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Article

Neutrosophic \mathcal{N} -Structures Applied to BCK/BCI -Algebras

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Received: 12 September 2017; Accepted: 6 October 2017; Published: 16 October 2017

Abstract: Neutrosophic \mathcal{N} -structures with applications in BCK/BCI -algebras is discussed. The notions of a neutrosophic \mathcal{N} -subalgebra and a (closed) neutrosophic \mathcal{N} -ideal in a BCK/BCI -algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal are considered, and relations between a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal are stated. Conditions for a neutrosophic \mathcal{N} -ideal to be a closed neutrosophic \mathcal{N} -ideal are provided.

Keywords: neutrosophic \mathcal{N} -structure; neutrosophic \mathcal{N} -subalgebra; (closed) neutrosophic \mathcal{N} -ideal

MSC: 06F35, 03G25, 03B52

1. Introduction

BCK -algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D -posets (MV -algebras). Additionally, Iséki introduced the notion of a BCI -algebra, which is a generalization of a BCK -algebra (see [2]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \rightarrow \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A . So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0, 1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed \mathcal{N} -structures. Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/ neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

$$(t, i, f) = (\text{truth, indeterminacy, falsehood})$$

For more details, refer to the following site:

<http://fs.gallup.unm.edu/FlorentinSmarandache.htm>

In this paper, we discuss a neutrosophic \mathcal{N} -structure with an application to BCK/BCI -algebras. We introduce the notions of a neutrosophic \mathcal{N} -subalgebra and a (closed) neutrosophic \mathcal{N} -ideal in a BCK/BCI -algebra, and investigate related properties. We consider characterizations of a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal. We discuss relations between a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal. We provide conditions for a neutrosophic \mathcal{N} -ideal to be a closed neutrosophic \mathcal{N} -ideal.

2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. A BCI -algebra refers to a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

- (I) $((x * y) * (x * z)) * (z * y) = \theta$,
- (II) $(x * (x * y)) * y = \theta$,
- (III) $x * x = \theta$,
- (IV) $x * y = y * x = \theta \Rightarrow x = y$.

for all $x, y, z \in X$. If a BCI -algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a BCK -algebra. We can define a partial ordering \preceq by

$$(\forall x, y \in X) (x \preceq y \Rightarrow x * y = \theta)$$

In a BCK/BCI -algebra X , the following hold:

$$(\forall x \in X) (x * \theta = x) \tag{1}$$

$$(\forall x, y, z \in X) ((x * y) * z = (x * z) * y) \tag{2}$$

A non-empty subset S of a BCK/BCI -algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI -algebra X is called an *ideal* of X if it satisfies the following:

- (I1) $0 \in I$,
- (I2) $(\forall x, y \in X)(x * y \in I, y \in I \Rightarrow x \in I)$.

We refer the reader to the books [6,7] for further information regarding BCK/BCI -algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

$$\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

We denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, \mathcal{N} -function on X). An \mathcal{N} -structure refers to an ordered pair (X, f) of X and an \mathcal{N} -function f on X (see [3]). In what follows, we let X denote the nonempty universe of discourse unless otherwise specified.

A *neutrosophic \mathcal{N} -structure* over X (see [8]) is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_{\mathbf{N}}, I_{\mathbf{N}}, F_{\mathbf{N}})} = \left\{ \frac{x}{(T_{\mathbf{N}}(x), I_{\mathbf{N}}(x), F_{\mathbf{N}}(x))} \mid x \in X \right\} \tag{3}$$

where T_N, I_N and F_N are \mathcal{N} -functions on X , which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X .

We note that every neutrosophic \mathcal{N} -structure X_N over X satisfies the condition:

$$(\forall x \in X) (-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

3. Application in BCK/BCI-Algebras

In this section, we take a BCK/BCI-algebra X as the universe of discourse unless otherwise specified.

Definition 1. A neutrosophic \mathcal{N} -structure X_N over X is called a neutrosophic \mathcal{N} -subalgebra of X if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(x * y) \leq \vee \{T_N(x), T_N(y)\} \\ I_N(x * y) \geq \wedge \{I_N(x), I_N(y)\} \\ F_N(x * y) \leq \vee \{F_N(x), F_N(y)\} \end{pmatrix} \tag{4}$$

Example 1. Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.

*	θ	a	b	c
θ	θ	θ	θ	θ
a	a	θ	θ	a
b	b	a	θ	b
c	c	c	c	θ

The neutrosophic \mathcal{N} -structure

$$X_N = \left\{ \frac{\theta}{(-0.7, -0.2, -0.6)}, \frac{a}{(-0.5, -0.3, -0.4)}, \frac{b}{(-0.5, -0.3, -0.4)}, \frac{c}{(-0.3, -0.8, -0.5)} \right\}$$

over X is a neutrosophic \mathcal{N} -subalgebra of X .

Let X_N be a neutrosophic \mathcal{N} -structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^\alpha &:= \{x \in X \mid T_N(x) \leq \alpha\} \\ I_N^\beta &:= \{x \in X \mid I_N(x) \geq \beta\} \\ F_N^\gamma &:= \{x \in X \mid F_N(x) \leq \gamma\} \end{aligned}$$

The set

$$X_N(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ) -level set of X_N . Note that

$$X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$$

Theorem 1. Let X_N be a neutrosophic \mathcal{N} -structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic \mathcal{N} -subalgebra of X , then the nonempty (α, β, γ) -level set of X_N is a subalgebra of X .

Proof. Let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$ and $X_N(\alpha, \beta, \gamma) \neq \emptyset$. If $x, y \in X_N(\alpha, \beta, \gamma)$, then $T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma, T_N(y) \leq \alpha, I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (4) that

$$\begin{aligned} T_N(x * y) &\leq \vee\{T_N(x), T_N(y)\} \leq \alpha, \\ I_N(x * y) &\geq \wedge\{I_N(x), I_N(y)\} \geq \beta, \text{ and} \\ F_N(x * y) &\leq \vee\{F_N(x), F_N(y)\} \leq \gamma. \end{aligned}$$

Hence, $x * y \in X_N(\alpha, \beta, \gamma)$, and therefore $X_N(\alpha, \beta, \gamma)$ is a subalgebra of X . \square

Theorem 2. Let X_N be a neutrosophic \mathcal{N} -structure over X and assume that T_N^α, I_N^β and F_N^γ are subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then X_N is a neutrosophic \mathcal{N} -subalgebra of X .

Proof. Assume that there exist $a, b \in X$ such that $T_N(a * b) > \vee\{T_N(a), T_N(b)\}$. Then $T_N(a * b) > t_\alpha \geq \vee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $a * b \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(x * y) \leq \vee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(a * b) < \wedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then

$$I_N(a * b) < t_\beta < \wedge\{I_N(a), I_N(b)\}$$

where $t_\beta := \frac{1}{2} \{I_N(a * b) + \wedge\{I_N(a), I_N(b)\}\}$. Thus $a, b \in I_N^{t_\beta}$ and $a * b \notin I_N^{t_\beta}$, which is a contradiction. Therefore

$$I_N(x * y) \geq \wedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(a * b) > t_\gamma \geq \vee\{F_N(a), F_N(b)\}$$

Then $a, b \in F_N^{t_\gamma}$ and $a * b \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(x * y) \leq \vee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore X_N is a neutrosophic \mathcal{N} -subalgebra of X . \square

Because $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3. If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic \mathcal{N} -subalgebras of X , then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.

Proposition 1. If a neutrosophic \mathcal{N} -structure X_N over X is a neutrosophic \mathcal{N} -subalgebra of X , then $T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$.

Proof. Straightforward. \square

Theorem 4. Let X_N be a neutrosophic \mathcal{N} -subalgebra of X . If there exists a sequence $\{a_n\}$ in X such that $\lim_{n \rightarrow \infty} T_N(a_n) = -1, \lim_{n \rightarrow \infty} I_N(a_n) = 0$ and $\lim_{n \rightarrow \infty} F_N(a_n) = -1$, then $T_N(\theta) = -1, I_N(\theta) = 0$ and $F_N(\theta) = -1$.

Proof. By Proposition 1, we have $T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Hence $T_N(\theta) \leq T_N(a_n), I_N(a_n) \leq I_N(\theta)$ and $F_N(\theta) \leq F_N(a_n)$ for every positive integer n . It follows that

$$\begin{aligned}
 -1 &\leq T_N(\theta) \leq \lim_{n \rightarrow \infty} T_N(a_n) = -1 \\
 0 &\geq I_N(\theta) \geq \lim_{n \rightarrow \infty} I_N(a_n) = 0 \\
 -1 &\leq F_N(\theta) \leq \lim_{n \rightarrow \infty} F_N(a_n) = -1
 \end{aligned}$$

Hence $T_N(\theta) = -1, I_N(\theta) = 0$ and $F_N(\theta) = -1$. \square

Proposition 2. *If every neutrosophic \mathcal{N} -subalgebra X_N of X satisfies:*

$$T_N(x * y) \leq T_N(y), I_N(x * y) \geq I_N(y), F_N(x * y) \leq F_N(y) \tag{5}$$

for all $x, y \in X$, then X_N is constant.

Proof. Using Equations (1) and (5), we have $T_N(x) = T_N(x * \theta) \leq T_N(\theta), I_N(x) = I_N(x * \theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x * \theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta), I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore X_N is constant. \square

Definition 2. *A neutrosophic \mathcal{N} -structure X_N over X is called a neutrosophic \mathcal{N} -ideal of X if the following assertion is valid:*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(\theta) \leq T_N(x) \leq \vee\{T_N(x * y), T_N(y)\} \\ I_N(\theta) \geq I_N(x) \geq \wedge\{I_N(x * y), I_N(y)\} \\ F_N(\theta) \leq F_N(x) \leq \vee\{F_N(x * y), F_N(y)\} \end{array} \right) \tag{6}$$

Example 2. *The neutrosophic \mathcal{N} -structure X_N over X in Example 1 is a neutrosophic \mathcal{N} -ideal of X .*

Example 3. *Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, *, \theta)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let X_N be a neutrosophic \mathcal{N} -structure over X given by*

$$X_N = \left\{ \frac{x}{(\alpha, 0, \gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \frac{x}{(0, \beta, 0)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then X_N is a neutrosophic \mathcal{N} -ideal of X .

Proposition 3. *Every neutrosophic \mathcal{N} -ideal X_N of X satisfies the following assertions:*

$$(x, y \in X) (x \preceq y \Rightarrow T_N(x) \leq T_N(y), I_N(x) \geq I_N(y), F_N(x) \leq F_N(y)) \tag{7}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$\begin{aligned}
 T_N(x) &\leq \vee\{T_N(x * y), T_N(y)\} = \vee\{T_N(\theta), T_N(y)\} = T_N(y) \\
 I_N(x) &\geq \wedge\{I_N(x * y), I_N(y)\} = \wedge\{I_N(\theta), I_N(y)\} = I_N(y) \\
 F_N(x) &\leq \vee\{F_N(x * y), F_N(y)\} = \vee\{F_N(\theta), F_N(y)\} = F_N(y)
 \end{aligned}$$

This completes the proof. \square

Proposition 4. *Let X_N be a neutrosophic \mathcal{N} -ideal of X . Then*

- (1) $T_N(x * y) \leq T_N((x * y) * y) \Leftrightarrow T_N((x * z) * (y * z)) \leq T_N((x * y) * z)$
- (2) $I_N(x * y) \geq I_N((x * y) * y) \Leftrightarrow I_N((x * z) * (y * z)) \geq I_N((x * y) * z)$
- (3) $F_N(x * y) \leq F_N((x * y) * y) \Leftrightarrow F_N((x * z) * (y * z)) \leq F_N((x * y) * z)$

for all $x, y, z \in X$.

Proof. Note that

$$((x * (y * z)) * z) * z \preceq (x * y) * z \tag{8}$$

for all $x, y, z \in X$. Assume that $T_N(x * y) \leq T_N((x * y) * y)$, $I_N(x * y) \geq I_N((x * y) * y)$ and $F_N(x * y) \leq F_N((x * y) * y)$ for all $x, y \in X$. It follows from Equation (2) and Proposition 3 that

$$\begin{aligned} T_N((x * z) * (y * z)) &= T_N((x * (y * z)) * z) \\ &\leq T_N(((x * (y * z)) * z) * z) \\ &\leq T_N((x * y) * z) \end{aligned}$$

$$\begin{aligned} I_N((x * z) * (y * z)) &= I_N((x * (y * z)) * z) \\ &\geq I_N(((x * (y * z)) * z) * z) \\ &\geq I_N((x * y) * z) \end{aligned}$$

and

$$\begin{aligned} F_N((x * z) * (y * z)) &= F_N((x * (y * z)) * z) \\ &\leq F_N(((x * (y * z)) * z) * z) \\ &\leq F_N((x * y) * z) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose

$$\begin{aligned} T_N((x * z) * (y * z)) &\leq T_N((x * y) * z) \\ I_N((x * z) * (y * z)) &\geq I_N((x * y) * z) \\ F_N((x * z) * (y * z)) &\leq F_N((x * y) * z) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. If we substitute z for y in Equation (9), then

$$\begin{aligned} T_N(x * z) &= T_N((x * z) * \theta) = T_N((x * z) * (z * z)) \leq T_N((x * z) * z) \\ I_N(x * z) &= I_N((x * z) * \theta) = I_N((x * z) * (z * z)) \geq I_N((x * z) * z) \\ F_N(x * z) &= F_N((x * z) * \theta) = F_N((x * z) * (z * z)) \leq F_N((x * z) * z) \end{aligned}$$

for all $x, z \in X$ by using (III) and Equation (1). \square

Theorem 5. Let X_N be a neutrosophic \mathcal{N} -structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic \mathcal{N} -ideal of X , then the nonempty (α, β, γ) -level set of X_N is an ideal of X .

Proof. Assume that $X_N(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Clearly, $\theta \in X_N(\alpha, \beta, \gamma)$. Let $x, y \in X$ be such that $x * y \in X_N(\alpha, \beta, \gamma)$ and $y \in X_N(\alpha, \beta, \gamma)$. Then $T_N(x * y) \leq \alpha$, $I_N(x * y) \geq \beta$, $F_N(x * y) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee \{T_N(x * y), T_N(y)\} \leq \alpha \\ I_N(x) &\geq \bigwedge \{I_N(x * y), I_N(y)\} \geq \beta \\ F_N(x) &\leq \bigvee \{F_N(x * y), F_N(y)\} \leq \gamma \end{aligned}$$

so that $x \in X_N(\alpha, \beta, \gamma)$. Therefore $X_N(\alpha, \beta, \gamma)$ is an ideal of X . \square

Theorem 6. Let X_N be a neutrosophic \mathcal{N} -structure over X and assume that T_N^α, I_N^β and F_N^γ are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then X_N is a neutrosophic \mathcal{N} -ideal of X .

Proof. If there exist $a, b, c \in X$ such that $T_N(\theta) > T_N(a), I_N(\theta) < I_N(b)$ and $F_N(\theta) > F_N(c)$, respectively, then $T_N(\theta) > a_t \geq T_N(a), I_N(\theta) < b_i \leq I_N(b)$ and $F_N(\theta) > c_f \geq F_N(c)$ for some $a_t, c_f \in [-1, 0)$ and $b_i \in (-1, 0]$. Then $\theta \notin T_N^{a_t}, \theta \notin I_N^{b_i}$ and $\theta \notin F_N^{c_f}$. This is a contradiction. Hence, $T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_N(a_t) > \bigvee\{T_N(a_t * b_t), T_N(b_t)\}, I_N(a_i) < \bigwedge\{I_N(a_i * b_i), I_N(b_i)\}$ and $F_N(a_f) > \bigvee\{F_N(a_f * b_f), F_N(b_f)\}$. Then there exist $s_t, s_f \in [-1, 0)$ and $s_i \in (-1, 0]$ such that

$$\begin{aligned} T_N(a_t) &> s_t \geq \bigvee\{T_N(a_t * b_t), T_N(b_t)\} \\ I_N(a_i) &< s_i \leq \bigwedge\{I_N(a_i * b_i), I_N(b_i)\} \\ F_N(a_f) &> s_f \geq \bigvee\{F_N(a_f * b_f), F_N(b_f)\} \end{aligned}$$

It follows that $a_t * b_t \in T_N^{s_t}, b_t \in T_N^{s_t}, a_i * b_i \in I_N^{s_i}, b_i \in I_N^{s_i}, a_f * b_f \in F_N^{s_f}$ and $b_f \in F_N^{s_f}$. However, $a_t \notin T_N^{s_t}, a_i \notin I_N^{s_i}$ and $a_f \notin F_N^{s_f}$. This is a contradiction, and so

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \\ I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \\ F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore X_N is a neutrosophic \mathcal{N} -ideal of X . \square

Proposition 5. For any neutrosophic \mathcal{N} -ideal X_N of X , we have

$$(\forall x, y, z \in X) \left(x * y \preceq z \Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right) \tag{10}$$

Proof. Let $x, y, z \in X$ be such that $x * y \preceq z$. Then $(x * y) * z = \theta$, and so

$$\begin{aligned} T_N(x * y) &\leq \bigvee\{T_N((x * y) * z), T_N(z)\} = \bigvee\{T_N(\theta), T_N(z)\} = T_N(z) \\ I_N(x * y) &\geq \bigwedge\{I_N((x * y) * z), I_N(z)\} = \bigwedge\{I_N(\theta), I_N(z)\} = I_N(z) \\ F_N(x * y) &\leq \bigvee\{F_N((x * y) * z), F_N(z)\} = \bigvee\{F_N(\theta), F_N(z)\} = F_N(z) \end{aligned}$$

It follows that

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\} \end{aligned}$$

This completes the proof. \square

Theorem 7. In a BCK-algebra, every neutrosophic \mathcal{N} -ideal is a neutrosophic \mathcal{N} -subalgebra.

Proof. Let X_N be a neutrosophic \mathcal{N} -ideal of a BCK-algebra X . For any $x, y \in X$, we have

$$\begin{aligned}
 T_N(x * y) &\leq \bigvee \{T_N((x * y) * x), T_N(x)\} = \bigvee \{T_N((x * x) * y), T_N(x)\} \\
 &= \bigvee \{T_N(\theta * y), T_N(x)\} = \bigvee \{T_N(\theta), T_N(x)\} \\
 &\leq \bigvee \{T_N(x), T_N(y)\}
 \end{aligned}$$

$$\begin{aligned}
 I_N(x * y) &\geq \bigwedge \{I_N((x * y) * x), I_N(x)\} = \bigwedge \{I_N((x * x) * y), I_N(x)\} \\
 &= \bigwedge \{I_N(\theta * y), I_N(x)\} = \bigwedge \{I_N(\theta), I_N(x)\} \\
 &\geq \bigwedge \{I_N(y), I_N(x)\}
 \end{aligned}$$

and

$$\begin{aligned}
 F_N(x * y) &\leq \bigvee \{F_N((x * y) * x), F_N(x)\} = \bigvee \{F_N((x * x) * y), F_N(x)\} \\
 &= \bigvee \{F_N(\theta * y), F_N(x)\} = \bigvee \{F_N(\theta), F_N(x)\} \\
 &\leq \bigvee \{F_N(x), F_N(y)\}
 \end{aligned}$$

Hence X_N is a neutrosophic \mathcal{N} -subalgebra of a BCK-algebra X . \square

The converse of Theorem 7 may not be true in general, as seen in the following example.

Example 4. Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.

*	θ	1	2	3	4
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let X_N be a neutrosophic \mathcal{N} -structure over X , which is given as follows:

$$X_N = \left\{ \begin{array}{l} \frac{\theta}{(-0.8, 0, -1)}, \frac{1}{(-0.8, -0.2, -0.9)}, \\ \frac{2}{(-0.2, -0.6, -0.5)}, \frac{3}{(-0.7, -0.4, -0.7)}, \frac{4}{(-0.4, -0.8, -0.3)} \end{array} \right\}$$

Then X_N is a neutrosophic \mathcal{N} -subalgebra of X , but it is not a neutrosophic \mathcal{N} -ideal of X as $T_N(2) = -0.2 > -0.7 = \bigvee \{T_N(2 * 3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge \{I_N(4 * 3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee \{F_N(4 * 3), F_N(3)\}$.

Theorem 7 is not valid in a BCI-algebra; that is, if X is a BCI-algebra, then there is a neutrosophic \mathcal{N} -ideal that is not a neutrosophic \mathcal{N} -subalgebra, as seen in the following example.

Example 5. Consider the neutrosophic \mathcal{N} -ideal X_N of X in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (\theta, 0) * (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence

$$\begin{aligned}
 T_N(x * y) &= 0 > \alpha = \bigvee \{T_N(x), T_N(y)\} \\
 I_N(x * y) &= \beta < 0 = \bigwedge \{I_N(x), I_N(y)\} \text{ or} \\
 F_N(x * y) &= 0 > \gamma = \bigvee \{F_N(x), F_N(y)\}
 \end{aligned}$$

Therefore X_N is not a neutrosophic \mathcal{N} -subalgebra of X .

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$\begin{aligned} X_N^{\omega_t} &:= \{x \in X \mid T_N(x) \leq T_N(\omega_t)\} \\ X_N^{\omega_i} &:= \{x \in X \mid I_N(x) \geq I_N(\omega_i)\} \\ X_N^{\omega_f} &:= \{x \in X \mid F_N(x) \leq F_N(\omega_f)\} \end{aligned}$$

Clearly, $\omega_t \in X_N^{\omega_t}, \omega_i \in X_N^{\omega_i}$ and $\omega_f \in X_N^{\omega_f}$.

Theorem 8. Let ω_t, ω_i and ω_f be any elements of X . If X_N is a neutrosophic \mathcal{N} -ideal of X , then $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X .

Proof. Clearly, $\theta \in X_N^{\omega_t}, \theta \in X_N^{\omega_i}$ and $\theta \in X_N^{\omega_f}$. Let $x, y \in X$ be such that $x * y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ and $y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$. Then

$$\begin{aligned} T_N(x * y) &\leq T_N(\omega_t), T_N(y) \leq T_N(\omega_t) \\ I_N(x * y) &\geq I_N(\omega_i), I_N(y) \geq I_N(\omega_i) \\ F_N(x * y) &\leq F_N(\omega_f), F_N(y) \leq F_N(\omega_f) \end{aligned}$$

It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee \{T_N(x * y), T_N(y)\} \leq T_N(\omega_t) \\ I_N(x) &\geq \bigwedge \{I_N(x * y), I_N(y)\} \geq I_N(\omega_i) \\ F_N(x) &\leq \bigvee \{F_N(x * y), F_N(y)\} \leq F_N(\omega_f) \end{aligned}$$

Hence $x \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$, and therefore $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X . \square

Theorem 9. Let $\omega_t, \omega_i, \omega_f \in X$ and let X_N be a neutrosophic \mathcal{N} -structure over X . Then

(1) If $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X , then the following assertion is valid:

$$(\forall x, y, z \in X) \left(\begin{aligned} T_N(x) \geq \bigvee \{T_N(y * z), T_N(z)\} &\Rightarrow T_N(x) \geq T_N(y) \\ I_N(x) \leq \bigwedge \{I_N(y * z), I_N(z)\} &\Rightarrow I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee \{F_N(y * z), F_N(z)\} &\Rightarrow F_N(x) \geq F_N(y) \end{aligned} \right) \tag{11}$$

(2) If X_N satisfies Equation (11) and

$$(\forall x \in X) (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}$$

then $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X for all $\omega_t \in \text{Im}(T_N), \omega_i \in \text{Im}(I_N)$ and $\omega_f \in \text{Im}(F_N)$.

Proof. (1) Assume that $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X for $\omega_t, \omega_i, \omega_f \in X$. Let $x, y, z \in X$ be such that $T_N(x) \geq \bigvee \{T_N(y * z), T_N(z)\}, I_N(x) \leq \bigwedge \{I_N(y * z), I_N(z)\}$ and $F_N(x) \geq \bigvee \{F_N(y * z), F_N(z)\}$. Then $y * z \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ and $z \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$, where $\omega_t = \omega_i = \omega_f = x$. It follows from (12) that $y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ for $\omega_t = \omega_i = \omega_f = x$. Hence $T_N(y) \leq T_N(\omega_t) = T_N(x), I_N(y) \geq I_N(\omega_i) = I_N(x)$ and $F_N(y) \leq F_N(\omega_f) = F_N(x)$.

(2) Let $\omega_t \in \text{Im}(T_N)$, $\omega_i \in \text{Im}(I_N)$ and $\omega_f \in \text{Im}(F_N)$ and suppose that X_N satisfies Equations (11) and (12). Clearly, $\theta \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ by Equation (12). Let $x, y \in X$ be such that $x * y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ and $y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$. Then

$$\begin{aligned} T_N(x * y) &\leq T_N(\omega_t), T_N(y) \leq T_N(\omega_t) \\ I_N(x * y) &\geq I_N(\omega_i), I_N(y) \geq I_N(\omega_i) \\ F_N(x * y) &\leq F_N(\omega_f), F_N(y) \leq F_N(\omega_f) \end{aligned}$$

which implies that $\bigvee\{T_N(x * y), T_N(y)\} \leq T_N(\omega_t)$, $\bigwedge\{I_N(x * y), I_N(y)\} \geq I_N(\omega_i)$, and $\bigvee\{F_N(x * y), F_N(y)\} \leq F_N(\omega_f)$. It follows from Equation (11) that $T_N(\omega_t) \geq T_N(x)$, $I_N(\omega_i) \leq I_N(x)$ and $F_N(\omega_f) \geq F_N(x)$. Thus, $x \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$, and therefore $X_N^{\omega_t}$, $X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X . \square

Definition 3. A neutrosophic \mathcal{N} -ideal X_N of X is said to be closed if it is a neutrosophic \mathcal{N} -subalgebra of X .

Example 6. Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table.

*	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let X_N be a neutrosophic \mathcal{N} -structure over X which is given as follows:

$$X_N = \left\{ \frac{\theta}{(-0.9, -0.3, -0.8)}, \frac{1}{(-0.7, -0.4, -0.7)}, \frac{a}{(-0.6, -0.8, -0.3)}, \frac{b}{(-0.2, -0.6, -0.3)}, \frac{c}{(-0.2, -0.8, -0.5)} \right\}$$

Then X_N is a closed neutrosophic \mathcal{N} -ideal of X .

Theorem 10. Let X be a BCI-algebra, For any $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [-1, 0]$ and $\beta_1, \beta_2 \in (-1, 0]$ with $\alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\beta_1 > \beta_2$, let $X_N := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic \mathcal{N} -structure over X given as follows:

$$\begin{aligned} T_N : X &\rightarrow [-1, 0], x \mapsto \begin{cases} \alpha_1 & \text{if } x \in X_+ \\ \alpha_2 & \text{otherwise} \end{cases} \\ I_N : X &\rightarrow [-1, 0], x \mapsto \begin{cases} \beta_1 & \text{if } x \in X_+ \\ \beta_2 & \text{otherwise} \end{cases} \\ F_N : X &\rightarrow [-1, 0], x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X_+ \\ \gamma_2 & \text{otherwise} \end{cases} \end{aligned}$$

where $X_+ = \{x \in X \mid \theta \preceq x\}$. Then X_N is a closed neutrosophic \mathcal{N} -ideal of X .

Proof. Because $\theta \in X_+$, we have $T_N(\theta) = \alpha_1 \leq T_N(x)$, $I_N(\theta) = \beta_1 \geq I_N(x)$ and $F_N(\theta) = \gamma_1 \leq F_N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\begin{aligned} T_N(x) &= \alpha_1 \leq \bigvee\{T_N(x * y), T_N(y)\} \\ I_N(x) &= \beta_1 \geq \bigwedge\{I_N(x * y), I_N(y)\} \\ F_N(x) &= \gamma_1 \leq \bigvee\{F_N(x * y), F_N(y)\} \end{aligned}$$

Suppose that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we have

$$\begin{aligned} T_N(x) &= \alpha_2 = \bigvee \{T_N(x * y), T_N(y)\} \\ I_N(x) &= \beta_2 = \bigwedge \{I_N(x * y), I_N(y)\} \\ F_N(x) &= \gamma_2 = \bigvee \{F_N(x * y), F_N(y)\} \end{aligned}$$

For any $x, y \in X$, if any one of x and y does not belong to X_+ , then

$$\begin{aligned} T_N(x * y) &\leq \alpha_2 = \bigvee \{T_N(x), T_N(y)\} \\ I_N(x * y) &\geq \beta_2 = \bigwedge \{I_N(x), I_N(y)\} \\ F_N(x * y) &\leq \gamma_2 = \bigvee \{F_N(x), F_N(y)\} \end{aligned}$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence

$$\begin{aligned} T_N(x * y) &= \alpha_1 = \bigvee \{T_N(x), T_N(y)\} \\ I_N(x * y) &= \beta_1 = \bigwedge \{I_N(x), I_N(y)\} \\ F_N(x * y) &= \gamma_1 = \bigvee \{F_N(x), F_N(y)\} \end{aligned}$$

Therefore X_N is a closed neutrosophic \mathcal{N} -ideal of X . \square

Proposition 6. Every closed neutrosophic \mathcal{N} -ideal X_N of a BCI-algebra X satisfies the following condition:

$$(\forall x \in X) (T_N(\theta * x) \leq T_N(x), I_N(\theta * x) \geq I_N(x), F_N(\theta * x) \leq F_N(x)) \tag{13}$$

Proof. Straightforward. \square

We provide conditions for a neutrosophic \mathcal{N} -ideal to be closed.

Theorem 11. Let X be a BCI-algebra. If X_N is a neutrosophic \mathcal{N} -ideal of X that satisfies the condition of Equation (13), then X_N is a neutrosophic \mathcal{N} -subalgebra and hence is a closed neutrosophic \mathcal{N} -ideal of X .

Proof. Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Equations (10) and (13), we have

$$\begin{aligned} T_N(x * y) &\leq \bigvee \{T_N(x), T_N(\theta * y)\} \leq \bigvee \{T_N(x), T_N(y)\} \\ I_N(x * y) &\geq \bigwedge \{I_N(x), I_N(\theta * y)\} \geq \bigwedge \{I_N(x), I_N(y)\} \\ F_N(x * y) &\leq \bigvee \{F_N(x), F_N(\theta * y)\} \leq \bigvee \{F_N(x), F_N(y)\} \end{aligned}$$

Hence X_N is a neutrosophic \mathcal{N} -subalgebra and is therefore a closed neutrosophic \mathcal{N} -ideal of X . \square

Author Contributions: In this paper, Y. B. Jun conceived and designed the main idea and wrote the paper, H. Bordbar performed the idea, checking contents and finding examples, F. Smarandache analyzed the data and checking language.

Conflicts of Interest: The authors declare no conflict of interest.

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