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Abstract: This paper proposed the idea of Neutrosophic norm in a linear space. An attempt has been made to find some related results in Neutrosophic normed linear space and study the Cauchy sequence and completeness in this structure.

Keywords: Linear space, Norm, Co-norm, Fuzzy Set, Fuzzy Norm, Neutrosophic norm, Neutrosophic normed linear space.

1. Introduction

This section gives the basic introduction about the present work starting with Literature survey, Scope and objective and chapter distribution.

1.1. Literature Survey:

The notion of normed linear space plays a major role in Functional Analysis. Dimension in normed linear space has attracted researchers to a greater extend. $G\ddot{a}$ hler (1965) took effort in developing the structure of 2-normed linear space and n-normed linear space. Recently many researchers have engaged themselves in developing the theory of n-normed linear space. Zadeh (1965) [40], introduced fuzzy set in his pioneering work which is a remarkable theory to deal with uncertainty. He stated that a fuzzy set assigns a membership value to each element of a given crisp universe set from [0, 1]. This notion laid the foundation for a wide range usage of Mathematics and also applied to a great variety of real-life scenarios. Later Atanassov (1986) [11-13], focused intuitionistic fuzzy set, which is characterized by a membership function and non-membership function for each in the Universe and then Smarandache (1998-2005) [2 - 4] developed another idea called Neutrosophic set by adding an intermediate membership. Maji (2013) also dealt about this Neutrosophic concept. Felbin (1992) [19,20,21] assigned a fuzzy real number to each element of the linear space and introduction another idea of fuzzy norm on a linear space and also proved that a finite dimensional fuzzy normed linear space has a unique fuzzy norm on it up to fuzzy equivalence. Further in 1993 he discussed about the completion of fuzzy normed linear spaces and in 1993 he proved that any finite dimensional fuzzy normed linear space is necessarily complete.

Beg & Samanta (2003) [14 - 17] introduced a definition of fuzzy norm on a linear space. They also provided a decomposition theorem of fuzzy norms into a family of crisp norms and studied the properties of finite dimensional fuzzy normed linear spaces. This paper motivated Narayanan et.al to develop the theory of fuzzy n-normed linear space. Santhosh & Ramakrishnan (2011) [36] introduced the concepts of norm and inner product on fuzzy linear spaces over fuzzy fields.
Then Vijayabalaji (2008) [38, 39] et.al studied the idea of interval valued fuzzy n-normed linear spaces. Later Vijayabalaji (2007) et.al, Samanta (2009) et.al, and Issac (2012) [25] et.al dealt the

concepts of normed linear spaced with intuitionistic fuzzy settings. Recently Sandeep Kumar (2018) discussed some results on Interval valued intuitionistic fuzzy n-normed linear space.

1.2. Scope and Objective of the Present Investigation:

The present study is aimed to extend the structures of fuzzy normed linear space into Neutrosophic normed linear space. An attempt has been made to study some elegant results in this structure through Neutrosophic norm and analyze the Cauchy sequences on Neutrosophic Normed linear space. The paper is classified into the following sections: Section 1 shows the introduction and section 2 gives some basic definitions and properties of linear space, fuzzy set, t-norm, t-conorm , fuzzy normed linear space etc., Section 3 deals the Neutrosophic normed linear space and discussed their properties. Section 4, ends with concluding remarks and future scope of the study.

2. Preliminaries

This section recalls the basis definitions and results that are necessary for the present work.

Definition 2.1. [14] A linear space (or vector space) V over a field F consist of the following

1. A field F of scalars.
2. A set V of objects called vectors
3. A rule (or operation) called vector addition which associates with each pair of vectors, $u , v \in V$ a vector $u + v \in V$ called the sum of u and v in such a way that
   - Addition is commutative,
   - Addition is associative
   - There is unique vector in u in V called the zero vector, such that $u + 0 = u \ \forall u \in V$
   - For each vector $u \in V$, there is unique vectors $-u \in V$ such that $u + (-u) = 0$.
4. A rule (or operation) called scalars multiplication which associates with each scalar $a \in F$ and vector and $u \in V$ in such a way that
   - $1.u = u \ \forall u \in V$ and $1 \in F$
   - $ab(u) = a(bu) \ \forall a , b \in F$ and $\forall u \in V$
   - $a(u + v) = au + av \ \forall a \in F$ and $\forall u , v \in V$
   - $(a + b)u = au + bu \ \forall a , b \in F$ and $\forall u \in V$

It is denoted as $(V , + , \cdot)$ is a linear space.

Definition 2.2. [14]A nonnegative function on a linear vector space V , $\| \cdot \| : V \to [0 , \infty)$ is called a norm if

1. $\| x \| = 0$ if and only if $x = 0$;
2. $\| x + y \| \leq \| x \| + \| y \|$ for all $x , y \in V$ (the triangular inequality)
3. $\| \alpha x \| = | \alpha | \| x \|$ for all $x \in V$ and $\alpha \in F$

Definition 2.3. [14]A normed linear space is a linear space V with a norm $\| \cdot \|_V$ on it.

Definition 2.4. [40] A fuzzy set A in X is defined as an object of the form $A = \{ (x , \mu_A(x)) : x \in X \}$, where $\mu_A(x)$ is called the membership function of x in X which maps X to the unit interval $I = [0 , 1]$.

Definition 2.5. [11]An intuitionistic fuzzy set A in a nonempty set X is defined as an objects of the form $A = \{ (x , \mu_A(x) , \vartheta_v(x)) : x \in X \}$ where the functions $\mu_A : X \to [0 , 1]$ and $\vartheta_A : X \to [0 , 1]$ defined the degree of membership and degree of non-membership of the element $x \in X$ respectively, and for $0 \leq \mu_v(x) + \vartheta_v(x) \leq 1 \ \forall x \in X$.

An ordinary fuzzy set A in X may be viewed as special intuitionistic fuzzy set with the non-membership function $\vartheta_A(x) = 1 - \mu_A(x)$.

Definition 2.6. Let [I] be the set of all closed sub intervals of the interval [0,1] and M = $[M_L, M_U]$ ∈ [I] where M_L *and* M_U are the lower extreme and upper extreme, respectively. For a set X, an IVFS (Interval Valued Fuzzy Set) A on X given by

$$A = \{\langle x, M_A(x) \rangle / \ x \in X\}$$

where the function $M_A : X \rightarrow [0,1]$ defines the degree of membership of an element x on A, and $M_A(x) = [M_{AL}(x), M_{AU}(x)]$ called an interval valued fuzzy number.

Definition 2.7. For a set X, an IVIFS (Interval Valued Intuitionistic Fuzzy Set) A on X is an objects having the form A = $\{\langle x, M_A(x), N_A(x)\rangle / \ x \in X\}$ where $M_A : X \rightarrow [I]$ and $N_A : X \rightarrow [I]$ represents the degree of membership and non-membership $0 \leq sup(M_A(x)) + sup(N_A(x)) \leq 1$ for every x ∈ X $M_A(x) = [M_{AL}(x), M_{AU}(x)]$ and $N_A(x) = [N_{AL}(x), N_{AU}(x)]$
Hence A =$\{[M_{AL}(x), M_{AU}(x)], [N_{AL}(x), N_{AU}(x)]\}$ is called IVIFS.

Definition 2.8. [14] Let X be a linear space over the field F (real or complex) and $*$ is a continuous t-norm. A fuzzy subset N on X × ℝ (R-set of all real numbers) is called a fuzzy norm on X if and only if for x,y ∈ X and c ∈ F,

(N1) ∀ t ∈ R with t ≤ 0, N(x,t) = 0

(N2) ∀ t ∈ R with t > 0 N(x,t) = 1, iff x = 0

(N3) t ∈ R, t > 0

$$N(cx,t) = N(x,\frac{t}{|c|}). \text{ If } ,c \neq 0$$

(N4) ∀ s,t ∈ R, x,y ∈ X,

$$N(x+y, t+s) \geq N(x,t) * N(y,s)$$

(N5) $\lim_{t\to\infty} N(x,t) = 1$.

The triplet (X,N,∗) will be referred to as a fuzzy normed linear space.

Definition 2.9. [25] A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is commutative and associative
2. $*$ is continuous
3. $a * 1 = a$, for all $a \in [0,1]$
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.10. A binary operation $\lozenge : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-co-norm if $\lozenge$ satisfies the following conditions:

1. $\lozenge$ is commutative and associative
2. $\lozenge$ is continuous
3. $a \lozenge 0 = a$, for all $a \in [0,1]$
4. $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.11 Let $*$ be a continuous t-norm, $\lozenge$ be a continuous t-co-norm, and V be a linear space over the field $F\ (= R \ or \ C)$. An intuitionistic fuzzy norm or in short IFN on V is an object of the form $A = \{((x,t), N(x,t), M(x,t)) : (x,t) \in V \times \mathbb{R}^+$, where N, M are fuzzy sets on $V \times \mathbb{R}^+$,N denotes the degree of membership and M denotes the degree of non-membership $(x,t) \in V \times \mathbb{R}^+$ satisfying the following conditions:

1. $N(x,t) + M(x,t) \leq 1 \ \forall \ (x,t) \in V \times \mathbb{R}^+$
2. $N(x,t) > 0$
3. $N(x,t) = 1$ if and only if $x = 0$

4. $N(cx,t) = N\left(x, \frac{t}{|c|}\right), c \neq 0, c \in F$
5. $N(x,s) * N(y,t) \leq N(x+y, s+t)$
6. $N(x, \cdot)$ is non – decreasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} N(x,t) = 1$
7. $M(x,t) > 0$
8. $M(x,t) = 0$ if and only if $x = 0$
9. $M(cx,t) = M\left(x, \frac{t}{|c|}\right), c \neq 0, c \in F$
10. $M(x,s) \lozenge M(y,t) \geq M(x+y, s+t)$
11. $M(x, \cdot)$ is non – increasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} M(x,t) = 0$.

Then the quadruple $(V, A, *, \lozenge)$ will be referred as a intuitionistic fuzzy normed linear space.

3. Neutrosophic Approach on Normed Linear Space

This section introduces the idea of Neutrosophic normed linear space using the notion of Neutrosophic set. Further, some result related to Cauchy sequence on Neutrosophic normed linear space are also dealt.

3.1 Neutrosophic Norm:

Here Neutrosophic norm is defined with suitable example. Further the convergence of sequence in NNLS and some properties also studied.

Definition 3.1. [33]Let S be a space of points (objects). A NS N on S is characterized by a truth-membership function ρ, an indeterminacy membership function ξ, and a falsity-membership function η, where $\rho(x), \xi(x) and\, \eta(x)$ and real standard and non-standard subset of $]^-0,1^+[$ i.e., ρ, ξ, η : X→ $]^-0,1^+[$. Thus the NS N over S is defined as:

$$N = \{< x, (\rho(x), \xi(x), \eta(x)) > | \, x \in S\}$$

On the same of $\rho(x), \xi(x) and\, \eta(x)$ there is no restriction and so $^-0 \leq sup\rho(x) + sup\xi(x) + sup\,\eta(x) \leq 3^+$. Here $1^+ = 1 + \epsilon$, where 1 is its standard part and ϵ its non-standard part. Also, $^-0 = 0 - \epsilon$where 0 is its standard part and ϵ its non-standard part.

From philosophical point of view, a NS takes the value from real standard or nonstandard subsets of] $^-0,1^+$[. But to practice in real scientific and engineering areas, it is difficult to use NS with value from real standard or nonstandard subset of] $^-0$, 1^+[. Hence, we consider the NS which takes the value from the subset of [0, 1].

Definition 3.2. Let V be a linear space field $F = (\mathbb{R}\; or\; \mathbb{C})$ and $*$ be a continuous t – norm, $\lozenge$ be a continuous t – co – norm. Then, a Neutrosophic subset $N : \langle \rho, \xi, \eta \rangle\; on\; V \times F$ is called a Neutrosophic norm on V if for $x, y \in V$ and $c \in F$ (c being scalar), if the following conditions hold.

1. $0 \leq \rho(x,t), \xi(x,t), \eta(x,t) \leq 1, \forall\; t \in R$
2. $0 \leq \rho(x,t) + \xi(x,t) + \eta(x,t) \leq 3, \forall\, t \in R$
3. $\rho(x,t) = 0$ with $t \leq 0$
4. $\rho(x,t) = 1\; with\; t > 0\; iff\; x = 0$, the null vector
5. $\rho(cx,t) = \rho\left(x, \frac{t}{|c|}\right), \forall\, c \neq 0, t > 0$
6. $\rho(x,s) * \rho(y,t) \leq \rho(x+y, s+t)\; \forall\, s, t \in R$
7. $\rho(x, \cdot)$ is continuous non – decreasing function for $t > 0$, $\lim_{t\to\infty} \rho(x,t) = 1$
8. $\xi(x,t) = 1\; with,\; t \leq 0$
9. $\xi(x,t) = 0\; with\; t > 0\; iff\; x = 0$, the null vector

10. $\xi(cx,t) = \xi\left(x, \frac{t}{|c|}\right), \forall\, c \neq 0, t > 0$
11. $\xi(x,s) \Diamond \xi(y,t) \geq \xi(x+y, s+t)\ \forall\, s,t \in R$
12. $\xi(x,\cdot)$ is a continuous non-increasing function for t > 0, $\lim_{t\to\infty} \xi(x,t) = 0$
13. $\eta(x,t) = 1\ with,\ t \leq 0;$
14. $\eta(x,t) = 0\ with\ t > 0\ iff\ x = 0$, the null vector;
15. $\eta(cx,t) = \eta\left(x, \frac{t}{|c|}\right), \forall\, c \neq 0, t > 0$
16. $\eta(x,s) \Diamond \eta(y,t) \geq \eta(x+y, s+t)\ \forall\, s,t \in R$
17. $\xi(x,\cdot)$ is a continuous non-increasing function for t > 0, $\lim_{t\to\infty} \eta(x,t) = 0;$

Further $(V, N, *, \Diamond)$ is Neutrosophic normed linear space (NNLS).

Example3.3.

Let $(V, \|\cdot\|)$ be a normed linear space. Take $a * \mathrm{b} = ab\ and\ a \Diamond b = a + b - ab$. Define,

$$\rho(\mathrm{x,t}) = \begin{cases} \frac{t}{t+||x||} & if\ \ t > ||x|| \\ 0 & otherwise. \end{cases}$$

$$\xi(\mathrm{x,t}) = \begin{cases} \frac{x}{t+||x||} & if\ \ t > ||x|| \\ 1 & otherwise. \end{cases}$$

$$\eta(\mathrm{x,t}) = \begin{cases} \frac{||x||}{t} & if\ \ t > ||x|| \\ 1 & otherwise. \end{cases}$$

, Then $(V, N, *, \Diamond)$ is an NNLS.

Proof:

All the conditions are obvious except the condition (6), (11), (16). For $s,t > 0$ because these are clearly true for $s,t \leq 0$.

Now, $\rho(x+y, s+t) - \rho(x,s) * \rho(y,t)$

$$= \frac{s+t}{s+t+||x+y||} - \frac{st}{(s+||x||)(t+||y||)}$$

$$\geq \frac{s+t}{s+t+||x+y||} - \frac{st}{(s+||x||)(t+||y||)}$$

$$= \{(s+t)(s+||x||)(t+||y||) - st(s+t+||x||+||y||)\}/\aleph$$

Where $\aleph = (s+t+||x||+||y||)(s+||x||)(t+||y||)$

$$= \{t^2||x||\, s^2||y|| + (s+t)||xy||\}/\aleph \geq 0.$$

Hence, $\rho(x,s) * \rho(y,t) \leq \rho(x+y, s+t), \forall\, s,t \in R$

$\xi(x,s) \Diamond \xi(y,t) - \xi(x+y, s+t)$

$$= \frac{||x||}{s+||x||} + \frac{||y||}{t+||y||} - \frac{||xy||}{(s+||x||)\ (t+||y||)} - \frac{x+y}{||x+y||+s+t}$$

$$= \frac{||xy|| + t||x|| + s||y||}{(s+||x||)\ (t+||y||)} - \frac{||x+y||}{||x+y||+s+t}$$

$$= \{(||x+y||+s+t)(t||x||+s||x||+||xy||) - ||x+y||(s+||x||)\ (t+||y||)\}/D$$

Where $D = (s+t+||x+y||)(s+||x||)(t+||y||)$

$$= \{(s+t)(t||x|| + s||y|| + ||xy||) - st||x+y||\}/D$$

$$\geq \{(s+t)(t||x|| + s||y|| + ||xy||) - st(||x|| + ||y||)\}/D$$

$$= \{t^2||x|| + s||y|| + (s+t)||xy||\}/D \geq 0.$$

Hence, $\xi(x,s) \lozenge \xi(y,t) \geq \xi(x+y, s+t), \forall s, t \in R.$

Finally $\eta(x,s) \lozenge \eta(y,t) \geq (x+y, s+t)$

$$= \frac{||x||}{s} + \frac{||y||}{t} - \frac{||xy||}{st} - \frac{||x+y||}{s+t}$$

$$= \frac{t||x|| + s||y|| - ||xy||}{st} - \frac{||x+y||}{s+t}$$

$$\geq \{s^2||y|| + t^2||x|| - (s+t)||xy||\}/st(s+t)$$

$$= \{s||y||(s - ||x||) + t||x||(t - ||y||)\} \ / st(s+t) \geq 0, (as\ s > ||x||, t > ||y||).$$

Thus, $\eta(x,s) \lozenge \eta(y,t) \geq (x+y, s+t), \forall s, t \in R.$ This completes the proof.

Definition 3.4. Let $\{x_n\}$ be a sequence of points in a NNLS $(V, N, *, \lozenge)$. Then the sequence converges to a point$x \in V$ if and only if for given $r \in (0,1), t > 0$ there exist $n_0 \in N$ (the set of natural numbers) such that

$$\rho(x_n - x, t) > 1 - r, \xi(x_n - x, t) < r, \eta(x_n - x, t) < r, \forall n \geq n_0.$$

(or)

$$\lim_{n\to\infty} \rho(x_n - x, t) = 1, \lim_{n\to\infty} \xi(x_n - x, t) = 0, \lim_{n\to\infty} \eta(x_n - x, t) = 0, t \to \infty$$

Then the sequence $\{x_n\}$ is called a convergent sequence in the NNLS $(V, N, *, \lozenge)$.

Theorem 3.5.

If the sequence $\{x_n\}$ in a NNLS $(V, N, *, \lozenge)$ is convergent, then the point of convergence is unique.

Proof:

Let $\lim_{n\to\infty} x_n = x\ and\ \lim_{n\to\infty} x_n = y. for\ x \neq y.\ Then\ for\ s, t > 0,$

$$\lim_{n\to\infty} \rho(x_n - x, s) = 1, \lim_{n\to\infty} \xi(x_n - x, s) = 0, \lim_{n\to\infty} \eta(x_n - x, s) = 0, as\ s \to \infty\ and$$

$$\lim_{n\to\infty} \rho(x_n - x, t) = 1, \lim_{n\to\infty} \xi(x_n - x, t) = 0, \lim_{n\to\infty} \eta(x_n - x, t) = 0, as\ t \to \infty$$

Now,

$$\rho(x - y, s+t) = \rho(x - x_n + x_n - y, s+t) \leq \rho(x_n - x, s) * \rho(x_n - y, t)$$

Taking limit as $n \to \infty$ and for s, t $n \to \infty$,

$$\rho(x - y, s+t) \geq 1 * 1 = 1\ i.e., \rho(x - y, s+t) = 1$$

Further,

$$\xi(x - y, s+t) = \xi(x - x_n + x_n - y, s+t) \leq \xi(x_n - x, s) \lozenge \xi(x_n - y, t)$$

Taking limit as $n \to \infty$ and for s, t $n \to \infty$,

$$\xi(x - y, s+t) \leq 0 \lozenge 0 = 0 i.e., \xi(x - y, s+t) = 0$$

Similarly, $\eta(x-y,s+t)=0$
Hence, $x=y$ and this complete the proof.

Theorem 3.6.
In an NNLS $(V,N,*,\Diamond)$, if $\lim_{n\to\infty}(x_n)=x$ and $\lim_{n\to\infty}(y_n)=y$ then $\lim_{n\to\infty}(x_n+y_n)=x+y$
Proof:
Here, for $s,t>0$

$$\lim_{n\to\infty}\rho(x_n-x,s)=1,\lim_{n\to\infty}\xi(x_n-x,s)=0,\lim_{n\to\infty}\eta(x_n-x,s)=0, as\ s\to\infty\ and$$

$$\lim_{n\to\infty}\rho(y_n-y,t)=1,\lim_{n\to\infty}\xi(y_n-y,t)=0,\lim_{n\to\infty}\eta(y_n-y,t)=0, as\ t\to\infty.$$

Now, $\lim_{n\to\infty}\rho[(x_n+y_n)-(x+y),s+t)]=\lim_{n\to\infty}\rho[(x_n-x)+(y_n-y),s+t)]$,

$$\geq\lim_{n\to\infty}\rho(x_n-x,s)*\lim_{n\to\infty}\rho(y_n-y,t)[by\ (6) in\ Definition\ 3.2]$$

$$=1*1=1\ as\ s,t\to\infty$$

Hence $\lim_{n\to\infty}\rho[(x_n-y_n)-(x+y),s+t)]=1\ as, s,t\to\infty.\ Again$

$$\lim_{n\to\infty}\xi[(x_n+y_n)-(x+y),s+t)]=\lim_{n\to\infty}\xi[(x_n-x)+(y_n-y),s+t)]$$

$$\geq\lim_{n\to\infty}\xi(x_n-x,s)\Diamond\lim_{n\to\infty}\xi(y_n-y,t)\ [by\ (11) in\ Definition\ 3.2]$$

$$=0\Diamond 0=0\ as\ s,t\to\infty$$

$$So,\lim_{n\to\infty}\xi[(x_n+y_n)-(x+y),s+t)]=0\ as\ s,t\to\infty.$$

Similarly,

$$\lim_{n\to\infty}\eta[(x_n+y_n)-(x+y),s+t)]=0\ as\ s,t\to\infty. and\ this\ end\ the\ theorem.$$

Theorem 3.7.
If $\lim_{n\to\infty}x_n=x$ and $0\neq c\in F$, then $\lim_{n\to\infty}cx_n$ in an NNLS $(V,N,*,\Diamond)$.
Proof:
Here,

$$\lim_{n\to\infty}\rho(cx_n-cx,t)=\lim_{n\to\infty}\rho\left(x_n-x,\frac{t}{|c|}\right)=1, as\ \frac{t}{|c|}\to\infty.$$

$$\lim_{n\to\infty}\xi(cx_n-cx,t)=\lim_{n\to\infty}\xi\left(x_n-x,\frac{t}{|c|}\right)=1, as\ \frac{t}{|c|}\to\infty.$$

$$\lim_{n\to\infty}\eta(cx_n-cx,t)=\lim_{n\to\infty}\eta\left(x_n-x,\frac{t}{|c|}\right)=1, as\ \frac{t}{|c|}\to\infty.$$

Thus, the theorem is proved.

3.2. Completeness on Neutrosophic Normed Linear Space:

Here the Cauchy sequence in NNLS and complete NNLS are introduced. Further several structural characteristics of complete NNLS also studied. .

Definition 3.8. A sequence $\{x_n\}$ of points in an NNLS $(V,N,*,\Diamond)$ is said to be bounded for $r\in(0,1)$ and $t>0$. if the following hold:

$$\rho(x_n, t) > 1 - r, \xi(x_n, t) < r, \eta(x_n, t) < r, \forall n \in N. (the\ set\ of\ all\ natural\ numbers\ ).$$

Definition 3.9.

1. A sequence $\{x_n\}$ of points in an NNLS $(V, N, *, \Diamond)$.is said to be a Cauchy sequence if giver$\in (0,1), t > 0$ there exist $n_0 \in N$ $(the\ set\ of\ all\ natural\ numbers\ )\ such\ that$
$$\rho(x_n - x_m, t) > 1 - r, \xi(x_n - x_m, t) < r, \eta(x_n - x_m, t) < r \,\forall m, n \in n_0.$$
$$(or)$$
$$\lim_{n,m\to\infty} \rho(x_n - x_m, t) = 1, \lim_{n,m\to\infty} \xi(x_n - x_m, t) = 0, \lim_{n,m\to\infty} \eta(x_n - x_m, t) = 0, as\ t \to \infty$$

2. Let $\{x_n\}$ be Cauchy sequence of points in a normed linear space $(V, ||\bullet||)$. Then $\lim_{n,m\to\infty} ||x_n - x_m|| = 0$ hold.

Example 3.10. For $t > 0, let\ \rho(x, t) = \frac{t}{t+||x||}$, $\xi(x, t) = \frac{||x||}{t+||x||}$, $\eta(x, t) = \frac{||x||}{t}$. Then $(V, N, *, \Diamond)$ is an NNLS. Now,

$$\lim_{n,m\to\infty} \frac{t}{t + ||x_n - x_m||} = 1\,, \lim_{n,m\to\infty} \frac{||x_n - x_m||}{t + ||x_n - x_m||} = 0\,, \lim_{n,m\to\infty} \frac{||x_n - x_m||}{t} = 0$$

$$\lim_{n,m\to\infty} \rho(x_n - x_m, t) = 1, \lim_{n,m\to\infty} \xi(x_n - x_m, t) = 0, \lim_{n,m\to\infty} \eta(x_n - x_m, t) = 0, as\ t \to \infty$$

This shows that $\{x_n\}$ is a Cauchy sequence in the NNLS $(V, N, *, \Diamond)$.

Theorem 3.11. Every convergent sequence of points in a NNLS $(V, N, *, \Diamond)$ is a Cauchy sequence.

Proof:

Let $\{x_n\}$ be a convergent sequence of a points in a NNLS $(V, N, *, \Diamond)$ so that $\lim_{n\to\infty} x_n = x$. Then for $t > 0$,

$$\lim_{n,m\to\infty} \rho(x_n - x_m, t) = \lim_{n,m\to\infty} \rho(x_n - x_m + x - x, t) = \lim_{n,m\to\infty} \rho[(x_n - x) + (x - x_m), t]),$$

$$\geq \lim_{n\to\infty} \rho\left(x_n - x, \frac{t}{2}\right) = * \lim_{m\to\infty} \rho\left(x - x_m, \frac{t}{2}\right) [by\ (6) in\ Definition\ 3.2]$$

$$= \lim_{n\to\infty} \rho\left(x_n - x, \frac{t}{2}\right) = * \lim_{m\to\infty} \rho\left(x_m - x, \frac{t}{2}\right) [by\ (5) in\ Definition\ 3.2]$$

$$= 1 * 1 = 1\ as\ t \to \infty.$$

So , $\lim_{n,m\to\infty} \rho(x_n - x_m, t) = 1.$

Again $\lim_{n,m\to\infty} \xi(x_n - x_m, t) = \lim_{n,m\to\infty} \xi(x_n - x_m + x - x, t)$
$$= \lim_{n,m\to\infty} \xi[(x_n - x) + (x - x_m), t])$$

$$\geq \lim_{n\to\infty} \xi\left(x_n - x, \frac{t}{2}\right) = \Diamond \lim_{m\to\infty} \xi\left(x - x_m, \frac{t}{2}\right) [by\ (11)\ in\ Definition\ 3.2]$$

$$= \lim_{n\to\infty} \xi\left(x_n - x, \frac{t}{2}\right) = \Diamond \lim_{m\to\infty} \xi\left(x_m - x, \frac{t}{2}\right) [by\ (10)\ in\ Definition\ 3.2]$$

$$= 0 \Diamond 0 = 0\ as\ t \to \infty.$$

So $\lim_{n,m\to\infty} \xi(x_n - x_m, t) = 0$ and similarly $\lim_{n,m\to\infty} \eta(x_n - x_m, t) = 0.$

Hence,$\{x_n\}$ is a Cauchy Sequence.

Example 3.12. The following example will clarify that the inverse of the Theorem 3.11 may not be true. Let $R_1 = \left\{\frac{1}{n}\middle| n \epsilon \mathrm{N}\right\}$ (the set of natural numbers) be a subset of real numbers and $\left||x|\right| = |x|$. With respect to the neutrosophic norm defined in Example.3.10, obviously (R,N,∗,◊) is an NNLS.
Now

$$\lim_{n,m\to\infty} \frac{t}{t + ||x_n - x_m||} = \lim_{n,m\to\infty} \frac{t}{t + \left|\frac{1}{n} - \frac{1}{m}\right|} = 1,$$

$$\lim_{n,m\to\infty} \frac{||x_n - x_m||}{t + ||x_n - x_m||} = \lim_{n,m\to\infty} \frac{\left|\frac{1}{n} - \frac{1}{m}\right|}{t + \left|\frac{1}{n} - \frac{1}{m}\right|} = 0,$$

$$and, \qquad \lim_{n,m\to\infty} \frac{||x_n - x_m||}{t} = \lim_{n,m\to\infty} \frac{\left|\frac{1}{n} - \frac{1}{m}\right|}{t} = 0,$$

Thus $\{x_n\}$ is a Cauchy Sequence of points in the NNLS (R, N,∗,◊). But

$$\lim_{n\to\infty} (x_n - x_k, t) = \lim_{n\to\infty} \frac{\left|\frac{1}{n} - \frac{1}{k}\right|}{t + \left|\frac{1}{n} - \frac{1}{k}\right|} \neq 0\,.$$

This shows that the Cauchy Sequence $\{x_n\}$ is not convergent in that NNLS.

Theorem 3.13. In an NNLS $(V, N,*,\lozenge)$, if $\{x_n\}, \{y_n\}$ are Cauchy Sequence of vectors and $\{\lambda_n\}$ is Cauchy Sequence of scalars in an NNLS $(V, N,*,\lozenge)$, then $\{x_n + y_n\}$ *and* $\{\lambda_n y_n\}$ are also Cauchy Sequence in NNLS $(V, N,*,\lozenge)$.

Proof:

For $t > 0$, we have,

$$\lim_{n,m\to\infty} \rho(x_n - x_m, t) = 1, \lim_{n,m\to\infty} \xi(x_n - x_m, t) = 0, \lim_{n,m\to\infty} \eta(x_n - x_m, t) = 0, as\ t \to \infty$$

And

$$\lim_{n,m\to\infty} \rho(y_n - y_m, t) = 1, \lim_{n,m\to\infty} \xi(y_n - y_m, t) = 0, \lim_{n,m\to\infty} \eta(y_n - y_m, t) = 0, as\ t \to \infty$$

$$\lim_{n,m\to\infty} \rho[(x_n + y_n) - (x_m + y_m), t)] = \lim_{n,m\to\infty} \rho[(x_n - x_m) + (y_n - y_m), t)]$$

$$\geq \lim_{n,m\to\infty} \rho\left(x_n - x_m, \frac{t}{2}\right) * \lim_{n,m\to\infty} \rho\left(y_n - y_m, \frac{t}{2}\right) = 1 * 1 = 1\ as\ t \to \infty$$

Hence, $\lim_{n,m\to\infty} \rho[(x_n + y_n) - (x_m + y_m), t)] = 1\ as\ t \to \infty$

$$\lim_{n,m\to\infty} \xi[(x_n + y_n) - (x_m + y_m), t)] = \lim_{n,m\to\infty} \xi[(x_n - x_m) + (y_n - y_m), t)]$$

$$\leq \lim_{n,m\to\infty} \xi\left(x_n - x_m, \frac{t}{2}\right) \lozenge \lim_{n,m\to\infty} \xi\left(y_n - y_m, \frac{t}{2}\right) = 0 \lozenge 0 = 0\ as\ t \to \infty$$

So, $\lim_{n,m\to\infty} \xi[(x_n + y_n) - (x_m + y_m), t)] = 0\ as\ t \to \infty$

Similarly,

$$\lim_{n,m\to\infty} \eta[(x_n + y_n) - (x_m + y_m), t)] = 0\ as\ t \to \infty$$

This ends the first part. For the next part,

$$\lim_{n,m\to\infty} \rho[(\lambda_m y_m - \lambda_n y_n), t] = \lim_{n,m\to\infty} \rho[(\lambda_m y_m - \lambda_n y_n) + (\lambda_m y_n - \lambda_m y_n), t]$$

$$= \lim_{n,m\to\infty} \rho[(\lambda_m(y_m - y_n) + y_n(\lambda_m - \lambda_n), t] \geq \lim_{n,m\to\infty} \rho[\left((y_m - y_n), \frac{t}{2|\lambda_m|}\right)] * \rho\left(y_n, \frac{t}{2|\lambda_m - \lambda_n|}\right)$$

Since $|\lambda_m - \lambda_n| \to 0 \; as \; m, n \to \infty, So \; |\lambda_m - \lambda_n| \neq 0$. Again $\{y_n\}$ being Cauchy sequence is bounded.

Hence , $\lim_{n,m\to\infty} \rho[(\lambda_m y_m - \lambda_n y_n), t] = 1 \; as \; t \to \infty. Further,$

$$\lim_{n,m\to\infty} \xi[(\lambda_m y_m - \lambda_n y_n), t] = \lim_{n,m\to\infty} \xi[(\lambda_m y_m - \lambda_n y_n) + (\lambda_m y_n - \lambda_m y_n), t]$$

$$= \lim_{n,m\to\infty} \xi[(\lambda_m(y_m - y_n) + y_n(\lambda_m - \lambda_n), t] \leq \lim_{n,m\to\infty} \xi[\left((y_m - y_n), \frac{t}{2|\lambda_m|}\right)] \Diamond \xi\left(y_n, \frac{t}{2|\lambda_m - \lambda_n|}\right)$$

By similar argument, $\lim_{n,m\to\infty} \xi[(\lambda_m y_m - \lambda_n y_n), t] = 0 \; as \; t \to \infty$ and finally,

$\lim_{n,m\to\infty} \eta[(\lambda_m y_m - \lambda_n y_n), t] = 0 \; as \; t \to \infty$

Hence, the 2nd part is complete.

Definition 3.14. Let $(V, N, *, \Diamond)$ be a NNLS and Δ_V be the collection of all points on V. Then $(V, N, *, \Diamond)$ is said to be a complete NNLS if every Cauchy sequence of points in Δ_V converges to a point of Δ_V.

Theorem 3.15. In an NNLS $(V, N, *, \Diamond)$, if every Cauchy sequence has a convergent subsequence then $(V, N, *, \Diamond)$ is a complete NNLS.

Proof: Let $\{x_{n_k}\}$ be a convergent subsequence of a Cauchy sequence $\{x_n\}$ in an NNLS $(V, N, *, \Diamond)$ such that $\{x_{n_k}\} \to x \in V$. Since $\{x_n\}$ be a Cauchy sequence in $(V, N, *, \Diamond)$, given $t > 0$

$$\lim_{n,k\to\infty} \rho\left(x_n - x_{n_k}, \frac{t}{2}\right) = 1, \lim_{n,k\to\infty} \xi\left(x_n - x_{n_k}, \frac{t}{2}\right) = 0, \lim_{n,k\to\infty} \eta\left(x_n - x_{n_k}, \frac{t}{2}\right) = 0, as \; t \to \infty$$

Again since $\{x_{n_k}\}$ converges to x, then

$$\lim_{n,k\to\infty} \rho\left(x_{n_k} - x, \frac{t}{2}\right) = 1, \lim_{n,k\to\infty} \xi\left(x_{n_k} - x, \frac{t}{2}\right) = 0, \lim_{n,k\to\infty} \eta\left(x_{n_k} - x, \frac{t}{2}\right) = 0, t \to \infty$$

Now,

$$\rho(x_n - x, t) = \rho(x_n - x_{n_k} + x_{n_k} - x, t) \geq \rho\left(x_n - x_{n_k}, \frac{t}{2}\right) * \rho\left(x_{n_k} - x, \frac{t}{2}\right).$$

It implies

$$\lim_{n\to\infty} \rho(x_n - x, t) = 1$$

Further,

$$\xi(x_n - x, t) = \xi(x_n - x_{n_k} + x_{n_k} - x, t) \leq \xi\left(x_n - x_{n_k}, \frac{t}{2}\right) \Diamond \xi\left(x_{n_k} - x, \frac{t}{2}\right).$$

It implies $\lim_{n\to\infty} \xi(x_n - x, t) = 0$.

It implies $\lim_{n\to\infty} \eta(x_n - x, t) = 0$.

This shows that x_n converges to $x \in V$ and thus the theorem is proved.

Theorem 3.16. In an NNLS $(V, N, *, \Diamond)$, every convergent sequence is a Cauchy sequence.

Proof: Let $\{x_n\}$ be a convergent sequence in the NNLS $(V, N, *, \Diamond)$ with $\lim_{n\to\infty} x_n = x$. Let $s, t \in \mathbb{R}^+$ and p = 1,2,3,…, we have

$$\rho(x_{n+p} - x_n, s + t) = \rho(x_{n+p} - x + x - x_n, s + t)$$

$$\geq \rho\left(x_{n+p} - x, s\right) * \rho(x - x_n, t)$$

$$= \rho\left(x_{n+p} - x, s\right) * \rho(x_n - x, t)$$

Taking limit, we have

$$\lim_{n\to\infty} \rho\left(x_{n+p} - x_n, s + t\right) \geq \lim_{n\to\infty} \rho\left(x_{n+p} - x, s\right) * \lim_{n\to\infty} \rho(x_n - x, t)$$
$$= 1 * 1 = 1$$

$$\lim_{n\to\infty} \rho\left(x_{n+p} - x_n, s + t\right) = 1 \; \forall s, t \to \infty \; and \; p = 1,2,3 \ldots.$$

Again,

$$\xi\left(x_{n+p} - x_n, s + t\right) \geq \xi(x_{n+p} - x + x - x_n, s + t)$$

$$\geq \xi\left(x_{n+p} - x, s\right) \lozenge \xi(x - x_n, t)$$

$$= \xi\left(x_{n+p} - x, s\right) \lozenge \xi(x_n - x, t)$$

Taking limit, we have

$$\lim_{n\to\infty} \xi\left(x_{n+p} - x_n, s + t\right) \geq \lim_{n\to\infty} \xi\left(x_{n+p} - x, s\right) \lozenge \lim_{n\to\infty} \xi(x_n - x, t)$$
$$= 0 \lozenge 0 = 0$$

$$\lim_{n\to\infty} \xi\left(x_{n+p} - x_n, s + t\right) = 0 \; \forall s, t \to \infty \; and \; p = 1,2,3 \ldots.$$

Similarly,

$$\lim_{n\to\infty} \eta\left(x_{n+p} - x_n, s + t\right) = 0 \; \forall s, t \to \infty \; and \; p = 1,2,3 \ldots.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the NNLS $(V, N, *, \lozenge)$.

Theorem 3.17. Let $(V, N, *, \lozenge)$ be an NNLS, such that every Cauchy sequence in $(V, N, *, \lozenge)$has a convergent sebsequence. Then $(V, N, *, \lozenge)$ is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in $(V, N, *, \lozenge)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ the converges to $x \in V \; and \; t > 0$. Since $\{x_n\}$ is a Cauchy sequence in $(V, N, *, \lozenge)$, we have

$$\lim_{n,k\to\infty} \rho\left(x_n - x_k, \frac{t}{2}\right) = 1, \lim_{n,k\to\infty} \xi\left(x_n - x_k, \frac{t}{2}\right) = 0, \lim_{n,k\to\infty} \eta\left(x_n - x_k, \frac{t}{2}\right) = 0$$

Again since $\{x_{n_k}\}$ converges to x, we have

$$\lim_{k\to\infty} \rho\left(x_{n_k} - x, \frac{t}{2}\right) = 1, \lim_{k\to\infty} \xi\left(x_{n_k} - x, \frac{t}{2}\right) = 0, \lim_{n,k\to\infty} \eta\left(x_{n_k} - x, \frac{t}{2}\right) = 0.$$

Now,

$$\rho(x_n - x, t) = \rho(x_n - x_{n_k} + x_{n_k} - x, t)$$

$$\geq \rho\left(x_n - x_{n_k}, \frac{t}{2}\right) * \rho\left(x_{n_k} - x, \frac{t}{2}\right)$$

$$\lim_{n\to\infty} \rho(x_n - x, t) = 1$$

Again, we see that

$$\xi(x_n - x, t) = \xi(x_n - x_{n_k} + x_{n_k} - x, t)$$

$$\leq \xi\left(x_n - x_{n_k}, \frac{t}{2}\right) \lozenge \xi\left(x_{n_k} - x, \frac{t}{2}\right)$$

$$\lim_{n\to\infty} \xi(x_n - x, t) = 0$$

Similarly, $\lim_{n\to\infty} \eta(x_n - x, t) = 0$
Thus, $\{x_n\}$ converges to x in $(V, N, *, \Diamond)$ and hence is complete.

Theorem 3.18. Every finite dimensional NNLS satisfying the condition.

$$\left.\begin{matrix} a \Diamond a = a \\ a * a = a \end{matrix}\right\} \forall a \in [0,1] \dots\dots\dots\dots (1)$$

$\rho(x, t) > 0 \ \forall t > 0 \rightarrow x = 0$ ……………(2) is complete.

Proof: Let $(V, N, *, \Diamond)$ be a finite dimensional NNLS satisfying the condition (1) and (2). Also, let dim V =k and $e_1, e_2, \dots, e_k$ be a basic of V.
Consider $\{x_n\}$ as an arbitrary Cauchy sequence in (V,A).
Let $x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \cdots + \beta_k^{(n)} e_k$ where $\beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_k^{(n)}$suitable scalars are. Then by the same calculation, there exist $\beta_1, \beta_2, \dots, \beta_k \in F$ such that the sequence $\{\beta_i^{(n)}\}_n$ converges to $\beta_i \ for \ i = 1,2, . . , k$. clearly $x = \rho(\sum_{i=1}^{k} \beta_i^{(n)} e_i \in V$

$$\begin{aligned} \rho(x_n - x, t) &= \rho(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i \, e_i, t) \\ &= \rho(\sum_{i=1}^{k} (\beta_i^{(n)} - \beta_i) \, e_i, t) \\ &\geq \rho\left((\beta_1^{(n)} - \beta_1) e_i, \frac{t}{k}\right) * \dots * \ \rho\left((\beta_k^{(n)} - \beta_k) e_k, \frac{t}{k}\right) \\ &= \rho\left(e_1, \frac{t}{k|\beta_1^{(n)} - \beta_1|}\right) * \dots * \rho\left(e_k, \frac{t}{k|\beta_k^{(n)} - \beta_k|}\right) \end{aligned}$$

Since $\lim_{n\to\infty} \frac{t}{k|\beta_i^{(n)} - \beta_i|} = \infty$, we see that $\lim_{n\to\infty} \rho\left(e_i, \frac{t}{k|\beta_i^{(n)} - \beta_i|}\right) = 1$

$\lim_{n\to\infty} \rho(x_n - x, t) \geq 1 * \dots * 1 = 1 \ \forall \ t > 0$

$\lim_{n\to\infty} \rho(x_n - x, t) = 1 \ \forall \ t > 0.$

Again, for all $t > 0$

$$\begin{aligned} \xi(x_n - x, t) &= \xi(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i \, e_i, t) \\ &= \xi(\sum_{i=1}^{k} (\beta_i^{(n)} - \beta_i) \, e_i, t) \\ &\leq \xi\left((\beta_1^{(n)} - \beta_1) e_i, \frac{t}{k}\right) \Diamond \dots \Diamond \ \xi\left((\beta_k^{(n)} - \beta_k) e_k, \frac{t}{k}\right) \\ &= \xi\left(e_1, \frac{t}{k|\beta_1^{(n)} - \beta_1|}\right) \Diamond \dots \Diamond \ \xi\left(e_k, \frac{t}{k|\beta_k^{(n)} - \beta_k|}\right) \end{aligned}$$

Since $\lim_{n\to\infty} \frac{t}{k|\beta_i^{(n)} - \beta_i|} = \infty$, we see that $\lim_{n\to\infty} \xi\left(e_i, \frac{t}{k|\beta_i^{(n)} - \beta_i|}\right) = 0$

$\lim_{n\to\infty} \xi(x_n - x, t) \leq 0 \Diamond \dots \Diamond 0 = 0 \ \forall \ t > 0$

$\lim_{n\to\infty} \xi(x_n - x, t) = 0 \ \forall \ t > 0.$

Similarly, Since $\lim_{n\to\infty} \frac{t}{k|\beta_i^{(n)}-\beta_i|} = \infty$, we see that $\lim_{n\to\infty} \eta\left(e_i, \frac{t}{k|\beta_i^{(n)}-\beta_i|}\right) = 0$

Thus, we see that $\{x_n\}$ is an arbitrary Cauchy Sequence that converges to x ∊ V, Hence the NNLS $(V, N,*,\lozenge)$ is complete.

Theorem 3.19. Let $(V, N,*,\lozenge)$be an NNLS satisfying the condition equation (1). Every Cauchy sequence in $(V, N,*,\lozenge)$ is bounded.

Proof: Let $\{x_n\}$ be a Cauchy sequence in the NNLS $(V, N,*,\lozenge)$. Then we have

$$\left.\begin{array}{l}\lim_{n\to\infty} \rho(x_{n+p} - x, t) = 1\\ \lim_{n\to\infty} \xi(x_{n+p} - x, t) = 0\\ \lim_{n\to\infty} \eta(x_{n+p} - x, t) = 0\end{array}\right\} \forall t > 0, p = 1,2, \dots$$

Choose a fixed r_0 with $0 < r_0 < 1$. Now we see that

$$\lim_{n\to\infty} \rho(x_n - x_{n+p}, t) = 1 > r_0 \; \forall t > 0, p = 1,2, \dots$$

For $t' > 0 \; \exists \; n_0 = n_0(t')$ such that $\rho(x_n - x_{n+p}, t') > r_0 \; \forall n \geq n_0, \; p = 1,2, \dots$

Since, $\lim_{n\to\infty} \rho(x, t) = 1$, we have for each $x \; \epsilon \; t > 0$ such that

$$\rho(x_n, t) > r_0 \; \forall t > t_i, n = 1,2, \dots$$

Let $t_0 = t' + max\{t_1, t_2, \dots, t_{n_0}\}$ Then,

$$\rho(x_n, t_0) \geq \rho(x_n, t' + t_{n_0})$$

$$= \rho(x_n - x_{n_0} + x_{n_0}, t' + t_{n_0})$$

$$\geq \rho(x_n - x_{n_0}, t') * \rho(x_{n_0}, t_{n_0})$$

$$> \quad r_0 * r_0 = r_0 \; \forall \; n \geq n_0$$

Thus, we have

$$\rho(x_n, t_0) > \; r_0 \; \forall \; n \geq n_0$$

$$Also, \qquad \rho(x_n, t_0) \geq \rho(x_n, t_n) > \; r_0 \; \; \forall n = 1,2, \dots, n_0$$

So, we have,

$$\rho(x_n, t_0) > \; r_0 \; \forall \; n = 1,2, \dots \dots \dots \dots \dots \dots .. (1)$$

$$Now, \; \lim_{n\to\infty} \xi(x_n - x_{n+p}, t) = 0 < (1 - r_0) \; \; \forall t > 0, p = 1,2, \dots$$

For $t' > 0 \; \exists \; n_0' = n_0'(t')$ such that $\xi(x_n - x_{n+p}, t') < (1 - r_0) \; \forall \; n \geq n_0', \; p = 1,2, \dots$

Since, $\lim_{n\to\infty} \xi(x, t) = 0$, we have for each $x_i \; \exists \; t_i' > 0$ such that

$$\xi(x_n, t) < (1 - r_0) \; \forall t > \; t_i', n = 1,2, \dots$$

Let $t_0' = t' + \; max\{t_1', t_2', \dots, t_{n_0}'\}$ Then,

$$\xi(x_n, t_0') \leq \xi(x_n, t' + t_{n_0}')$$

$$= \xi\left(x_n - x_{n_0'} + x_{n_0'}, t' + t_{n_0}'\right)$$

$$\leq \xi\left(x_n - x_{n_0'}, t'\right) \lozenge \; \xi(x_{n_0'}, t_{n_0}')$$

$$< \ (1-r_0) \lozenge (1-r_0) = (1-r_0) \ \forall \ n > n_0'$$

Thus, we have

$$\xi(x_n, t_0') < (1-r_0) \ \forall n > n_0'$$

$$Also, \quad \xi(x_n, t_0') \le \xi(x_n, t_n') < (1-\ r_0) \ \ \forall n = 1,2, \dots, n_0'$$

So, we have,

$$\xi(x_n, t_0') < (1-\ r_0) \ \forall \ n = 1,2, \dots \dots \dots \dots \dots \dots .. (2)$$

Similarly, we prove

$$\eta(x_n, t_0') < (1-\ r_0) \ \forall \ n = 1,2, \dots \dots \dots \dots \dots \dots .. (3)$$

Let $t_0'' = \max\{t_0, t_0'\}$. Hence from (1),(2), and (3) we see that

$$\left.\begin{array}{l} \rho(x_n, t_0'') \quad > \quad r_0 \\ \xi(x_n, t_0'') < (1-r_0) \\ \eta(x_n, t_0'') < (1-r_0) \end{array}\right\} \forall \ n = 1,2, \dots$$

This implies that $\{x_n\}$ is bounded in $(V, N, *, \lozenge)$.

4. Conclusion

4.1 Concluding Remarks:

The aim of the present work is to introduce a Neutrosophic norm on a linear space. Also, the convergence of sequence, characteristic of Cauchy sequence in NNLS (Neutrosophic normed linear space) have been studied here. These are illustrated by suitable examples. Their related properties and structural characteristic have been discussed.

4.2 Future Scope:

This studied provides the structure of NNLS (Neutrosophic normed linear space) on a NLS (Normed linear space) with help of NS (Neutrosophic Set). In future this study leads to the extension of the following ideas:

- Neutrosophic-n-Normed Linear Space
- Finite Dimensional Neutrosophic-n-Normed Linear Space
- Neutrosophic Metric Space

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