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Neutrosophic Bitopological Spaces

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Abstract: In this study, bitopological structure which is a more general structure than topological spaces is built on neutrosophic sets. The necessary arguments which are pairwise neutrosophic open set, pairwise neutrosophic closed set, pairwise neutrosophic closure, pairwise neutrosophic interior are defined and their basic properties are presented. The relations of these concepts with their counterparts in neutrosophic topological spaces are given and many examples are presented.

Keywords: Neutrosophic set; neutrosophic bitopological space; pairwise neutrosophic open (closed) set; pairwise neutrosophic interior; pairwise neutrosophic closure; pairwise neutrosophic neighbourhood.

1. Introduction

In recent years, the major factor in the progress of natural sciences and its sub-branches is the construction of new set structures in mathematics. It is the fuzzy set theory defined by Zadeh [19] that leads to these set structures. This set structure is followed by intuitionistic set theory [7], intuitionistic fuzzy set theory [1] and soft set theory [15]. Later, as a generalization of fuzzy set and intuitionistic fuzzy set, Samarandache [17] introduced neutrosophic set. Neutrosophic set $N$ consist of three independent object called truth-membership $T_N(x)$, interminancy-membership $I_N(x)$ and falsity-membership $F_N(x)$ whose values are real standard or non-standard subset of unit interval $]-0,1+[$. Scientists continue to intensively study in different fields with this set structure [3, 4, 8, 14, 15, 17, 18, 19, 20, 21, 22]. These set structures have been studied by some authors in topology [2, 5, 6, 16, 18].

The concept of bitopological spaces was introduced by Kelly [13] as an extension of topological spaces in 1963. This concept has been studied with interest in other set structures [10, 12]. Therefore, we find it necessary and important to construct a bitopological spaces on the neutrosophic set structure.

In this study, we presented bitopological spaces on neutrosophic set structure and some basic notions of this spaces, open (closed) set, closure, interior, neighbourhood systems are defined. In addition, the theorems required for this structure are proved and their relations with neutrosophic topological spaces are investigated.

2. Preliminary

In this section, we will give some preliminary information for the present study.

Definition 2.1 [23] Let $X$ be a non empty set, then $N = \{(x,T_N(x),I_N(x),F_N(x)) : x \in X\}$ is called a neutrosophic set on $X$, where $-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$ for all $x \in X$, $T_N(x)$, $I_N(x)$ and $F_N(x) \in ]-0,1+[$ are the degree of membership (namely $T_N(x)$), the degree of indeterminacy (namely...
$I_N(x)$ and the degree of non membership (namely $F_N(x)$) of each $x \in X$ to the set $N$ respectively. For $X$, $\mathcal{N}(X)$ denotes the collection of all neutrosophic sets of $X$.

**Definition 2.2 [23]** The following statements are true for neutrosophic sets $N$ and $M$ on $X$:

i) $T_N(x) \leq T_M(x)$, $I_N(x) \leq I_M(x)$ and $F_N(x) \geq F_M(x)$ for all $x \in X$ iff $N \subseteq M$.

ii) $N \subseteq M$ and $M \subseteq N$ iff $N = M$.

iii) $N \cap M = \{(x, \min(T_N(x), T_M(x)), \min(I_N(x), I_M(x)), \max(F_N(x), F_M(x))): x \in X\}$.

iv) $N \cup M = \{(x, \max(T_N(x), T_M(x)), \max(I_N(x), I_M(x)), \min(F_N(x), F_M(x))): x \in X\}$.

More generally, the intersection and the union of a collection of neutrosophic sets $\{N_i\}_{i \in I}$ are defined by:

\[
\cap_{i \in I} N_i = \{x, \inf(T_{N_i}(x)), \inf(I_{N_i}(x)), \sup(F_{N_i}(x))): x \in X\},
\]

\[
\cup_{i \in I} N_i = \{x, \sup(T_{N_i}(x)), \inf(I_{N_i}(x)), \inf(F_{N_i}(x))): x \in X\}.
\]

v) $N$ is called neutrosophic universal set, denoted by $1_X$, if $T_N(x) = 1$, $I_N(x) = 1$ and $F_N(x) = 0$ for all $x \in X$.

vi) $N$ is called neutrosophic empty set, denoted by $0_X$, if $T_N(x) = 0$, $I_N(x) = 0$ and $F_N(x) = 1$ for all $x \in X$.

vii) $\mathcal{N}(X)$ is said to be a neutrosophic supra topology on $X$ if the sets $1_X$, $0_X$ and $\mathcal{N}(X)$ are defined as:

\[
(1_x)^c = 1_x \setminus 1_x = (x, 0, 0, 1) = 0_x,
\]

\[
(0_x)^c = 1_x \setminus 0_x = (x, 1, 1, 0) = 1_x.
\]

**Proposition 2.1 [23]** Let $N_1$, $N_2$, $N_3$ and $N_4 \in \mathcal{N}(X)$. Then followings hold:

i) $N_1 \cap N_3 \subseteq N_2 \cap N_4$ and $N_1 \cup N_3 \subseteq N_2 \cup N_4$, if $N_1 \subseteq N_2$ and $N_3 \subseteq N_4$.

ii) $(N_1)^c = N_1$ and $N_1 \subseteq N_2$, if $N_1 \subseteq N_2$ and $N_3 \subseteq N_4$.

iii) $(N_1 \cap N_2)^c = N_1^c \cup N_2^c$ and $(N_1 \cup N_2)^c = N_1^c \cap N_2^c$.

**Definition 2.3 [22]** Let $\alpha$, $\beta$, $\gamma$ be real standard or non standard subsets of $[0,1]$, then the neutrosophic set $x_{\alpha, \beta, \gamma}$ is called a neutrosophic point in given by

\[
x_{\alpha, \beta, \gamma}(y) = \begin{cases} (\alpha, \beta, y), & \text{if } x = y \\ (0,0,1), & \text{if } x \neq y \end{cases}
\]

for $y \in X$ is called the support of $x_{\alpha, \beta, \gamma}$.

It is clear that every neutrosophic set is the union of its neutrosophic points.

**Definition 2.4 [9]** Let $X$ be a non empty set. If $\mathcal{N}(X)$ is a neutrosophic set on $X$ then neutrosophic set $x_{\alpha, \beta, \gamma}$ is called a neutrosophic point in given by

\[
x_{\alpha, \beta, \gamma}(y) = \begin{cases} (\alpha, \beta, y), & \text{if } x = y \\ (0,0,1), & \text{if } x \neq y \end{cases}
\]

for $y \in X$ is called the support of $x_{\alpha, \beta, \gamma}$.

It is clear that every neutrosophic set is the union of its neutrosophic points.

**Definition 2.5 [9]** Let $N \in \mathcal{N}(X)$. We say that $x_{\alpha, \beta, \gamma} \in N$ read as belonging to the neutrosophic set $N$ whenever $\alpha \leq T_N(x)$, $\beta \leq I_N(x)$ and $\gamma \geq F_N(x)$.

**Definition 2.6 [11]** A subcollection $\tau_n^\ast$ of neutrosophic sets on a non empty set $X$ is said to be a neutrosophic supr topology on $X$ if the sets $1_X$, $0_X \in \tau_n^\ast$ and $\bigcup_{i=1}^{\infty} N_i \in \tau_n^\ast$ for $\{N_i\}_{i=1}^{\infty} \in \tau_n^\ast$. Then $(X, \tau_n^\ast)$ is called neutrosophic supr topological space on $X$.

3. Neutrosophic Bitopological Spaces
Definition 3.1 Let \((X, \tau_1^n, \tau_2^n)\) be the two different neutrosophic topologies on X. Then \((X, \tau_1^n, \tau_2^n)\) is called a neutrosophic bitopological space.

Definition 3.2 Let \((X, \tau_1^n, \tau_2^n)\) be a neutrosophic bitopological space. A neutrosophic set \(N = \{(x, T_N(x), I_N(x), F_N(x)) : x \in X\}\) over X is said to be a pairwise neutrosophic open set in \((X, \tau_1^n, \tau_2^n)\) if there exist a neutrosophic open set \(N_1 = \{(x, T_{N_1}(x), I_{N_1}(x), F_{N_1}(x)) : x \in X\}\) in \(\tau_1^n\) and a neutrosophic open set \(N_2 = \{(x, T_{N_2}(x), I_{N_2}(x), F_{N_2}(x)) : x \in X\}\) in \(\tau_2^n\) such that \(N = N_1 \cup N_2 = \{(x, \max(T_{N_1}(x), T_{N_2}(x)), \max(I_{N_1}(x), I_{N_2}(x)), \min(F_{N_1}(x), F_{N_2}(x))) : x \in X\}\).

Definition 3.3 Let \((X, \tau_1^n, \tau_2^n)\) be a neutrosophic bitopological space. A neutrosophic set \(N\) over X is said to be a pairwise neutrosophic closed set in \((X, \tau_1^n, \tau_2^n)\) if its neutrosophic complement is a pairwise neutrosophic open set in \((X, \tau_1^n, \tau_2^n)\). Obviously, a neutrosophic set \(C = \{(x, T_C(x), I_C(x), F_C(x)) : x \in X\}\) over X is a pairwise neutrosophic closed set in \((X, \tau_1^n, \tau_2^n)\) if there exist a neutrosophic closed set \(C_1 = \{(x, T_{C_1}(x), I_{C_1}(x), F_{C_1}(x)) : x \in X\}\) in \((\tau_1^n)^c\) and a neutrosophic closed set \(C_2 = \{(x, T_{C_2}(x), I_{C_2}(x), F_{C_2}(x)) : x \in X\}\) in \((\tau_2^n)^c\) such that \(C = C_1 \cap C_2 = \{(x, \min(T_{C_1}(x), T_{C_2}(x)), \min(I_{C_1}(x), I_{C_2}(x)), \max(F_{C_1}(x), F_{C_2}(x))) : x \in X\}\), where \((\tau_1^n)^c = \{N^c \in \mathcal{P}(X) : N \in \tau_1^n\}, i = 1, 2\).

The family of all pairwise neutrosophic open (closed) sets in \((X, \tau_1^n, \tau_2^n)\) is denoted by \(\text{PNO}(X, \tau_1^n, \tau_2^n)\) [\(\text{PNC}(X, \tau_1^n, \tau_2^n)\)], respectively.

Example 3.1 Let \(X = \{a, b, c\}\). We think that following neutrosophic set over X.
\[
N_1 = \{(a, 0.3, 0.2, 0.5), (b, 0.6, 0.5, 0.3), (c, 0.7, 0.1, 0.9)\},
\]
\[
N_2 = \{(a, 0.4, 0.1, 0.3), (b, 0.2, 0.6, 0.7), (c, 0.1, 0.3, 0.4)\},
\]
\[
N_3 = \{(a, 0.3, 0.1, 0.5), (b, 0.2, 0.5, 0.7), (c, 0.1, 0.1, 0.9)\},
\]
\[
N_4 = \{(a, 0.4, 0.2, 0.3), (b, 0.6, 0.6, 0.3), (c, 0.7, 0.3, 0.4)\},
\]
and
\[
M_1 = \{(a, 0.1, 0.2, 0.3), (b, 0.2, 0.1, 0.4), (c, 0.5, 0.2, 0.4)\},
\]
\[
M_2 = \{(a, 0.7, 0.3, 0.1), (b, 0.7, 0.8, 0.2), (c, 0.9, 0.8, 0.3)\}.
\]
Then \((X, \tau_1^n, \tau_2^n)\) is a neutrosophic bitopological space, where
\[
\tau_1^n = \{0_X, 1_X, N_1, N_2, N_3, N_4\},
\]
\[
\tau_2^n = \{0_X, 1_X, M_1, M_2\}.
\]
Obviously,
\[
\tau_1^{n_2} = \tau_1^n \cup \tau_2^n \cup \{N_1 \cup M_2, N_2 \cup M_1, N_3 \cup M_4\}
\]
because the neutrosophic sets \(N_1 \cup M_2, N_2 \cup M_1, N_3 \cup M_4\) do not belong to either \(\tau_1^n\) nor \(\tau_2^n\).

Theorem 3.1 Let \((X, \tau_1^n, \tau_2^n)\) be a neutrosophic bitopological space. Then,

1. \(0_X\) and \(1_X\) are pairwise neutrosophic open sets and pairwise neutrosophic closed sets.
2. An arbitrary neutrosophic union of pairwise neutrosophic open sets is a pairwise neutrosophic open set.
3. An arbitrary neutrosophic intersection of pairwise neutrosophic closed sets is a pairwise neutrosophic closed set.

Proof. 1. Since \(0_X \in \tau_1^n \cap \tau_2^n\) and \(0_X \cup 0_X = 0_X\), then \(0_X\) is a pairwise neutrosophic open set. Similarly, \(1_X\) is a pairwise neutrosophic open set.
2. Let \(\{\{N_i\} : i \in I\} \subseteq \text{PNO}(X, \tau_1^n, \tau_2^n).\) Then \(N_i\) is a pairwise neutrosophic open set for all \(i \in I\), therefore there exist \(N_1^i \in \tau_1^n\) and \(N_2^i \in \tau_2^n\) such that \(N_i = N_1^i \cup N_2^i\) for all \(i \in I\) which implies that
\[
\bigcup_{i \in I} N_1^i = \bigcup_{i \in I} N_2^i = \bigcup_{i \in I} N_i.
\]
3. It is immediate from the Definition 9, Proposition 1.
Corollary 3.1 Let \((X, \tau^1, \tau^2)\) be a neutrosophic bitopological space. Then, the family of all pairwise neutrosophic open sets is a supra neutrosophic topology on \(X\). This supra neutrosophic topology we denoted by \(\tau^{12}\).

Remark 3.1 The Example 1 show that:
1. \(\tau^{12}\) is not neutrosophic topology in general.
2. The finite neutrosophic intersection of pairwise neutrosophic open sets need not be a pairwise neutrosophic open set.
3. The arbitrary neutrosophic union of pairwise neutrosophic closed sets need not be a pairwise neutrosophic closed set.

Theorem 3.2 Let \((X, \tau^1, \tau^2)\) be a neutrosophic bitopological space. Then,

1. Every \(\tau^i\) — open neutrosophic set is a pairwise neutrosophic open set \(i = 1, 2\), i.e., \(\tau^i \cup \tau^j \subseteq \tau^{12}\).
2. Every \(\tau^i\) — closed neutrosophic set is a pairwise neutrosophic closed set \(i = 1, 2\), i.e., \((\tau^i)^c \cup (\tau^j)^c \subseteq (\tau^{12})^c\).
3. If \(\tau^1 \subseteq \tau^2\), then \(\tau^1 = \tau^2\) and \((\tau^{12})^c = (\tau^2)^c\).

Proof. Straightforward.

Definition 3.4 Let \((X, \tau^1, \tau^2)\) be a neutrosophic bitopological space and \(N \in \mathcal{K}(X)\). The pairwise neutrosophic closure of \(N\), denoted by \(cl^p(N)\), is the neutrosophic intersection of all pairwise neutrosophic closed super sets of \(N\), i.e.,

\[ cl^p(N) = \bigcap \{C \in (\tau^{12})^c : N \subseteq C\}. \]

It is clear that \(cl^p(N)\) is the smallest pairwise neutrosophic closed set containing \(N\).

Example 3.2 Let \((X, \tau^1, \tau^2)\) be the same as in Example 1 and \(G = \{(a,0,7,0,8,0.7), (b,0.5,0,4,0,6), (c,0,8,0,7,0.5)\}\) be a neutrosophic set over \(X\).

Now, we need to determine pairwise neutrosophic closed sets in \((X, \tau^1, \tau^2)\) to find \(cl^p(G)\). Then,

\[ N_1^c = \{(a,0.7,0,8,0.5), (b,0.4,0,5,0,7), (c,0.3,0,9,0,1)\}, \]
\[ N_2^c = \{(a,0.6,0,9,0,7), (b,0.8,0,4,0,3), (c,0.9,0,7,0,6)\}, \]
\[ N_3^c = \{(a,0.7,0,9,0,5), (b,0.8,0,5,0,3), (c,0.9,0,9,0,1)\}, \]
\[ N_4^c = \{(a,0.6,0,8,0,7), (b,0.4,0,4,0,7), (c,0.3,0,7,0,6)\}, \]
\[ M_1^c = \{(a,0.9,0,8,0,7), (b,0.8,0,9,0,6), (c,0.5,0,8,0,6)\}, \]
\[ M_2^c = \{(a,0.3,0,7,0,9), (b,0.3,0,2,0,8), (c,0.1,0,2,0,7)\}. \]

and

\[ (N_1 \cup M_1)^c = \{(a,0.7,0,8,0,7), (b,0.4,0,5,0,7), (c,0.3,0,8,0,6)\} \]
\[ (N_2 \cup M_1)^c = \{(a,0.6,0,8,0,7), (b,0.8,0,4,0,6), (c,0.5,0,7,0,6)\} \]
\[ (N_3 \cup M_1)^c = \{(a,0.7,0,8,0,7), (b,0.8,0,5,0,6), (c,0.5,0,8,0,6)\}. \]

In here, the pairwise neutrosophic closed sets which contains \(G\) are \(N_3^c\) and \(1_X\) it follows that \(cl^p(G) = N_3^c \cap 1_X\). Therefore, \(cl^p(G) = N_3^c\).

Theorem 3.3 Let \((X, \tau^1, \tau^2)\) be a neutrosophic bitopological space and \(N, M \in \mathcal{K}(X)\). Then,

1. \(cl^p(0_X) = 0_X\) and \(cl^p(1_X) = 1_X\).
2. \(N \subseteq cl^p(N)\).
3. \(N\) is a pairwise neutrosophic closed set if \(cl^p(N) = N\).
4. \(N \subseteq M \Rightarrow cl^p(N) \subseteq cl^p(M)\).
5. \(cl^p(N) \cup cl^p(M) \subseteq cl^p(N \cup M)\).
6. \(cl^p[cl^p(N)] = cl^p(N)\), i.e., \(cl^p(N)\) is a pairwise neutrosophic closed set.

Proof. Straightforward.
Theorem 3.4 Let \((X, \tau^*_1, \tau^*_2)\) be a neutrosophic bitopological space and \(N \in \mathcal{N}(X)\). Then,

\[
x_{\alpha,\beta,Y} \in cl^*_p(N) \Leftrightarrow U_{x_{\alpha,\beta,Y}} \cap N \neq \emptyset, \forall U_{x_{\alpha,\beta,Y}} \in \tau^*_2(x_{\alpha,\beta,Y}),
\]

where \(U_{x_{\alpha,\beta,Y}}\) is any pairwise neutrosophic open sets contains \(x_{\alpha,\beta,Y}\) and \(\tau^*_2(x_{\alpha,\beta,Y})\) is the family of all pairwise neutrosophic open sets contains \(x_{\alpha,\beta,Y}\).

Proof. Let \(x_{\alpha,\beta,Y} \in cl^*_p(N)\) and suppose that there exists \(U_{x_{\alpha,\beta,Y}} \in \tau^*_2(x_{\alpha,\beta,Y})\) such that \(U_{x_{\alpha,\beta,Y}} \cap N = \emptyset\). Then \(N \subseteq (U_{x_{\alpha,\beta,Y}})^c\), thus \(cl^*_p(N) \subseteq cl^*_p(U_{x_{\alpha,\beta,Y}})^c = (U_{x_{\alpha,\beta,Y}})^c\) which implies \(cl^*_p(N) \cap U_{x_{\alpha,\beta,Y}} = \emptyset\), a contradiction.

Conversely, assume that \(x_{\alpha,\beta,Y} \notin cl^*_p(N)\), then \(x_{\alpha,\beta,Y} \in cl^*_p(N)^c\). Thus, \([cl^*_p(N)]^c \in \tau^*_2(x_{\alpha,\beta,Y})\), so, by hypothesis, \([cl^*_p(N)]^c \cap N \neq \emptyset\), a contradiction.

Theorem 3.5 Let \((X, \tau^*_1, \tau^*_2)\) be a neutrosophic bitopological space. A neutrosophic set \(N\) over \(X\) is a pairwise neutrosophic closed set iff \(N = cl^*_{t_1}(N) \cap cl^*_{t_2}(N)\).

Proof. Suppose that \(N\) is a pairwise neutrosophic closed set and \(x_{\alpha,\beta,Y} \notin N\). Then, \(x_{\alpha,\beta,Y} \notin cl^*_p(N)\). Thus, [by Theorem 4], there exists \(U_{x_{\alpha,\beta,Y}} \in \tau^*_2(x_{\alpha,\beta,Y})\) such that \(U_{x_{\alpha,\beta,Y}} \cap N = \emptyset\). Since \(x_{\alpha,\beta,Y} \in \tau^*_2(x_{\alpha,\beta,Y})\), then there exists \(M_1 \in \tau^*_1(x_{\alpha,\beta,Y})\) such that \(U_{x_{\alpha,\beta,Y}} = M_1 \cup M_2\). Hence, \((M_1 \cup M_2) \cap N = \emptyset\). Since \(x_{\alpha,\beta,Y} \in \tau^*_2(x_{\alpha,\beta,Y})\), \(x_{\alpha,\beta,Y} \in M_1\) or \(x_{\alpha,\beta,Y} \in M_2\) implies, \(x_{\alpha,\beta,Y} \notin cl^*_p(N)\) or \(x_{\alpha,\beta,Y} \notin cl^*_p(N)\). Therefore, \(x_{\alpha,\beta,Y} \notin cl^*_p(N) \cap cl^*_p(N)\). Thus, \(cl^*_p(N) \cap cl^*_p(N) \subseteq N\). On the other hand, we have \(N \subseteq cl^*_p(N) \cap cl^*_p(N)\). Hence, \(N = cl^*_p(N) \cap cl^*_p(N)\).

Conversely, suppose that \(N = cl^*_p(N) \cap cl^*_p(N)\). Since, \(cl^*_p(N)\) is a neutrosophic closed set in \((X, \tau^*_1)\) and \(cl^*_p(N)\) is a neutrosophic closed set in \((X, \tau^*_2)\), then, [by Definition 9], \(cl^*_p(N) \cap cl^*_p(N)\) is a pairwise neutrosophic closed set in \((X, \tau^*_1, \tau^*_2)\), so \(N\) is a pairwise neutrosophic closed set.

Corollary 3.2 Let \((X, \tau^*_1, \tau^*_2)\) be a neutrosophic bitopological space. Then,

\[
cl^*_p(N) = cl^*_p_{t_1}(N) \cap cl^*_p_{t_2}(N), \forall N \in \mathcal{N}(X).
\]

Definition 3.5 An operator \(\Psi: \mathcal{N}(X) \rightarrow \mathcal{N}(X)\) is called a neutrosophic supra closure operator if it satisfies the following conditions for all \(N, M \in \mathcal{N}(X)\).

1. \(\Psi(0_X) = 0_X\).
2. \(N \subseteq \Psi(N)\).
3. \(\Psi(N) \cup \Psi(M) \subseteq \Psi(N \cup M)\).
4. \(\Psi(\Psi(N)) = \Psi(N)\).

Theorem 3.6 Let \((X, \tau^*_1, \tau^*_2)\) be a neutrosophic bitopological space. Then, the operator \(cl^*_p: \mathcal{N}(X) \rightarrow \mathcal{N}(X)\) which defined by

\[
cl^*_p(N) = cl^*_p_{t_1}(N) \cap cl^*_p_{t_2}(N)
\]

is neutrosophic supra closure operator and it is induced, a unique neutrosophic supra topology given by \(\{N \in \mathcal{N}(X): cl^*_p(N^c) = N^c\}\) which is precisely \(\tau^*_2\).

Proof. Straightforward.
Definition 3.6 Let \((X, \tau^p_1, \tau^p_2)\) be a neutrosophic bitopological space and \(N \in \mathcal{N}(X)\). The pairwise neutrosophic interior of \(N\), denoted by \(\text{int}^p_\cap(N)\), is the neutrosophic union of all pairwise neutrosophic open subsets of \(N\), i.e.,
\[
\text{int}^p_\cap(N) = \bigcup \{M \in \tau^p_1 \cup \tau^p_2 : M \subseteq N\}.
\]
Obviously, \(\text{int}^p_\cap(N)\) is the biggest pairwise neutrosophic open set contained in \(N\).

Example 3.3 Let \((X, \tau^p_1, \tau^p_2)\) be the same as in Example 1 and \(M = \{(a,0.3,0.4,0.2), (b,0.5,0.7,0.1), (c,0.8,0.7,0.3)\}\) be a neutrosophic set over \(X\). Then the pairwise neutrosophic open sets which containing in \(M\) are \(N_3\), \(M_1\), \(N_3 \cup M_1\) and \(0_X\). Therefore,
\[
\begin{align*}
\text{int}^p_\cap(M) &= N_3 \cup M_1 \cup (N_3 \cup M_1) \cup 0_X \\
&= N_3 \cup M_1 \\
&= \{(a,0.3,0.2,0.3), (b,0.2,0.5,0.4), (c,0.5,0.2,0.4)\}.
\end{align*}
\]

Theorem 3.7 Let \((X, \tau^p_1, \tau^p_2)\) be a neutrosophic bitopological space and \(N, M \in \mathcal{N}(X)\). Then,
\begin{enumerate}
\item \(\text{int}^p_\cap(0_X) = 0_X\) and \(\text{int}^p_\cap(1_X) = 1_X\),
\item \(\text{int}^p_\cap(N) \subseteq N\),
\item \(N\) is a pairwise neutrosophic open set iff \(\text{int}^p_\cap(N) = N\),
\item \(N \subseteq M \Rightarrow \text{int}^p_\cap(N) \subseteq \text{int}^p_\cap(M)\),
\item \(\text{int}^p_\cap(N \cap M) \subseteq \text{int}^p_\cap(N) \cap \text{int}^p_\cap(M)\),
\item \(\text{int}^p_\cap[\text{int}^p_\cap(N)] = \text{int}^p_\cap(N)\).
\end{enumerate}
Proof. Starightforward.

Theorem 3.8 Let \((X, \tau^p_1, \tau^p_2)\) be a neutrosophic bitopological space and \(N \in \mathcal{N}(X)\). Then, \(x_{a,b,y} \in \text{int}^p_\cap(N) \Leftrightarrow \exists U_{x_{a,b,y}} \in \tau^p_{12}(x_{a,b,y})\) such that \(U_{x_{a,b,y}} \subseteq N\).

Proof. Starightforward.

Theorem 3.9 Let \((X, \tau^p_1, \tau^p_2)\) be a neutrosophic bitopological space. A neutrosophic set \(N\) over \(X\) is a pairwise neutrosophic open set iff \(N = \text{int}^p_\cap(1_X) \cup \text{int}^p_\cap(2_X)\).

Proof. Let \(N\) be a pairwise neutrosophic open set. Since, \(\text{int}^p_\cap(N) \subseteq N\), \(i = 1,2\), then \(\text{int}^p_{\tau_1}(N) \cup \text{int}^p_{\tau_2}(N) \subseteq N\). Now, let \(x_{a,b,y} \in N\). Then, there exists \(U_{x_{a,b,y}} \in \tau^p_{1}\) such that \(U_{x_{a,b,y}} \subseteq N\) or there exists \(U_{x_{a,b,y}} \in \tau^p_{2}\) such that \(U_{x_{a,b,y}} \subseteq N\). Hence, \(x_{a,b,y} \in \text{int}^p_{\tau_1}(N) \cup \text{int}^p_{\tau_2}(N)\). Therefore, \(N = \text{int}^p_{\tau_1}(N) \cup \text{int}^p_{\tau_2}(N)\).

Coversely, since \(\text{int}^p_{\tau_1}(N)\) is a neutrosophic open set in \((X, \tau^p_1)\) and \(\text{int}^p_{\tau_2}(N)\) is a neutrosophic open set in \((X, \tau^p_2)\), then, [by Definition 3.8], \(\text{int}^p_{\tau_1}(N) \cup \text{int}^p_{\tau_2}(N)\) is a pairwise neutrosophic open set in \((X, \tau^p_1, \tau^p_2)\). Thus, \(N\) is a pairwise neutrosophic open set.

Corollary 3.3 Let \((X, \tau^p_1, \tau^p_2)\) be a neutrosophic bitopological space. Then,
\[
\text{int}^p_\cap(N) = \text{int}^p_{\tau_1}(N) \cup \text{int}^p_{\tau_2}(N).
\]

Definition 3.7 An operator \(I : \mathcal{N}(X) \rightarrow \mathcal{N}(X)\) is called a neutrosophic supra interior operator if it satisfies the following conditions for all \(N, M \in \mathcal{N}(X)\).
\begin{enumerate}
\item \(I(0_X) = 0_X\),
\item \(I(N) \subseteq N\),
\item \(I(N \cap M) \subseteq I(N) \cap I(M)\)
\item \(I(I(N)) = I(N)\).
\end{enumerate}
Theorem 3.10 Let \((X, \tau^n_1, \tau^n_2)\) be a neutrosophic bitopological space. Then, the operator \(\text{int}^n_p; \mathcal{K}(X) \to \mathcal{K}(X)\) which defined by

\[
\text{int}^n_p(N) = \text{int}^n_p(N) \cup \text{int}^n_q(N)
\]

is neutrosophic supra interior operator and it is induced, a unique neutrosophic supra topology given by \(\{N \in \mathcal{K}(X): \text{int}^n_p(N) = N\}\) which is precisely \(\tau^n_{12}\).

Proof. Straightforward.

Theorem 3.11 Let \((X, \tau^n_1, \tau^n_2)\) be a neutrosophic bitopological space and \(N \in \mathcal{K}(X)\). Then,

1. \(\text{int}^n_p(N) = (\text{cl}^n_p(N^c))^c\).
2. \(\text{cl}^n_p(N) = (\text{int}^n_q(N^c))^c\).

Proof. Starightforward.

Definition 3.8 Let \((X, \tau^n_1, \tau^n_2)\) be a neutrosophic bitopological space, \(N \in \mathcal{K}(X)\) and \(x_{a,\beta,\gamma} \in \mathcal{K}(X)\). Then \(N\) is said to be a pairwise neutrosophic neighborhood of \(x_{a,\beta,\gamma}\), if there exists a pairwise neutrosophic open set \(U\) such that \(x_{a,\beta,\gamma} \in U \subseteq N\). The family of pairwise neutrosophic neighborhood of neutrosophic point \(x_{a,\beta,\gamma}\) denoted by \(N_{\tau^n_{12}}(x_{a,\beta,\gamma})\).

Theorem 3.12 Let \((X, \tau^n_1, \tau^n_2)\) be a neutrosophic bitopological space and \(N \in \mathcal{K}(X)\). Then \(N\) is pairwise neutrosophic open set if \(N\) is a pairwise neutrosophic neighborhood of its neutrosophic points.

Proof. Let \(N\) be a pairwise neutrosophic open set and \(x_{a,\beta,\gamma} \in N\). Then \(x_{a,\beta,\gamma} \in N \subseteq N\). Therefore \(N\) is a pairwise neutrosophic neighborhood of \(x_{a,\beta,\gamma}\) for each \(x_{a,\beta,\gamma} \in N\).

Conversely, suppose that \(N\) is a pairwise neutrosophic neighborhood of its neutrosophic points and \(x_{a,\beta,\gamma} \in N\). Then there exists a pairwise neutrosophic open set \(U\) such that \(x_{a,\beta,\gamma} \in U \subseteq N\). Since

\[
N = \bigcup_{x_{a,\beta,\gamma} \in N} \{x_{a,\beta,\gamma}\} \subseteq U \subseteq \bigcup_{x_{a,\beta,\gamma} \in N} \{x_{a,\beta,\gamma}\} = N
\]

it follows that \(N\) is an union of pairwise neutrosophic open sets. Hence, \(N\) is a pairwise neutrosophic open set.

Proposition 3.2 Let \((X, \tau^n_1, \tau^n_2)\) be a neutrosophic bitopological space and \(\{N_{\tau^n_{12}}(x_{a,\beta,\gamma}): x_{a,\beta,\gamma} \in \mathcal{K}(X)\}\) be a system of pairwise neutrosophic neighborhoods. Then,

1. For every \(N \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\), \(x_{a,\beta,\gamma} \in N\);
2. \(N \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\) and \(M \subseteq M \Rightarrow M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\);
3. \(N \in N_{\tau^n_{12}}(x_{a,\beta,\gamma}) \Rightarrow \exists M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\) such that \(M \subseteq N\) and \(M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\), for every \(y_{\alpha,\beta,\gamma} \in \mathcal{X}\), \(y_{\alpha,\beta,\gamma} \in M\).

Proof. Proofs of 1 and 2 are straightforward.

3. Let \(N\) be a pairwise neutrosophic neighborhood of \(x_{a,\beta,\gamma}\), then there exists a pairwise neutrosophic open set \(M \in \tau^n_{12}\) such that \(x_{a,\beta,\gamma} \in M \subseteq N\). Since \(x_{a,\beta,\gamma} \in M \subseteq M\) can be written, then \(M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\). From the Theorem 12, if \(M\) is pairwise neutrosophic open set then \(N\) is a pairwise neutrosophic neighborhood of its neutrosophic points, i.e., \(M \in N_{\tau^n_{12}}(y_{\alpha,\beta,\gamma})\), for every \(y_{\alpha,\beta,\gamma} \in \mathcal{X}\), \(y_{\alpha,\beta,\gamma} \in M\).

Remark 3.2 Generally, \(N, M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma}) \Rightarrow N \cap M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\). Actually, if \(N, M \in N_{\tau^n_{12}}(x_{a,\beta,\gamma})\), there exist \(U_1, U_2 \in \tau^n_{12}\) such that \(x_{a,\beta,\gamma} \in U_1 \subseteq N\) and \(x_{a,\beta,\gamma} \in U_2 \subseteq M\). But \(U_1 \cap U_2\) need not be a
pairwise neutrosophic open set. Therefore, \( N \cap M \) need not be a pairwise neutrosophic neighborhood of \( x_{a\beta\gamma} \).

**Theorem 3.13** Let \((X, \tau^1_\nu, \tau^2_\nu)\) be a neutrosophic bitopological space. Then

\[
N_{\tau^k_\nu}(x_{a\beta\gamma}) = N_{\tau^1_\nu}(x_{a\beta\gamma}) \cup N_{\tau^2_\nu}(x_{a\beta\gamma})
\]

for each \( x_{a\beta\gamma} \in \mathcal{N}(X) \).

Proof. Let \( x_{a\beta\gamma} \in \mathcal{N}(X) \) be any neutrosophic point and \( N \in N_{\tau^k_\nu}(x_{a\beta\gamma}) \). Then there exists a pairwise neutrosophic open set \( M \in \tau^k_\nu \) such that \( x_{a\beta\gamma} \in M \subseteq N \). If \( M \in \tau^1_\nu \) and \( M \in \tau^2_\nu \) such that \( M = M_1 \cup M_2 \). Since \( x_{a\beta\gamma} \in M_1 \cup M_2 \), then \( x_{a\beta\gamma} \in M_1 \) or \( x_{a\beta\gamma} \in M_2 \). So, \( x_{a\beta\gamma} \in M_1 \subseteq M \subseteq N \) or \( x_{a\beta\gamma} \in M_2 \subseteq M \subseteq N \). In this case, \( N \in N_{\tau^1_\nu}(x_{a\beta\gamma}) \) or \( N \in N_{\tau^2_\nu}(x_{a\beta\gamma}) \), i.e., \( N \in N_{\tau^k_\nu}(x_{a\beta\gamma}) \).

Conversely, suppose that \( N \in N_{\tau^1_\nu}(x_{a\beta\gamma}) \cup N_{\tau^2_\nu}(x_{a\beta\gamma}) \). Then \( N \in N_{\tau^1_\nu}(x_{a\beta\gamma}) \) or \( N \in N_{\tau^2_\nu}(x_{a\beta\gamma}) \). Hence, there exists \( x_{a\beta\gamma} \in M_1 \subseteq N \) or \( x_{a\beta\gamma} \in M_2 \subseteq N \) such that \( x_{a\beta\gamma} \in M_1 \subseteq N \) and \( x_{a\beta\gamma} \in M_2 \subseteq N \). As a result, \( x_{a\beta\gamma} \in M_1 \cup M_2 = M \subseteq N \) such that \( M \in \tau^k_\nu \). i.e., \( N \in N_{\tau^k_\nu}(x_{a\beta\gamma}) \).

**Definition 3.9** An operator \( v: \mathcal{N}(X) \to \mathcal{N}(X) \) is called a neutrosophic supra neighborhood operator if it satisfies the following conditions for all \( N, M \in \mathcal{N}(X) \).

1. \( \forall N \in v(x_{a\beta\gamma}), x_{a\beta\gamma} \in N \);
2. \( N \in v(x_{a\beta\gamma}) \) and \( N \subseteq M \Rightarrow M \in v(x_{a\beta\gamma}) \);
3. \( N \in v(x_{a\beta\gamma}) \Rightarrow \exists M \in v(x_{a\beta\gamma}) \) such that \( N \subseteq M \) and \( M \in v(y_{\alpha^\prime\beta^\prime\gamma^\prime}) \), \( y_{\alpha^\prime\beta^\prime\gamma^\prime} \in M \).

**Theorem 3.14** Let \((X, \tau^1_\nu, \tau^2_\nu)\) be a neutrosophic bitopological space. Then, the operator \( N_{\tau^k_\nu}: \mathcal{N}(X) \to \mathcal{N}(X) \) which defined by

\[
N_{\tau^k_\nu}(x_{a\beta\gamma}) = N_{\tau^1_\nu}(x_{a\beta\gamma}) \cup N_{\tau^2_\nu}(x_{a\beta\gamma})
\]

is neutrosophic supra neighborhood operator and it is induced, a unique neutrosophic supra topology given by \( \{ N \in \mathcal{N}(X): \forall x_{a\beta\gamma} \in N \forall M \in N_{\tau^k_\nu}(x_{a\beta\gamma}) \} \) which is precisely \( \tau^k_\nu \).

4. Conclusions

In this paper, neutrosophic bitopological spaces are presented. By defining open (closed) sets, interior, closure and neighbourhood systems, fundamentals theorems for neutrosophic bitopological spaces are proved and some examples on the subject are given. This paper is just a beginning of a new structure and we have studied a few ideas only, it will be necessary to carry out more theoretical research to establish a general framework for the practical application. In the future, using these notions, various classes of mappings on neutrosophic bitopological space, separation axioms on the neutrosophic bitopological spaces and many researchers can be studied.

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**Conflicts of Interest**

The authors declare no conflict of interest.
References


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