8-26-1955

A Proposal for Two Computers Dealing With the Convolution Integral

Charles S. Williams Jr.

PROPOSAL FOR TWO COMPUTERS DEALING WITH THE CONVOLUTION INTEGRAL — WILLIAMS

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A PROPOSAL FOR TWO COMPUTERS
DEALING WITH THE CONVOLUTION INTEGRAL

By

Charles S. Williams, Jr.

A Thesis

In partial fulfillment of the

Requirements for the Degree of

Master of Science in Electrical Engineering

The University of New Mexico

1955

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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DATE

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CHAIRMAN

J. L. Ellis

R. B. Moore

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CHAPTER I

CONVOLUTION COMPUTERS AND THEIR USES

1.1 Purpose of the computers. The integral

$$f(x) = \int_{-\infty}^{\infty} g(x - y)h(y)dy \qquad (1.1\text{-}1)$$

is known as a convolution integral. The first computer described herein will determine the function f when the functions g and h are known. The second computer will determine the function h when f and g are known; in using the second computer to find h, the first computer is used for auxiliary computations.

1.2 Uses of the convolution integral. The convolution integral and similar integrals of products where one function is delayed (x represents delay) find wide application. Two are described here.

1.2(a) Application to linear systems. The convolution integral probably finds its greatest use in relating the output to the input of linear systems in the following way. Let h(t) be the response of a linear system to a unit impulse, where t is real time. Let g(t) be the input time function. Then f(t) is the output time function.[1] See Fig. 1.2-a.

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1 Murray F. Gardner and John L. Barnes, Transients in Linear Systems, (New York: John Wiley & Sons, Inc., 1942), Vol. I, p. 262.

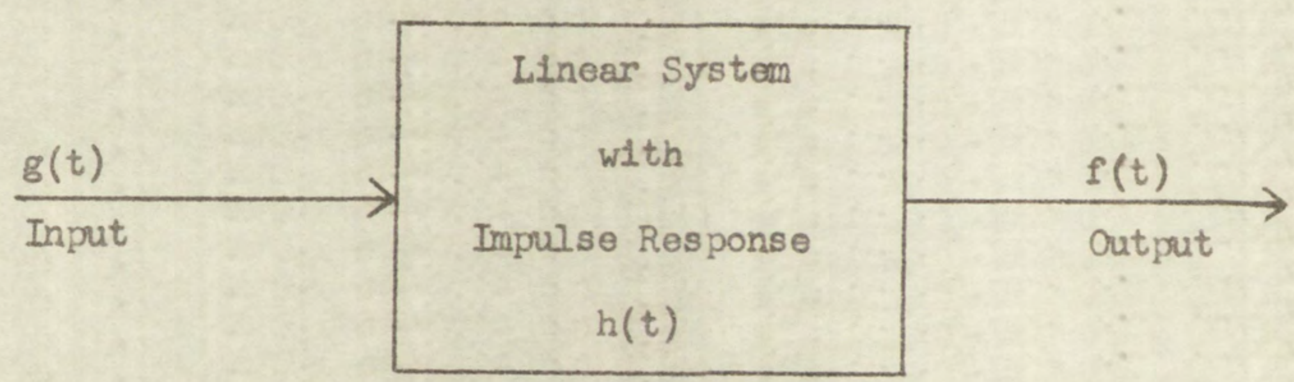

Figure 1.2-a

Block Diagram for Equation (1.1-1)

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Most electrical circuits are treated as (and usually approximate) linear systems. All circuits that contain only R, L, and C elements are linear, provided the R's, L's, and C's are constants. Many circuits that contain vacuum tubes and other nonlinear elements are linear to a first approximation. For all such electrical systems, the functions $f(t)$ and $g(t)$ must represent either current or voltage since these are the quantities that occur linearly in the differential equations that describe the system. Further, $h(t)$ is the current response, if $f(t)$ represents current, to a unit impulse of voltage, if $g(t)$ represents voltage.

1.2(b) Application to communications. In the study of the transmission of information, correlation functions are frequently calculated to aid in both statistical and harmonic analysis.[2] The autocorrelation function of a function $f_1(t)$ is defined by

$$\varphi_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, f_1(t+\tau)\, dt,$$

and the cross-correlation function of two functions $f_1(t)$ and $f_2(t)$ is defined by

$$\varphi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, f_2(t+\tau)\, dt.$$

It is evident that these integrals have essentially the form of the integral of equation (1.1-1). The argument of the function g differs

2 Y. W. Lee and J. W. Wiesner, "Correlation Functions and Communication Applications," Electronics, Vol. 23, pp. 87-88, June, 1950.

from that of the second function in the definitions of φ_{11} and φ_{12} above. However, it will become clear with the description of the computers that the argument of g in equation (1.1-1) could be readily changed to x + y insofar as the computers are concerned.

1.3 Review of the literature. Many devices for performing convolutions (or correlations) have been proposed and constructed. Some of the early computers use optical principles. The functions g and h of equation (1.1-1) are cut out of cardboard[3] or put on film[4] in such a way that the functions are represented by the line between the portion of the cardboard or film which passes light and that which does not. Delay, i.e., the value of x in equation (1.1-1), is obtained by physically shifting one cutout or film with respect to the other. These computers are capable of handling integrals where the function g has the more general form g(x,y). Of course, many cutouts or films have to be prepared since each function used requires one.

A mechanical correlator has been devised and constructed.[5] Integration is accomplished by mechanical principles. Two operators are required, one each to follow the functions g and h with styli. A run

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3 T. S. Gray, "A Photo-Electric Integraph," Journal of the Franklin Institute, Vol. 212, pp. 77-102, July, 1931.

4 H. L. Hazen and G. S. Brown, "The Cinema Integraph," Journal of the Franklin Institute, Vol. 230, pp. 19-44, pp. 183-205, August, 1940.

5 H. R. Seiwell, "A New Mechanical Autocorrelator," The Review of Scientific Instruments, Vol. 21, pp. 481-484, May, 1950.

is required for each value of delay, x. Graphical and numerical methods have also been found useful.[6,7,8]

More automatic systems with ability to handle a wide variety of functions include a digital correlator,[9] an analogue correlator using a delay line,[10] and an analogue system using drum function generators whose relative angular positions are shifted to obtain delay.[11]

The systems described above will find an unknown function of the integrand, provided this function is related in some simple way to the integral itself which is also unknown, by a method of successive approximations. For example, if h(y) = f'(y), a guess is made for f which is then differentiated and put into the convolution as h. The output of the computer gives a new f which is also differentiated and put back in the machine as a new h. For a large class of functions this is a

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6 Robert T. Jones, "Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances," Journal of the Aeronautical Sciences, Vol. 3, pp. 421-422, Oct., 1936.

7 E. V. Laitone and J. T. Ablin, "A Simple Graphical Solution of the Duhamel Integral," Journal of the Aeronautical Sciences, Vol. 18, pp. 142-143, Feb., 1951.

8 R. A. Hessemer, Jr., and C. S. Williams, Jr., "Solution of the Integral Equation Which Determines Radar-Cross-Section for a Scattering Ground," (Technical Memorandum 207-54-54 of Sandia Corporation, Albuquerque, 1954), pp. 7-8.

9 Henry E. Singleton, "A Digital Electronic Correlator," Proceedings of the I.R.E., Vol. 38, pp. 1422-1428, Dec., 1950.

10 Harold Bell, Jr., and V. C. Rideout, "A High-Speed Correlator," Transactions of the I.R.E., Professional Group on Electronic Computers, Vol. EC-3, No. 2, pp. 30-36, June, 1954.

11 T. O. Ellis, M. R. Davis, and L. L. Grandi, "A Machine for Computing the Convolution Integral," Conference Paper No. CP 55-515 of the American Institute of Electrical Engineers, 1955.

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6 Robert T. Jones, "Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances," *Journal of the Aeronautical Sciences*, Vol. 3, pp. 421-422, Oct., 1936.

7 E. V. Laitone and J. T. Allen, "A Simple Graphical Solution of the Duhamel Integral," *Journal of the Aeronautical Sciences*, Vol. 18, pp. 142-143, Feb., 1951.

8 R. A. Hessemer, Jr., and O. B. Williams, Jr., "Solution of the Integral Equation Which Determines Radar Cross-section for a Distributed Ground," (Technical Memorandum 20-54-54 of Sandia Corporation, Albuquerque, 1954), pp. 1-8.

9 Henry E. Singleton, "A Digital Electronic Correlator," *Proceedings of the I.R.E.*, Vol. 38, pp. 1422-1428, Dec., 1950.

10 Harold Bell, Jr., and V. C. Rideout, "A High-Speed Correlator," *Transactions of the I.R.E.*, Professional Group on Electronic Computers, Vol. EC-3, No. 2, pp. 30-36, June, 1954.

11 T. D. Ellis, M. R. Davis, and L. A. Gould, "A Machine for Computing the Convolution Integral," Conference Paper No. CP 53-219 of the American Institute of Electrical Engineers, 1953.

convergent process and leads to a solution for the function f. (The reason for the requirement of a simple relation between h and f is evident; if it is not simple, another device would be required to find h once the convolution computer had produced an f.) Two of the references give examples of such problems.[12,13] The graphical method of Laitone and Ahlin[14] will find a function H(y), related to h(y) by $H'(y) = h(y)$, when the functions f and g are known.

R. A. Hessemer, Jr., and C. S. Williams, Jr.[15] proposed a computer to find the function h when the functions f and g of equation (1.1-1) are known. The same computer will also find f when g and h are known. Williams described this computer in more detail and proposed modifications of the output circuits to improve the operation.[16] The computer was designed in detail and fabricated at the Physical Science Laboratory of the New Mexico College of Agriculture and Mechanic Arts under the direction of Mr. Kenneth "E" Smith,[17] who is also supervising its testing which, at this date, is not complete.

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12 Gray, op. cit., pp. 96-98.

13 Hazen and Brown, op. cit., appendix contributed by W. R. Hedeman, Jr., pp. 193-205.

14 Laitone and Ahlin, loc. cit.

15 Hessemer and Williams, op. cit., p. 10.

16 C. S. Williams, Jr., "A Computer to Solve One Form of Volterra's Integral Equation," (Technical Memorandum 104-55-54 of Sandia Corporation, Albuquerque, 1955), pp. 5-22.

17 Kenneth "E" Smith, "Preliminary Report on Convolution Integral Computer," (AER 14-S, Physical Science Laboratory, New Mexico College of Agriculture and Mechanic Arts, State College, 1955), pp. 3-27.

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12 Gray, op. cit., pp. [illegible].

13 Hazen and Brown, op. cit., Appendix contributed by W. R. Hedeman, Jr., pp. [illegible]-205.

14 Ledtone and Anlin, loc. cit.

15 Hessemer and Williams, op. cit., p. 40.

16 C. R. Williams, Jr., "A Computer to Solve the Faltung Integral Equation," (Technical Memorandum 104-55-54, Sandia Corporation, Albuquerque, 1955), pp. 5-22.

17 Kenneth "S" Smith, "Preliminary Report on a Convolution Integral Computer," (AER 14-B, Physical Science Laboratory, New Mexico College of Agriculture and Mechanic Arts, State College, 1955), pp. 1-2.

The computer that is described herein in Chapter III is taken directly from the computer mentioned just above, but it is considerably simplified. It performs only the operation of finding the function f when the functions g and h are specified.

The computer that is described in Chapter IV is an outgrowth of the work already done by Hessemer,[18] Smith,[19] and Williams.[18,20] It, too, finds the function h when the functions f and g are known. However, the method by which it finds h is not the same as that used in the computer first proposed. It gives the solution immediately after the proper quantities are entered, whereas the original computer approaches the solution through repeated manual operations that minimize a meter reading.

1.4 Definition of symbols and terms. The symbols and terms that are used in this paper are those that are standard in electrical engineering. However, for the sake of clarity, several of them are defined and discussed here.

j: This is the imaginary unit; $j = \sqrt{-1}$.

ω: This symbol is used throughout this paper only to designate angular velocity in radians per second. It is related to frequency f by $\omega = 2\pi f$.

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18 Hessemer and Williams, *op. cit.*, p. 10.

19 Smith, *loc.* cit.

20 Williams, *loc. cit.*

s: This is commonly used for the independent variable of the Laplace transform, and it is so used here. It is complex and its real and imaginary parts are given by $s = \sigma + j\omega$, where ω is defined above. The imaginary axis in the s-plane is the so-called real-frequency axis since it corresponds to $\sigma = 0$.

$\stackrel{\Delta}{=}$: This means "is defined by". It includes, of course, equality $=$, and identity $\equiv$.

Unit impulse: The unit impulse, sometimes called the Dirac delta-function, is designated by $\delta(x)$. It has the properties $\delta(x) = 0$, $(x \neq 0)$, $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. It can be constituted in various ways. One method is to take the limit of a rectangular function at the origin, with width Δx and amplitude A, as $\Delta x \rightarrow 0$ while $A\Delta x = 1$.

The functional notation that is used herein is standard. To avoid confusion, parentheses are used to enclose the independent variable, and are nowhere used to separate factors to be multiplied. Thus, $f(M\Delta y + N\Delta y)$ can mean only the value of the function f when its argument has the value $M\Delta y + N\Delta y$. It is worth mentioning that only continuous and differentiable functions are considered in this paper, except that such functions may have a finite number of finite discontinuities in a finite interval.

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CHAPTER II

APPROXIMATING THE CONVOLUTION INTEGRAL

The computers described herein make use of approximations to the integral. These approximations can be made as close as desired to the exact values; practical considerations will dictate the degree of accuracy to be used. In this chapter, the method of approximation is defined, and an illustration of the manner of using the approximation is given.

2.1 An approximation to the convolution integral. The integral of equation (1.1-1) can be approximated by the sum

$$f(x) = \Delta y \sum_{n=-\infty}^{n=+\infty} g(x - n\Delta y + \tfrac{1}{2}\Delta y)h(n\Delta y - \tfrac{1}{2}\Delta y), \qquad (2.1\text{-}1)$$

where n takes on integer values only. In equation (2.1-1), the limit of the sum as $\Delta y \rightarrow 0$ is the integral of equation (1.1-1). (See any elementary calculus text for definition of the definite integral.) Hence, the smaller Δy is taken, the better is the approximation to the integral. The interval Δy is determined by the degree of accuracy desired. In general, no large variation in g(y) or h(y) should occur in an interval Δy, unless, of course, there is a step discontinuity in one of the functions.

Since values of g(y) and h(y) are selected at discrete intervals, Δy, of the independent variable, the variable x in equation (2.1-1) should also be discrete, not continuous. Hence, let $x = m\Delta y$, where m is

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an integer. Then

$$f(m\Delta y) = \Delta y \sum_{n=-\infty}^{n=+\infty} g(m\Delta y - n\Delta y + \tfrac{1}{2}\Delta y)h(n\Delta y - \tfrac{1}{2}\Delta y). \qquad (2.1\text{-}2)$$

The restrictions that will be applied to the functions g(y) and h(y) together with the resultant restrictions of f(y) are:

$$\left.\begin{aligned} &f(y) = 0,\ (y \le 0,\ y \ge M\Delta y + N\Delta y),\\ &g(y) = 0,\ (y < 0,\ y > M\Delta y),\\ &h(y) = 0,\ (y < 0,\ y > N\Delta y), \end{aligned}\right\} \qquad (2.1\text{-}3)$$

where M and N are integers. This changes the lower limit of summation in equation (2.1-2) to n = 1 and the upper limit to n = m.

To simplify the notation let

$$\left.\begin{aligned} f_k &\triangleq f(k\Delta y)\\ &\triangleq 0,\ (k < 1,\ k > M + N - 1),\\ g_k &\triangleq g(k\Delta y - \tfrac{1}{2}\Delta y)\\ &\triangleq 0,\ (k < 1,\ k > M),\\ h_k &\triangleq h(k\Delta y - \tfrac{1}{2}\Delta y)\\ &\triangleq 0,\ (k < 1,\ k > N), \end{aligned}\right\} \qquad (2.1\text{-}4)$$

where k has integer values only. The ranges for which f_k, g_k, and h_k are nonzero follow from their definitions and from the restrictions of equations (2.1-3).

Now, by virtue of equations (2.1-4), equation (2.1-2) may be written

$$f_m = \Delta y \sum_{n=1}^{m} g_{m-n+1}\, h_n, \qquad (2.1\text{-}5)$$

where

$$m = 1, 2, \ldots, M + N - 1.$$

This is the desired approximation to the convolution integral and it is the form with which the computers deal.

2.2 Illustration of the convolution integral approximation. In order to illustrate the use of the approximation of equations (2.1-5) to the integral (1.1-1), let

$$g(x) = 1, \quad (0 \leq x < 1.2),$$

$$= 0, \quad (x < 0,\ 1.2 \leq x),$$

$$h(x) = e^{-x}, \ (0 \leq x),$$

$$= 0, \quad (x < 0).$$

Then, from equation (1.1-1),

$$f(x) = 0, \qquad (x \leq 0),$$

$$= \int_0^x e^{-y}\, dy, \qquad (0 \leq x \leq 1.2),$$

$$= \int_{x-1.2}^x e^{-y}\, dy, \ (1.2 \leq x).$$

Upon evaluation

$$f(x) = 0, \qquad (x \le 0),$$
$$= 1 - e^{-x}, \qquad (0 \le x \le 1.2),$$
$$= \left[e^{1.2} - 1\right] e^{-x}, \quad (1.2 \le x).$$

To evaluate $f(x)$ by equations (2.1-5), choose

$$\Delta y = 0.4.$$

Then,

$$h_1 = h(.2), \quad h_2 = h(.6), \text{ etc.},$$
$$g_1 = g(.2), \quad g_2 = g(.6), \text{ etc.}$$

These values are illustrated in Fig. 2.2-a. Next, the f_m are computed by equations (2.1-5); the values of the f_m as so calculated are compared with the function $f(x)$, as found analytically, in the following table:

m	f_m	$f(m\Delta y)$	m	f_m	$f(m\Delta y)$
0	0	0	10	.042	.042
1	.327	.330	11	.028	.028
2	.547	.551	12	.019	.019
3	.695	.699	13	.013	.013
4	.466	.468	14	.008	.009
5	.312	.313	15	.006	.006
6	.209	.210	16	.004	.004
7	.140	.141	17	.002	.003
8	.094	.094	18	.001	.002
9	.063	.063	19	.000	.001

The agreement is quite good and may be improved by taking a smaller value of Δy.

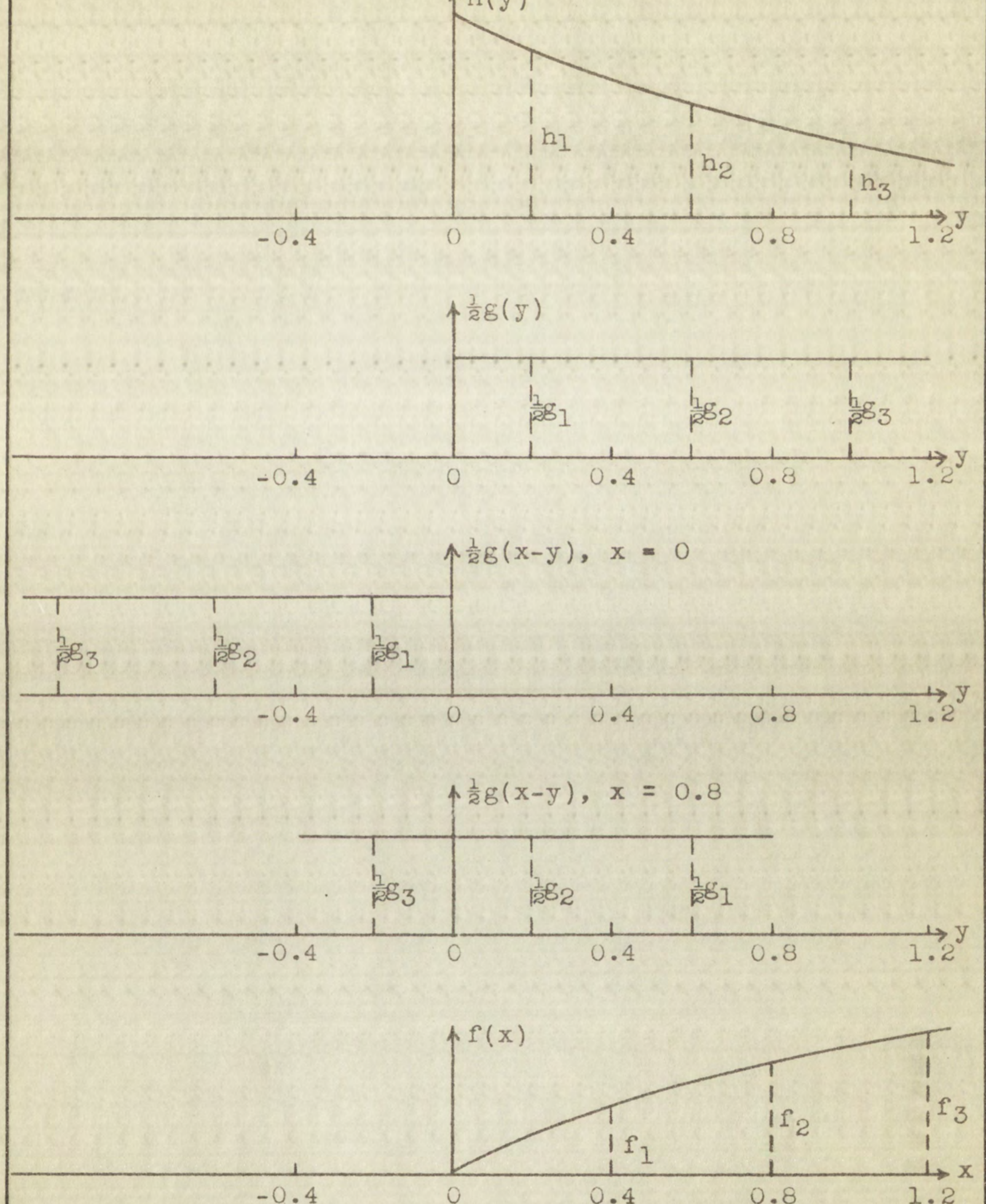

Figure 2.2-a

Illustration of Convolution

CHAPTER III

CONVOLUTION COMPUTER

The computer described in this chapter finds the function f of equation (1.1-1) when g and h are known. In this respect, it is the same as the computer proposed by Hessemer and Williams,[21,22] and it uses exactly the same type of switch. However, it is a great deal simpler since it does not solve the integral equation for h when f and g are known.

3.1 <u>Circuit of the convolution computer</u>. The computer uses the approximation of equations (2.1-5) to represent the integral (1.1-1). In order to illustrate the computer in Fig. 3.1-a, M, the number of ordinates taken to represent g(x), is 3; N, the number of ordinates of h(x), is 4. Then, from equations (2.1-4), the number of ordinates of f(x) will be M + N - 1 = 6. With these restrictions, equations (2.1-5) may be written:

$$\left.\begin{aligned}
f_1 &= \Delta y(g_1h_1), \\
f_2 &= \Delta y(g_2h_1 + g_1h_2), \\
f_3 &= \Delta y(g_3h_1 + g_2h_2 + g_1h_3), \\
f_4 &= \Delta y(g_3h_2 + g_2h_3 + g_1h_4), \\
f_5 &= \Delta y(g_3h_3 + g_2h_4), \\
f_6 &= \Delta y(g_3h_4).
\end{aligned}\right\} \qquad (3.1\text{-}1)$$

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21 Hessemer and Williams, <u>op</u>. <u>cit</u>., p. 10.

22 Williams, <u>loc</u>. <u>cit</u>.

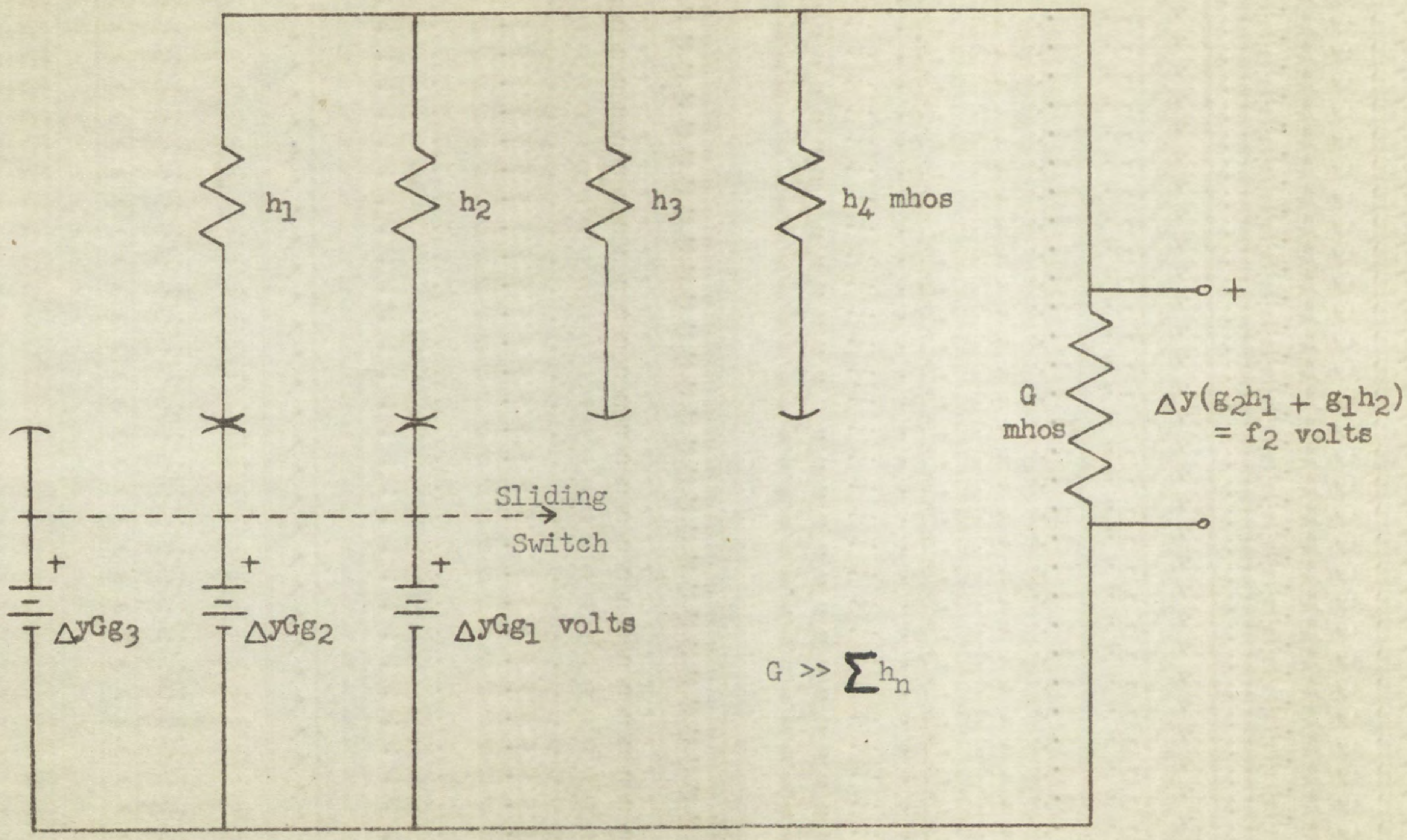


Figure 3.1-a

Circuit of Convolution Computer

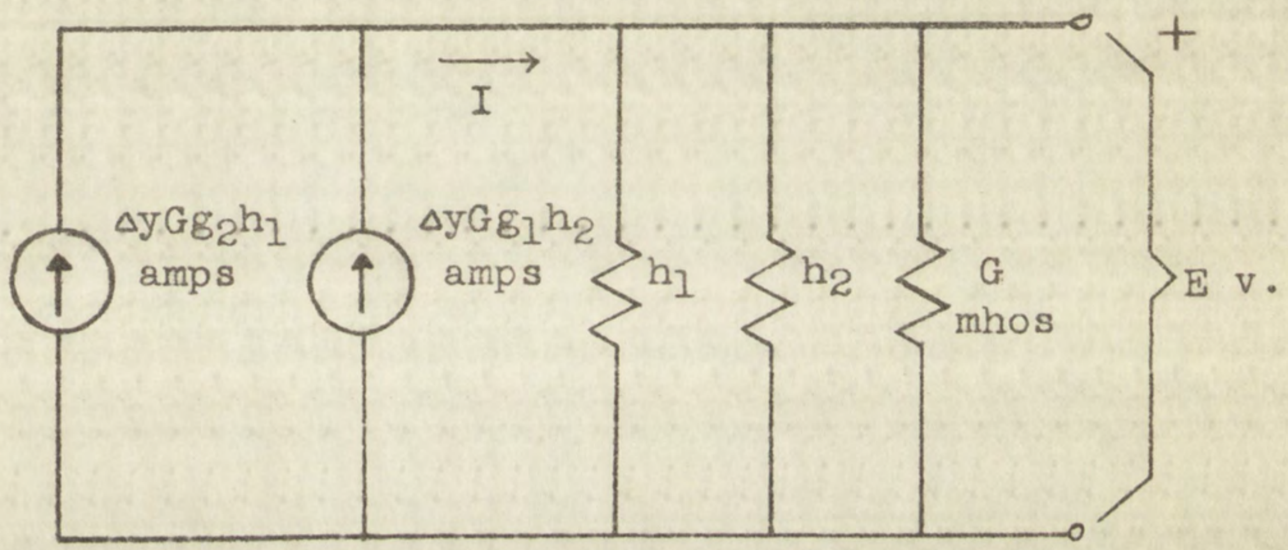
Now, in Fig. 3.1-a, the h_n are represented by conductances, and the quantities ΔyGg_n are represented by voltages. Thus, in the switch position shown, the voltage across the conductance G is f_2 provided $G >> \Sigma h_n$, as reference to Fig. 3.1-b shows. It is evident, then, that the various switch positions give the ordinates of $f(x)$; i.e., as the switch moves from left to right, the output voltages are f_1, f_2, f_3, f_4, f_5, and f_6.

In order to satisfy the condition $G >> \Sigma h_n$, it may be necessary to use a feedback amplifier at the output to give an effectively large conductance. See Sec. 4.4 concerning admittance conversion by amplifiers.

Although it is not shown in Fig. 3.1-a, there must be some method of varying the conductances h_n and the voltages ΔyGg_n so that various functions $g(x)$ and $h(x)$ may be inserted in the computer. The methods of entering the functions are several. It is possible to use a voltmeter for the ΔyGg_n and a mhometer for the h_n. A better, and more expensive, system would use precision variable conductances and potential dividers so that the functions could be entered by dial settings only.

The output voltages f_m may be read with a voltmeter and recorded as the switch is moved manually from position to position. If a rotating, motor-driven switch is used so that the output function repeats in time, it may be displayed on an oscilloscope.[23] An intermediate

23 Smith, loc. cit.



$$
\begin{aligned}
I &= \Delta yG\left[g_2h_1 + g_1h_2\right] , \\
E &= I/G , \quad (\text{since } G \gg h_1 + h_2), \\
&= \Delta y\left[g_2h_1 + g_1h_2\right] , \\
&= f_2 .
\end{aligned}
$$

Figure 3.1-b

Equivalent Circuit of Figure 3.1-a
for Switch Position Shown

system would use an electromechanical pen recorder whose paper drive is synchronized with the motion of the switch. The switch would then be moved manually through its positions and the pen recorder would record the output.

In the circuit of Fig. 3.1-a, it is evident that it is an easy matter to handle negative values of the function g(x). Entering negative values of h(x) is another matter and is discussed in the next section.

3.2 Negative values of h(x). In Fig. 3.1-a, each combination of ΔyGg_{m-n+1} volts with an h_n mhos acts as a current source of value $\Delta yGg_{m-n+1}\, h_n$ amperes with a shunt conductance of h_n mhos as Fig. 3.1-b shows. Because $G >> \Sigma h_n$, the shunt conductance is neglected.

If an h_n is negative, it will be represented by a conductance of $-h_n$ mhos, since conductance is necessarily positive. This is shown in Fig. 3.2-a. An amplifier of gain -1 is placed in the line between the switch contact and the conductance. (See Sec. 4.5 concerning amplifiers.) Then, the current source as seen at terminals a-b has value $\Delta yGg_{m-n+1}\, h_n$, which is the value desired. See Fig. 3.2-b. The sign of the shunt conductance is without import; since $G >> \Sigma|h_n|$, it is negligible in the circuit as a whole.

Note that it is important that the amplifier be on the same side of the switch as the conductance $-h_n$; otherwise, the effect would be to change the sign of ΔyGg_{m-n+1} rather than that of $-h_n$.

system would use an electromechanical [illegible] whose paper drive is synchronized with the motion of the switch. The switch would then be moved manually through its positions and the [illegible] would record the output.

In the circuit of Fig. 3.1-a it is evident that it is a simple matter to handle negative values of the function f(x). Handling negative values of h(x) is another matter and is discussed in the next section.

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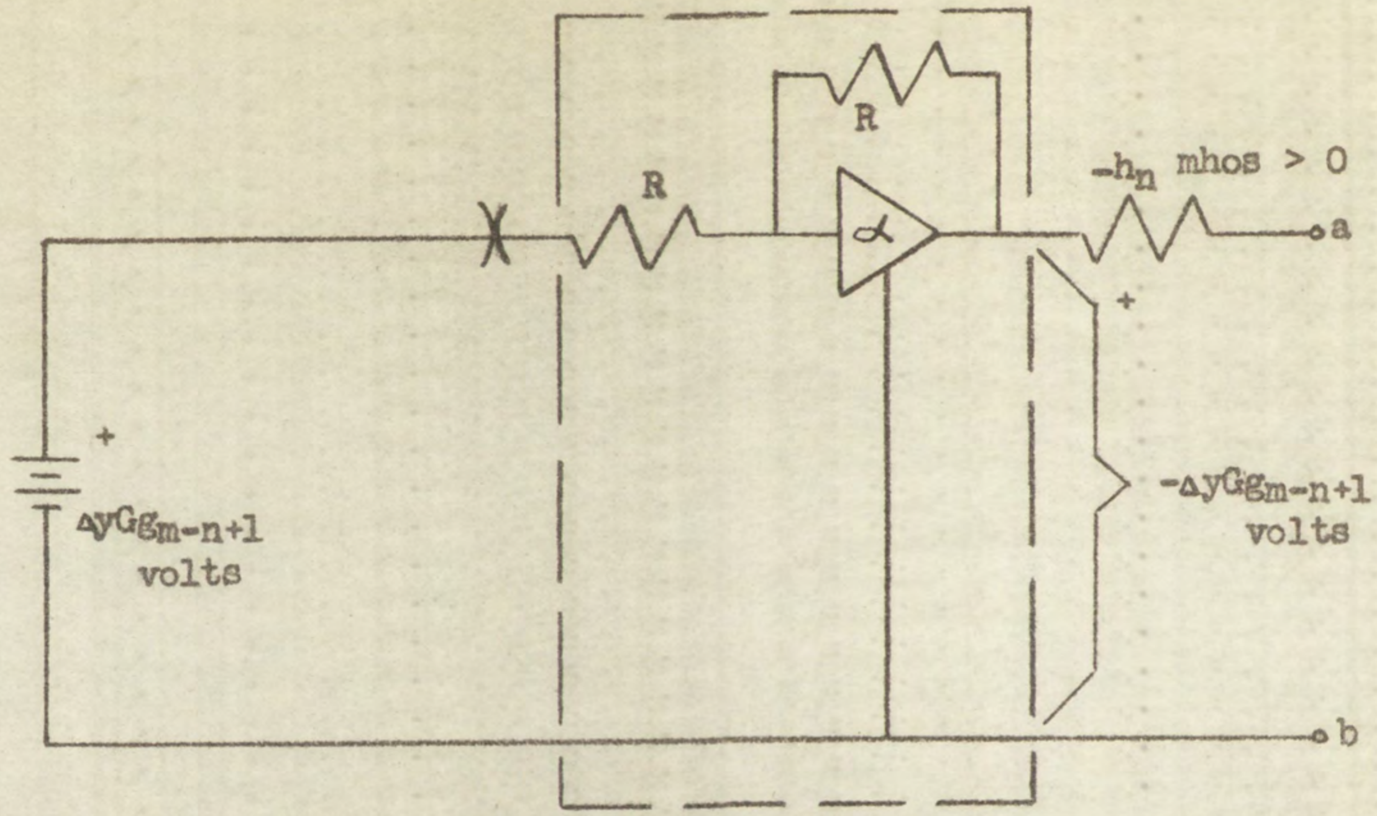

Triangular symbol indicates amplifier of voltage gain α, $|\alpha| >> 1$.

Voltage gain of network enclosed in dashed lines is -1.

Figure 3.2-a

Circuit for Entering Negative Values of Function h(t)

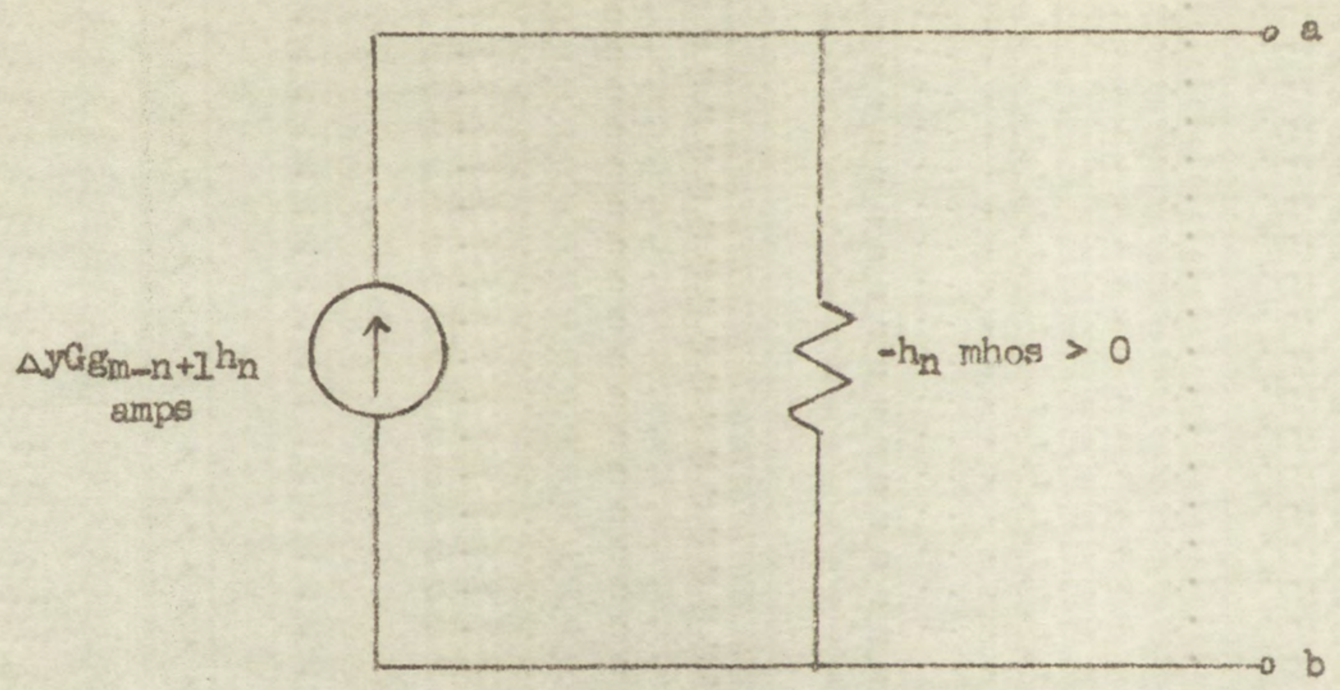

Figure 3.2-b

Current-Source Equivalent of Figure 3.2-a

CHAPTER IV

A COMPUTER TO SOLVE THE CONVOLUTION INTEGRAL EQUATION

The computer discussed in this chapter will find the function h when the functions f and g are known in the convolution integral of equation (1.1-1). The computer proposed by Hessemer and Williams[24] does this, but the computer described in this chapter does it more directly. Needless to say, this problem is more difficult than that of performing the convolution.

4.1 <u>Method of solution</u>. Examination of equations (2.1-5), with the f_k and the g_k known, shows that it represents a system of M + N - 1 equations with N unknowns h_k. If it is possible to find a set of h_k, (k = 1, 2, ..., N), that satisfy the M + N - 1 equations, then the equations are not independent; in fact, only N of them are independent.

In most applications of the convolution integral, this lack of independence in the equations is to be expected. The convolution represents a physical process in which the functions g and h generate f. In the generation of the first N of the M + N - 1 equations, all information concerning h is used; hence, the remaining M - 1 equations are not independent of the first N equations.

In an application of the computer, the functions f and g are known through measurement, and h is to be found. If the measurements are

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24 Hessemer and Williams, *op. cit.*, p. 10.

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24 Hancock and Williams, op. cit., p. 10.

perfect, and if the convolution integral exactly represents the physical process, only N of the equations are needed to find the N quantities h_k. In fact, an easy method exists for finding the h_k: Solve the first of equations (2.1-5) for h_1; use this value of h_1 in the second equation and solve for h_2; continue the process until the N^{th} equation gives h_N. (Equations (3.1-1) make it clear that only h_1 appears in the first equation; only h_1 and h_2 in the second; etc.)

Hessemer and Williams[25] tried the process for finding the h_k described in the preceding paragraph without success. In certain cases the method might be successful, but in others, where either the f_k and the g_k are not known exactly (i.e., the measurement of these quantities is not perfect), or the convolution integral does not exactly describe the physical process, the method will fail. For that reason, the method of least squares as described in the remainder of this section is chosen to find the h_k.

Let

$$F(h_1, h_2, \ldots, h_N) \triangleq \sum_{m=1}^{M+N-1} \left[f_m - \Delta y \sum_{n=1}^{m} g_{m-n+1}\, h_n \right]^2. \qquad (4.1\text{-}1)$$

If a set of h_k can be found for which $F = 0$, all of equations (2.1-5) will be satisfied. Since such a set of h_k may not exist, that set of h_k that minimizes F will be taken as the solution. Such a set of h_k

25 Hessemer and Williams, op. cit., pp. 7-10.

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Let

$$F(h_1, h_2, \ldots, h_N) = \sum_{m=1}^{M+N-1} \left[f_m - \Delta_t \sum_{n=1}^{N} g_{m-n+1}\, h_n \right]^2 \qquad (4.1-2)$$

If a set of h_k can be found for which $F = 0$, all of equations (2.1-5) will be satisfied. Since such a set of h_k may not exist, that set of h_k that minimizes F will be taken as the solution. Such a set of h_k

25 Hessemer and Williams, *op. cit.*, pp. 7-10.

causes the quantities on the right-hand side of equations (2.1-5) to fit those on the left, i.e., the f_m, in a least-squares sense.

The set of h_k that produces the minimum of F must simultaneously satisfy the equations

$$\frac{\partial F}{\partial h_k} = 0, \quad (k = 1, 2, \ldots, N).$$

Thus, there are N linear equations in N unknowns h_k and a unique solution exists. Differentiating (4.1-1) yields

$$\frac{\partial F}{\partial h_k} = \sum_{m=1}^{M+N-1} \left\{ 2 \left[f_m - \Delta y \sum_{n=1}^{m} g_{m-n+1}\, h_n \right] \left[- \Delta y\, g_{m-k+1} \right] \right\}$$

where

$$k = 1, 2, \ldots, N.$$

Since, from equations (2.1-4),

$$f_m = g_n = h_k = 0 \text{ for } m > M + N - 1,\ n > M,\ k > N,$$

the upper limit of all summation signs above can be taken as $+\infty$. Then rearrangement gives

$$\frac{\partial F}{\partial h_k} = -2\Delta y \left\{ \sum_{m=1}^{\infty} f_m g_{m-k+1} - \Delta y \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} g_{m-n+1}\, g_{m-k+1} \right] h_n \right\}$$

Setting $\frac{\partial F}{\partial h_k} = 0$ leads to

$$\sum_{n=1}^{N}\left[\sum_{m=1}^{\infty} g_{m-n+1}\, g_{m-k+1}\right] h_n = \frac{1}{\Delta y} \sum_{m=1}^{\infty} f_m g_{m-k+1} \tag{4.1-2}$$

where

$$k = 1,\ 2,\ \ldots,\ N.$$

The sum on n stops at N since $h_n = 0$ for $n > N$. The limits on m in the other summations can also be restricted because of the conditions on the f_k and the g_k, but there is no advantage in doing so at this time.

Evidently, (4.1-2) represents N linear equations in the N unknowns h_n. Solution of these equations gives the function h(x).

For simplicity of notation, let the coefficients of the h_n in equations (4.1-2) be a_{kn}, and let b_k represent the right-hand sides. Then

$$\sum_{n=1}^{N} a_{kn}h_n = b_k,\ (k = 1,\ 2,\ \ldots,\ N), \tag{4.1-3}$$

where

$$a_{kn} \triangleq \sum_{m=1}^{\infty} g_{m-n+1}\, g_{m-k+1}, \tag{4.1-4}$$

$$b_k \triangleq \frac{1}{\Delta y}\sum_{m=1}^{\infty} f_m\, g_{m-k+1}. \tag{4.1-5}$$

Note that

$$a_{kn} = a_{nk} \tag{4.1-6}$$

so that the matrix of the coefficients of equations (4.1-3) is symmetric. Further

$$a_{k+1,\ n+1} = \sum_{m=1}^{\infty} g_{m-n}\, g_{m-k}$$

$$= \sum_{m=0}^{\infty} g_{m-n+1}\, g_{m-k+1}$$

$$= \sum_{m=1}^{\infty} g_{m-n+1}\, g_{m-k+1},$$

or,

$$a_{k+1,\ n+1} = a_{kn} \tag{4.1-7}$$

When equations (4.1-3) are written in more complete form, the result is

$$\left.\begin{aligned}
&a_{11}h_1 + a_{12}h_2 + \dots + a_{1N}h_N = b_1,\\
&a_{21}h_1 + a_{22}h_2 + \dots + a_{2N}h_N = b_2,\\
&\cdot\\
&\cdot\\
&\cdot\\
&a_{N1}h_1 + a_{N2}h_2 + \dots + a_{NN}h_N = b_N,
\end{aligned}\right\} \tag{4.1-8}$$

where some of the a_{kn} may be zero. The solution $(h_1, h_2, \dots, h_N)$ of the system of equations (4.1-8) is that set of h_n which fits the quantities

$$\Delta y \sum_{n=1}^{m} g_{m-n+1} h_n$$

to the f_m in a least-squares sense. See equations (2.1-5).

4.2 The computer. Consider a network with N + 1 nodes or junctions numbered from 0 to N. Junction 0 is designated the reference junction and all potentials are referred to it. Let

$$\left.\begin{aligned} Y_{kn} &\stackrel{\Delta}{=} (\text{admittance directly joining nodes k and n, } k \neq n), \\ Y_{kk} &\stackrel{\Delta}{=} (\text{sum of admittances directly connected to node k}), \\ E_n &\stackrel{\Delta}{=} (\text{potential at node n}), \\ I_n &\stackrel{\Delta}{=} (\text{current flowing into node n from current sources}). \end{aligned}\right\} \quad (4.2\text{-}1)$$

Then these quantities are related by the equations

$$\left.\begin{aligned} Y_{11}E_1 - Y_{12}E_2 - \cdots - Y_{1N}E_N &= I_1, \\ -Y_{21}E_1 + Y_{22}E_2 - \cdots - Y_{2N}E_N &= I_2, \\ &\vdots \\ -Y_{N1}E_1 - Y_{N2}E_2 - \cdots + Y_{NN}E_N &= I_N, \end{aligned}\right\} \quad (4.2\text{-}2)$$

where

$$Y_{kn} = Y_{nk}$$

by virtue of the definition in (4.2-1).[26]

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26 Wilbur R. LePage and Samuel Seely, General Network Analysis, (New York: McGraw-Hill Book Company, Inc., 1952), pp. 47-49.

Comparison of equations (4.1-8) with equations (4.2-2) shows that if a network is constructed in which

$$\left.\begin{aligned} Y_{kn} &= ja_{kn}, \ (k \neq n), \\ Y_{kk} &= -ja_{kk}, \\ I_n &= b_n, \end{aligned}\right\} \qquad (4.2\text{-}3)$$

then

$$E_n = jh_n \qquad (4.2\text{-}4)$$

That is, the voltages at the various nodes give the function $jh(x)$. This network is the computer.

It is important to note that the network can only be used to solve equations which have a symmetric matrix since the network equations (4.2-2) have a symmetric matrix, i.e., since $Y_{kn} = Y_{nk}$. The equations (4.1-8) satisfy this condition since $a_{kn} = a_{nk}$. Further, in using the network to solve equations (4.1-8), the fact that $a_{k+1,\ n+1} = a_{kn}$ reduces the total number of <u>different</u> admittances needed. That is, since

$$Y_{11} = Y_{22} = \dots = Y_{N-1,\ N-1} = Y_{NN},$$

$$Y_{12} = Y_{23} = \dots = Y_{N-1,\ N},$$

$$\vdots$$

$$Y_{1,\ N-1} = Y_{2N},$$

the admittances which are equal can perhaps all be adjusted simultaneously to the desired value by turning one shaft.

Comparison of equations (4.1-4) and (4.1-5), which define the coefficients a_{kn} and the terms b_k, with equations (2.1-5) indicates considerable similarity. In fact, a few minor changes of notation show that the computer of Chapter III may be used to compute the a_{kn} and the b_k.

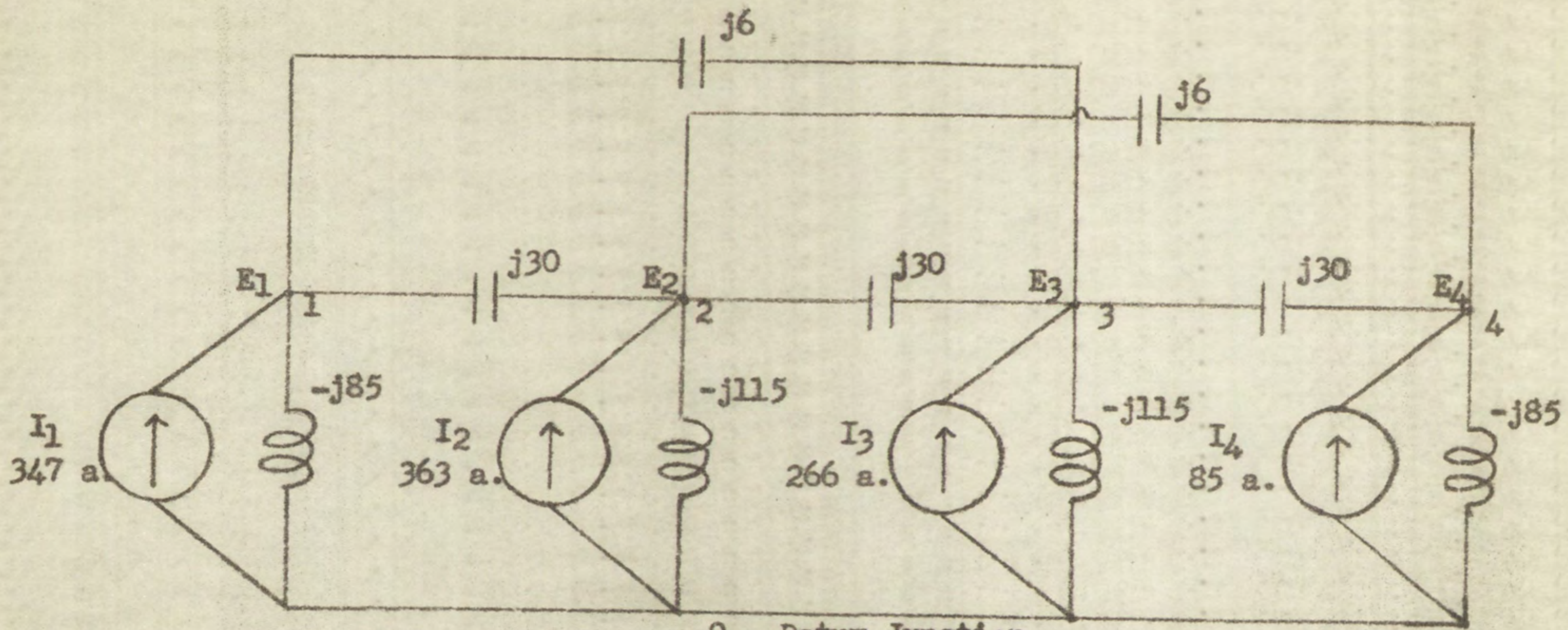
4.3 Application of computer to a system of equations. Arbitrary values are assigned to the a_{kn} and the b_k in equations (4.1-8) and N is given the value 4 to yield the equations

$$\begin{aligned} 49h_1 + 30h_2 + 6h_3 \qquad\quad &= 347, \\ 30h_1 + 49h_2 + 30h_3 + 6h_4 &= 363, \\ 6h_1 + 30h_2 + 49h_3 + 30h_4 &= 266, \\ 6h_2 + 30h_3 + 49h_4 &= 85. \end{aligned} \tag{4.3-1}$$

The condition of symmetry required by equations (4.1-6) and the condition of equations (4.1-7) that makes the coefficients the same along any diagonal (downward, left to right) have been met.

Now consider the circuit of Fig. 4.3-a. It is described by the equations

$$\begin{aligned} -j49E_1 - j30E_2 - j6E_3 \qquad\qquad &= 347, \\ -j30E_1 - j49E_2 - j30E_3 - j6E_4 &= 363, \\ -j6E_1 - j30E_2 - j49E_3 - j30E_4 &= 266, \\ -j6E_2 - j30E_3 - j49E_4 &= 85. \end{aligned}$$



0 - Datum Junction

Elements specified in mhos.

Figure 4.3-a

Convolution Integral Equation Computer

It is evident that these equations are identical to equations (4.3-1) provided

$$E_1 = jh_1,$$

$$E_2 = jh_2,$$

$$E_3 = jh_3,$$

$$E_4 = jh_4.$$

This relationship between the E_n and the h_n is that of equation (4.2-4). The circuit of Fig. 4.3-a was derived, of course, from equations (4.3-1) through the relations (4.2-3). Note that, since

$$Y_{11} = Y_{10} + Y_{12} + Y_{13},$$

then

$$Y_{10} = Y_{11} - Y_{12} - Y_{13}.$$

The other Y_{kk} are found similarly.

This example illustrates the fact that only inductive and capacitive elements are required, provided the coefficients a_{kn} in equations (4.1-8) are real. Further, all the current generators have the same phase (or differ by exactly π radians) provided the b_k are real. (Both the a_{kn} and the b_k are real as they have been defined in equations (4.1-4) and (4.1-5).) For convenience, all quantities may be suitably scaled so that they can more readily be realized. Moreover, a frequency can be chosen that brings the capacitors and inductors into a reasonable range.

For some applications it is known that the function h(x) is everywhere positive; in such cases, the potentials E_n need only be read

for magnitude. In others, $h(x)$ may take on both positive and negative values. In this case, it will be necessary to read phase and amplitude of the E_n. This should not be difficult since all of the E_n either have the same phase or differ by exactly π radians.

The components that are selected for the computer network should have low losses. For exact results, the admittances should be purely capacitive or purely inductive. It is easy to obtain low-loss capacitors. It is considerably more difficult to obtain low-loss inductors; however, if the ratio $\omega L/R$ for the inductors is 100 or more, the accuracy of the results should be better than $\pm$ 1%.

In order to prepare the computer for various problems, it is necessary to be able to adjust the Y_{kn}. Since

$$Y_{k+1,\ n+1} = Y_{kn},$$

it should be possible to link the adjusting mechanism in some way so that a single setting would fix several admittances at their proper value. It is reasonably easy to calibrate capacitors and put several on a common shaft so that they all have nearly equal values at any setting. It is more difficult to do so with inductors; for that reason, another method of varying admittances is discussed in the next section.

4.4 <u>Admittance conversion; stability</u>. Consider the circuit of Fig. 4.4-a. The triangular symbol indicates an amplifier with infinite input impedance and an amplification $A(j\omega)$. Due to the infinite input

for magnitude. In others, K(x) may take on both positive and negative values. In this case, it will be necessary to treat phase and magnitude of the K_n. This should not be difficult since all of the K_n either have the same phase or differ by exactly π radians.

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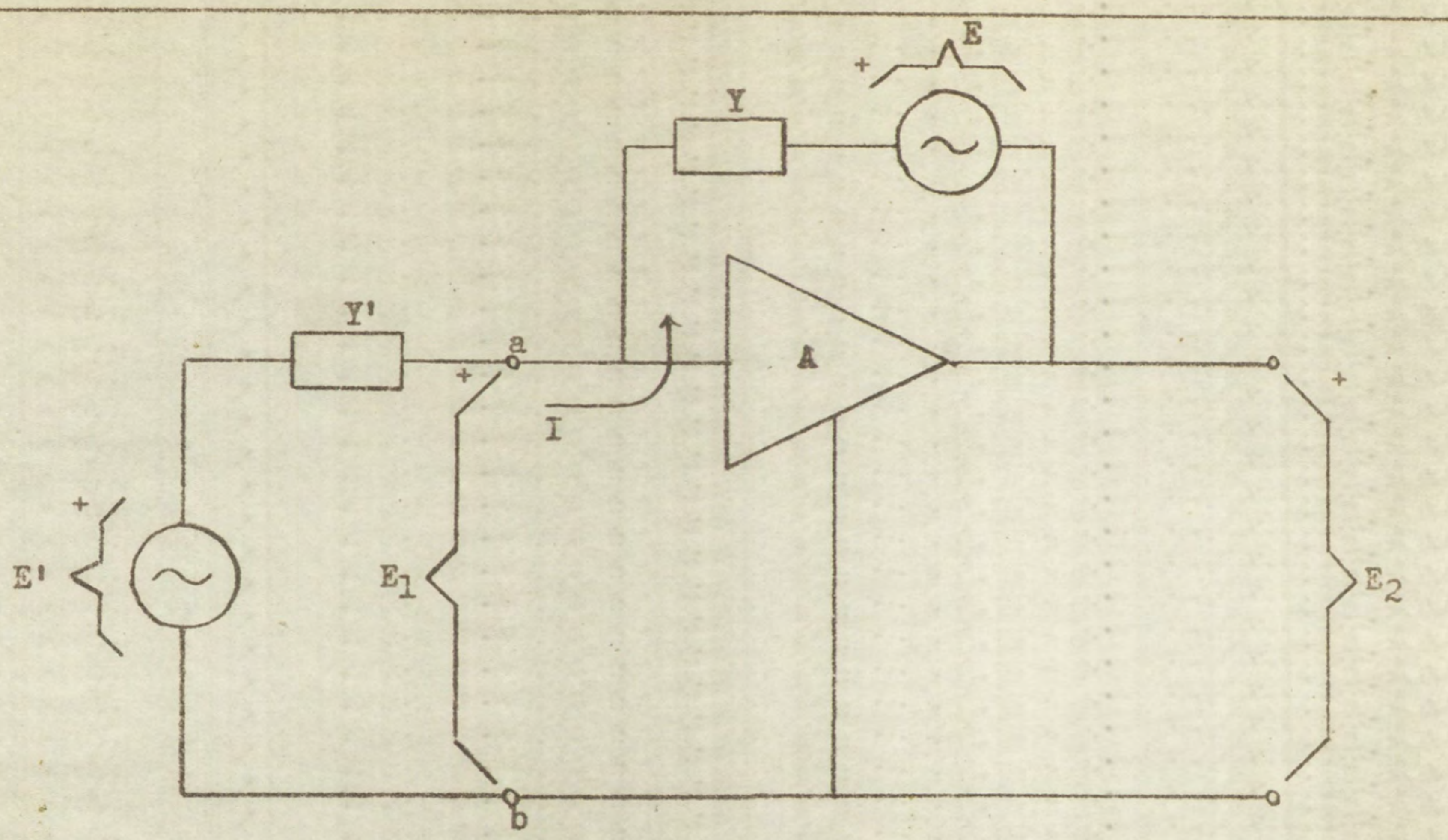

Figure 4.4-a

Feedback Amplifier for Admittance Conversion

impedance, no current flows into the amplifier but takes the path through the admittance Y and the emf E as indicated. From these facts and Fig. 4.4-a, it follows that

$$\left.\begin{aligned} I &= \Big[E_1 - E - E_2\Big]Y, \\ I &= \Big[E' - E_1\Big]Y', \\ E_2 &= AE_1. \end{aligned}\right\} \qquad (4.4\text{-}1)$$

The voltages E', E, E_1, and E_2, the current I, the amplification A, and the admittances Y and Y' are all functions of $j\omega$ and, hence, are complex.

In order to find an equivalent circuit for that part of Fig. 4.4-a to the right of terminals a-b, the first and third of equations (4.4-1) may be solved to obtain

$$I = E_1\, Y \Big[\, 1 - A \Big] - EY. \qquad (4.4\text{-}2)$$

This equation may be represented by the circuit of Fig. 4.4-b and is the desired equivalent circuit since, for the same input voltage and current, it is described by the same equation as the original circuit.

Since feedback is used in the circuit of Fig. 4.4-a, the question of stability arises. To examine this point, equations (4.4-1) are solved to find the dependence of E_2 $(j\omega)$ on E' $(j\omega)$ and E $(j\omega)$. The result is

$$E_2\,(j\omega) = \frac{A\Big[Y'E' + YE\Big]/\Big[Y + Y'\Big]}{1 - YA/\Big[Y + Y'\Big]}, \qquad (4.4\text{-}3)$$

where, of course, the quantities A, E, E', Y, and Y' are all functions

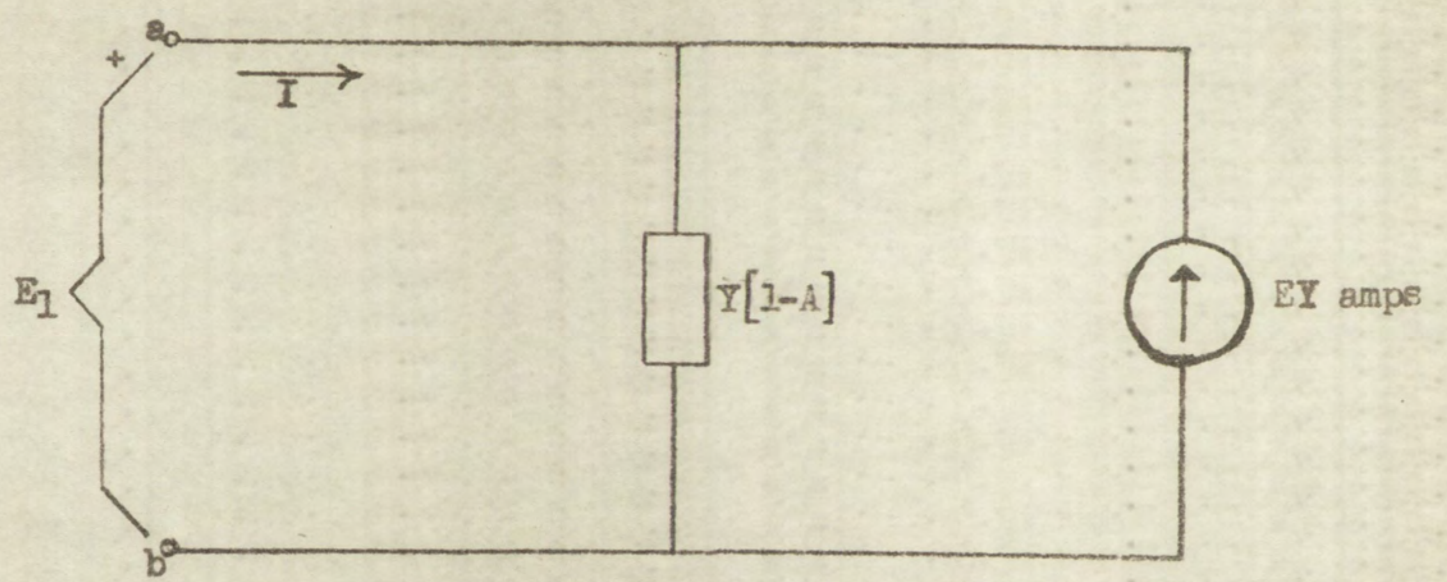

Figure 4.4-b

Current-Source Equivalent for Figure 4.4-a

of $j\omega$. In equation (4.4-3), if $s = \sigma + j\omega$ replaces $j\omega$, the function E_2 (s) is the Laplace transform of the time function of the output voltage of Fig. 4.4-a, for quiescent initial conditions. For such a circuit to be stable, E_2 (s) must not have any poles in the right half of the s-plane.

It is reasonable to assume that the numerator, i.e., the quantity above the main fraction bar, of equation (4.4-3) does not have poles in the right half of the s-plane. That is, the amplifier without feedback is presumed stable, the bracket involving E' and E is without poles for stable values of these voltages, and the quantity Y + Y' does not have zeros in the right half-plane. Hence, the poles of E_2 (s) are the zeros of the denominator of equation (4.4-3). To determine whether or not such zeros exist, the quantity $-YA/\left[Y + Y'\right]$ is plotted on the complex plane as a function of $s = j\omega$ (i.e., with $\sigma = 0$). If the point $-1 + j0$ is not encircled as ω varies from $-\infty$ to $+\infty$, the denominator of (4.4-3) does not have zeros in the right half of the s-plane. Usually, if

$$\mathrm{Re}\left[\frac{YA}{Y + Y'}\right] < 0,$$

the system will be stable. In a later section, consideration will be given to systems that may be unstable.

It is evident from Fig. 4.4-b that the portion of Fig. 4.4-a to the right of terminals a-b can be represented by a current source EY and a shunt admittance $Y\left[1 - A\right]$. Hence, it is possible to have an input admittance which differs from the feedback admittance by the factor 1 - A.

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For admittance conversion only, it may be convenient to remove the source E.

4.4(a) Capacitance conversion. In Fig. 4.4-b, let

$$Y = j\omega C$$

so that

$$Y_{in} \triangleq E_1/I = j\omega \left[1 - A \right] C.$$

Thus, if A is real, the input admittance is a pure capacitance whose value is determined by 1 - A. Then, if A can be precisely controlled, a wide range of values of capacitance can be obtained with only one fixed capacitor. Methods of controlling A will be considered later. To increase input capacitance over C, $A < 0$; to decrease it, $0 < A < 1$.

4.4(b) Inductance conversion. In Fig. 4.4-b, let

$$Y = 1/j\omega L$$

so that

$$Y_{in} \triangleq E_1/I = \frac{1}{j\omega L/\left[1-A \right]} .$$

Then, if A is real, the input admittance is a pure inductance. To make the input inductance greater than L, $0 < A < 1$; to make it less, $A < 0$.

4.5 Precise control of amplification. The circuit of Fig. 4.5-a is the same as that of Fig. 4.4-a with E_a replacing E', R_1 replacing 1/Y', R_2 replacing 1/Y, α replacing A, and E removed. Hence, from equation (4.4-3), the resultant amplification of the circuit of Fig. 4.5-a is

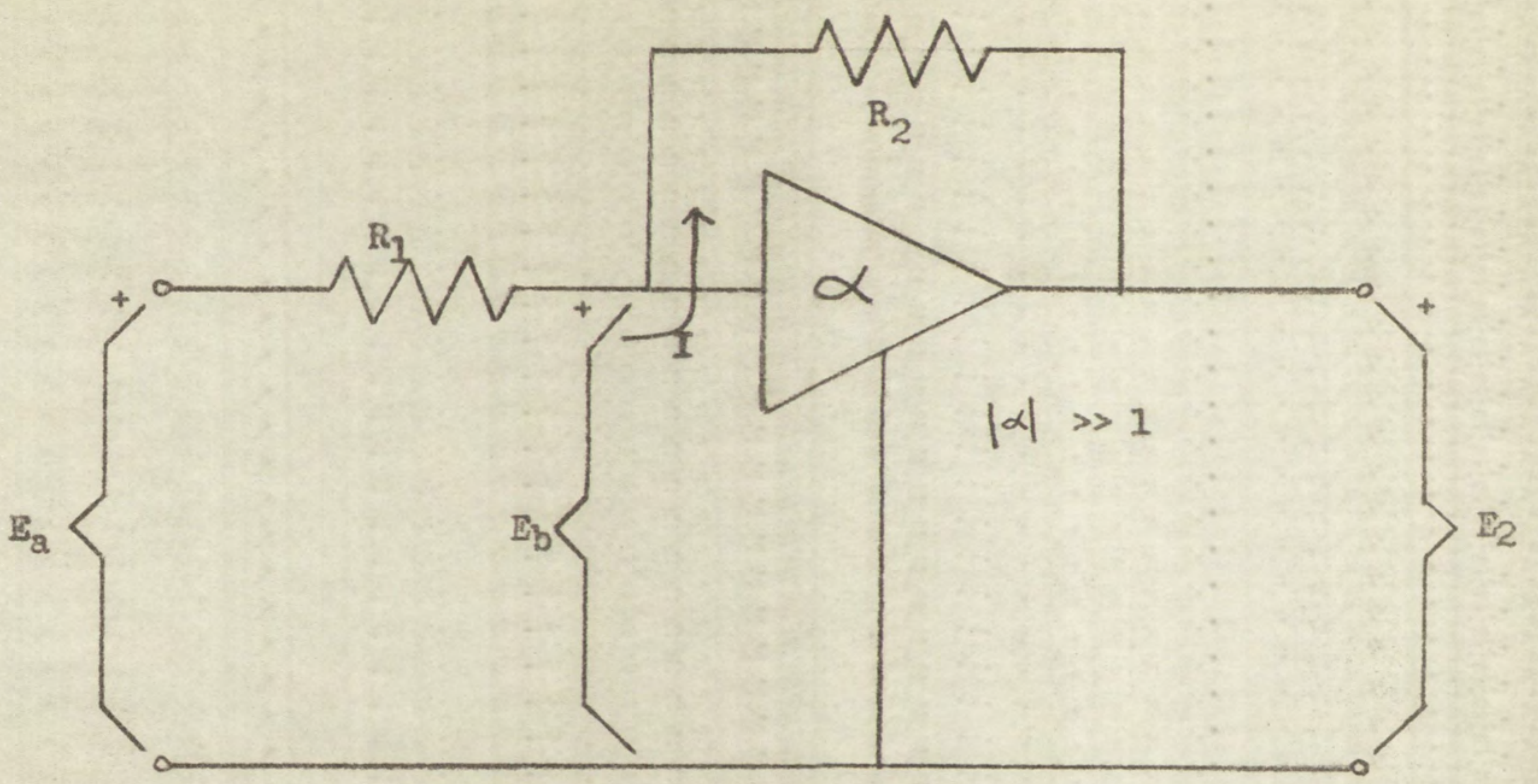

Figure 4.5-a

Feedback Amplifier to Provide Precise Values of Gain

$$A \triangleq E_2/E_a = \frac{\alpha R_2}{\left[1-\alpha\right] R_1 + R_2} . \qquad (4.5\text{-}1)$$

Now if α is real, $\alpha < 0, |\alpha| \gg 1$, and $\left|\alpha R_1\right| \gg R_2$, equation (4.5-1) becomes

$$A = -R_2/R_1 .$$

Since there are inexpensive, commercially-available amplifiers with $\alpha \doteq -20{,}000$ over wide frequency ranges, the conditions on α are easy to meet. Hence, the resultant amplification A can be controlled by external precision resistors R_1 and R_2. These feedback circuits can be used to obtain the required values of A to convert capacitance and inductance. To obtain positive values of amplification, two such circuits can be placed in cascade. Note that the overall circuit will have to be isolated from the input in order to give the required extremely high input impedance.

4.6 Computer circuit using negative conductances; instability. It was originally intended that the computer network should consist of only real elements, i.e., of conductances so that d-c could be used. This required negative conductances to represent certain coefficients. To obtain them, a circuit such as Fig. 4.4-a was used with $Y' = G'$ and $Y = G$. Figure 4.4-b shows that A must be real and greater than unity at $\omega = 0$ in order that $Y_{in} = G_{in} < 0$.

To satisfy the conditions on A, a two-stage amplifier may be assumed, with A having the form

$$A \triangleq \frac{E_2}{E_1} = \frac{aR_2}{(1-a)R_1 + R_2} \qquad (4.5\text{-}1)$$

Now if a is real, $a < 0$, $|a| >> 1$, and $|aR_1| >> R_2$, equation (4.5-1) becomes

$$A \approx -R_2/R_1 .$$

Since there are inexpensive, commercially-available amplifiers with $a = -20,000$ over wide frequency ranges, the conditions on a are easy to meet. Hence, the resultant amplification A can be controlled by external precision resistors R_1 and R_2. These feedback circuits can be used to obtain the required values of A to prevent capacitance and inductance. To obtain positive values of amplification, two such circuits can be placed in cascade. Note that the overall circuit will have to be isolated from the input in order to give the required extremely high input impedance.

4.6 Computer circuit using negative conductance instability. It was originally intended that the computer network should consist of only real elements, i.e., of conductances so that d-c could be used. This required negative conductances to represent certain coefficients. To obtain them, a circuit such as Fig. 4.4-a was used with $Y' = 1$ and $Y = 0$. Figure 4.4-b shows that A must be real and greater than unity at $\omega = 0$ in order that $Y'_{in} = Y_{in} < 0$.

To satisfy the conditions on A, a two-stage amplifier may be assumed, with A having the form

$$A(j\omega) = A_o / \left[1 + j\omega T\right]^2 , \; A_o > 1.$$

Then, from equation (4.4-3), if the quantity

$$- \frac{GA_o}{\left[1 + j\omega T\right]^2 \left[G + G'\right]} ,$$

when plotted on the complex plane, does not encircle the point -1 + j0 as ω varies from $-\infty$ to $+\infty$, the circuit will be stable. This plot is shown in Fig. 4.6-a. It is clear that the point -1 to j0 will not be encircled if

$$GA_o / \left[G + G'\right] < 1,$$

that is, if

$$-G \left[1 - A_o\right] < G'. \qquad (4.6\text{-}1)$$

The inequality (4.6-1) states that the conductance connected across terminals a-b in Fig. 4.4-a must be greater than the absolute value of the input conductance seen at the terminals. In terms of Fig. 4.4-b, it states that the conductance seen by the current generator must be positive. (In Figs. 4.4-a and 4.4-b, the admittances Y and Y' should be replaced by G and G' insofar as this section is concerned.)

When the feedback amplifier was used in the computer network to provide negative conductances and current generators, it developed that only a very limited class of equations could be solved since, for most systems of equations, at least one of the current generators would see a negative conductance, and, hence, the circuit would be unstable.

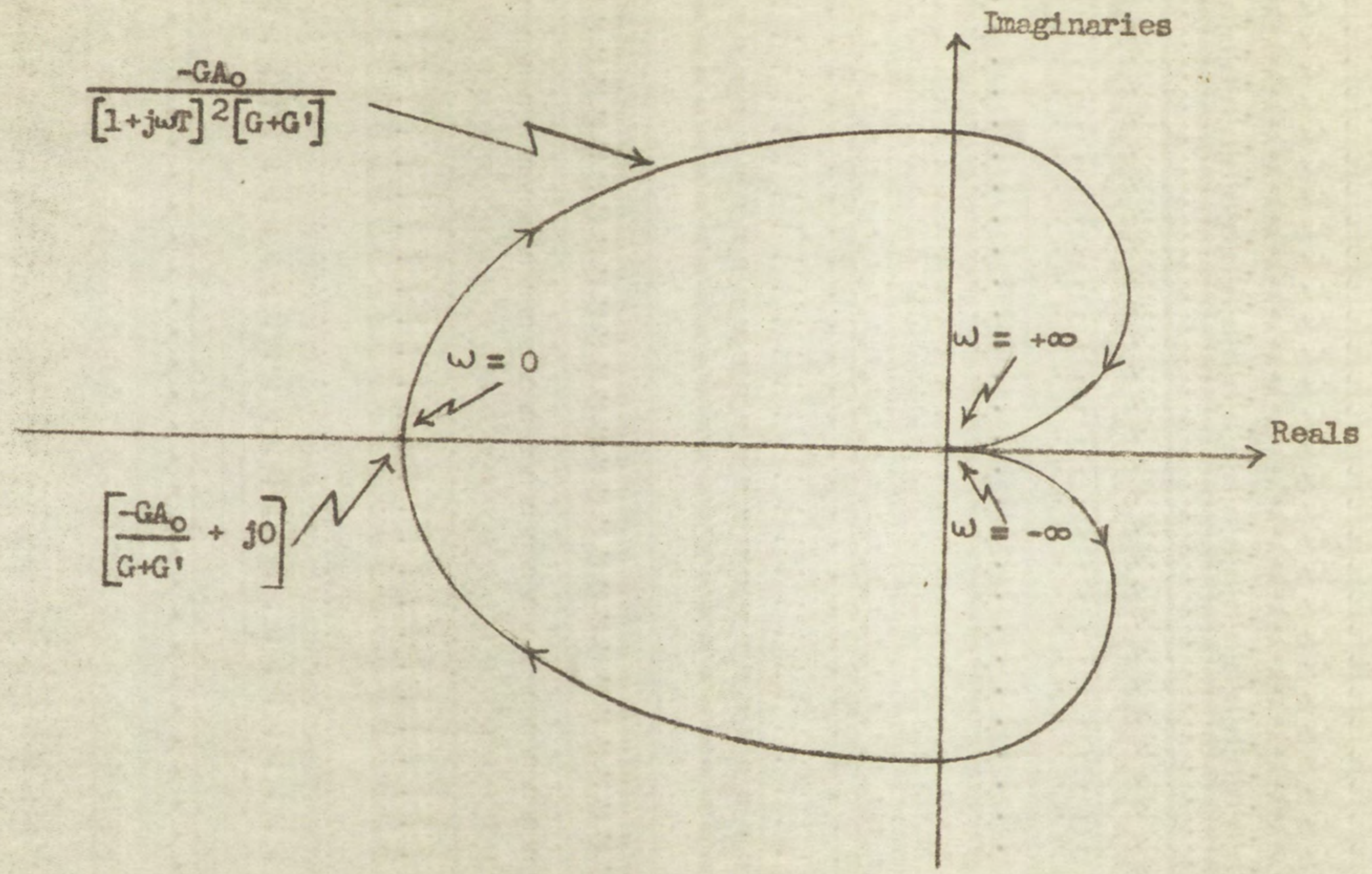

Figure 4.6-a

Complex Plane Plot of Loop Gain of an Amplifier

with Positive Feedback

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary. A rapidly growing need for speedy methods of analyzing linear systems has led to the development of various computational devices to evaluate the convolution integral. These same devices will also find the correlation function of a pair of functions; in fact, a number of them were designed for this purpose due to the increasing application of correlation to the study of communication systems. Many of these computers evaluate the integral on a continuous basis. A few, however, use a numerical approximation to the integral; the convolution computer described in this paper does so, and the method of approximating the integral is described in an early chapter.

The convolution computer depends on a switch which allows the delay to be advanced (or retarded). The functions are entered as voltages and conductances; a scheme is described that makes it possible to have both negative and positive values of the functions. The value of the integral is read at the output as a voltage; there is a value for each delay, or switch position.

It appears that no computers have been advanced which will find one of the functions of the convolution integral, when the other function and the integral itself are known, except the computer proposed

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It appears that no computers have been designed which will find one of the functions of the convolution integral when the other function and the integral itself are known, except the computer proposed

by Hessemer and Williams[27] and a graphical scheme which does not give direct results. Such a computer has as wide a usage as the convolution computer itself since the problem of determining the input, given the output and system response, arises as often as that of finding the output. It has the further application of finding the system response with any input and resultant output; this obviates impulse or step inputs--often difficult to obtain--in finding system response.

The computer described herein in Chapter IV does the job described in the preceding paragraph, but its method is entirely different from that used by the computer proposed by Hessemer and Williams. The computer is quite simple and could readily be constructed from commercial components.

5.2 Conclusions. The work done by Smith[28] leaves no doubt about the ability of the convolution computer that is described in this paper to perform with an accuracy of ±1%. However, the accuracy of the results depend on the number of ordinates that are used to represent the functions within the integral. Each ordinate requires a switch contact, and this is the limitation of the computer. Nonetheless, since the straightforward operation of convolution does not require fast switch operation, it should not be difficult to fabricate a switch with the order of 100 contacts. Such a convolution computer should be appreciably cheaper than most of those which have been

27 Hessemer and Williams, *op. cit.*, p. 10.

28 Smith, *loc. cit.*

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27 Hessemer and Williams, op. cit., p. 10.

28 Smith, loc. cit.

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The need for solving the convolution integral equation for a function of the integrand led to the computer that is proposed herein in Chapter IV. Since it does only that task, it is less versatile than the Hessemer-Williams computer which also performs convolutions. However, it finds the solution directly without the minimizing process required by the other.

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