

8-26-1955

A Proposal for Two Computers Dealing With the Convolution Integral

Charles S. Williams Jr.

Follow this and additional works at: https://digitalrepository.unm.edu/ece_etds



Part of the [Electrical and Computer Engineering Commons](#)

Recommended Citation

Williams, Charles S. Jr.. "A Proposal for Two Computers Dealing With the Convolution Integral." (1955).
https://digitalrepository.unm.edu/ece_etds/397

This Thesis is brought to you for free and open access by the Engineering ETDs at UNM Digital Repository. It has been accepted for inclusion in Electrical and Computer Engineering ETDs by an authorized administrator of UNM Digital Repository. For more information, please contact disc@unm.edu.

UNIVERSITY OF NEW MEXICO-UNIVERSITY LIBRARIES



A14429 095229

378.789

Un 3 Ow

1956

cop. 2

PROPOSAL FOR TWO COMPUTERS DEALING WITH THE CONVOLUTION INTEGRAL — WILLIAM'S

THE LIBRARY
UNIVERSITY OF NEW MEXICO

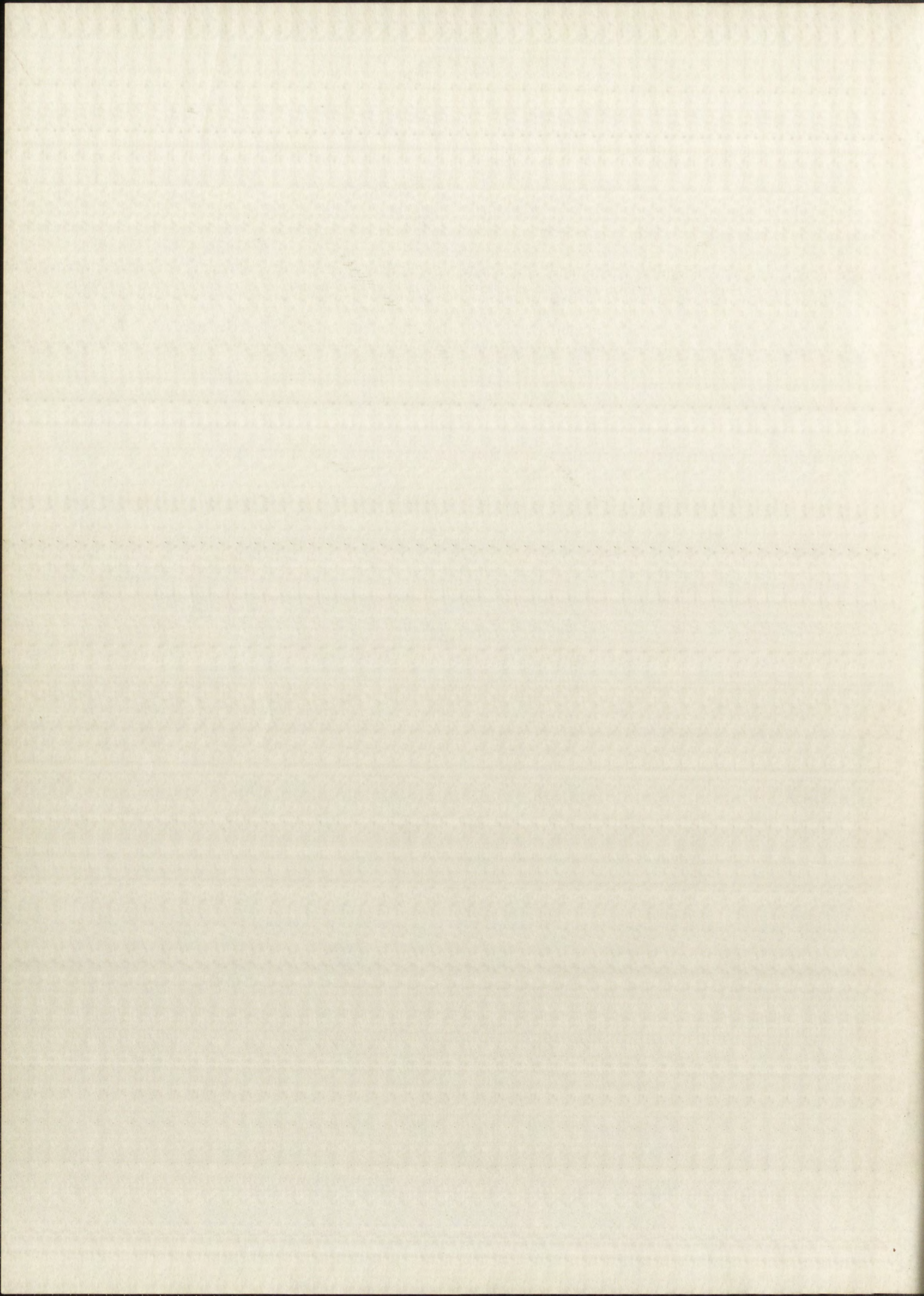


Call No.
378.789
Un30w
1956
cop.2

Accession
Number
209782

IMPORTANT!

Special care should be taken to prevent loss or damage of this volume. If lost or damaged, it must be paid for at the current rate of typing.



UNIVERSITY OF NEW MEXICO LIBRARY

MANUSCRIPT THESES

Unpublished theses submitted for the Master's and Doctor's degrees and deposited in the University of New Mexico Library are open for inspection, but are to be used only with due regard to the rights of the authors. Bibliographical references may be noted, but passages may be copied only with the permission of the authors, and proper credit must be given in subsequent written or published work. Extensive copying or publication of the thesis in whole or in part requires also the consent of the Dean of the Graduate School of the University of New Mexico.

This thesis byCharles S. Williams, Jr.
has been used by the following persons, whose signatures attest their acceptance of the above restrictions.

A Library which borrows this thesis for use by its patrons is expected to secure the signature of each user.

NAME AND ADDRESS	DATE
Dr. Hansjoerg Niehden Waleways Exp. Station Vicksburg Min.	5. March 1965

MANUSCRIPT 11111

The author of this manuscript has the honor to acknowledge the
great and generous aid which has been rendered to him by the
open for manuscript and the right of the author to publish his
right of the author to publish his work in any form and in
passages may be copied and used in any form and in any
proper credit to the author in any form and in any
work. Extensive copying of this work in any form and in any
particulars and the author of this work in any form and in any
of the University of New Mexico.

This thesis by
has been used by the author in his work in any form and in any
acceptance of the author in any form and in any
A further while the author of this work in any form and in any
expected the author of this work in any form and in any

NAME AND ADDRESS

DATE

A PROPOSAL FOR TWO COMPUTERS
DEALING WITH THE CONVOLUTION INTEGRAL

By

Charles S. Williams, Jr.

A Thesis

In partial fulfillment of the
Requirements for the Degree of
Master of Science in Electrical Engineering

The University of New Mexico
1955



A PROPOSAL FOR THE CONSTRUCTION
OF A NEW BUILDING FOR THE
DEPARTMENT OF CHEMISTRY

Submitted by
Charles S. Gillingham, Jr.

In partial fulfillment of the
requirements for the degree of
Master of Science in Chemical Engineering

The University of Illinois at Urbana-Champaign

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

E. H. Castetter

DEAN

8/26/1955

DATE

Thesis committee

R. A. Hersenberger

CHAIRMAN

J. L. Ellis

R. B. Moore

This thesis directed and approved by the
University of New Mexico in partial fulfillment
means for the degree of

MASTER OF ARTS

[Signature]

8/26/1933

Thesis committee

[Signature] R.A. Starnes

[Signature]

[Signature]

378.789
Un30w
1956
cop. 2

TABLE OF CONTENTS

CHAPTER	PAGE
I. CONVOLUTION COMPUTERS AND THEIR USES	1
1.1 Purpose of the computers	1
1.2 Uses of the convolution integral	1
1.2(a) Application to linear systems	1
1.2(b) Application to communications	3
1.3 Review of the literature	4
1.4 Definition of symbols and terms	7
II. APPROXIMATING THE CONVOLUTION INTEGRAL	9
2.1 An approximation to the convolution integral	9
2.2 Illustration of the convolution integral approximation	11
III. CONVOLUTION COMPUTER	14
3.1 Circuit of the convolution computer	14
3.2 Negative values of $h(x)$	18
IV. A COMPUTER TO SOLVE THE CONVOLUTION INTEGRAL EQUATION	21
4.1 Method of solution	21
4.2 The computer	26
4.3 Application of computer to a system of equations	28
4.4 Admittance conversion; stability	30
4.4(a) Capacitance conversion	36
4.4(b) Inductance conversion	36
4.5 Precise control of amplification	36
4.6 Computer circuit using negative conductances; instability	38

I.	CONVOLUTION COMPUTERS AND THEIR USES	1
1.1	Purpose of the computer	1
1.2	Uses of the convolution computer	1
1.2(a)	Application to linear systems	1
1.2(b)	Application to communication	3
1.3	Review of the literature	4
1.4	Definition of symbols and terms	4
II.	APPROXIMATING THE CONVOLUTION INTEGRAL	7
2.1	An approximation to the convolution integral	7
2.2	Illustration of the convolution integral approximation	11
III.	CONVOLUTION COMPUTERS	14
3.1	Characteristics of the convolution computer	14
3.2	Negative values of $x(t)$	18
IV.	A COMPUTER TO SOLVE THE CONVOLUTION INTEGRAL	21
4.1	Method of solution	21
4.2	The computer	23
4.3	Application of computer to system of equations	25
4.4	Advantages of computer at display	27
4.4(a)	Capacitor conversion	27
4.4(b)	Inductor conversion	31
4.5	Precise control of amplification	33
4.6	Computer circuit using negative capacitance	35
	Instability	35

CHAPTER	PAGE
V. SUMMARY AND CONCLUSIONS	41
5.1 Summary	41
5.2 Conclusions	42
BIBLIOGRAPHY	44

PAGE

vi

vi

45

44

CHAPTER

V. SUMMARY AND CONCLUSIONS

2.1 Summary

2.2 Conclusions

BIBLIOGRAPHY

LIST OF FIGURES

FIGURE	PAGE
1.2-a Block Diagram for Equation (1.1-1)	2
2.2-a Illustration of Convolution	13
3.1-a Circuit of Convolution Computer	15
3.1-b Equivalent Circuit of Figure 3.1-a for Switch Position Shown	17
3.2-a Circuit for Entering Negative Values of Function $h(t)$	19
3.2-b Current-Source Equivalent of Figure 3.2-a	20
4.3-a Convolution Integral Equation Computer	29
4.4-a Feedback Amplifier for Admittance Conversion	32
4.4-b Current-Source Equivalent for Figure 4.4-a	34
4.5-a Feedback Amplifier to Provide Precise Values of Gain	37
4.6-a Complex-Plane Plot of Loop Gain of an Amplifier with Positive Feedback	40

LIST OF FIGURES

PAGE	FIGURE
2	1.2-a Block Diagram for Equation (1.1-1)
3	2.2-a Illustration of Convolution
12	3.1-a Circuit of Convolution Computer
	3.1-b Equivalent Circuit of Figure 3.1-a for Section Position
17	Shown
19	3.2-a Circuit for Entering Negative Values of Parameter $\mu(t)$
20	3.2-b Current-Source Equivalent of Figure 3.2-a
29	4.3-a Convolution Integral Equation Computer
32	4.4-a Feedback Amplifier for Automatic Convolution
34	4.4-b Current-Source Equivalent for Figure 4.4-a
37	4.5-a Feedback Amplifier to Provide Preset Values of Gain
	4.5-b Complex-Plane Plot of Loop Gain of an Amplifier with
40	Positive Feedback

CHAPTER I

CONVOLUTION COMPUTERS AND THEIR USES

1.1 Purpose of the computers. The integral

$$f(x) = \int_{-\infty}^{\infty} g(x - y)h(y)dy \quad (1.1-1)$$

is known as a convolution integral. The first computer described herein will determine the function f when the functions g and h are known. The second computer will determine the function h when f and g are known; in using the second computer to find h , the first computer is used for auxiliary computations.

1.2 Uses of the convolution integral. The convolution integral and similar integrals of products where one function is delayed (x represents delay) find wide application. Two are described here.

1.2(a) Application to linear systems. The convolution integral probably finds its greatest use in relating the output to the input of linear systems in the following way. Let $h(t)$ be the response of a linear system to a unit impulse, where t is real time. Let $g(t)$ be the input time function. Then $f(t)$ is the output time function.¹ See Fig. 1.2-a.

¹ Murray F. Gardner and John L. Barnes, Transients in Linear Systems, (New York: John Wiley & Sons, Inc., 1942), Vol. I, p. 262.

CONSTRUCTION OF THE SYSTEM

1.1 Purpose of the system. The system

$$f(x) = \int_{-\infty}^{\infty} g(t) \delta(t-x) dt \quad (1.1-1)$$

is known as a convolution system. The first computer is used to determine the function $f(x)$ when $g(t)$ and $\delta(t-x)$ are known. The second computer will determine the function $g(t)$ when $f(x)$ and $\delta(t-x)$ are known. Using the second computer to find $g(t)$ when $f(x)$ and $\delta(t-x)$ are known is an auxiliary computation.

1.2 Uses of the convolution system. The convolution system

and similar integrals of products where the function is delayed (a time resents delay) find wide application. They are described here.

1.2(a) Application to linear systems. The convolution integral

probably finds its greatest use in relating the output to the input of linear systems in the following way. Let $y(t)$ be the response of a linear system to a unit impulse, where $\delta(t)$ is a unit impulse. Let $g(t)$ be the input time function. Then $f(t)$ is the output time function.

1.2-a.

Murray F. Garman and John L. Garman, Transients in Linear Systems, (New York: John Wiley & Sons, Inc., 1954), Vol. 1, p. 102.

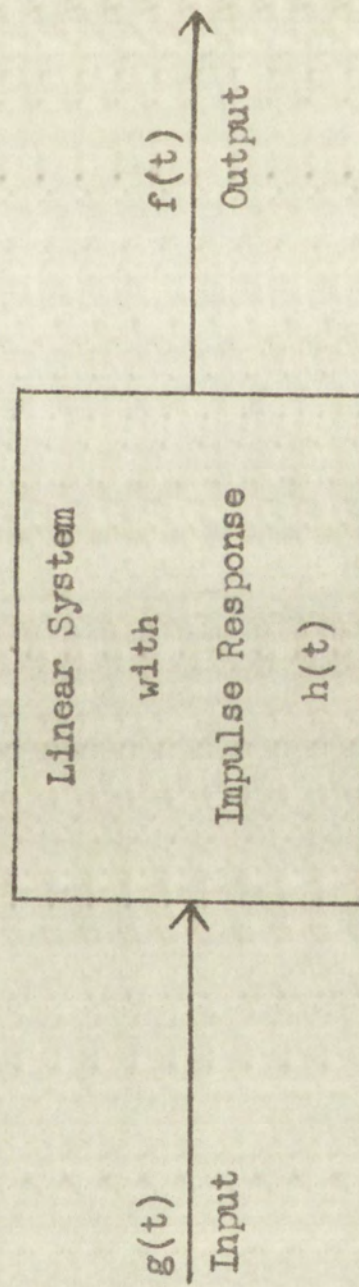


Figure 1.2-a
Block Diagram for Equation (1.1-1)

Most electrical circuits are treated as (and usually approximate) linear systems. All circuits that contain only R, L, and C elements are linear, provided the R's, L's, and C's are constants. Many circuits that contain vacuum tubes and other nonlinear elements are linear to a first approximation. For all such electrical systems, the functions $f(t)$ and $g(t)$ must represent either current or voltage since these are the quantities that occur linearly in the differential equations that describe the system. Further, $h(t)$ is the current response, if $f(t)$ represents current, to a unit impulse of voltage, if $g(t)$ represents voltage.

1.2(b) Application to communications. In the study of the transmission of information, correlation functions are frequently calculated to aid in both statistical and harmonic analysis.² The autocorrelation function of a function $f_1(t)$ is defined by

$$\phi_{11}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_1(t) f_1(t + \tau) dt,$$

and the cross-correlation function of two functions $f_1(t)$ and $f_2(t)$ is defined by

$$\phi_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_1(t) f_2(t + \tau) dt.$$

It is evident that these integrals have essentially the form of the integral of equation (1.1-1). The argument of the function g differs

² Y. W. Lee and J. W. Wiesner, "Correlation Functions and Communication Applications," Electronics, Vol. 23, pp. 87-88, June, 1950.

from that of the second function in the definitions of ϕ_{11} and ϕ_{12} above. However, it will become clear with the description of the computers that the argument of g in equation (1.1-1) could be readily changed to $x + y$ insofar as the computers are concerned.

1.3 Review of the literature. Many devices for performing convolutions (or correlations) have been proposed and constructed. Some of the early computers use optical principles. The functions g and h of equation (1.1-1) are cut out of cardboard³ or put on film⁴ in such a way that the functions are represented by the line between the portion of the cardboard or film which passes light and that which does not. Delay, i.e., the value of x in equation (1.1-1), is obtained by physically shifting one cutout or film with respect to the other. These computers are capable of handling integrals where the function g has the more general form $g(x,y)$. Of course, many cutouts or films have to be prepared since each function used requires one.

A mechanical correlator has been devised and constructed.⁵ Integration is accomplished by mechanical principles. Two operators are required, one each to follow the functions g and h with styli. A run

³ T. S. Gray, "A Photo-Electric Integrator," Journal of the Franklin Institute, Vol. 212, pp. 77-102, July, 1931.

⁴ H. L. Hazen and G. S. Brown, "The Cinema Integrator," Journal of the Franklin Institute, Vol. 230, pp. 19-44, pp. 183-205, August, 1940.

⁵ H. R. Seiwel, "A New Mechanical Autocorrelator," The Review of Scientific Instruments, Vol. 21, pp. 481-484, May, 1950.

from that of the actual function in the definition of g and h above. However, it will become clear with the description of the computer that the argument of g in equation (1.1-1) could be readily changed to $x + y$ insofar as the computers are concerned.

1.3 Review of the literature. Many devices for performing convolutions (or correlations) have been proposed and constructed. Some of the early computers use optical principles. The functions g and h of equation (1.1-1) are cut out of cardboard³ or put on film⁴ in such a way that the functions are represented by the line between the portion of the cardboard or film which passes light and that which does not. Delay, i.e., the value of x in equation (1.1-1), is obtained by physically shifting one output or film with respect to the other. These computers are capable of handling integrals where the function g has the more general form $g(x, y)$. Of course, many outputs or films have to be prepared since each function used requires one.

A mechanical correlator has been devised and constructed.⁵ Information is accomplished by mechanical principles. Two operators are required, one each to follow the functions g and h with stylus. A run

3. F. S. Gray, "A Photo-Electric Integrator," Journal of the Franklin Institute, Vol. 212, pp. 77-102, July, 1931.
4. H. L. Essex and G. S. Brown, "The Cinema Integrator," Journal of the Franklin Institute, Vol. 230, pp. 19-44, pp. 183-205, August, 1940.
5. H. K. Selwell, "A New Mechanical Autocorrelator," The Review of Scientific Instruments, Vol. 21, pp. 431-434, May, 1950.

is required for each value of delay, x . Graphical and numerical methods have also been found useful.^{6,7,8}

More automatic systems with ability to handle a wide variety of functions include a digital correlator,⁹ an analogue correlator using a delay line,¹⁰ and an analogue system using drum function generators whose relative angular positions are shifted to obtain delay.¹¹

The systems described above will find an unknown function of the integrand, provided this function is related in some simple way to the integral itself which is also unknown, by a method of successive approximations. For example, if $h(y) = f'(y)$, a guess is made for f which is then differentiated and put into the convolution as h . The output of the computer gives a new f which is also differentiated and put back in the machine as a new h . For a large class of functions this is a

6 Robert T. Jones, "Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances," Journal of the Aeronautical Sciences, Vol. 3, pp. 421-422, Oct., 1936.

7 E. V. Laitone and J. T. Ablin, "A Simple Graphical Solution of the Duhamel Integral," Journal of the Aeronautical Sciences, Vol. 18, pp. 142-143, Feb., 1951.

8 R. A. Hessemer, Jr., and C. S. Williams, Jr., "Solution of the Integral Equation Which Determines Radar-Cross-Section for a Scattering Ground," (Technical Memorandum 207-54-54 of Sandia Corporation, Albuquerque, 1954), pp. 7-8.

9 Henry E. Singleton, "A Digital Electronic Correlator," Proceedings of the I.R.E., Vol. 38, pp. 1422-1428, Dec., 1950.

10 Harold Bell, Jr., and V. C. Rideout, "A High-Speed Correlator," Transactions of the I.R.E., Professional Group on Electronic Computers, Vol. EC-3, No. 2, pp. 30-36, June, 1954.

11 T. O. Ellis, M. R. Davis, and L. L. Grandi, "A Machine for Computing the Convolution Integral," Conference Paper No. CP 55-515 of the American Institute of Electrical Engineers, 1955.

is required for each value of t , $\phi(t)$ and $\psi(t)$ must be

have also been found useful.

More automatic systems have been proposed for the

fractional calculus and integral calculus, and these have been

delay line, ¹⁰ and an analog system using delay lines.

whose relative angular positions are subject to constant delay.

The systems described above will find an important application in the

integrals, provided this function is related in some simple way to the

integral itself which is also known, by a method of successive ap-

proximations. For example, if $y(t) = f(t)$, then the method for the

is then differentiated and put into the convolution integral. The output

of the computer gives a new $y(t)$ which is then differentiated and put back

in the machine as a new $y(t)$. For a large class of functions $y(t)$ a

Robert T. Jones, "Differentiation of the α -th order of a function
under the influence of irregular distributions," *Journal of the
Mathematical Sciences*, Vol. 3, pp. 421-425, Oct., 1953.

Y. E. V. Lashin and J. E. Lashin, "On the α -th order of a function
the Riemann integral," *Journal of the Mathematical Sciences*, Vol. 3,
pp. 142-143, Feb., 1953.

8. R. A. Messner, Jr., and C. E. Rife, Jr., "On the α -th order of a
integral equation which determines the α -th order of a function,"
Growth," (Technical Memorandum 50-54, RAND Corporation,
Albuquerque, 1954), pp. 1-10.

9. Henry E. Blagovest, "On the α -th order of a function,"
conditions of the I.R.S., Vol. 3, pp. 142-143, Feb., 1953.

10. Harold Bell, Jr., and V. G. W. Jones, "On the α -th order of a
Transactions of the I.R.S., Professional Group on Mathematical Computing,
Vol. 3, No. 2, pp. 30-32, June, 1954.

11. T. O. Ellis, M. R. Davis, and J. E. Lashin, "On the α -th order of
Computing the Convolution Integral," *Mathematics Paper No. 5-54*, of
the American Institute of Electrical Engineers, 1954.

convergent process and leads to a solution for the function f . (The reason for the requirement of a simple relation between h and f is evident; if it is not simple, another device would be required to find h once the convolution computer had produced an f .) Two of the references give examples of such problems.^{12,13} The graphical method of Laitone and Ahlin¹⁴ will find a function $H(y)$, related to $h(y)$ by $H'(y) = h(y)$, when the functions f and g are known.

R. A. Hessemer, Jr., and C. S. Williams, Jr.¹⁵ proposed a computer to find the function h when the functions f and g of equation (1.1-1) are known. The same computer will also find f when g and h are known. Williams described this computer in more detail and proposed modifications of the output circuits to improve the operation.¹⁶ The computer was designed in detail and fabricated at the Physical Science Laboratory of the New Mexico College of Agriculture and Mechanic Arts under the direction of Mr. Kenneth "E" Smith,¹⁷ who is also supervising its testing which, at this date, is not complete.

¹² Gray, op. cit., pp. 96-98.

¹³ Hazen and Brown, op. cit., appendix contributed by W. R. Hedeman, Jr., pp. 193-205.

¹⁴ Laitone and Ahlin, loc. cit.

¹⁵ Hessemer and Williams, op. cit., p. 10.

¹⁶ C. S. Williams, Jr., "A Computer to Solve One Form of Volterra's Integral Equation," (Technical Memorandum 104-55-54 of Sandia Corporation, Albuquerque, 1955), pp. 5-22.

¹⁷ Kenneth "E" Smith, "Preliminary Report on Convolution Integral Computer," (AER 14-S, Physical Science Laboratory, New Mexico College of Agriculture and Mechanic Arts, State College, 1955), pp. 3-27.

convergent process and leads to a solution of the functional equation (1.1).
 reason for the requirement of a unique solution is that if a
 solution exists, it is not unique, and any other solution would be rejected as a
 solution of the convolution equation (1.1). The functional method of
 Laitone and Allen¹⁴ will find a solution (1.1), which is known.
 $H'(y) = H(y)$, when the functions H and H' are known.

H. A. Hessemer, Jr., and G. C. Williams, Jr.¹⁵ proposed a com-
 puter to find the function H when the functions H' and G are known.
 (1.1-1) are known. The same computer will also find H when H' and G are
 known. Williams designed the computer in more detail and proposed
 modifications of the design to improve the operation.¹⁶ The
 computer was designed in detail and fabricated at the Physical Sciences
 Laboratory of the New Mexico College of Agriculture and Mechanical Arts
 under the direction of Mr. Kenneth "B" Smith,¹⁷ who is also supervising
 the testing which, at this date, is not complete.

¹² Gray, op. cit., pp. 2-3.

¹³ Hessemer and Brown, op. cit., Appendix A, pp. 1-20.

¹⁴ Laitone and Allen, op. cit., pp. 1-10.

¹⁵ Hessemer and Williams, op. cit., p. 10.

¹⁶ G. C. Williams, Jr., "Computer to solve the Fredholm integral equation of the second kind," (Technical Memorandum 10-57-54, Air Force Research Office, Dayton, Ohio, 1957), pp. 1-32.

¹⁷ Kenneth "B" Smith, "Preliminary report on the design and construction of the computer," (AFR 14-5, Physical Sciences Laboratory, New Mexico College of Agriculture and Mechanical Arts, Las Alamos, New Mexico, 1957), p. 1-2.

The computer that is described herein in Chapter III is taken directly from the computer mentioned just above, but it is considerably simplified. It performs only the operation of finding the function f when the functions g and h are specified.

The computer that is described in Chapter IV is an outgrowth of the work already done by Hessemer,¹⁸ Smith,¹⁹ and Williams.^{18,20} It, too, finds the function h when the functions f and g are known. However, the method by which it finds h is not the same as that used in the computer first proposed. It gives the solution immediately after the proper quantities are entered, whereas the original computer approaches the solution through repeated manual operations that minimize a meter reading.

1.4 Definition of symbols and terms. The symbols and terms that are used in this paper are those that are standard in electrical engineering. However, for the sake of clarity, several of them are defined and discussed here.

j : This is the imaginary unit; $j = \sqrt{-1}$.

ω : This symbol is used throughout this paper only to designate angular velocity in radians per second. It is related to frequency f by $\omega = 2\pi f$.

¹⁸ Hessemer and Williams, op. cit., p. 10.

¹⁹ Smith, loc. cit.

²⁰ Williams, loc. cit.

The computer that is described herein in Chapter III is based directly from the computer described first above, and it is considered simplified. It performs only the operation of finding the function f when the functions g and h are specified.

The computer that is described in Chapter IV is based upon the work already done by Hassamer, Smith, and Williams, 18, 19, 20. It finds the function f when the functions g and h are known. However, the method by which it finds f is not the same as that used in the computer first proposed. It gives the function f immediately after the proper quantities are entered, whereas the original computer approaches the solution through repeated manual input and output a meter reading.

1.4 Definition of symbols and terms. The symbols and terms that are used in this paper are those that are standard in electrical engineering. However, for the sake of clarity, several of them are defined and discussed here.

j : This is the imaginary unit, $j = \sqrt{-1}$.

ω : This symbol is used throughout this paper only to designate angular velocity in radians per second. It is related to frequency f by $\omega = 2\pi f$.

18 Hassamer and Williams, op. cit., p. 11.
 19 Smith, loc. cit.
 20 Williams, loc. cit.

s : This is commonly used for the ⁱⁿ⁻dependent variable of the Laplace transform, and it is so used here. It is complex and its real and imaginary parts are given by $s = \sigma + j\omega$, where ω is defined above. The imaginary axis in the s -plane is the so-called real-frequency axis since it corresponds to $\sigma = 0$.

\triangleq : This means "is defined by". It includes, of course, equality $=$, and identity \equiv .

Unit impulse: The unit impulse, sometimes called the Dirac delta-function, is designated by $\delta(x)$. It has the properties $\delta(x) = 0$, ($x \neq 0$), $\int_{-\infty}^{\infty} \delta(x) dx = 1$. It can be constituted in various ways. One method is to take the limit of a rectangular function at the origin, with width Δx and amplitude A , as $\Delta x \rightarrow 0$ while $A\Delta x = 1$.

The functional notation that is used herein is standard. To avoid confusion, parentheses are used to enclose the independent variable, and are nowhere used to separate factors to be multiplied. Thus, $f(M\Delta y + N\Delta y)$ can mean only the value of the function f when its argument has the value $M\Delta y + N\Delta y$. It is worth mentioning that only continuous and differentiable functions are considered in this paper, except that such functions may have a finite number of finite discontinuities in a finite interval.

No special notation has been adopted for complex numbers. Where complex quantities arise, it is pointed out in the text.

as: This is commonly used for the representation of the
 Laplace transform, and it is a real number. It is written
 and the real and imaginary parts are given by $\sigma + j\omega$,
 where ω is defined as the angular frequency in radians per second.
 a-plane is the so-called real-frequency plane and
 corresponds to $\sigma = 0$.

Δ : This is the Laplace transform of the function, $f(t)$, and is written
 as $F(s)$, where $s = \sigma + j\omega$.

Unit impulse: The unit impulse, $\delta(t)$, is a function of time t which is zero everywhere except at $t = 0$, where it is infinite. It is defined by the equation

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$
 in various ways. One method is to take the limit of a
 rectangular function of width Δt and height $1/\Delta t$ as $\Delta t \rightarrow 0$.
 amplitude A , as $A \rightarrow \infty$ while $A\Delta t = 1$.

The functional notation $f(t)$ is used to denote a function of time t . In
 contrast, parentheses are used to denote the independent variable, and
 are nowhere used to denote functions to be differentiated. Thus,
 $f(t_1 + t_2)$ can mean only the value of the function f at the argument
 $t_1 + t_2$. It is a common mistake to think that any combination
 and differentiable functions are indicated in this paper, except that
 such functions may have a finite number of finite discontinuities in
 a finite interval.

No special notation has been adopted for complex numbers, where
 complex quantities arise, it is placed over the letter.

CHAPTER II

APPROXIMATING THE CONVOLUTION INTEGRAL

The computers described herein make use of approximations to the integral. These approximations can be made as close as desired to the exact values; practical considerations will dictate the degree of accuracy to be used. In this chapter, the method of approximation is defined, and an illustration of the manner of using the approximation is given.

2.1 An approximation to the convolution integral. The integral of equation (1.1-1) can be approximated by the sum

$$f(x) = \Delta y \sum_{n=-\infty}^{n=+\infty} g(x - n\Delta y + \frac{1}{2}\Delta y)h(n\Delta y - \frac{1}{2}\Delta y), \quad (2.1-1)$$

where n takes on integer values only. In equation (2.1-1), the limit of the sum as $\Delta y \rightarrow 0$ is the integral of equation (1.1-1). (See any elementary calculus text for definition of the definite integral.) Hence, the smaller Δy is taken, the better is the approximation to the integral. The interval Δy is determined by the degree of accuracy desired. In general, no large variation in $g(y)$ or $h(y)$ should occur in an interval Δy , unless, of course, there is a step discontinuity in one of the functions.

Since values of $g(y)$ and $h(y)$ are selected at discrete intervals, Δy , of the independent variable, the variable x in equation (2.1-1) should also be discrete, not continuous. Hence, let $x = m\Delta y$, where m is

THE INTEGRAL

CHAPTER 1

APPROXIMATION OF THE INTEGRAL

The computer is used to approximate the value of the integral. These approximations are obtained by using the method of rectangles. The method of rectangles is a numerical method for approximating the value of a definite integral. In this method, the area under the curve is approximated by the sum of the areas of several rectangles. The width of each rectangle is Δx , and the height is the value of the function at the midpoint of the interval. The approximation is given by:

2.1 Approximation of the Integral

of equation (1.1-1) can be approximated by the sum

$$I(x) = \Delta x \sum_{i=1}^{n-1} f(x_i) \quad (2.1-1)$$

where n is the number of rectangles. The value of Δx is chosen such that the sum of the areas of the rectangles is a good approximation of the area under the curve. The error in the approximation is given by the difference between the exact value of the integral and the approximation. The error is small when Δx is small. The interval Δx is determined by the number of rectangles. In general, the larger the number of rectangles, the smaller the error. The interval Δx is determined by the number of rectangles. In general, the larger the number of rectangles, the smaller the error. The interval Δx is determined by the number of rectangles. In general, the larger the number of rectangles, the smaller the error.

Since the value of $f(x)$ is known at the midpoint of each interval, the value of the function at the midpoint of each interval is used to approximate the value of the function at the midpoint of each interval. The value of the function at the midpoint of each interval is used to approximate the value of the function at the midpoint of each interval. The value of the function at the midpoint of each interval is used to approximate the value of the function at the midpoint of each interval.

an integer. Then

$$f(m\Delta y) = \Delta y \sum_{n=-\infty}^{n=+\infty} g(m\Delta y - n\Delta y + \frac{1}{2}\Delta y) h(n\Delta y - \frac{1}{2}\Delta y). \quad (2.1-2)$$

The restrictions that will be applied to the functions $g(y)$ and $h(y)$ together with the resultant restrictions of $f(y)$ are:

$$\left. \begin{aligned} f(y) &= 0, \quad (y \leq 0, y \geq M\Delta y + N\Delta y), \\ g(y) &= 0, \quad (y < 0, y > M\Delta y), \\ h(y) &= 0, \quad (y < 0, y > N\Delta y), \end{aligned} \right\} \quad (2.1-3)$$

where M and N are integers. This changes the lower limit of summation in equation (2.1-2) to $n = 1$ and the upper limit to $n = m$.

To simplify the notation let

$$\left. \begin{aligned} f_k &\triangleq f(k\Delta y) \\ &\triangleq 0, \quad (k < 1, k > M + N - 1), \\ g_k &\triangleq g(k\Delta y - \frac{1}{2}\Delta y) \\ &\triangleq 0, \quad (k < 1, k > M), \\ h_k &\triangleq h(k\Delta y - \frac{1}{2}\Delta y) \\ &\triangleq 0, \quad (k < 1, k > N), \end{aligned} \right\} \quad (2.1-4)$$

where k has integer values only. The ranges for which f_k , g_k , and h_k are nonzero follow from their definitions and from the restrictions of equations (2.1-3).

Now, by virtue of equations (2.1-4), equation (2.1-2) may be written

an integer. Then

$$(2.1-2) \quad l(m) = \sum_{n=0}^{m-1} g(m-n) \cdot h(n) \quad (2.1-2)$$

The restrictions that will be applied to the functions $g(n)$ and

$h(n)$ together with the resultant restrictions of $l(n)$ are:

$$(2.1-3) \quad \left\{ \begin{array}{l} l(n) = 0, \quad (n \leq 0, n \geq M+N) \\ g(n) = 0, \quad (n < 0, n > M) \\ h(n) = 0, \quad (n < 0, n > N) \end{array} \right.$$

where M and N are integers. This changes the lower limit of summation

in equation (2.1-2) to $n = 1$ and the upper limit to $n = m$.

To simplify the notation let

$$(2.1-4) \quad \left\{ \begin{array}{l} l_k \triangleq l(k) \\ g_k \triangleq g(k) \\ h_k \triangleq h(k) \end{array} \right. \quad \left\{ \begin{array}{l} \triangleq 0, \quad (k < 1, k > M+N-1) \\ \triangleq 0, \quad (k < 1, k > M) \\ \triangleq 0, \quad (k < 1, k > N) \end{array} \right.$$

where k has integer values only. The ranges for g_k and h_k

are nonzero follow from their definitions and from the restrictions of

equations (2.1-3).

Now, by virtue of equations (2.1-4), equation (2.1-2) may be

written

$$f_m = \Delta y \sum_{n=1}^m g_{m-n+1} h_n, \quad (2.1-5)$$

where

$$m = 1, 2, \dots, M + N - 1.$$

This is the desired approximation to the convolution integral and it is the form with which the computers deal.

2.2 Illustration of the convolution integral approximation. In order to illustrate the use of the approximation of equations (2.1-5) to the integral (1.1-1), let

$$g(x) = 1, \quad (0 \leq x < 1.2),$$

$$= 0, \quad (x < 0, 1.2 \leq x),$$

$$h(x) = e^{-x}, \quad (0 \leq x),$$

$$= 0, \quad (x < 0).$$

Then, from equation (1.1-1),

$$f(x) = 0, \quad (x \leq 0),$$

$$= \int_0^x e^{-y} dy, \quad (0 \leq x \leq 1.2),$$

$$= \int_{x-1.2}^x e^{-y} dy, \quad (1.2 \leq x).$$

(2.1-2)

$$T = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i + 1}$$

where

$$x_i = 1, 2, \dots, n-1$$

This is the desired approximation to the convolution integral and is in the form with which the computer deals.

2.2 Illustration of the convolution integral approximation

Order to illustrate the use of the approximation in equation (2.1-2) is the integral (1.1-1), i.e.

$$g(x) = 1, \quad (0 \leq x < 1),$$

$$= 0, \quad (x < 0, 1 \leq x),$$

$$h(x) = e^{-x}, \quad (0 \leq x),$$

$$= 0, \quad (x < 0).$$

Then, from equation (1.1-1),

$$r(x) = 0, \quad (x \geq 0),$$

$$= \int_0^x e^{-v} dv, \quad (0 \leq x < 1),$$

$$= \int_{x-1}^x e^{-v} dv, \quad (1 \leq x).$$

Upon evaluation

$$\begin{aligned}
 f(x) &= 0, & (x \leq 0), \\
 &= 1 - e^{-x}, & (0 \leq x \leq 1.2), \\
 &= [e^{1.2} - 1] e^{-x}, & (1.2 \leq x).
 \end{aligned}$$

To evaluate $f(x)$ by equations (2.1-5), choose

$$\Delta y = 0.4.$$

Then,

$$\begin{aligned}
 h_1 &= h(.2), \quad h_2 = h(.6), \text{ etc.}, \\
 g_1 &= g(.2), \quad g_2 = g(.6), \text{ etc.}
 \end{aligned}$$

These values are illustrated in Fig. 2.2-a. Next, the f_m are computed by equations (2.1-5); the values of the f_m as so calculated are compared with the function $f(x)$, as found analytically, in the following table:

m	f_m	$f(m\Delta y)$	m	f_m	$f(m\Delta y)$
0	0	0	10	.042	.042
1	.327	.330	11	.028	.028
2	.547	.551	12	.019	.019
3	.695	.699	13	.013	.013
4	.466	.468	14	.008	.009
5	.312	.313	15	.006	.006
6	.209	.210	16	.004	.004
7	.140	.141	17	.002	.003
8	.094	.094	18	.001	.002
9	.063	.063	19	.000	.001

The agreement is quite good and may be improved by taking a smaller value of Δy .

Upon evaluation

$$f(x) = 0, \quad (x \geq 0)$$

$$f(x) = 1 - e^{-x}, \quad (0 \leq x < \infty)$$

$$f(x) = \begin{cases} 1 - e^{-x} & (0 \leq x < \infty) \\ 0 & (x < 0) \end{cases}$$

To evaluate $f(x)$ by equation (2.1-5), we set

$$x = 0.1$$

Then,

$$f_1 = f(x) = 1 - e^{-x} = 1 - e^{-0.1} = 0.09516$$

$$f_2 = f(x) = 1 - e^{-x} = 1 - e^{-0.1} = 0.09516$$

These values are illustrated in Fig. 2.1-5. Next, the f are computed

by equation (2.1-5); the values of the f are no longer needed and are dropped.

With the function $f(x)$, as found previously, in the f listing below:

x	f	$f(x)$	f	$f(x)$
0	0	0	0	0
1	.367	.367	.367	.367
2	.747	.747	.747	.747
3	.950	.950	.950	.950
4	.982	.982	.982	.982
5	.993	.993	.993	.993
6	.997	.997	.997	.997
7	.999	.999	.999	.999
8	.999	.999	.999	.999
9	.999	.999	.999	.999

The agreement is quite good and may be improved by taking a smaller

value of Δx .

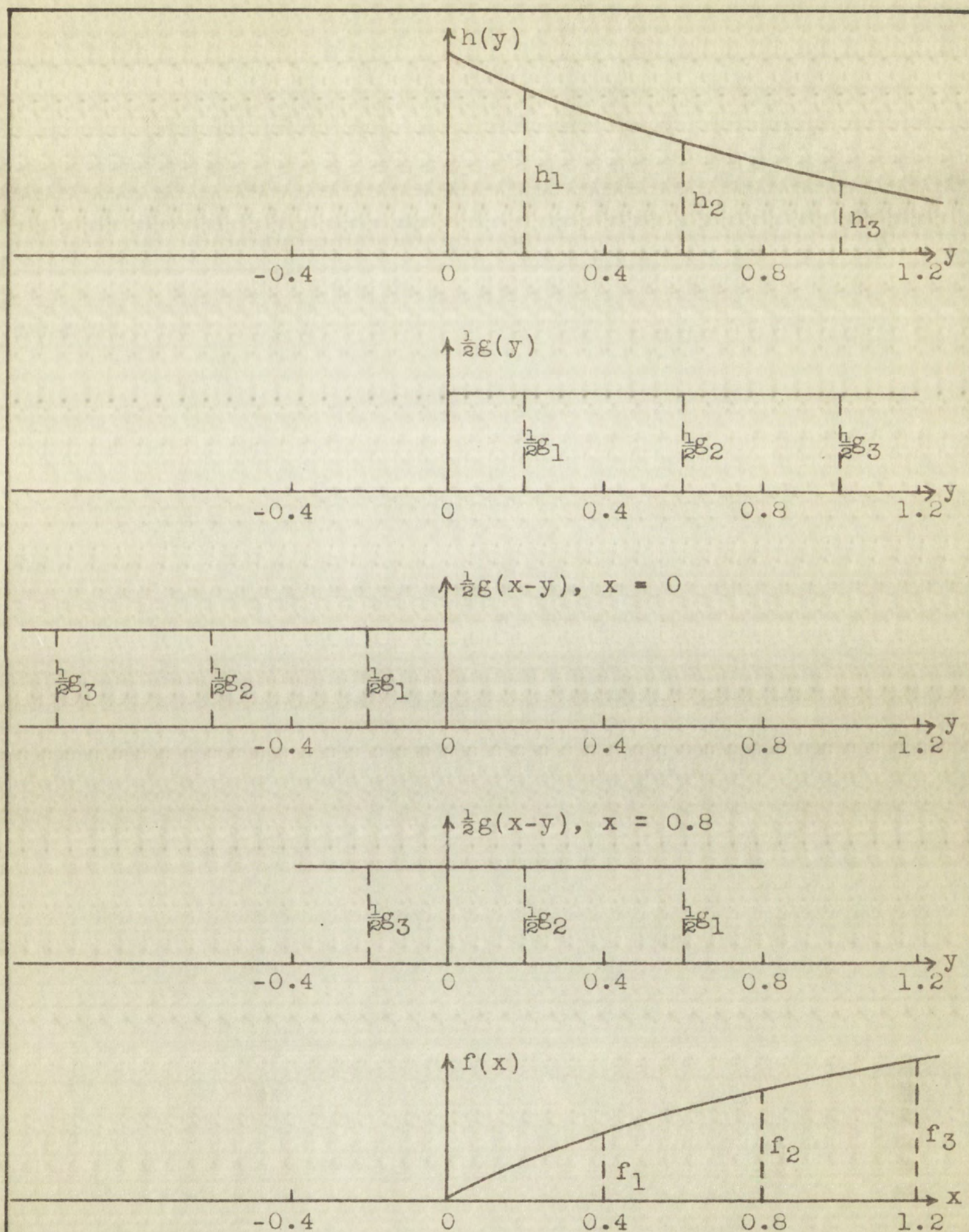
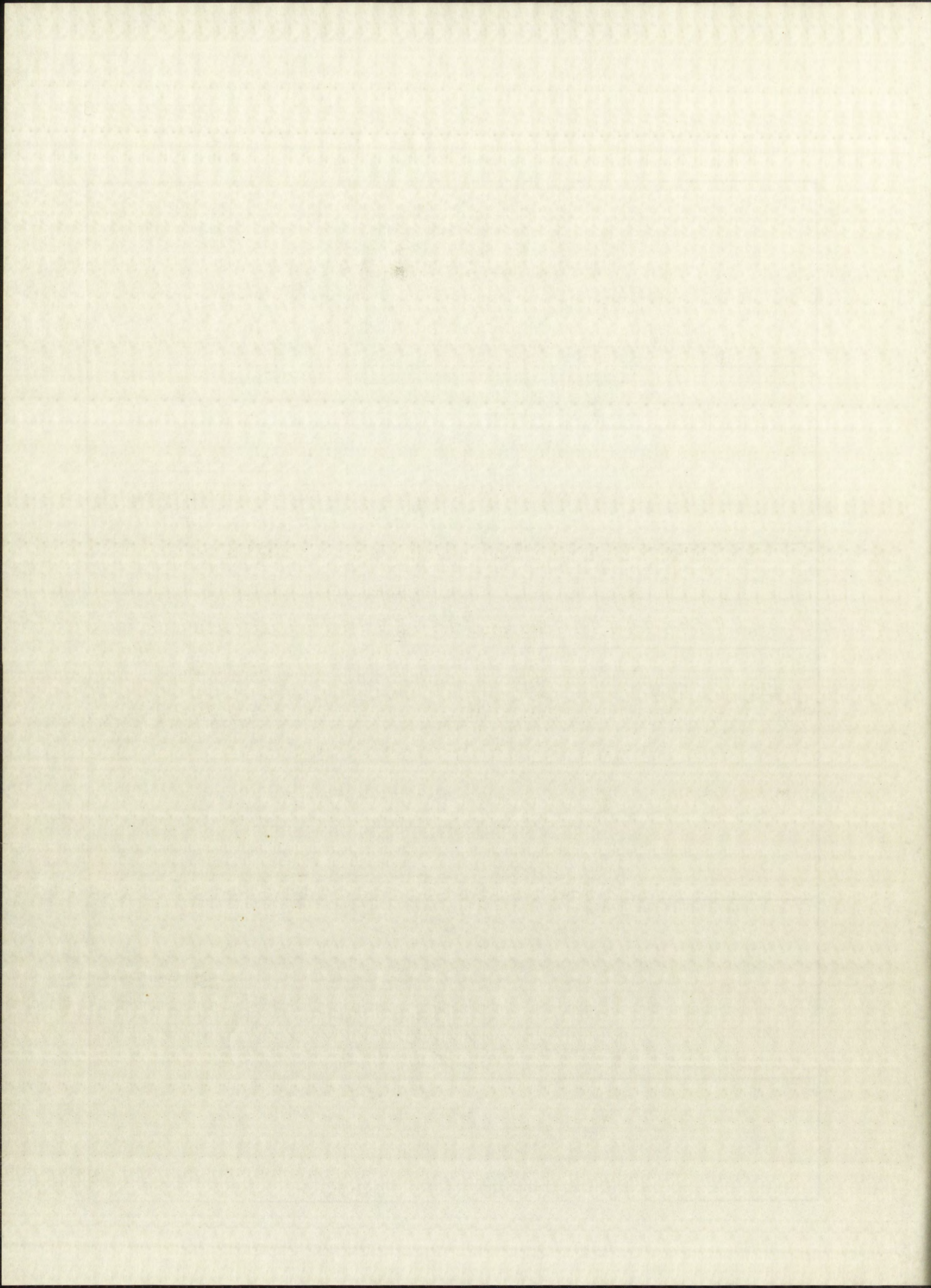


Figure 2.2-a

Illustration of Convolution



CHAPTER III

CONVOLUTION COMPUTER

The computer described in this chapter finds the function f of equation (1.1-1) when g and h are known. In this respect, it is the same as the computer proposed by Hessemer and Williams,^{21,22} and it uses exactly the same type of switch. However, it is a great deal simpler since it does not solve the integral equation for h when f and g are known.

3.1 Circuit of the convolution computer. The computer uses the approximation of equations (2.1-5) to represent the integral (1.1-1). In order to illustrate the computer in Fig. 3.1-a, M , the number of ordinates taken to represent $g(x)$, is 3; N , the number of ordinates of $h(x)$, is 4. Then, from equations (2.1-4), the number of ordinates of $f(x)$ will be $M + N - 1 = 6$. With these restrictions, equations (2.1-5) may be written:

$$\left. \begin{aligned} f_1 &= \Delta y(g_1 h_1), \\ f_2 &= \Delta y(g_2 h_1 + g_1 h_2), \\ f_3 &= \Delta y(g_3 h_1 + g_2 h_2 + g_1 h_3), \\ f_4 &= \Delta y(g_3 h_2 + g_2 h_3 + g_1 h_4), \\ f_5 &= \Delta y(g_3 h_3 + g_2 h_4), \\ f_6 &= \Delta y(g_3 h_4). \end{aligned} \right\} \quad (3.1-1)$$

21 Hessemer and Williams, op. cit., p. 10.

22 Williams, loc. cit.

CHAPTER III

CONVOLUTION OPERATOR

The computer described in this chapter is a convolution operator. It takes as input a function $f(x)$ and a kernel $h(x)$ and produces as output the convolution of $f(x)$ and $h(x)$. The convolution of two functions $f(x)$ and $h(x)$ is defined by the equation (1.1-1) which is a six known. In this respect, it is the same as the computer proposed by Hassner and Williams, [1, 2], and it has exactly the same type of analysis. However, it is a more general operator since it does not solve the integral equation (1.1-1) when $f(x)$ and $h(x)$ are known.

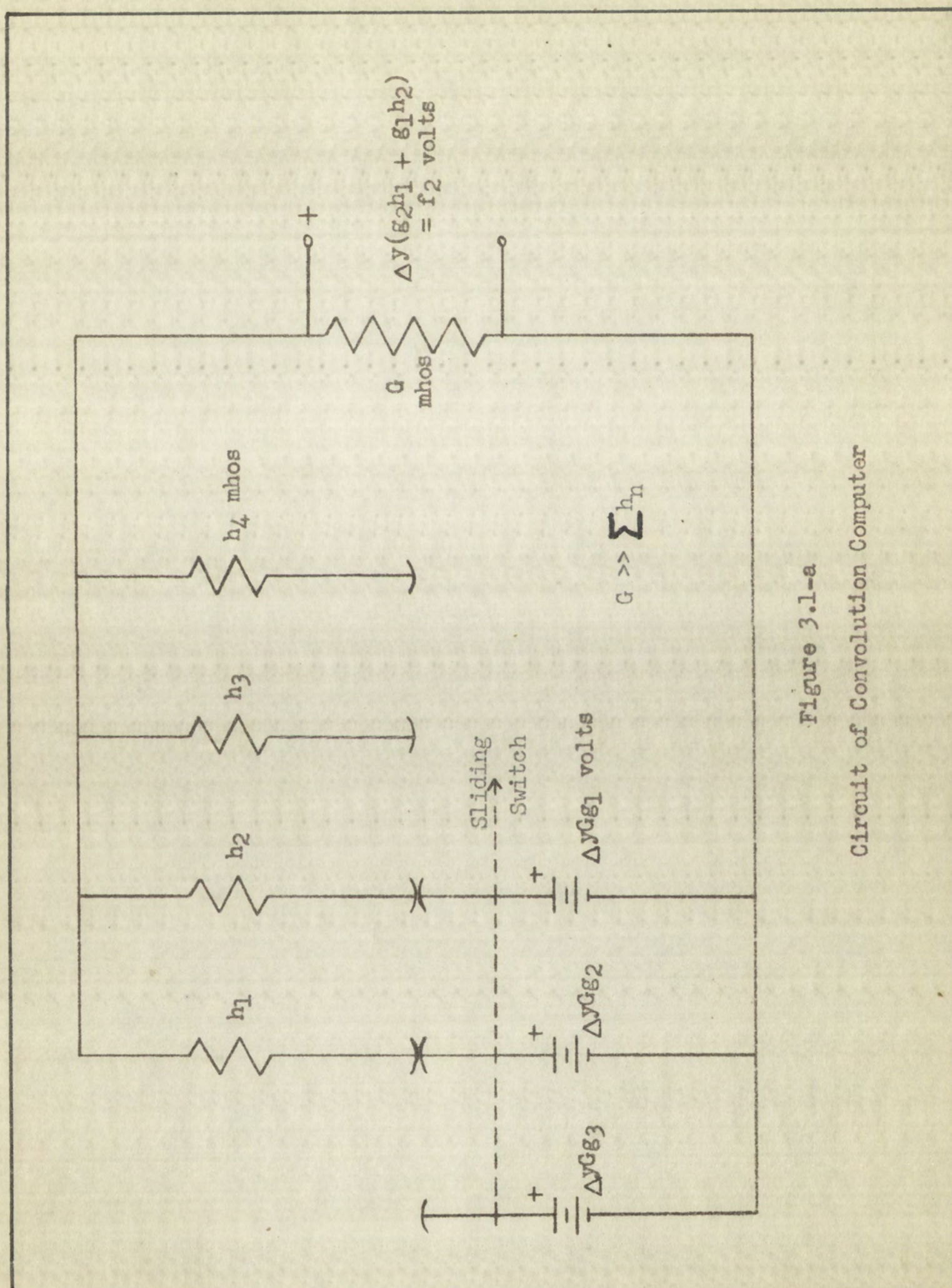
3.1. Circuit of the convolution operator. The computer is a form

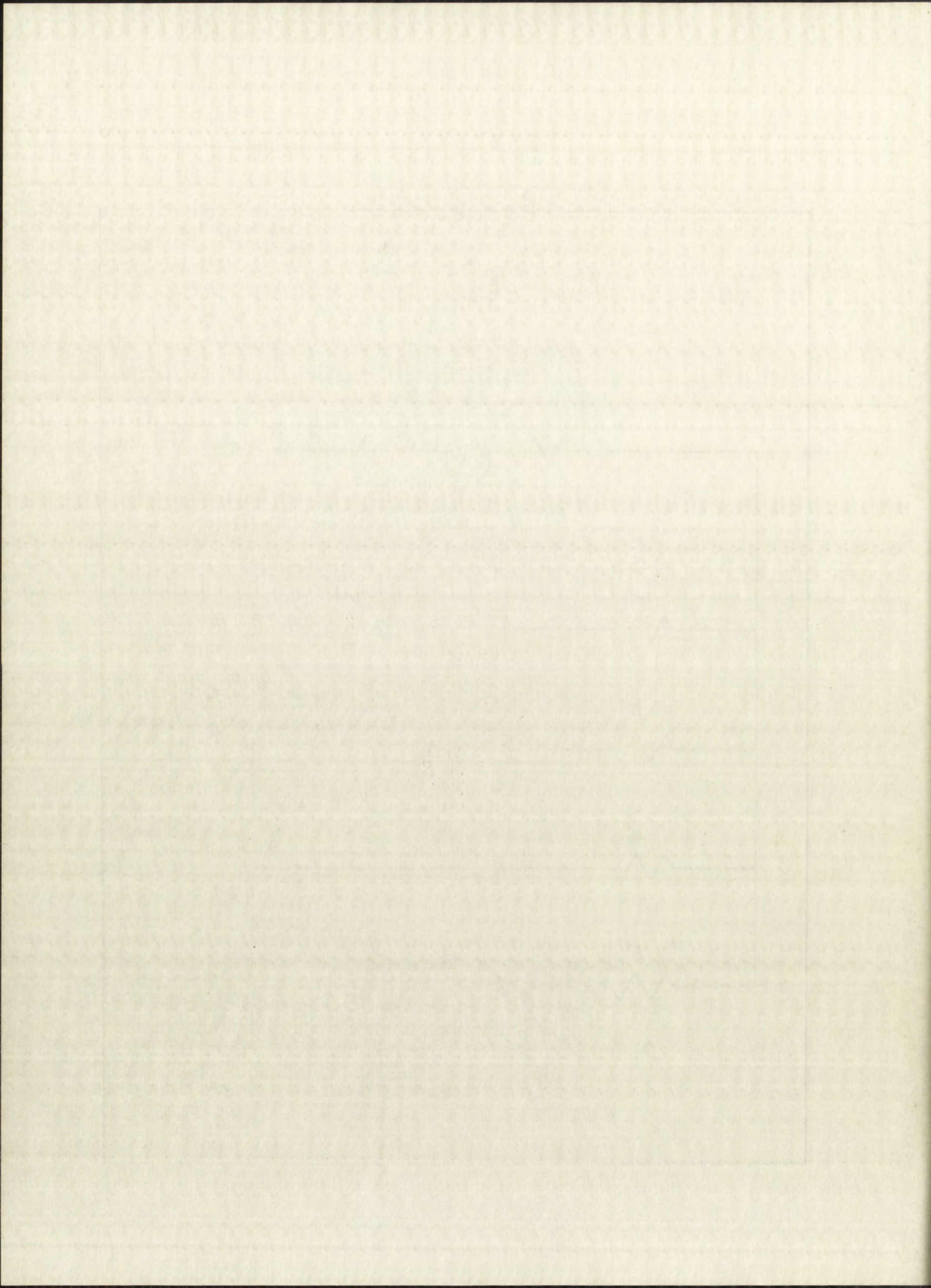
approximation of equation (2.1-1) to represent the integral (1.1-1). In order to illustrate the computer in Fig. 3.1-1, let the number of samples N be 10, the number of samples of $f(x)$ be 5, the number of samples of $h(x)$ be 5, the number of samples of $g(x)$ be 10, and the number of samples of $b(x)$ be 10. Then, from equation (2.1-1), the number of samples of $f(x)$ will be $M + N - 1 = 10$. With these restrictions, equation (2.1-1) may be written:

$$\begin{aligned} g_1 &= f_1 h_1 \\ g_2 &= f_1 h_2 + f_2 h_1 \\ g_3 &= f_1 h_3 + f_2 h_2 + f_3 h_1 \\ g_4 &= f_2 h_3 + f_3 h_2 + f_4 h_1 \\ g_5 &= f_3 h_3 + f_4 h_2 + f_5 h_1 \\ g_6 &= f_4 h_3 + f_5 h_2 \\ g_7 &= f_5 h_3 \end{aligned} \quad (3.1-1)$$

[1] Hassner and Williams, *op. cit.*, p. 10.

[2] Williams, *op. cit.*, p. 11.





Now, in Fig. 3.1-a, the h_n are represented by conductances, and the quantities $\Delta y G_n$ are represented by voltages. Thus, in the switch position shown, the voltage across the conductance $\overset{G}{\wedge}$ is f_2 provided $G \gg \Sigma h_n$, as reference to Fig. 3.1-b shows. It is evident, then, that the various switch positions give the ordinates of $f(x)$; i.e., as the switch moves from left to right, the output voltages are f_1, f_2, f_3, f_4, f_5 , and f_6 .

In order to satisfy the condition $G \gg \Sigma h_n$, it may be necessary to use a feedback amplifier at the output to give an effectively large conductance. See Sec. 4.4 concerning admittance conversion by amplifiers.

Although it is not shown in Fig. 3.1-a, there must be some method of varying the conductances h_n and the voltages $\Delta y G_n$ so that various functions $g(x)$ and $h(x)$ may be inserted in the computer. The methods of entering the functions are several. It is possible to use a voltmeter for the $\Delta y G_n$ and a rhometer for the h_n . A better, and more expensive, system would use precision variable conductances and potential dividers so that the functions could be entered by dial settings only.

The output voltages f_m may be read with a voltmeter and recorded as the switch is moved manually from position to position. If a rotating, motor-driven switch is used so that the output function repeats in time, it may be displayed on an oscilloscope.²³ An intermediate

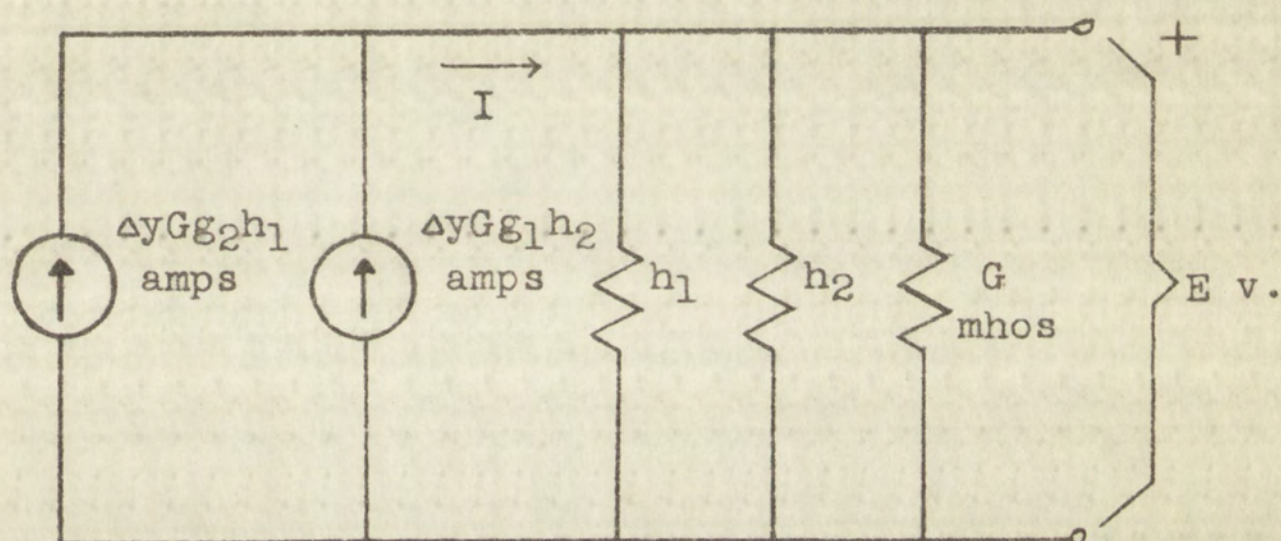
²³ Smith, loc. cit.

Now, in Fig. 1, the various conductances G_1, G_2, \dots, G_n are represented by voltages V_1, V_2, \dots, V_n in the switch position shown. The voltage across the conductance G_i is V_i , as referred to Fig. 1.1-1. In an equivalent circuit, the various switch positions give the voltages V_1, V_2, \dots, V_n as the switch moves from left to right. The voltages V_1, V_2, \dots, V_n are V_1, V_2, \dots, V_n and V_n .

In order to satisfy the condition $G > 0$, it may be necessary to use a feedback amplifier at the output to give an effectively large conductance. See Sec. 4.1. Some other advantages of this circuit are:

Although it is not shown in Fig. 1.1-1, there must be some means of varying the conductances G_i and the voltages V_i as a function of the functions $g(x)$ and $a(x)$ may be inserted in the conductance. The use of entering the functions are essential. It is possible to use a voltage meter for the V_i and a microammeter for the G_i . A meter, and more expensive, system would use a variable conductance and a central divider so that the functions could be entered by dial settings only.

The output voltages V_i may be read with a voltmeter and recorded as the switch is moved manually from position to position. In a more complex, motor-driven switch, it may be that the output function register in time, it may be displayed on a oscilloscope. In an analog



$$I = \Delta y G [g_2 h_1 + g_1 h_2] ,$$

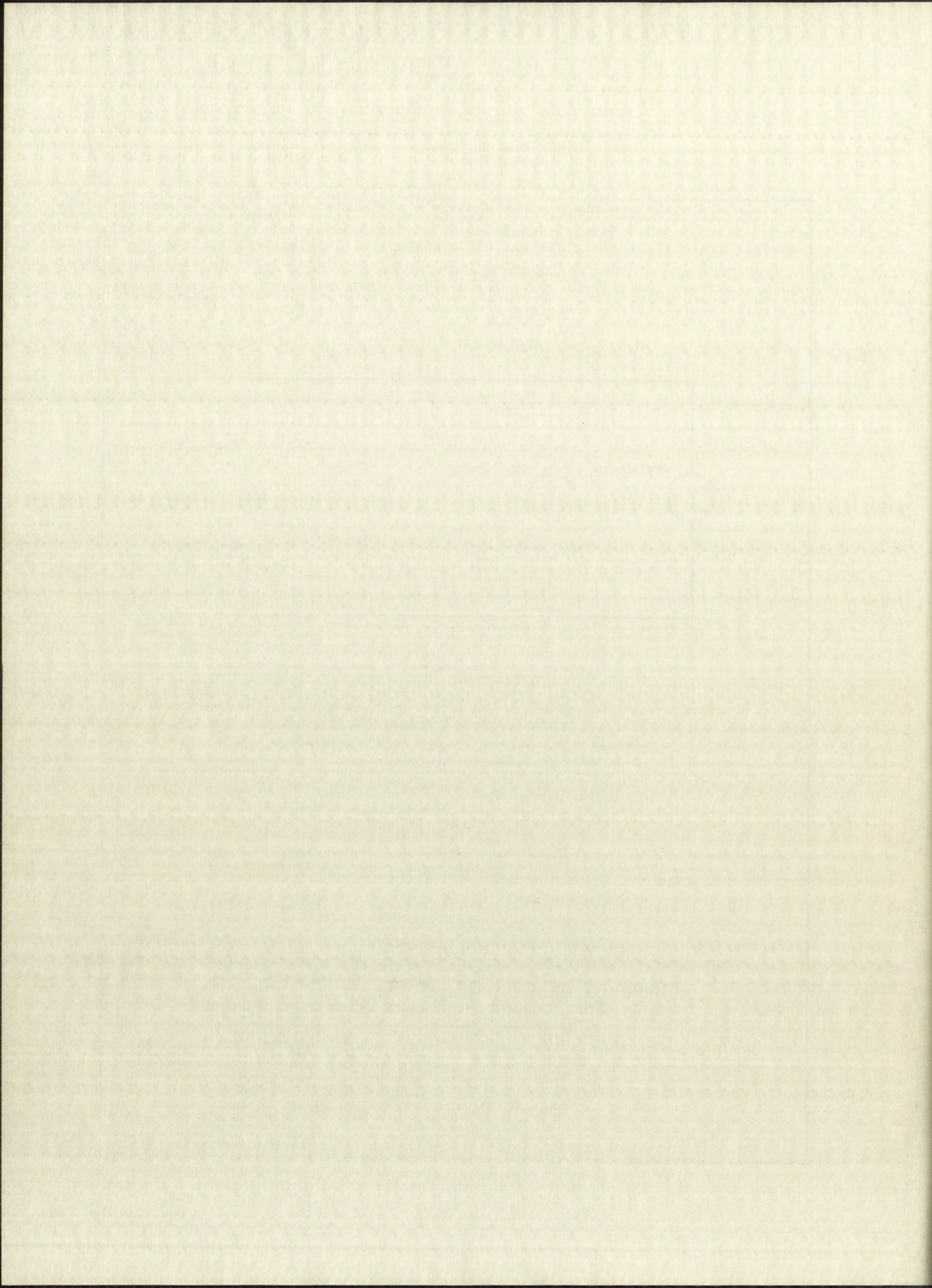
$$E = I/G , \quad (\text{since } G \gg h_1 + h_2),$$

$$= \Delta y [g_2 h_1 + g_1 h_2] ,$$

$$= f_2 .$$

Figure 3.1-b

Equivalent Circuit of Figure 3.1-a
for Switch Position Shown



system would use an electromechanical pen recorder whose paper drive is synchronized with the motion of the switch. The switch would then be moved manually through its positions and the pen recorder would record the output.

In the circuit of Fig. 3.1-a, it is evident that it is an easy matter to handle negative values of the function $g(x)$. Entering negative values of $h(x)$ is another matter and is discussed in the next section.

3.2 Negative values of $h(x)$. In Fig. 3.1-a, each combination of $\Delta y G_{m-n+1}$ volts with an h_n mhos acts as a current source of value $\Delta y G_{m-n+1} h_n$ amperes with a shunt conductance of h_n mhos as Fig. 3.1-b shows. Because $G \gg \sum h_n$, the shunt conductance is neglected.

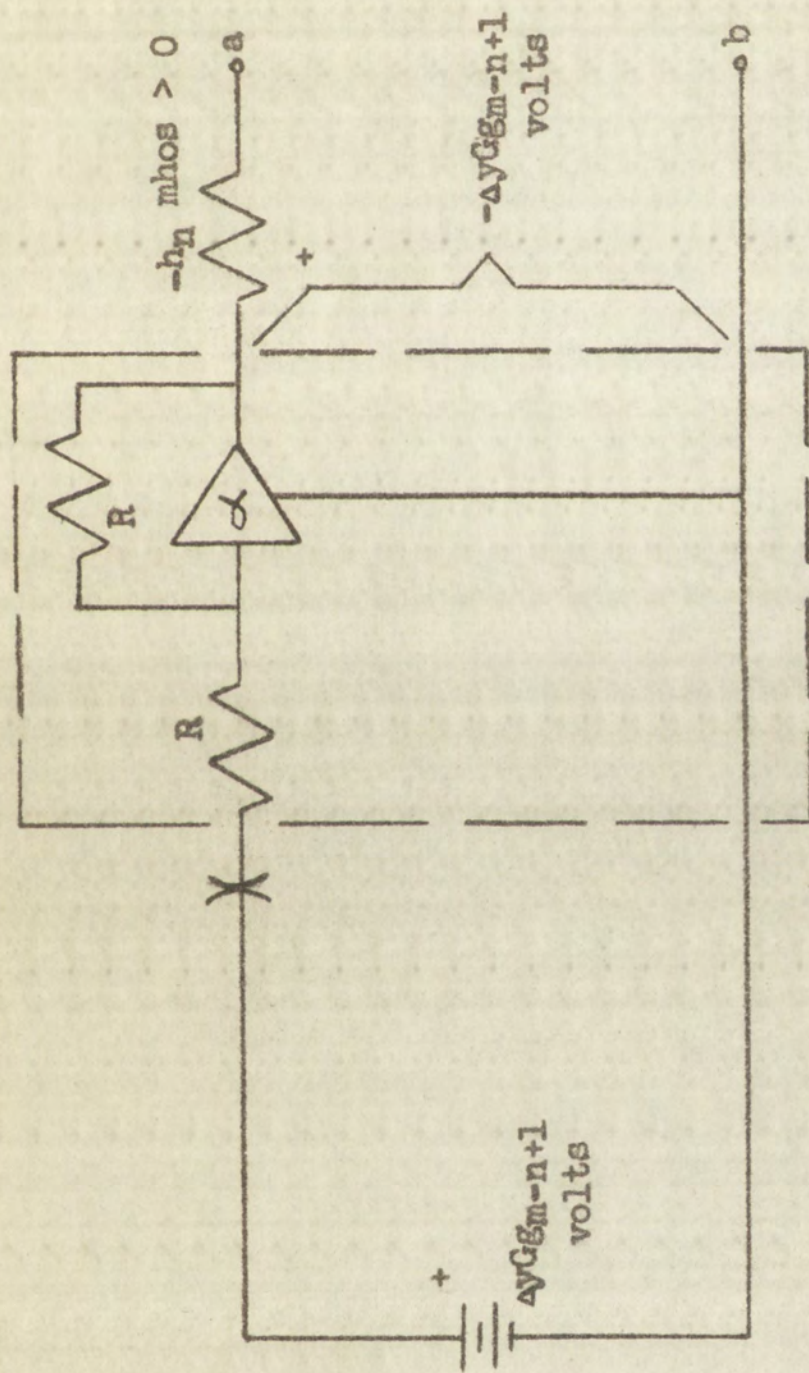
If an h_n is negative, it will be represented by a conductance of $-h_n$ mhos, since conductance is necessarily positive. This is shown in Fig. 3.2-a. An amplifier of gain -1 is placed in the line between the switch contact and the conductance. (See Sec. 4.5 concerning amplifiers.) Then, the current source as seen at terminals a-b has value $\Delta y G_{m-n+1} h_n$, which is the value desired. See Fig. 3.2-b. The sign of the shunt conductance is without import; since $G \gg \sum |h_n|$, it is negligible in the circuit as a whole.

Note that it is important that the amplifier be on the same side of the switch as the conductance $-h_n$; otherwise, the effect would be to change the sign of $\Delta y G_{m-n+1}$ rather than that of $-h_n$.

system would use an electro-mechanical system which would be
synchronized with the motion of the dial. This dial would then be
moved manually through the mechanism and the output would be
the output.

In the circuit of Fig. 3-1-a, it is assumed that it is desired
to handle negative values of $x(t)$. In order to handle nega-
tive values of $x(t)$ as another matter and is discussed in the next
section.

3.2 Negative values of $x(t)$ In Fig. 3-1-b, a circuit is shown
of Δy_{n-1} with an n value as a constant current of value
 Δy_{n-1} and a switch with a constant current of n value as Fig. 3-1-a
shows. Because $G > 0$, the same mechanism is required.
If an n is negative, it will be represented by a constant
of $-n$ value, since conductance is necessarily positive. This is shown
in Fig. 3-2-a. An addition of y_{n-1} is placed in the line between
the switch contact and the conductance. (See Fig. 3-2-b, showing
amplifiers.) Then, the current source is seen as Δy_{n-1} and
value Δy_{n-1} , which is the value desired. See Fig. 3-2-b.
The sign of the current conductance is without sign, since $G > 0$.
It is negligible in the circuit as a whole.
Note that it is important that the amplifier be on the same side
of the switch as the conductance. Otherwise, the effect would be
to change the sign of Δy_{n-1} .

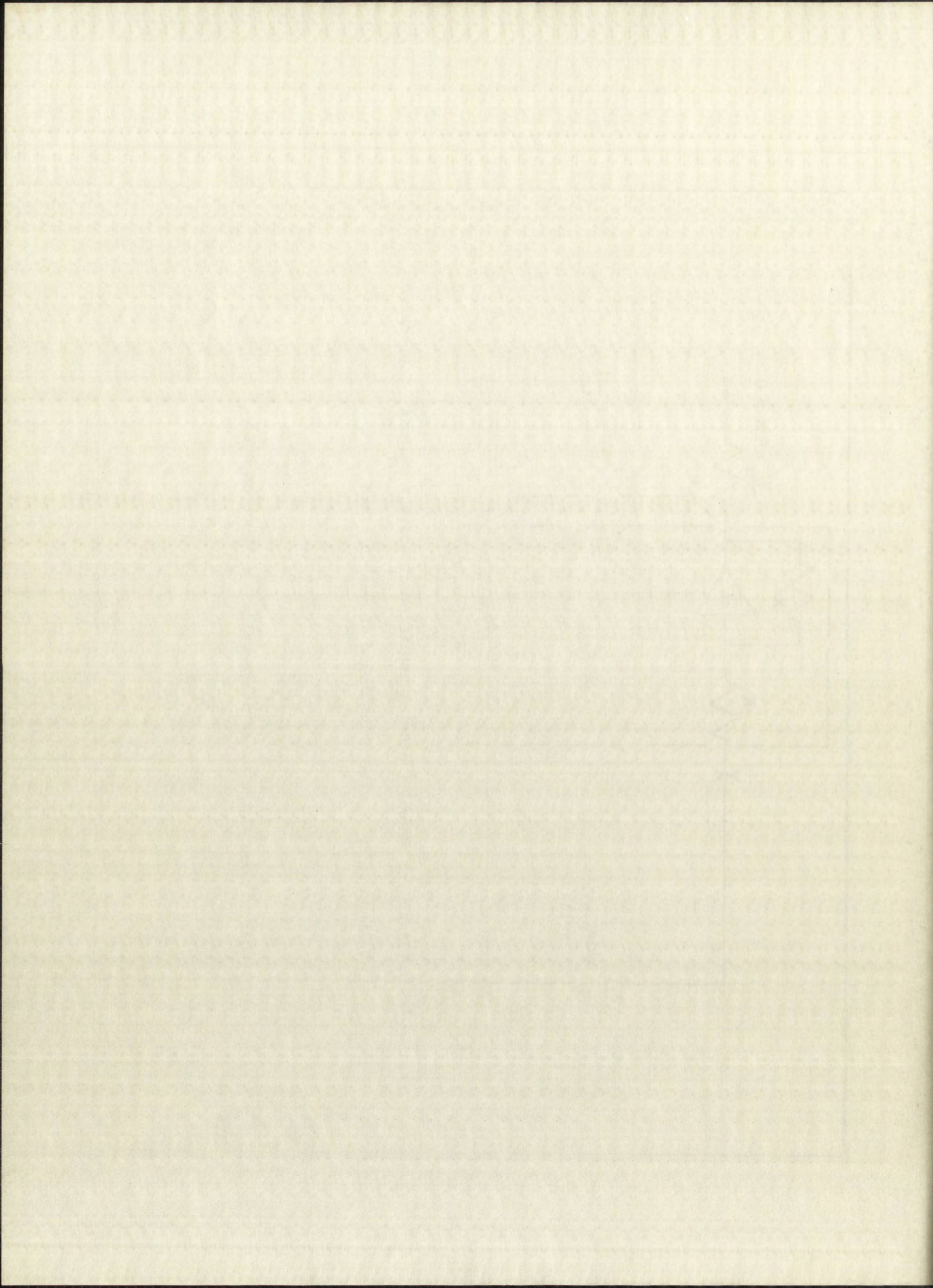


Triangular symbol indicates amplifier of voltage gain α , $|\alpha| \gg 1$.

Voltage gain of network enclosed in dashed lines is -1.

Figure 3.2-a

Circuit for Entering Negative Values of Function $h(t)$



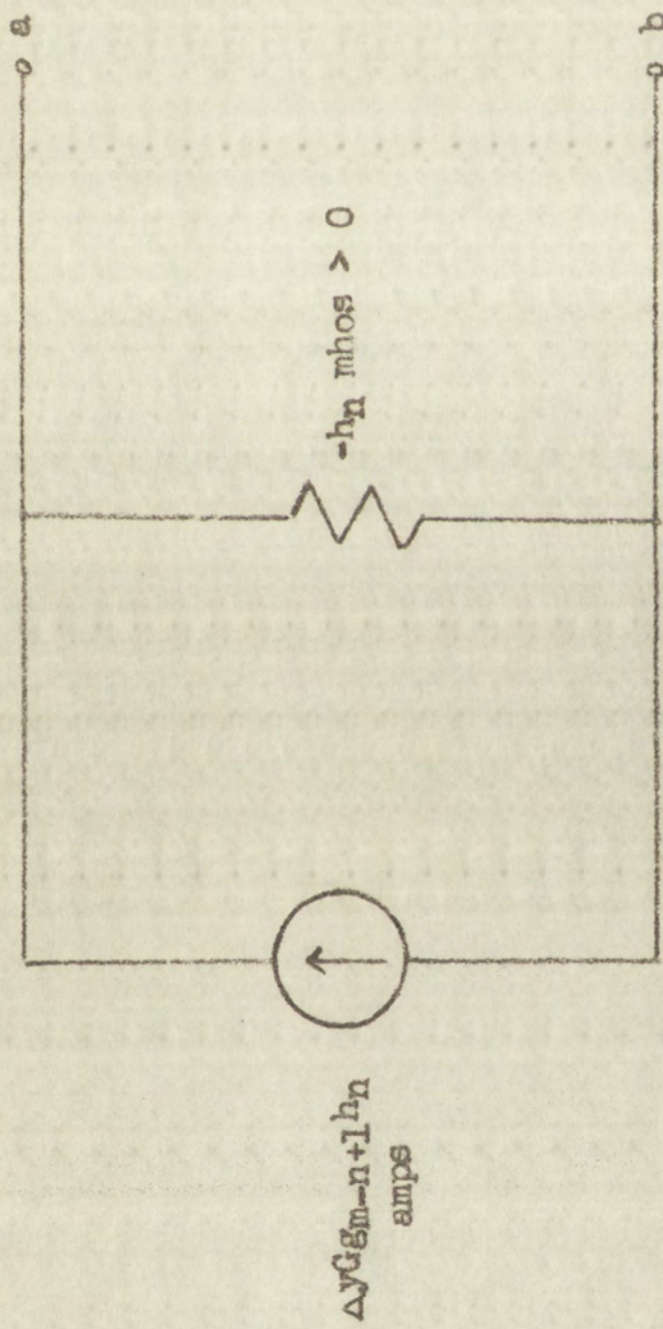
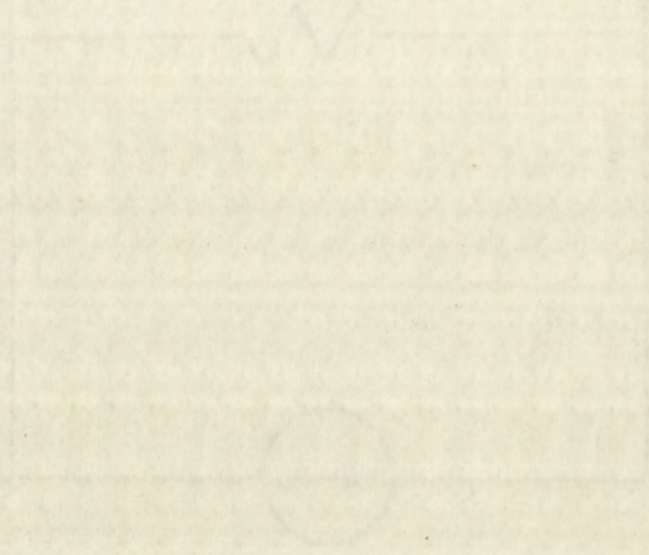


Figure 3.2-b
Current-Source Equivalent of Figure 3.2-a



CHAPTER IV

A COMPUTER TO SOLVE THE CONVOLUTION INTEGRAL EQUATION

The computer discussed in this chapter will find the function h when the functions f and g are known in the convolution integral of equation (1.1-1). The computer proposed by Hessemer and Williams²⁴ does this, but the computer described in this chapter does it more directly. Needless to say, this problem is more difficult than that of performing the convolution.

4.1 Method of solution. Examination of equations (2.1-5), with the f_k and the g_k known, shows that it represents a system of $M + N - 1$ equations with N unknowns h_k . If it is possible to find a set of h_k , ($k = 1, 2, \dots, N$), that satisfy the $M + N - 1$ equations, then the equations are not independent; in fact, only N of them are independent.

In most applications of the convolution integral, this lack of independence in the equations is to be expected. The convolution represents a physical process in which the functions g and h generate f . In the generation of the first N of the $M + N - 1$ equations, all information concerning h is used; hence, the remaining $M - 1$ equations are not independent of the first N equations.

In an application of the computer, the functions f and g are known through measurement, and h is to be found. If the measurements are

²⁴ Hessemer and Williams, op. cit., p. 10.

CHAPTER IV

A COMPUTER TO SOLVE THE DIFFERENTIAL EQUATIONS

The computer discussed in this chapter will solve the

when the functions f and g are known in the interval $[a, b]$.

equation (1.1-1). The computer requires of the user the

this, but the computer handles in this chapter as a mere detail.

Needless to say, this problem is more difficult than that of ordinary

the convolution.

2.1 Method of solution. The solution of equation (2.1-1), which

the f and the g known, shows that it is necessary to solve $N + 1$

equations with N unknowns. If N is a positive integer, then N

($i = 1, 2, \dots, N$), each having the form $N + 1$ equations, then the system

tions are not independent in fact, only N of them are independent.

In most applications of the convolution method, the last of

independence in the equations is to be expected. The convolution method

represents a physical process in which the functions f and g are given.

In the generation of the first N of the $N + 1$ equations, all known

information concerning f is used, hence, the remaining $N - 1$ equations are

not independent of the first N equations.

In an application of the computer, the functions f and g are given

through measurement, and a table is formed. If the measurement is

St. Hester, Inc., 1964, pp. 1-10.

Y. F. H. CO.

perfect, and if the convolution integral exactly represents the physical process, only N of the equations are needed to find the N quantities h_k . In fact, an easy method exists for finding the h_k : Solve the first of equations (2.1-5) for h_1 ; use this value of h_1 in the second equation and solve for h_2 ; continue the process until the N^{th} equation gives h_N . (Equations (3.1-1) make it clear that only h_1 appears in the first equation; only h_1 and h_2 in the second; etc.)

Hessemer and Williams²⁵ tried the process for finding the h_k described in the preceding paragraph without success. In certain cases the method might be successful, but in others, where either the f_k and the g_k are not known exactly (i.e., the measurement of these quantities is not perfect), or the convolution integral does not exactly describe the physical process, the method will fail. For that reason, the method of least squares as described in the remainder of this section is chosen to find the h_k .

Let

$$F(h_1, h_2, \dots, h_N) \triangleq \sum_{m=1}^{M+N-1} \left[f_m - \Delta y \sum_{n=1}^m g_{m-n+1} h_n \right]^2. \quad (4.1-1)$$

If a set of h_k can be found for which $F = 0$, all of equations (2.1-5) will be satisfied. Since such a set of h_k may not exist, that set of h_k that minimizes F will be taken as the solution. Such a set of h_k

²⁵ Hessemer and Williams, op. cit., pp. 7-10.

perfect, and if the convolution integral is not perfect, only W of the equations are needed to find the W elements. In fact, an easy method exists for finding W : solve the first n equations (2.1-5) for h_1 ; use this value of h_1 in the second equation and solve for h_2 ; continue the process until the N equations are solved. (Equations (2.1-1) make it clear that only n equations in the first equations; only h_1 and h_2 in the second, etc.)

Hessner and Williams² noted one aspect of finding the h coefficients in the preceding paragraph. In certain cases the method might be excessive, but in general, where either h_1 and h_2 are not known exactly (i.e., the assumption of linear practice is not perfect), or the convolution integral does not exactly resemble the physical process, the method will tell. For one reason, the method of least squares as described in the remainder of this section is chosen to find the h .

Let

$$T(h_1, h_2, \dots, h_N) = \sum_{n=1}^{N+1} \Delta t \left[\sum_{m=1}^{n-1} h_m \right] \quad (2.1-1)$$

If a set of h can be found for which $T = 0$, all of equations (2.1-5) will be satisfied. Since each h is a real number, the set of h that minimizes T will be taken as the solution. Such a set of h

² S. Hessner and Williams, op. cit., pp. 7-10.

causes the quantities on the right-hand side of equations (2.1-5) to fit those on the left, i.e., the f_m , in a least-squares sense.

The set of h_k that produces the minimum of F must simultaneously satisfy the equations

$$\frac{\partial F}{\partial h_k} = 0, \quad (k = 1, 2, \dots, N).$$

Thus, there are N linear equations in N unknowns h_k and a unique solution exists. Differentiating (4.1-1) yields

$$\frac{\partial F}{\partial h_k} = \sum_{m=1}^{M+N-1} \left\{ 2 \left[f_m - \Delta y \sum_{n=1}^m g_{m-n+1} h_n \right] \left[-\Delta y g_{m-k+1} \right] \right\}$$

where

$$k = 1, 2, \dots, N.$$

Since, from equations (2.1-4),

$$f_m = g_n = h_k = 0 \text{ for } m > M + N - 1, n > M, k > N,$$

the upper limit of all summation signs above can be taken as $+\infty$. Then rearrangement gives

$$\frac{\partial F}{\partial h_k} = -2\Delta y \left\{ \sum_{m=1}^{\infty} f_m g_{m-k+1} - \Delta y \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} g_{m-n+1} g_{m-k+1} \right] h_n \right\}$$

Setting $\frac{\partial F}{\partial h_k} = 0$ leads to

choose the quantities of the right-hand side of (1.1) to be
 the same on the left, i.e., let f_1, \dots, f_n be arbitrary
 The set of all such functions is denoted by \mathcal{F} and also by \mathcal{F}_n .
 satisfy the equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot \dot{x}_i \right) = 0 \quad (1.2)$$

Thus, there are n linear equations in n unknowns and a unique solution exists. Differentiating (1.2) yields

$$\frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot \dot{x}_i \right) = 0 \quad (1.3)$$

where

$$\dot{x}_i = \frac{dx_i}{dt} \quad i = 1, \dots, n$$

Since, from equations (1.1) and (1.2), it follows that f_1, \dots, f_n are
 the upper limit of all solutions above and below f_1, \dots, f_n . Then

rearrangement gives

$$\frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot \dot{x}_i \right) = 0 \quad (1.4)$$

Setting $\frac{\partial f}{\partial t} = 0$ leads to

$$\sum_{n=1}^N \left[\sum_{m=1}^{\infty} g_{m-n+1} g_{m-k+1} \right] h_n = \frac{1}{\Delta y} \sum_{m=1}^{\infty} f_m g_{m-k+1} \quad (4.1-2)$$

where

$$k = 1, 2, \dots, N.$$

The sum on n stops at N since $h_n = 0$ for $n > N$. The limits on m in the other summations can also be restricted because of the conditions on the f_k and the g_k , but there is no advantage in doing so at this time.

Evidently, (4.1-2) represents N linear equations in the N unknowns h_n . Solution of these equations gives the function $h(x)$.

For simplicity of notation, let the coefficients of the h_n in equations (4.1-2) be a_{kn} , and let b_k represent the right-hand sides.

Then

$$\sum_{n=1}^N a_{kn} h_n = b_k, \quad (k = 1, 2, \dots, N), \quad (4.1-3)$$

where

$$a_{kn} \triangleq \sum_{m=1}^{\infty} g_{m-n+1} g_{m-k+1}, \quad (4.1-4)$$

$$b_k \triangleq \frac{1}{\Delta y} \sum_{m=1}^{\infty} f_m g_{m-k+1}. \quad (4.1-5)$$

Note that

$$a_{kn} = a_{nk} \quad (4.1-6)$$

(4.1-1)

$$\sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$

where

$$N = 1, 2, \dots, \infty$$

The sum on the right at $N = \infty$ is 1 . The limit on the right can also be obtained because of the condition on the

$\frac{1}{n}$ and the $\frac{1}{n+1}$ but there is no advantage in doing so at this time.

Obviously, (4.1-1) is a telescoping series in the n term.

known. Calculation of the sum is given by the formula (4.1-1).

For simplicity of notation, let the summation of the n term

equation (4.1-1) be S_N , and let the expression on the right side

Then

(4.1-2)

$$\sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$

where

(4.1-3)

$$\sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$

(4.1-4)

$$\sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$

Note that

(4.1-5)

$$\sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$

so that the matrix of the coefficients of equations (4.1-3) is symmetric. Further

$$a_{k+1, n+1} = \sum_{m=1}^{\infty} g_{m-n} g_{m-k}$$

$$= \sum_{m=0}^{\infty} g_{m-n+1} g_{m-k+1}$$

$$= \sum_{m=1}^{\infty} g_{m-n+1} g_{m-k+1},$$

or,

$$a_{k+1, n+1} = a_{kn} \quad (4.1-7)$$

When equations (4.1-3) are written in more complete form, the result is

$$\left. \begin{aligned} a_{11}h_1 + a_{12}h_2 + \dots + a_{1N}h_N &= b_1, \\ a_{21}h_1 + a_{22}h_2 + \dots + a_{2N}h_N &= b_2, \\ &\vdots \\ a_{N1}h_1 + a_{N2}h_2 + \dots + a_{NN}h_N &= b_N, \end{aligned} \right\} \quad (4.1-8)$$

where some of the a_{kn} may be zero. The solution (h_1, h_2, \dots, h_N) of the system of equations (4.1-8) is that set of h_n which fits the quantities

so that the matrix of the coefficients of equations (4.1-3) is symmetric

that is, further

$$a_{ij} = a_{ji} \quad i, j = 1, 2, \dots, n$$

$$a_{ii} = a_{ii} \quad i = 1, 2, \dots, n$$

$$a_{ij} = a_{ji} \quad i, j = 1, 2, \dots, n$$

or,

$$(4.1-4) \quad a_{ij} = a_{ji} \quad i, j = 1, 2, \dots, n$$

When equations (4.1-3) are written in normal form, the result is

which is

$$(4.1-5) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

where some of the a_{ij} may be zero. The set (4.1-5) is

the system of equations (4.1-5) in that case of n which also the quantities

$$\Delta y \sum_{n=1}^m \epsilon_{m-n+1} h_n$$

to the f_m in a least-squares sense. See equations (2.1-5).

4.2 The computer. Consider a network with $N + 1$ nodes or junctions numbered from 0 to N . Junction 0 is designated the reference junction and all potentials are referred to it. Let

$$\left. \begin{aligned} Y_{kn} &\triangleq (\text{admittance directly joining nodes } k \text{ and } n, k \neq n), \\ Y_{kk} &\triangleq (\text{sum of admittances directly connected to node } k), \\ E_n &\triangleq (\text{potential at node } n), \\ I_n &\triangleq (\text{current flowing into node } n \text{ from current sources}). \end{aligned} \right\} (4.2-1)$$

Then these quantities are related by the equations

$$\left. \begin{aligned} Y_{11}E_1 - Y_{12}E_2 - \dots - Y_{1N}E_N &= I_1, \\ -Y_{21}E_1 + Y_{22}E_2 - \dots - Y_{2N}E_N &= I_2, \\ \vdots & \\ -Y_{N1}E_1 - Y_{N2}E_2 - \dots + Y_{NN}E_N &= I_N, \end{aligned} \right\} (4.2-2)$$

where

$$Y_{kn} = Y_{nk}$$

by virtue of the definition in (4.2-1).²⁶

²⁶ Wilbur R. LePage and Samuel Seely, General Network Analysis, (New York: McGraw-Hill Book Company, Inc., 1952), pp. 47-49.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

to the π is a least-squares error. The expression (2.1-1) is

4.2 The computer. Consider a network with N nodes or junctions

labeled numbered from 0 to N . Junction 0 is designated as reference

junction and all potentials are referred to it. Let

$$\left. \begin{aligned} Y_{kn} &= \text{admittance directly joining nodes } k \text{ and } n \\ Y_{kr} &= \text{sum of admittances joining node } k \text{ to reference node } r \\ E_n &= \text{potential of node } n \\ I_n &= \text{current flowing into node } n \text{ from an external source} \end{aligned} \right\} \quad (2.2-1)$$

Then these quantities are related by the equations

$$\left. \begin{aligned} Y_{11}V_1 + Y_{12}V_2 + \dots + Y_{1N}V_N &= -I_1 \\ Y_{21}V_1 + Y_{22}V_2 + \dots + Y_{2N}V_N &= -I_2 \\ &\vdots \\ Y_{N1}V_1 + Y_{N2}V_2 + \dots + Y_{NN}V_N &= -I_N \end{aligned} \right\} \quad (2.2-2)$$

where

$$Y_{kn} = \frac{1}{Z_{kn}}$$

by virtue of the definition in (2.1-1).

Comparison of equations (4.1-8) with equations (4.2-2) shows that if a network is constructed in which

$$\left. \begin{aligned} Y_{kn} &= ja_{kn}, (k \neq n), \\ Y_{kk} &= -ja_{kk}, \\ I_n &= b_n, \end{aligned} \right\} \quad (4.2-3)$$

then

$$E_n = jh_n \quad (4.2-4)$$

That is, the voltages at the various nodes give the function $jh(x)$.

This network is the computer.

It is important to note that the network can only be used to solve equations which have a symmetric matrix since the network equations (4.2-2) have a symmetric matrix, i.e., since $Y_{kn} = Y_{nk}$. The equations (4.1-8) satisfy this condition since $a_{kn} = a_{nk}$. Further, in using the network to solve equations (4.1-8), the fact that $a_{k+1, n+1} = a_{kn}$ reduces the total number of different admittances needed. That is, since

$$Y_{11} = Y_{22} = \dots = Y_{N-1, N-1} = Y_{NN},$$

$$Y_{12} = Y_{23} = \dots = Y_{N-1, N},$$

⋮

$$Y_{1, N-1} = Y_{2N},$$

Comparison of equations (4.1-5) with equations (4.1-2) shows

that if a network is constructed in which

$$(4.1-3) \quad \begin{cases} Y_{11} = \frac{1}{Z_{11}} \\ Y_{12} = -\frac{1}{Z_{12}} \\ Y_{21} = -\frac{1}{Z_{21}} \\ Y_{22} = \frac{1}{Z_{22}} \end{cases}$$

then

$$(4.1-4) \quad Y_{11} = \frac{1}{Z_{11}}$$

That is, the voltages at the various nodes give the function $Y(x)$.

This network is the equivalent.

It is important to note that the network can only be used to

solve equations which have a symmetric matrix since the network equa-

tions (4.1-2) have a symmetric matrix, i.e., since $Y_{12} = Y_{21}$. The

equations (4.1-3) satisfy this condition since $Y_{12} = Y_{21}$. Therefore, in

using the network to solve equations (4.1-3), the first two equations

reduce the total number of elements of the matrix to be solved. This is

since

VICTORIA BOND

$$Y_{11} = \frac{1}{Z_{11}}, \quad Y_{12} = -\frac{1}{Z_{12}}, \quad Y_{21} = -\frac{1}{Z_{21}}, \quad Y_{22} = \frac{1}{Z_{22}}$$

$$Y_{12} = Y_{21} = -\frac{1}{Z_{12}}$$

U.S.A.

VALLEY PARK, TENNESSEE

$$Y_{11} = \frac{1}{Z_{11}}, \quad Y_{12} = -\frac{1}{Z_{12}}, \quad Y_{21} = -\frac{1}{Z_{21}}, \quad Y_{22} = \frac{1}{Z_{22}}$$

the admittances which are equal can perhaps all be adjusted simultaneously to the desired value by turning one shaft.

Comparison of equations (4.1-4) and (4.1-5), which define the coefficients a_{kn} and the terms b_k , with equations (2.1-5) indicates considerable similarity. In fact, a few minor changes of notation show that the computer of Chapter III may be used to compute the a_{kn} and the b_k .

4.3 Application of computer to a system of equations. Arbitrary values are assigned to the a_{kn} and the b_k in equations (4.1-8) and N is given the value 4 to yield the equations

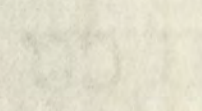
$$\begin{aligned} 49h_1 + 30h_2 + 6h_3 &= 347, \\ 30h_1 + 49h_2 + 30h_3 + 6h_4 &= 363, \\ 6h_1 + 30h_2 + 49h_3 + 30h_4 &= 266, \\ 6h_2 + 30h_3 + 49h_4 &= 85. \end{aligned} \tag{4.3-1}$$

The condition of symmetry required by equations (4.1-6) and the condition of equations (4.1-7) that makes the coefficients the same along any diagonal (downward, left to right) have been met.

Now consider the circuit of Fig. 4.3-a. It is described by the equations

$$\begin{aligned} -j49E_1 - j30E_2 - j6E_3 &= 347, \\ -j30E_1 - j49E_2 - j30E_3 - j6E_4 &= 363, \\ -j6E_1 - j30E_2 - j49E_3 - j30E_4 &= 266, \\ -j6E_2 - j30E_3 - j49E_4 &= 85. \end{aligned}$$

the admittances which are equal can be written as follows:
 transversely to the desired value by setting the limit.
 Comparison of equations (4.1-1) and (4.1-2) will show the
 coefficients a_m and the terms b_m with equations (4.1-1) identical
 considerably similarly. In fact, a further change of notation
 show that the computer of Chapter III may be used to compute a_m



a_m and the b_m .
 4.3 Application of computer to a system of equations (4.1-1)

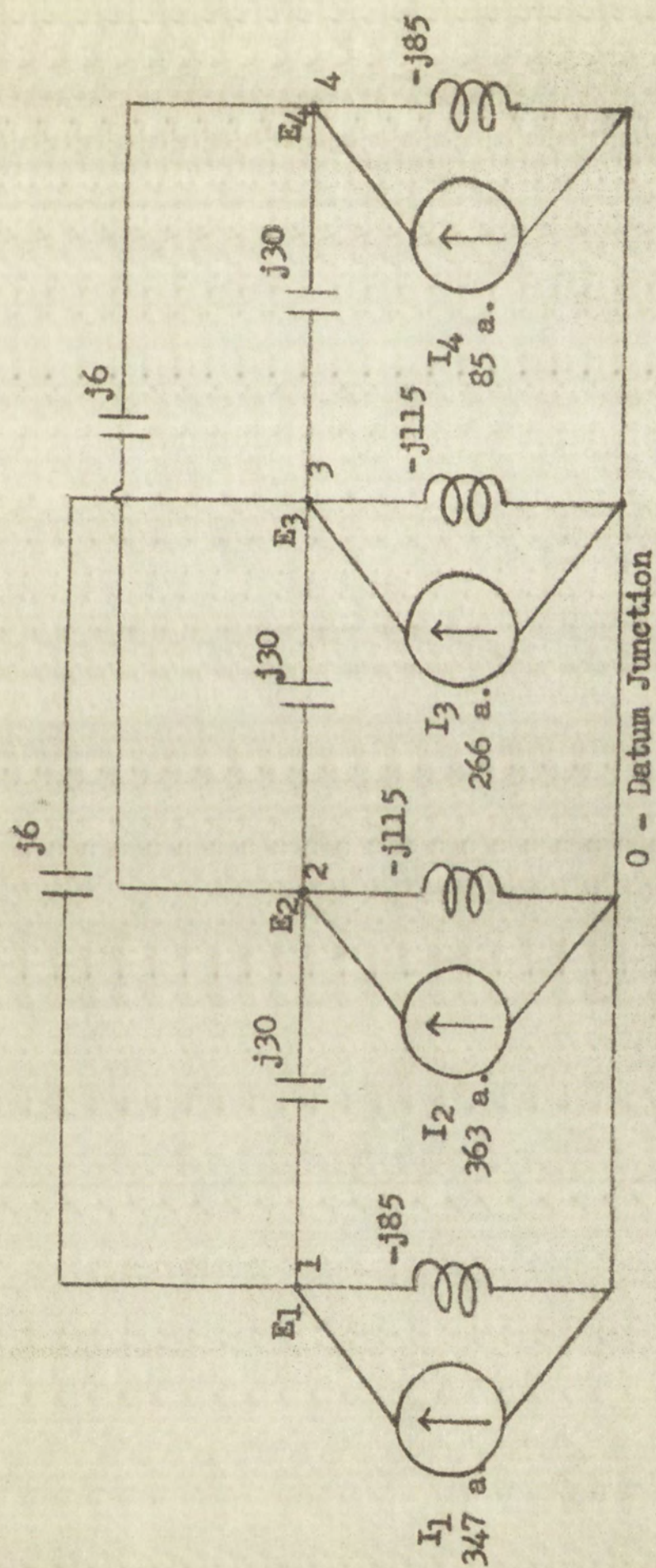
Cray values are assigned to the a_m and the b_m in equations (4.1-1)
 and N is given the value 4 to yield the equations

$$\begin{aligned} 4a_1 + 3a_2 + a_3 &= 34 \\ 3a_1 + 4a_2 + 3a_3 + a_4 &= 30 \\ a_1 + 3a_2 + 4a_3 + 3a_4 &= 24 \\ 3a_2 + 3a_3 + 4a_4 &= 0 \end{aligned} \quad (4.1-3)$$

The condition of symmetry required by equations (4.1-1) and the condition
 of equations (4.1-3) that makes the coefficients the same along any di-
 agonal (downward, left to right) have been met.
 Now consider the circuit of Fig. 4.1-4. It is described by the

equations

$$\begin{aligned} -4a_1 - 3a_2 - 3a_3 &= 34 \\ -3a_1 - 4a_2 - 3a_3 - a_4 &= 30 \\ -3a_1 - 3a_2 - 4a_3 - 3a_4 &= 24 \\ -3a_2 - 3a_3 - 4a_4 &= 0 \end{aligned}$$



Elements specified in mhos.

Figure 4.3-a

Convolution Integral Equation Computer

It is evident that these equations are identical to equations (4.3-1) provided

$$E_1 = jh_1,$$

$$E_2 = jh_2,$$

$$E_3 = jh_3,$$

$$E_4 = jh_4.$$

This relationship between the E_n and the h_n is that of equation (4.2-4). The circuit of Fig. 4.3-a was derived, of course, from equations (4.3-1) through the relations (4.2-3). Note that, since

$$Y_{11} = Y_{10} + Y_{12} + Y_{13},$$

then

$$Y_{10} = Y_{11} - Y_{12} - Y_{13}.$$

The other Y_{kk} are found similarly.

This example illustrates the fact that only inductive and capacitive elements are required, provided the coefficients a_{kn} in equations (4.1-8) are real. Further, all the current generators have the same phase (or differ by exactly π radians) provided the b_k are real. (Both the a_{kn} and the b_k are real as they have been defined in equations (4.1-4) and (4.1-5).) For convenience, all quantities may be suitably scaled so that they can more readily be realized. Moreover, a frequency can be chosen that brings the capacitors and inductors into a reasonable range.

For some applications it is known that the function $h(x)$ is everywhere positive; in such cases, the potentials E_n need only be read

It is evident that these operations are identical to equation (4.1-1)

provided

$$Y_1 = Y_2$$

$$Y_1 = Y_2$$

$$Y_1 = Y_2$$

$$Y_1 = Y_2$$

This relationship between Y_1 and Y_2 is the case of equation (4.1-1).

The circuit of Fig. 4.1-2 was derived, in contrast, from equation (4.1-1)

through the relations (4.1-2). However, since

$$Y_1 = Y_2 + Y_3$$

then

$$Y_1 = Y_2 + Y_3$$

The other Y_k are found similarly.

This example illustrates the fact that only linear and constant

elements are required, provided the coefficients a_k in equation

(4.1-3) are real. Further, all the circuit elements have the same phase

(or differ by exactly π radians) provided a_k are real. (Both a_k

and a_k^* are real as they have been defined in equation (4.1-1).

and (4.1-2). For convenience, all quantities may be suitably scaled

so that they can more readily be realized. Moreover, a frequency can

be chosen that brings the circuit elements and admittances into a reasonable

range.

For some applications it is known that the function $H(\omega)$ is

everywhere positive. In such cases, the function $H(\omega)$ need only be real

for magnitude. In others, $h(x)$ may take on both positive and negative values. In this case, it will be necessary to read phase and amplitude of the E_n . This should not be difficult since all of the E_n either have the same phase or differ by exactly π radians.

The components that are selected for the computer network should have low losses. For exact results, the admittances should be purely capacitive or purely inductive. It is easy to obtain low-loss capacitors. It is considerably more difficult to obtain low-loss inductors; however, if the ratio $\omega L/R$ for the inductors is 100 or more, the accuracy of the results should be better than $\pm 1\%$.

In order to prepare the computer for various problems, it is necessary to be able to adjust the Y_{kn} . Since

$$Y_{k+1, n+1} = Y_{kn},$$

it should be possible to link the adjusting mechanism in some way so that a single setting would fix several admittances at their proper value. It is reasonably easy to calibrate capacitors and put several on a common shaft so that they all have nearly equal values at any setting. It is more difficult to do so with inductors; for that reason, another method of varying admittances is discussed in the next section.

4.4 Admittance conversion; stability. Consider the circuit of Fig. 4.4-a. The triangular symbol indicates an amplifier with infinite input impedance and an amplification $A(j\omega)$. Due to the infinite input

for magnitude. In other, $k(x)$ may have a non-zero value and may be
negative values. In this case, it will be necessary to use a sign
code at the x . This would not be a problem if the sign code
have the same phase or differ by exactly π radians.

The components that are selected for the output should
have low losses. For exact results, the output should be purely
capacitive or purely inductive. It is easy to obtain low-loss
cores. It is considerably more difficult to obtain low-loss
however, if the ratio μ/ν for the laminations is too small, the
ratio of the results should be better than 1.

In order to prepare the computer for various problems, it is
necessary to be able to adjust the x value.

$$Y_{k+1} = Y_k + \Delta Y_k$$

It should be possible to link the adjusting mechanism in some way so
that a single setting would fix several adjustments in their proper
value. It is reasonably easy to adjust capacitors and the way to
on a common shaft so that they all have nearly equal values at any
setting. It is more difficult to do so with inductors. For this reason,
another method of varying inductance is discussed in the next section.

4.4 Inductance conversion; auxiliary. Consider the circuit of
Fig. 4-4-a. The triangular symbol indicates a variable inductor
input impedance and an amplification factor (μ) . The value of μ is

VALLEY PAPER

4-4-a

WILSON CAR

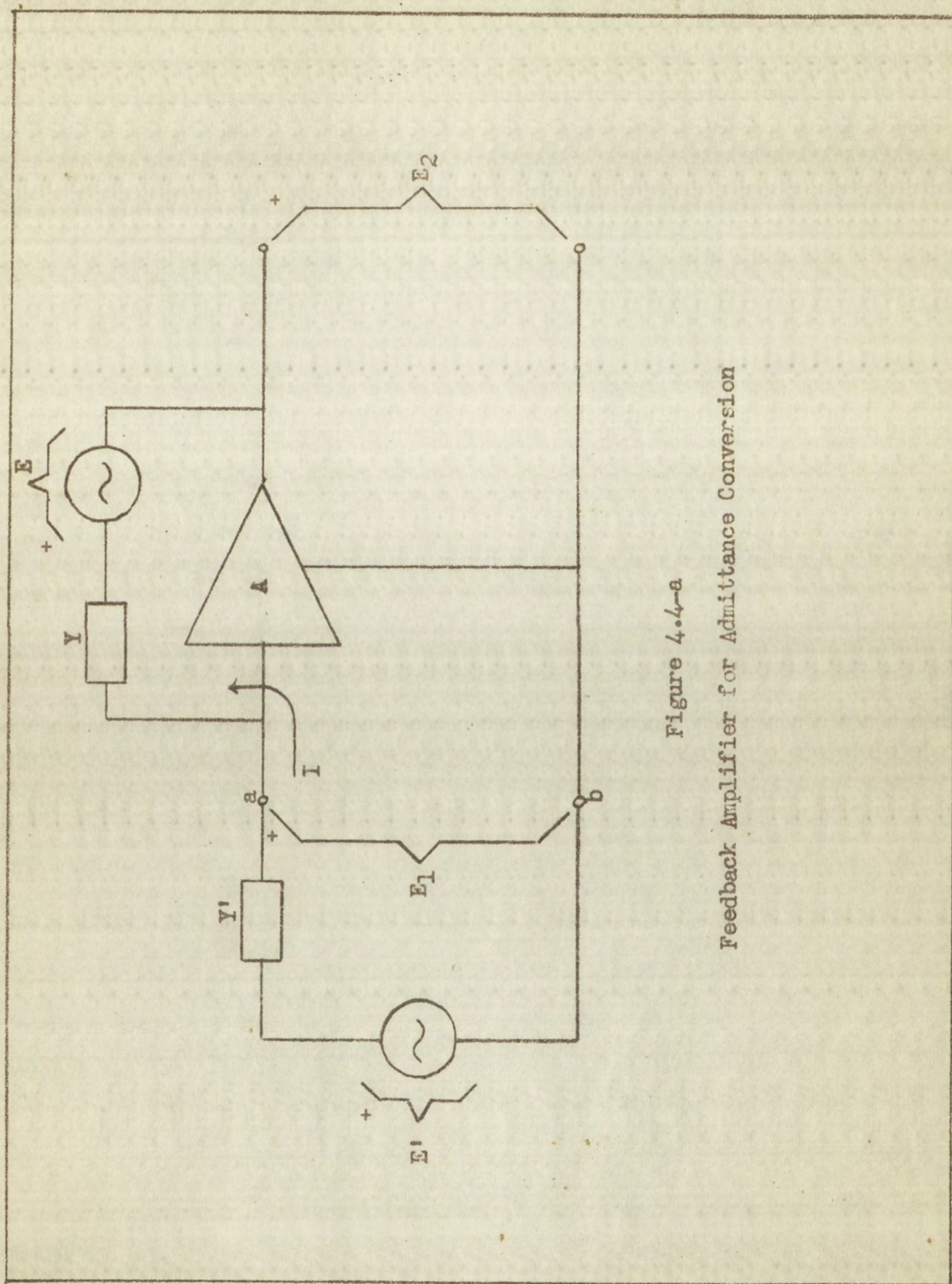
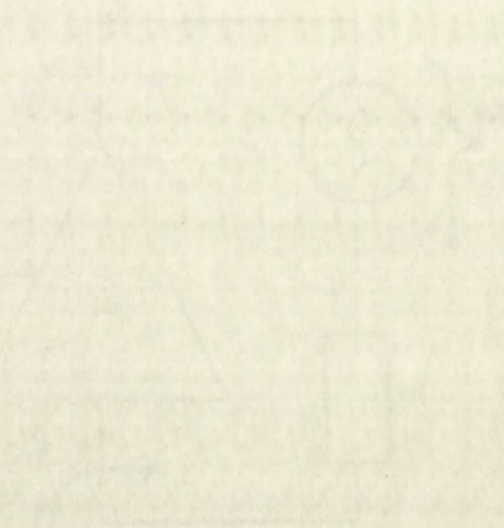


Figure 4.4-a
Feedback Amplifier for Admittance Conversion



impedance, no current flows into the amplifier but takes the path through the admittance Y and the emf E as indicated. From these facts and Fig. 4.4-a, it follows that

$$\left. \begin{aligned} I &= [E_1 - E - E_2] Y, \\ I &= [E' - E_1] Y', \\ E_2 &= AE_1. \end{aligned} \right\} \quad (4.4-1)$$

The voltages E' , E , E_1 , and E_2 , the current I , the amplification A , and the admittances Y and Y' are all functions of $j\omega$ and, hence, are complex.

In order to find an equivalent circuit for that part of Fig. 4.4-a to the right of terminals a-b, the first and third of equations (4.4-1) may be solved to obtain

$$I = E_1 Y [1 - A] - EY. \quad (4.4-2)$$

This equation may be represented by the circuit of Fig. 4.4-b and is the desired equivalent circuit since, for the same input voltage and current, it is described by the same equation as the original circuit.

Since feedback is used in the circuit of Fig. 4.4-a, the question of stability arises. To examine this point, equations (4.4-1) are solved to find the dependence of $E_2(j\omega)$ on $E'(j\omega)$ and $E(j\omega)$. The result is

$$E_2(j\omega) = \frac{A[Y'E' + YE]/[Y + Y']}{1 - YA/[Y + Y']}, \quad (4.4-3)$$

where, of course, the quantities A , E , E' , Y , and Y' are all functions

impedance, no current flows into the amplifier and across the load through the admittance Y and the load is in parallel. From Fig. 4.4-2 and Fig. 4.4-3, it follows that

$$\begin{cases} I = I_1 + I_2 \\ I = I_1 + I_2 \\ I = I_1 + I_2 \end{cases} \quad (4.4-1)$$

The voltages E_1 , E_2 , and E_3 , the current I , the admittance Y , and the admittances Y_1 and Y_2 are all functions of the input voltage and frequency. In order to find an equivalent circuit for the circuit of Fig. 4.4-2

to the right of terminals $a-b$, the first and third of equations (4.4-1) may be solved to obtain

$$I = E_1 Y_1 + E_2 Y_2 - E_3 Y \quad (4.4-2)$$

This equation may be represented by the circuit of Fig. 4.4-4 and is the desired equivalent circuit. For the same input voltage and current, it is described by the same equation as the original circuit. Since feedback is used in the circuit of Fig. 4.4-2, the question of stability arises. To examine this point, equations (4.4-1) are solved to find the dependence of E_3 on E_1 and E_2 and (4.4-2) is solved to find the dependence of I on E_1 and E_2 . The result is

$$E_3 = \frac{Y_1 Y_2}{Y_1 + Y_2 + Y} E_1 \quad (4.4-3)$$

where, of course, the admittances Y_1 , Y_2 , and Y are all functions

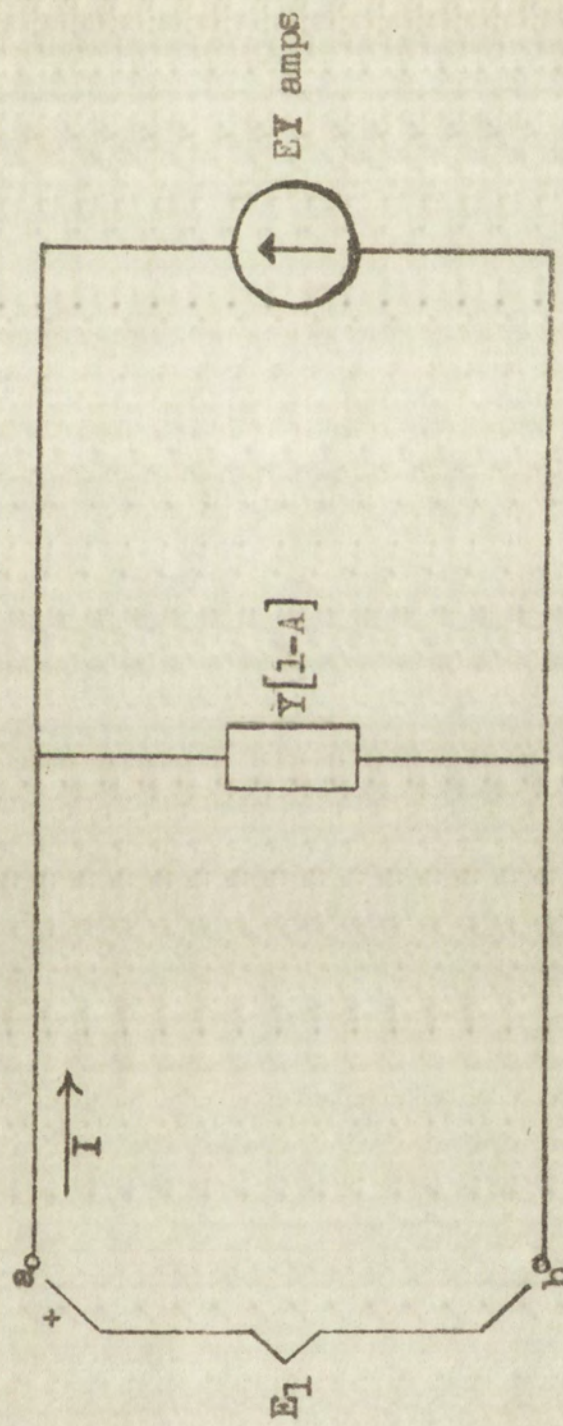
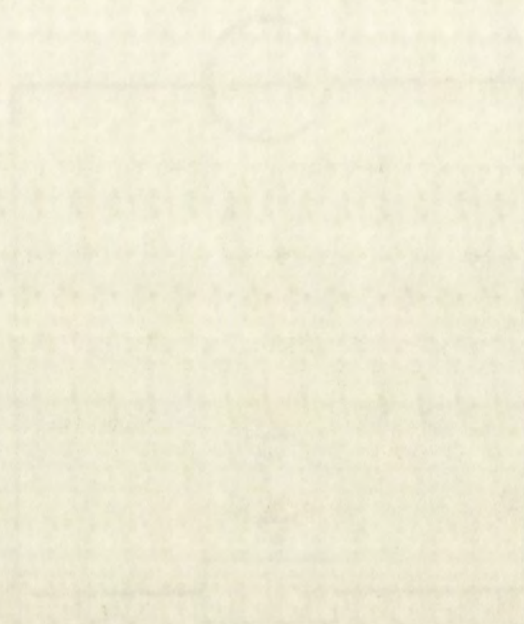


Figure 4.4-b
Current-Source Equivalent for Figure 4.4-a



of $j\omega$. In equation (4.4-3), if $s = \sigma + j\omega$ replaces $j\omega$, the function $E_2(s)$ is the Laplace transform of the time function of the output voltage of Fig. 4.4-a, for quiescent initial conditions. For such a circuit to be stable, $E_2(s)$ must not have any poles in the right half of the s -plane.

It is reasonable to assume that the numerator, i.e., the quantity above the main fraction bar, of equation (4.4-3) does not have poles in the right half of the s -plane. That is, the amplifier without feedback is presumed stable, the bracket involving E' and E is without poles for stable values of these voltages, and the quantity $Y + Y'$ does not have zeros in the right half-plane. Hence, the poles of $E_2(s)$ are the zeros of the denominator of equation (4.4-3). To determine whether or not such zeros exist, the quantity $-YA/[Y + Y']$ is plotted on the complex plane as a function of $s = j\omega$ (i.e., with $\sigma = 0$). If the point $-1 + j0$ is not encircled as ω varies from $-\infty$ to $+\infty$, the denominator of (4.4-3) does not have zeros in the right half of the s -plane. Usually, if

$$\operatorname{Re} \left[\frac{YA}{Y + Y'} \right] < 0,$$

the system will be stable. In a later section, consideration will be given to systems that may be unstable.

It is evident from Fig. 4.4-b that the portion of Fig. 4.4-a to the right of terminals a-b can be represented by a current source EY and a shunt admittance $Y [1 - A]$. Hence, it is possible to have an input admittance which differs from the feedback admittance by the factor $1 - A$.

For admittance conversion only, it may be convenient to remove the source E .

4.4(a) Capacitance conversion. In Fig. 4.4-b, let

$$Y = j\omega C$$

so that

$$Y_{in} \triangleq E_1/I = j\omega [1 - A] C.$$

Thus, if A is real, the input admittance is a pure capacitance whose value is determined by $1 - A$. Then, if A can be precisely controlled, a wide range of values of capacitance can be obtained with only one fixed capacitor. Methods of controlling A will be considered later. To increase input capacitance over C , $A < 0$; to decrease it, $0 < A < 1$.

4.4(b) Inductance conversion. In Fig. 4.4-b, let

$$Y = 1/j\omega L$$

so that

$$Y_{in} \triangleq E_1/I = \frac{1}{j\omega L [1 - A]}.$$

Then, if A is real, the input admittance is a pure inductance. To make the input inductance greater than L , $0 < A < 1$; to make it less, $A < 0$.

4.5 Precise control of amplification. The circuit of Fig. 4.5-a is the same as that of Fig. 4.4-a with E_a replacing E' , R_1 replacing $1/Y'$, R_2 replacing $1/Y$, α replacing A , and E removed. Hence, from equation (4.4-3), the resultant amplification of the circuit of Fig. 4.5-a is

VICTORIA EDWIN

For a complete description of the system, see the report of the

author.

APPENDIX A

4.1 (a) Generalized form of the system.

Y = 1/X

so that

$$Y_{in} = \frac{1}{1 - \frac{1}{X}}$$

Thus, if X is real, the input admittance is a pure capacitance with a value is determined by $1/X$. Then, if X is complex, the input admittance is a wide range of values of capacitance can be obtained with a fixed capacitor. Methods of controlling X will be considered later. To increase input capacitance over X , $0 < X < 1$.

4.1 (b) Inductance conversion. The input admittance is

$$Y = 1/X$$

so that

$$Y_{in} = \frac{1}{1 - \frac{1}{X}}$$

Then, if X is real, the input admittance is a pure inductance. To obtain the input inductance greater than X , $0 < X < 1$, to make X less than X .

4.2 Positive control of admittance. The circuit of Fig. 4.2 is

is the same as that of Fig. 4.1-a with X replaced by X' .

X' , R_2 replacing X , a negative X and R_2 removed. Hence, the admittance

tion (4.1-3), the resultant admittance of the circuit of Fig. 4.2 is

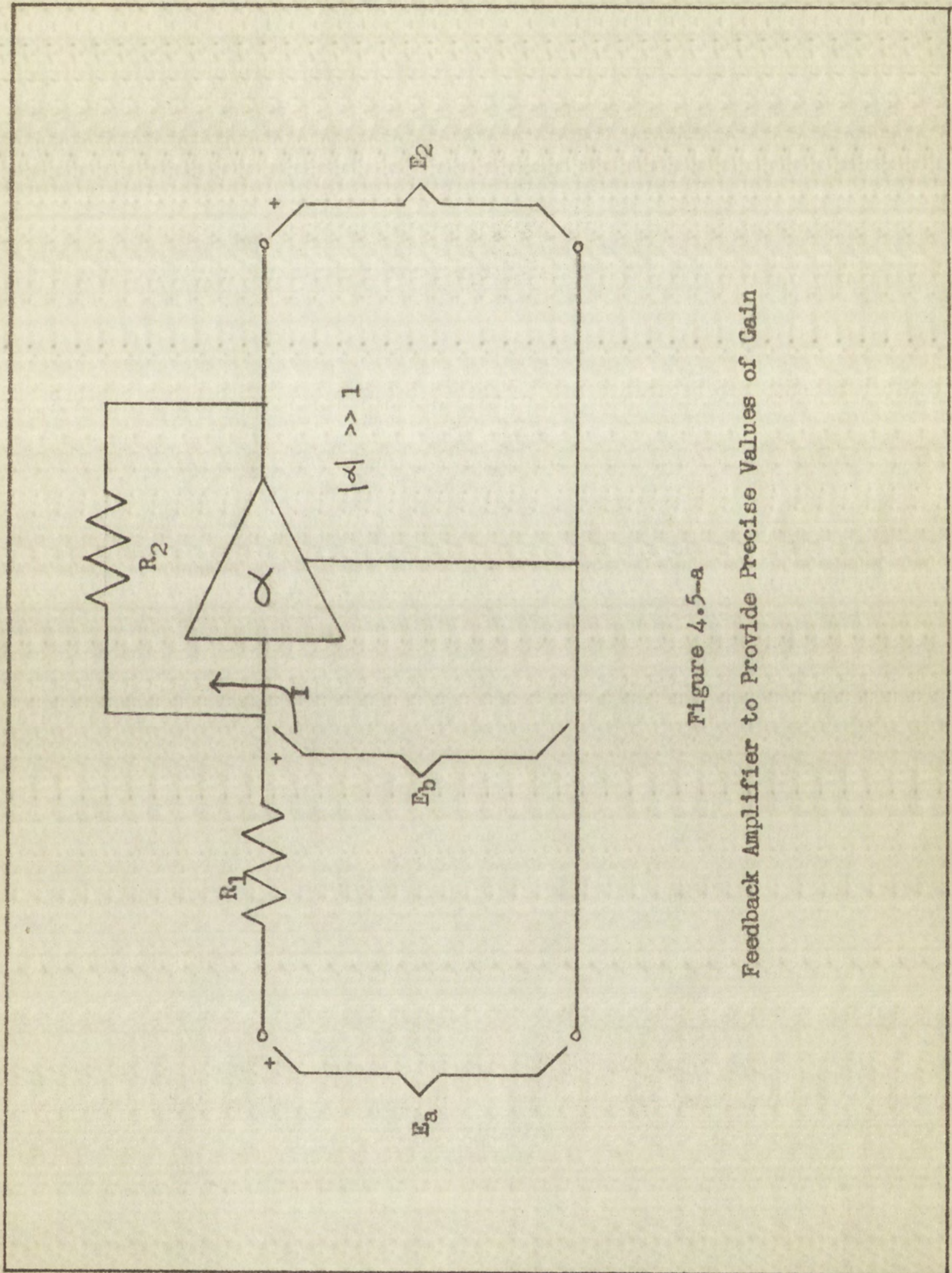
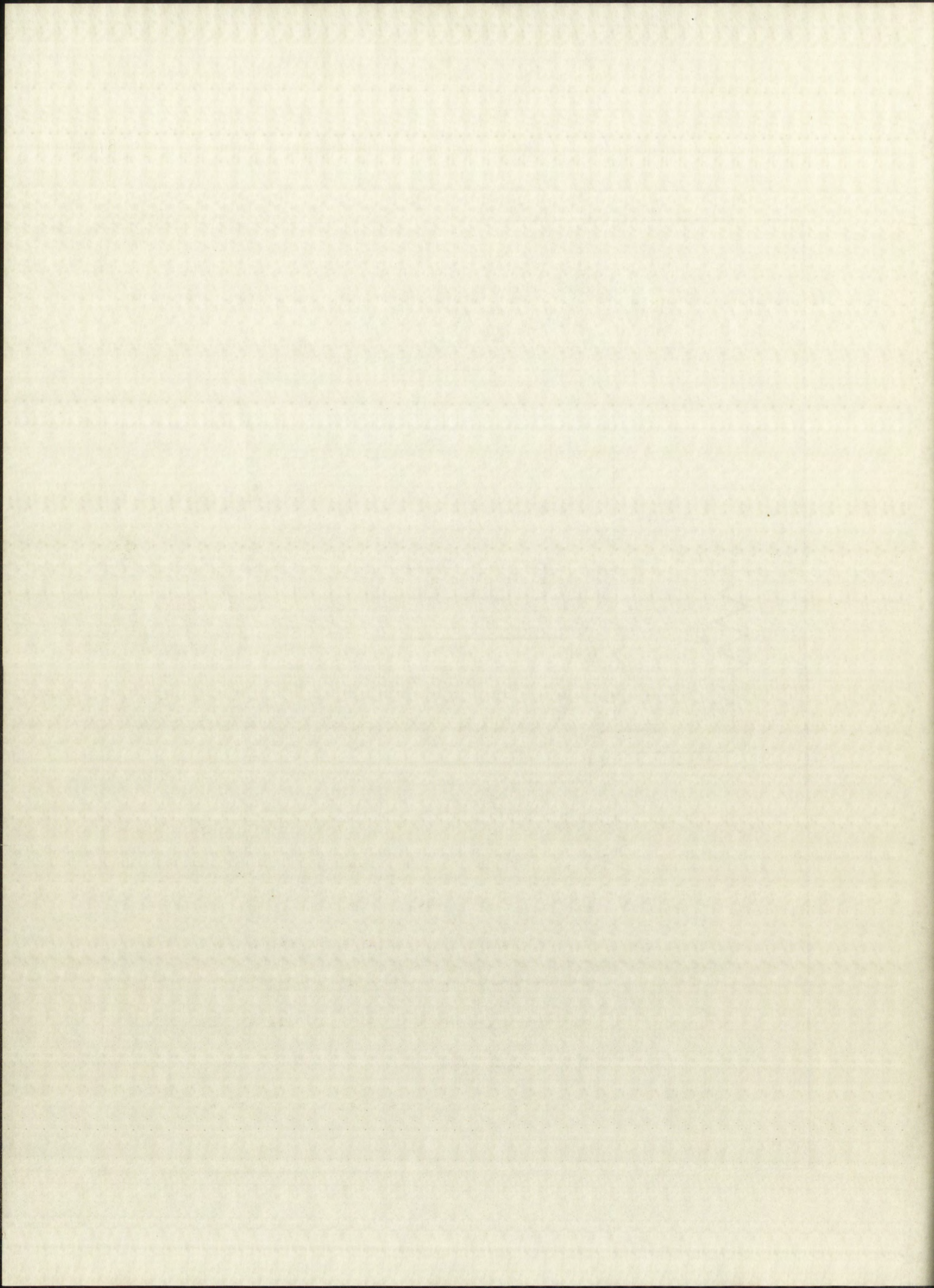


Figure 4.5--a
Feedback Amplifier to Provide Precise Values of Gain



$$A \triangleq E_2/E_a = \frac{\alpha R_2}{[1-\alpha] R_1 + R_2} . \quad (4.5-1)$$

Now if α is real, $\alpha < 0$, $|\alpha| \gg 1$, and $|\alpha R_1| \gg R_2$, equation (4.5-1) becomes

$$A = -R_2/R_1 .$$

Since there are inexpensive, commercially-available amplifiers with $\alpha = -20,000$ over wide frequency ranges, the conditions on α are easy to meet. Hence, the resultant amplification A can be controlled by external precision resistors R_1 and R_2 . These feedback circuits can be used to obtain the required values of A to convert capacitance and inductance. To obtain positive values of amplification, two such circuits can be placed in cascade. Note that the overall circuit will have to be isolated from the input in order to give the required extremely high input impedance.

4.6 Computer circuit using negative conductances; instability.

It was originally intended that the computer network should consist of only real elements, i.e., of conductances so that d-c could be used. This required negative conductances to represent certain coefficients. To obtain them, a circuit such as Fig. 4.4-a was used with $Y' = G'$ and $Y = G$. Figure 4.4-b shows that A must be real and greater than unity at $\omega = 0$ in order that $Y_{in} = G_{in} < 0$.

To satisfy the conditions on A , a two-stage amplifier may be assumed, with A having the form

$$A(j\omega) = A_0 / [1 + j\omega T]^2, \quad A_0 > 1.$$

Then, from equation (4.4-3), if the quantity

$$- \frac{GA_0}{[1 + j\omega T]^2 [G + G']},$$

when plotted on the complex plane, does not encircle the point $-1 + j0$ as ω varies from $-\infty$ to $+\infty$, the circuit will be stable. This plot is shown in Fig. 4.6-a. It is clear that the point $-1 + j0$ will not be encircled if

$$GA_0 / [G + G'] < 1,$$

that is, if

$$-G [1 - A_0] < G'. \quad (4.6-1)$$

The inequality (4.6-1) states that the conductance connected across terminals a-b in Fig. 4.4-a must be greater than the absolute value of the input conductance seen at the terminals. In terms of Fig. 4.4-b, it states that the conductance seen by the current generator must be positive. (In Figs. 4.4-a and 4.4-b, the admittances Y and Y' should be replaced by G and G' insofar as this section is concerned.)

When the feedback amplifier was used in the computer network to provide negative conductances and current generators, it developed that only a very limited class of equations could be solved since, for most systems of equations, at least one of the current generators would see a negative conductance, and, hence, the circuit would be unstable.

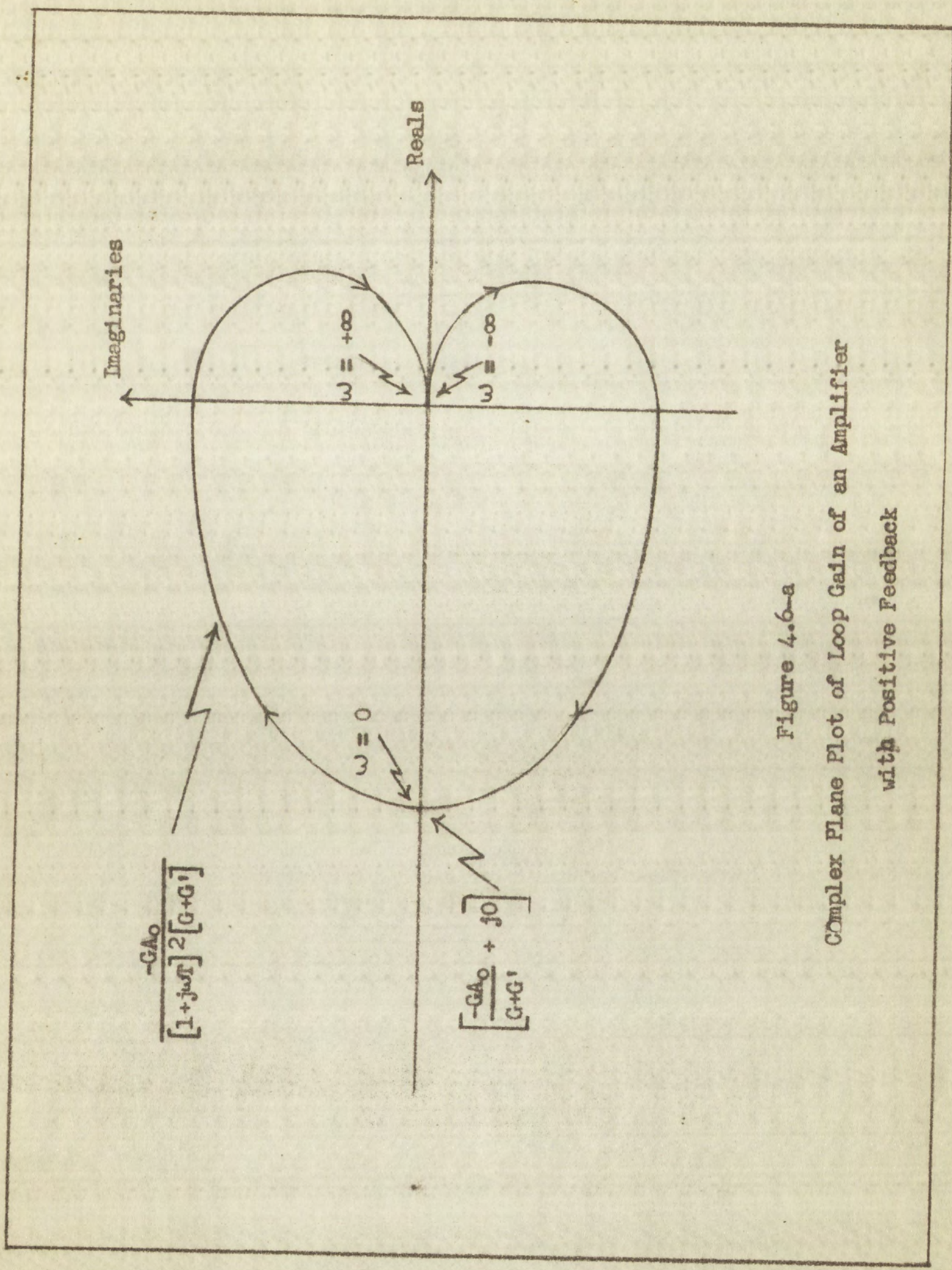
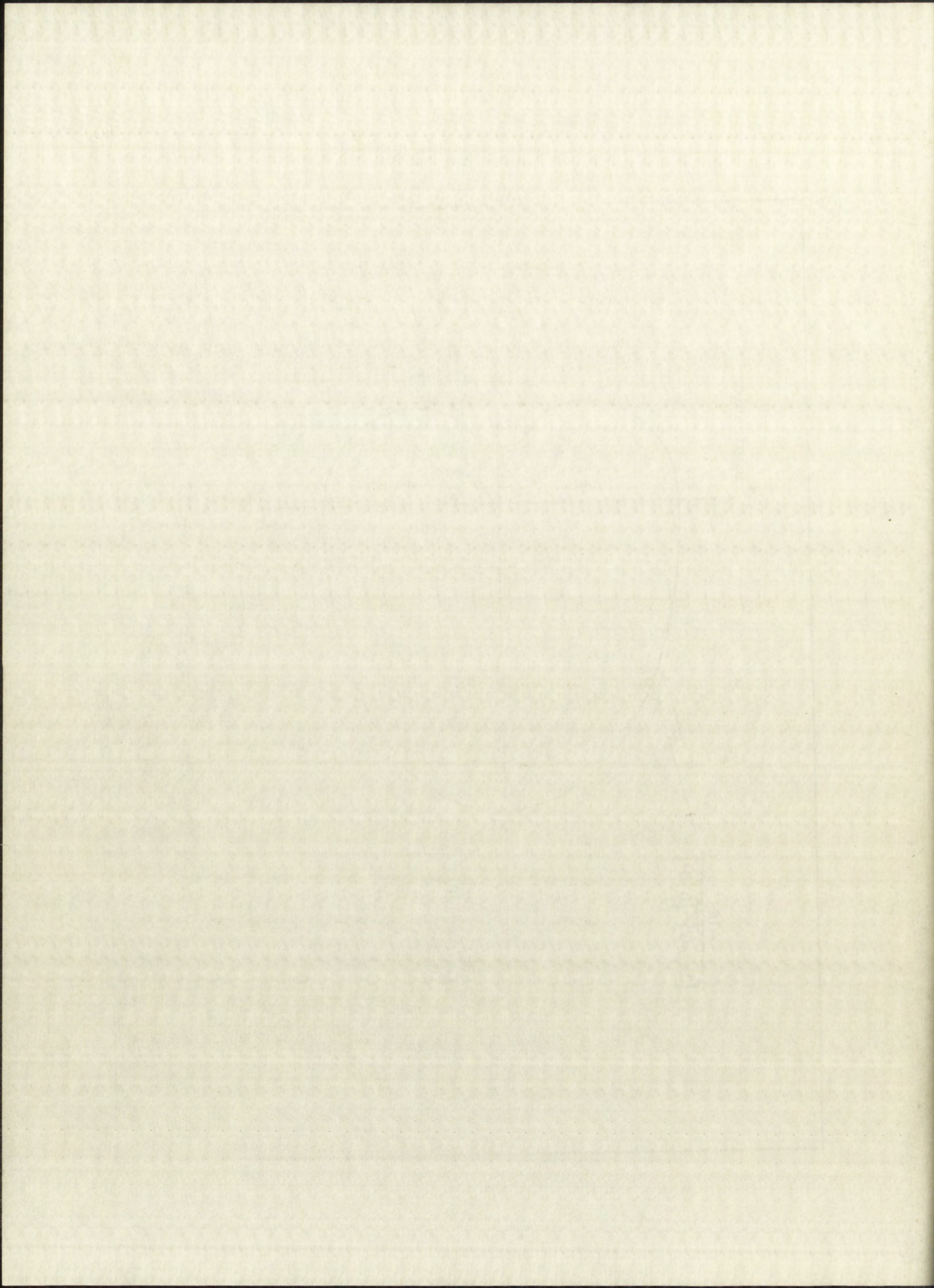


Figure 4.6-a
Complex Plane Plot of Loop Gain of an Amplifier
with Positive Feedback



CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary. A rapidly growing need for speedy methods of analyzing linear systems has led to the development of various computational devices to evaluate the convolution integral. These same devices will also find the correlation function of a pair of functions; in fact, a number of them were designed for this purpose due to the increasing application of correlation to the study of communication systems. Many of these computers evaluate the integral on a continuous basis. A few, however, use a numerical approximation to the integral; the convolution computer described in this paper does so, and the method of approximating the integral is described in an early chapter.

The convolution computer depends on a switch which allows the delay to be advanced (or retarded). The functions are entered as voltages and conductances; a scheme is described that makes it possible to have both negative and positive values of the functions. The value of the integral is read at the output as a voltage; there is a value for each delay, or switch position.

It appears that no computers have been advanced which will find one of the functions of the convolution integral, when the other function and the integral itself are known, except the computer proposed

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary. A rapidly growing need for means of analyzing linear systems has led to the development of several automatic devices to evaluate the convolution integral. These devices will also find the correlation function of a pair of functions; in fact, a number of them were designed for this purpose due to the increasing application of correlation to the study of communication systems. Many of these computers evaluate the integral of a continuous basis. A few, however, use a numerical approximation to the integral; the convolution computer described in this paper does so, and the method of approximation is described in an early chapter.

The convolution computer described is a device which allows the delay to be advanced (or retarded). The functions are entered as voltages and conductances; a current is generated and measured in a circuit to have both negative and positive values for the functions. The value of the integral is read as the current as a voltage; there is a value for each delay, or twice per delay.

It appears that no computer has been described which will find one of the functions of the convolution integral, with the other function and the integral itself are known, except the computer proposed.

by Hessemer and Williams²⁷ and a graphical scheme which does not give direct results. Such a computer has as wide a usage as the convolution computer itself since the problem of determining the input, given the output and system response, arises as often as that of finding the output. It has the further application of finding the system response with any input and resultant output; this obviates impulse or step inputs--often difficult to obtain--in finding system response.

The computer described herein in Chapter IV does the job described in the preceding paragraph, but its method is entirely different from that used by the computer proposed by Hessemer and Williams. The computer is quite simple and could readily be constructed from commercial components.

5.2 Conclusions. The work done by Smith²⁸ leaves no doubt about the ability of the convolution computer that is described in this paper to perform with an accuracy of $\pm 1\%$. However, the accuracy of the results depend on the number of ordinates that are used to represent the functions within the integral. Each ordinate requires a switch contact, and this is the limitation of the computer. Nonetheless, since the straightforward operation of convolution does not require fast switch operation, it should not be difficult to fabricate a switch with the order of 100 contacts. Such a convolution computer should be appreciably cheaper than most of those which have been

²⁷ Hessemer and Williams, op. cit., p. 10.

²⁸ Smith, loc. cit.

by Hassemer and Williams, and a graphical scheme which shows the
direct results. Such a scheme is shown in Figure 1. The
computer itself, since the function of determining the input, gives
the output and a system response, which is shown as the output of the
output. It has the function of determining the system response
with any input and response, and the system response of any
input-output difference is shown in Figure 2 as a system response.

The computer described herein in Chapter IV has the func-
tion of the preceding paragraph, and its method is entirely different
ent from that used by the computer described by Hassemer and Williams.
The computer is quite simple and could readily be constructed from
commercial components.

5.2 Conclusions. The work done by Hassemer and Williams in their
about the ability of the convolution computer was as described in
this paper to perform with an accuracy of 1%. However, the accuracy
of the results depend on the number of samples that are used to re-
present the functions within the integrals. Such a finite representation
switch concept, and this is the limitation of the computer. It is
less, since the straightforward operation of convolution does not
require fast switch operation, as would be the case with a convolution
a switch with the order of 100 contacts. Such a convolution computer
should be especially designed to be used with such a switch.

27. Hassemer and Williams, op. cit., p. 10.

28. Smith, loc. cit.

devised, and it would perform convolution and correlation with dispatch and accuracy.

The need for solving the convolution integral equation for a function of the integrand led to the computer that is proposed herein in Chapter IV. Since it does only that task, it is less versatile than the Hessemer-Williams computer which also performs convolutions. However, it finds the solution directly without the minimizing process required by the other.

devised, and it would permit conversion and correlation with the

pages and accuracy.

The need for a reliable and accurate method of

function of the integrated has to be compared with a proposed method

in Chapter IV. Since it does not only the same, it is less variable

than the Resonance-Width method which also is less variable.

However, it finds the relation between the minimum and

reported by the other.

BOUND

BIBLIOGRAPHY

RECEIVED

BIBLIOGRAPHY

- Ahlin, J. T., and E. V. Laitone, "A Simple Graphical Solution of the Duhamel Integral," Journal of the Aeronautical Sciences, Vol. 18, Feb., 1951, pp. 142-143.
- Barnes, John L., and Murray F. Gardner, Transients in Linear Systems, New York: John Wiley & Sons, Inc., 1942, 389 pp.
- Bell, Harold, Jr., and V. C. Rideout, "A High-Speed Correlator," Transactions of the I.R.E., Professional Group on Electronic Computers, Vol. EC-3, No. 2, June, 1954, pp. 30-36.
- Brown, G. S., and H. L. Hazen, "The Cinema Integrator," Journal of the Franklin Institute, Vol. 230, July, 1940, pp. 19-44, and Vol. 230, Aug., 1940, pp. 183-205.
- Davis, M. R., T. O. Ellis, and L. L. Grandi, "A Machine for Computing the Convolution Integral," Conference Paper No. CP 55-515 of the American Institute of Electrical Engineers, 1955, 10 pp.
- Ellis, T. O., M. R. Davis, and L. L. Grandi, (see under Davis, M. R.).
- Gardner, Murray F., and John L. Barnes, (see under Barnes, John L.).
- Grandi, L. L., T. O. Ellis, and M. R. Davis, (see under Davis, M. R.).
- Gray, T. S., "A Photo-Electric Integrator," Journal of the Franklin Institute, Vol. 212, July, 1931, pp. 77-102.
- Hazen, H. L., and G. S. Brown, (see under Brown, G. S.).
- Hessemer, R. A., Jr., and C. S. Williams, Jr., "Solution of the Integral Equation Which Determines Radar-Cross-Section for a Scattering Ground," Technical Memorandum 207-54-54 of Sandia Corporation, Albuquerque, 1954, 13 pp.
- Jones, Robert T., "Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances," Journal of the Aeronautical Sciences, Vol. 3, Oct., 1936, pp. 419-425.
- Laitone, E. V., and J. T. Ahlin, (see under Ahlin, J. T.).
- Lee, Y. W., and J. W. Wiesner, "Correlation Functions and Communication Applications," Electronics, Vol. 23, June, 1950, pp. 86-92.

REFERENCES

- Allen, J. T., and A. V. Laitone, "A Simple Graphical Method of the
Dissolution of Solids in Liquids," Journal of the American Chemical Society, Vol. 73,
Feb., 1951, pp. 142-143.
- Barnes, John L., and Arthur E. Garton, Viscosity and Flow of Liquids,
New York: John Wiley & Sons, Inc., 1942, 331 pp.
- Bell, Harold, Jr., and A. V. Laitone, "A Simple Graphical Method of the
Dissolution of Solids in Liquids," Journal of the American Chemical Society, Vol. 73,
June, 1951, pp. 30-31.
- Brown, G. S., and H. L. Hagen, "The Critical Temperature of the
Frenkel-Instability," Vol. 23, Jan., 1940, pp. 19-24, and Vol. 23,
Aug., 1940, pp. 18-20.
- Davis, M. R., T. O. Ellis, and L. I. Grand, "A Simple Graphical Method of the
Dissolution of Solids in Liquids," Journal of the American Chemical Society, Vol. 73,
June, 1951, pp. 30-31.
- Ellis, T. O., M. R. Davis, and L. I. Grand, (see under Davis, M. R.).
- Gardner, Murray R., and John L. Barnes, (see under Barnes, John L.).
- Grand, L. I., T. O. Ellis, and M. R. Davis, (see under Davis, M. R.).
- Gray, T. S., "A Photochemical Investigation," Journal of the American Chemical Society, Vol. 73,
June, 1951, pp. 30-31.
- Hagen, H. L., and G. S. Brown, (see under Brown, G. S.).
- Hessner, R. A., Jr., and G. S. Brown, "A Simple Graphical Method of the
Dissolution of Solids in Liquids," Journal of the American Chemical Society, Vol. 73,
June, 1951, pp. 30-31.
- Jones, Robert F., "Calculation of the Rate of Dissolution of Solids in Liquids,"
Journal of the American Chemical Society, Vol. 73, Oct., 1951, pp. 49-50.
- Laitone, A. V., and J. T. Allen, (see under Allen, J. T.).
- Lee, Y. W., and J. M. Wessner, "Calculation of the Rate of Dissolution of Solids in Liquids,"
Journal of the American Chemical Society, Vol. 73, Jan., 1951, pp. 19-24.

Lepage, Wilbur R., and Samuel Seely, General Network Analysis, New York: McGraw-Hill Book Company, Inc., 1952, 516 pp.

Rideout, V. C., and Harold Bell, Jr., (see under Bell, Harold, Jr.).

Seely, Samuel, and Wilbur R. Lepage, (see under Lepage, Wilbur R.).

Seiwell, H. R., "A New Mechanical Autocorrelator," The Review of Scientific Instruments, Vol. 21, May, 1950, pp. 481-484.

Singleton, Henry E., "A Digital Electronic Correlator," Proceedings of the I.R.E., Vol. 38, Dec., 1950, pp. 1422-1428.

Smith, Kenneth "E", "Preliminary Report on Convolution Integral Computer," AER 14-S, Physical Science Laboratory, New Mexico College of Agriculture and Mechanic Arts, State College, 1955, 27 pp.

Wiesner, J. W., and Y. W. Lee, (see under Lee, Y. W.).

Williams, C. S., Jr., and R. A. Hessemer, Jr., (see under Hessemer, R. A., Jr.).

Williams, C. S., Jr., "A Computer to Solve One Form of Volterra's Integral Equation," Technical Memorandum 104-55-54 of Sandia Corporation, Albuquerque, 1955, 22 pp.



Lepage, Wilbur E., and Samuel Seely, "General Features of the New York: McGraw-Hill Book Company, Inc., 1952, p. 144.

Ridgway, V. C., and Harold Bell, Jr., (see also Bell, Harold, Jr.).

Seely, Samuel, and Wilbur E. Lepage, (see also Lepage, Wilbur E.).

Salway, H. R., "A New Mechanical Instrumentation," See Section of Scientific Instruments, Vol. 2, No. 1, 1952, p. 144.

Stark, Henry E., "A Digital Electronic Computer," Proceedings of the I.R.E., Vol. 35, Dec., 1950, pp. 1422-1423.

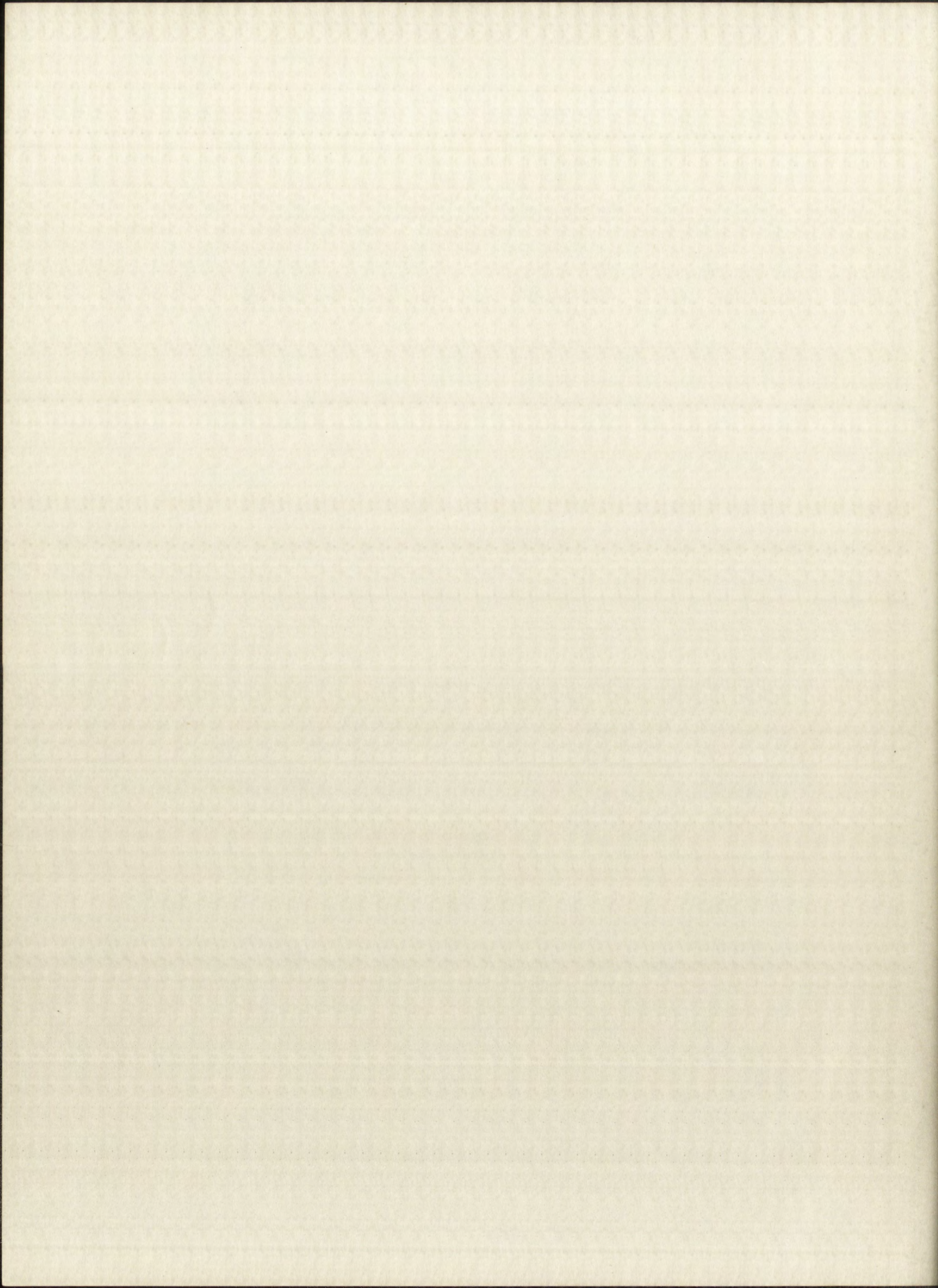
Smith, Kenneth "E", "Preliminary Report on Construction of a Computer," See Section of Scientific Instruments, Vol. 2, No. 1, 1952, p. 144.

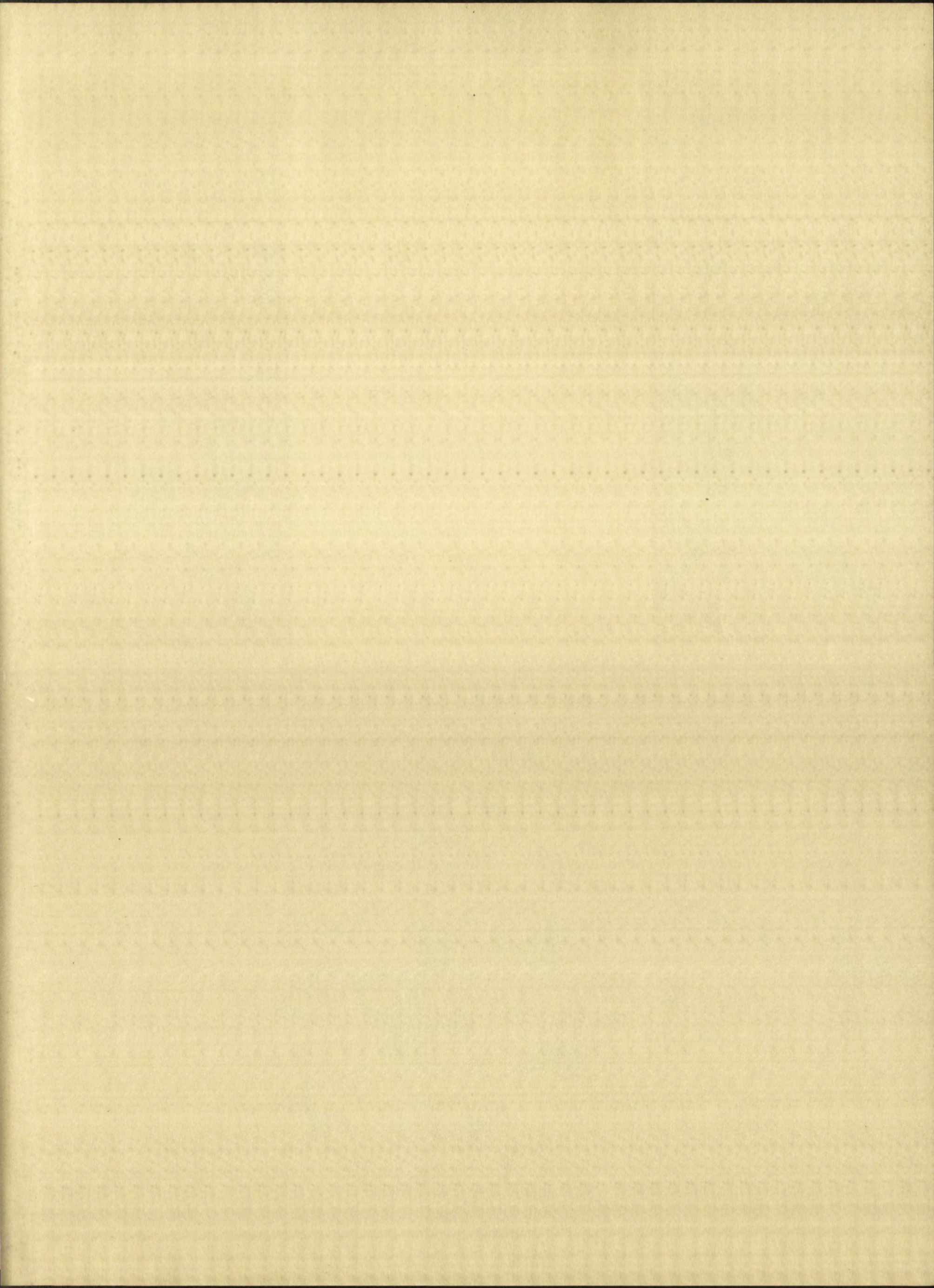
Wiener, J. W., and Y. W. Lee, (see also Lee, Y. W.).

Williams, C. E., Jr., and R. A. Thompson, Jr., (see also Thompson, R. A., Jr.).

Williams, C. E., Jr., "A Computer to Solve the Form of Volterra's Integral Equation," Technical Memorandum 104-25-24 of Santa Corporation, Albuquerque, 1952, 22 pp.

W





[illegible]

IMPORTANT!

Special care should be taken to prevent loss or damage of this volume. If lost or damaged, it must be paid for at the current rate of typing.

