theorem 4.7.** *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete neutrosophic* $CMS$ *and* $K$, $L$ *be self-mappings of* $\Sigma$. *Let* $K$ *and* $L$ *be sporadically weakly compatible. If* $\exists$ *a point* $c \in (0,1)$ *for all* $\epsilon_1, \epsilon_2 \in \Sigma$ *and* $r \gg 0_\Theta$

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ \Xi(L(\epsilon_1), K(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ \Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.18}$$

*for all* $\epsilon_1, \epsilon_2 \in \Sigma$, *where* $a, b > 0_\Theta$, $a + b > 1_\Theta$. *Then* $K$ *and* $L$ *have a unique common fixed point.*

*Proof.* Let the pairs $\{K, L\}$ be sporadicallyweakly compatible, so there is a point $\epsilon_1 \in \Sigma$ such that $K(\epsilon_1) = L(\epsilon_1)$. Suppose that there exists another point $\epsilon_2 \in \Sigma$ for which $K(\epsilon_2) = L(\epsilon_2)$. We claim that $G(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.17) and (4.18), we have

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq\, & a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ & \Xi(L(\epsilon_1), K(\epsilon_1), r), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\\ =\, & a\Xi(L(\epsilon_1), L(\epsilon_2), m) + b\,min\{\Xi(L(\epsilon_1), L(\epsilon_2), m),\\ & \Xi(L(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), L(\epsilon_2), m),\}\\ =\, & a + b\Xi(L(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq\, & a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ & \Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), r)\}\\ =\, & a\Theta(L(\epsilon_1), L(\epsilon_2), m) + b\,max\{\Theta(L(\epsilon_1), L(\epsilon_2), m),\\ & \Theta(L(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), L(\epsilon_2), m),\}\\ =\, & a + b\Theta(L(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

a contradiction, since $a + b > 1_\Theta$. Therefore $L(\epsilon_1) = L(\epsilon_2)$. Therefore $K(\epsilon_1) = K(\epsilon_2)$ and $K(\epsilon_1)$ is unique. From Lemma 2.2, $K$ and $L$ have a unique fixed point. □

## 5. Conclusion

In this paper, the concept of neutrosophic $CMS$ is introduced. Some fixed point theorems on neutrosophic $CMS$ are stated and proved.

## 6. Conflict of Interests

Regarding this manuscript, the authors declare that there is no conflict of interests.

## 7. Acknowledgments

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