10-2017

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Article

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Received: 16 September 2017; Accepted: 16 October 2017; Published: 18 October 2017

Abstract: The notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras is introduced, and several properties are investigated. Relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic commutative $\mathcal{N}$-ideal are considered.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-ideal; neutrosophic commutative $\mathcal{N}$-ideal

MSC: 06F35, 03G25, 03B52

1. Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set.

Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophy" [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy"/"intuitionistic fuzzy" logic/set and "neutrosophic" logic/set, i.e., the *included middle* component (Lupasco–Nicolescu's logic in philosophy), i.e., the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(t, i, f) = (\text{truth, indeterminacy, falsehood}).$$

For more details, refer to the site http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

Jun et al. [2] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [3] introduced the notion of neutrosophic $\mathcal{N}$-structure and applied it to a semigroup. Jun et al. [4] applied the notion of neutrosophic $\mathcal{N}$-structure to BCK/BCI-algebras. They introduced the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigated related properties. They also considered characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal, and discussed relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. They provided conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal. BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [5], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [6]).

In this paper, we introduce the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigate several properties. We consider relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We discuss characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

2. Preliminaries

By a *BCI*-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$
(II) $(x * (x * y)) * y = 0,$
(III) $x * x = 0,$
(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a *BCI*-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK*-algebra.

We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \;\Rightarrow\; x * y = 0).$$

In a *BCK/BCI*-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{1}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2}$$

A *BCK*-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x))\,. \tag{3}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{4}$$
$$(\forall x, y \in X)\,(x * y \in I,\ y \in I \;\Rightarrow\; x \in I)\,. \tag{5}$$

A subset I of a *BCK*-algebra X is called a *commutative ideal* of X if it satisfies (4) and

$$(\forall x, y, z \in X)\,((x * y) * z \in I,\ z \in I \;\Rightarrow\; x * (y * (y * x)) \in I)\,. \tag{6}$$

Lemma 1. *An ideal I is commutative if and only if the following assertion is valid.*

$$(\forall x, y \in X)\,(x * y \in I \;\Rightarrow\; x * (y * (y * x)) \in I)\,. \tag{7}$$

We refer the reader to the books [7,8] for further information regarding *BCK/BCI*-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure*, we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [2]). A *neutrosophic* $\mathcal{N}$*-structure* over a nonempty universe of discourse X (see [3]) is defined to be the structure

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{8}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

Note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)\,.$$

3. Neutrosophic Commutative $\mathcal{N}$-Ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 1 ([4]). *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid.*

$$(\forall x, y \in X) \begin{pmatrix} T_N(0) \leq T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \\ I_N(0) \geq I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \\ F_N(0) \leq F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \end{pmatrix}. \tag{9}$$

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic commutative $\mathcal{N}$-ideal of X if the following assertions are valid.*

$$(\forall x \in X)\,(T_N(0) \leq T_N(x),\ I_N(0) \geq I_N(x),\ F_N(0) \leq F_N(x))\,, \tag{10}$$

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x * (y * (y * x))) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} \\ I_N(x * (y * (y * x))) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} \\ F_N(x * (y * (y * x))) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} \end{pmatrix}. \tag{11}$$

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1.*

Table 1. Cayley table for the binary operation "*".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8, -0.2, -0.9)}, \frac{1}{(-0.3, -0.9, -0.5)}, \frac{2}{(-0.7, -0.7, -0.4)}, \frac{3}{(-0.3, -0.6, -0.7)}, \frac{4}{(-0.5, -0.3, -0.1)} \right\}$$

over X is a neutrosophic commutative $\mathcal{N}$-ideal of X.

Theorem 1. *Every neutrosophic commutative $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X. For every $x, z \in X$, we have

$$T_N(x) = T_N(x*(0*(0*x))) \leq \bigvee\{T_N((x*0)*z), T_N(z)\} = \bigvee\{T_N(x*z), T_N(z)\},$$
$$I_N(x) = I_N(x*(0*(0*x))) \geq \bigwedge\{I_N((x*0)*z), I_N(z)\} = \bigwedge\{I_N(x*z), I_N(z)\},$$
$$F_N(x) = F_N(x*(0*(0*x))) \leq \bigvee\{F_N((x*0)*z), F_N(z)\} = \bigvee\{F_N(x*z), F_N(z)\}$$

by putting $y = 0$ in (11) and using (1). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

The converse of Theorem 1 is not true in general as seen in the following example.

Example 2. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 2.*

Table 2. Cayley table for the binary operation "$*$"

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8,-0.1,-0.7)}, \frac{1}{(-0.7,-0.6,-0.6)}, \frac{2}{(-0.6,-0.2,-0.4)}, \frac{3}{(-0.3,-0.8,-0.4)}, \frac{4}{(-0.3,-0.8,-0.4)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-ideal of X. But it is not a neutrosophic commutative $\mathcal{N}$-ideal of X since $F_N(2*(3*(3*2)) = F_N(2) = -0.4 \nleq -0.7 = \bigvee\{F_N((2*3)*0), F_N(0)\}$.

We consider characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x*(y*(y*x))) \leq T_N(x*y), \\ I_N(x*(y*(y*x))) \geq I_N(x*y), \\ F_N(x*(y*(y*x))) \leq F_N(x*y) \end{array} \right). \tag{12}$$

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. The assertion (12) is by taking $z = 0$ in (11) and using (1) and (10).

Conversely, suppose that a neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the condition (12). Then,

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x*y) \leq \bigvee\{T_N((x*y)*z), T_N(z)\} \\ I_N(x*y) \geq \bigwedge\{I_N((x*y)*z), I_N(z)\} \\ F_N(x*y) \leq \bigvee\{F_N((x*y)*z), F_N(z)\} \end{array} \right). \tag{13}$$

It follows that the condition (11) is induced by (12) and (13). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

Lemma 2 ([4]). *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x*y \preceq z \Rightarrow \left\{ \begin{array}{l} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{array} \right. \right). \tag{14}$$

Theorem 3. *In a commutative BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic commutative $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a commutative BCK-algebra X. For any $x, y, z \in X$, we have

$$\begin{aligned}&((x*(y*(y*x)))*((x*y)*z))*z\\&=((x*(y*(y*x)))*z)*((x*y)*z)\\&\preceq (x*(y*(y*x)))*(x*y)\\&=(x*(x*y))*(y*(y*x))=0,\end{aligned}$$

that is, $(x*(y*(y*x)))*((x*y)*z) \preceq z$. It follows from Lemma 2 that

$$\begin{aligned}T_N(x*(y*(y*x))) &\leq \bigvee\{T_N((x*y)*z),\ T_N(z)\},\\ I_N(x*(y*(y*x))) &\geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\},\\ F_N(x*(y*(y*x))) &\leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}.\end{aligned}$$

Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets.

$$\begin{aligned}T_N^\alpha &:= \{x \in X \mid T_N(x) \leq \alpha\},\\ I_N^\beta &:= \{x \in X \mid I_N(x) \geq \beta\},\\ F_N^\gamma &:= \{x \in X \mid F_N(x) \leq \gamma\}.\end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)-*level set* of $X_{\mathbf{N}}$. It is clear that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma.$$

Theorem 4. *If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then T_N^α, I_N^β and F_N^γ are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ whenever they are nonempty.*

We call T_N^α, I_N^β and F_N^γ *level commutative ideals* of $X_{\mathbf{N}}$.

Proof. Assume that T_N^α, I_N^β and F_N^γ are nonempty for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $x \in T_N^\alpha$, $y \in I_N^\beta$ and $z \in F_N^\gamma$ for some $x, y, z \in X$. Thus, $T_N(0) \leq T_N(x) \leq \alpha$, $I_N(0) \geq I_N(y) \geq \beta$, and $F_N(0) \leq F_N(z) \leq \gamma$, that is, $0 \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Let $(x*y)*z \in T_N^\alpha$ and $z \in T_N^\alpha$. Then, $T_N((x*y)*z) \leq \alpha$ and $T_N(z) \leq \alpha$, which imply that

$$T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z), T_N(z)\} \leq \alpha,$$

that is, $x*(y*(y*x)) \in T_N^\alpha$. If $(a*b)*c \in I_N^\beta$ and $c \in I_N^\beta$, then $I_N((a*b)*c) \geq \beta$ and $I_N(c) \geq \beta$. Thus

$$I_N(a*(b*(b*c))) \geq \bigwedge\{I_N((a*b)*c), I_N(c)\} \geq \beta,$$

and so $a * (b * (b * c)) \in I_N^\beta$. Finally, suppose that $(u * v) * w \in F_N^\gamma$ and $w \in F_N^\gamma$. Then, $F_N((u * v) * w) \leq \gamma$ and $F_N(w) \leq \gamma$. Thus,

$$F_N(u * (v * (v * w))) \leq \bigvee \{F_N((u * v) * w), F_N(w)\} \leq \gamma,$$

that is, $u * (v * (v * w)) \in F_N^\gamma$. Therefore, T_N^α, I_N^β and F_N^γ are commutative ideals of X. □

Corollary 1. *Let $X_\mathbf{N}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_\mathbf{N}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_\mathbf{N}$ is a commutative ideal of X.*

Proof. Straightforward. □

Lemma 3 ([4]). *Let $X_\mathbf{N}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^α, I_N^β and F_N^γ are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_\mathbf{N}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Theorem 5. *Let $X_\mathbf{N}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^α, I_N^β and F_N^γ are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $X_\mathbf{N}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. If T_N^α, I_N^β and F_N^γ are commutative ideals of X, then they are ideals of X. Hence, $X_\mathbf{N}$ is a neutrosophic $\mathcal{N}$-ideal of X by Lemma 3. Let $x, y \in X$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ such that $T_N(x * y) = \alpha$, $I_N(x * y) = \beta$ and $F_N(x * y) = \gamma$. Then, $x * y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Since $T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$ is a commutative ideal of X, it follows from Lemma 1 that $x * (y * (y * x)) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.
Hence

$$\begin{aligned} T_N(x * (y * (y * x))) &\leq \alpha = T_N(x * y), \\ I_N(x * (y * (y * x))) &\geq \beta = I_N(x * y), \\ F_N(x * (y * (y * x))) &\leq \gamma = F_N(x * y). \end{aligned}$$

Therefore, $X_\mathbf{N}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 2. □

Theorem 6. *Let $f : X \to X$ be an injective mapping. Given a neutrosophic $\mathcal{N}$-structure $X_\mathbf{N}$ over X, the following are equivalent.*

(1) *$X_\mathbf{N}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the following condition.*

$$(\forall x \in X) \begin{pmatrix} T_N(f(x)) = T_N(x) \\ I_N(f(x)) = I_N(x) \\ F_N(f(x)) = F_N(x) \end{pmatrix}. \tag{15}$$

(2) *T_N^α, I_N^β and F_N^γ are commutative ideals of $X_\mathbf{N}$, satisfying the following condition.*

$$f(T_N^\alpha) = T_N^\alpha, f(I_N^\beta) = I_N^\beta, f(F_N^\gamma) = F_N^\gamma. \tag{16}$$

Proof. Let $X_\mathbf{N}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the condition (15). Then, T_N^α, I_N^β and F_N^γ are commutative ideals of $X_\mathbf{N}$ by Theorem 4. Let $\alpha \in \mathrm{Im}(T_N)$, $\beta \in \mathrm{Im}(I_N)$, $\gamma \in \mathrm{Im}(F_N)$ and $x \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $T_N(f(x)) = T_N(x) \leq \alpha$, $I_N(f(x)) = I_N(x) \geq \beta$ and $F_N(f(x)) = F_N(x) \leq \gamma$. Thus, $f(x) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$, which shows that $f(T_N^\alpha) \subseteq T_N^\alpha$, $f(I_N^\beta) \subseteq I_N^\beta$ and $f(F_N^\gamma) \subseteq F_N^\gamma$. Let $y \in X$ be such that $f(y) = x$. Then, $T_N(y) = T_N(f(y)) = T_N(x) \leq \alpha$, $I_N(y) = I_N(f(y)) = I_N(x) \geq \beta$

and $F_N(y) = F_N(f(y)) = F_N(x) \leq \gamma$, which imply that $y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Thus, $x = f(y) \in f(T_N^\alpha) \cap f(I_N^\beta) \cap f(F_N^\gamma)$, and so $T_N^\alpha \subseteq f(T_N^\alpha)$, $I_N^\beta \subseteq f(I_N^\beta)$ and $F_N^\gamma \subseteq f(F_N^\gamma)$. Therefore (16) is valid.

Conversely, assume that T_N^α, I_N^β and F_N^γ are commutative ideals of $X_{\mathbf{N}}$, satisfying the condition (16). Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 5. Let $x, y, z \in X$ be such that $T_N(x) = \alpha$, $I_N(y) = \beta$ and $F_N(z) = \gamma$. Note that

$$T_N(x) = \alpha \Longleftrightarrow x \in T_N^\alpha \text{ and } x \notin T_N^{\tilde{\alpha}} \text{ for all } \alpha > \tilde{\alpha},$$
$$I_N(y) = \beta \Longleftrightarrow y \in I_N^\beta \text{ and } y \notin I_N^{\tilde{\beta}} \text{ for all } \beta < \tilde{\beta},$$
$$F_N(z) = \gamma \Longleftrightarrow z \in F_N^\gamma \text{ and } z \notin F_N^{\tilde{\gamma}} \text{ for all } \gamma > \tilde{\gamma}.$$

It follows from (16) that $f(x) \in T_N^\alpha$, $f(y) \in I_N^\beta$ and $f(z) \in F_N^\gamma$. Hence, $T_N(f(x)) \leq \alpha$, $I_N(f(y)) \geq \beta$ and $F_N(f(z)) \leq \gamma$. Let $\tilde{\alpha} = T_N(f(x))$, $\tilde{\beta} = I_N(f(y))$ and $\tilde{\gamma} = F_N(f(z))$. If $\alpha > \tilde{\alpha}$, then $f(x) \in T_N^{\tilde{\alpha}} = f\left(T_N^{\tilde{\alpha}}\right)$, and thus $x \in T_N^{\tilde{\alpha}}$ since f is one to one. This is a contradiction. Hence, $T_N(f(x)) = \alpha = T_N(x)$. If $\beta < \tilde{\beta}$, then $f(y) \in I_N^{\tilde{\beta}} = f\left(I_N^{\tilde{\beta}}\right)$ which implies from the injectivity of f that $y \in I_N^{\tilde{\beta}}$, a contradiction. Hence, $I_N(f(x)) = \beta = I_N(x)$. If $\gamma > \tilde{\gamma}$, then $f(z) \in F_N^{\tilde{\gamma}} = f\left(F_N^{\tilde{\gamma}}\right)$. Since f is one to one, we have $z \in F_N^{\tilde{\gamma}}$ which is a contradiction. Thus, $F_N(f(x)) = \gamma = F_N(x)$. This completes the proof. □

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$X_{\mathbf{N}}^{\omega_t} := \{x \in X \mid T_N(x) \leq T_N(\omega_t)\},$$
$$X_{\mathbf{N}}^{\omega_i} := \{x \in X \mid I_N(x) \geq I_N(\omega_i)\},$$
$$X_{\mathbf{N}}^{\omega_f} := \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}.$$

Obviously, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Lemma 4 ([4]). *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Theorem 7. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X.*

Proof. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then it is a neutrosophic $\mathcal{N}$-ideal of X and so $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X by Lemma 4. Let $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for any $x, y \in X$. Then, $T_N(x * y) \leq T_N(\omega_t)$, $I_N(x * y) \geq T_N(\omega_i)$ and $F_N(x * y) \leq F_N(\omega_f)$. It follows from Theorem 2 that

$$T_N(x * (y * (y * x))) \leq T_N(x * y) \leq T_N(\omega_t),$$
$$I_N(x * (y * (y * x))) \geq I_N(x * y) \geq I_N(\omega_i),$$
$$F_N(x * (y * (y * x))) \leq F_N(x * y) \leq F_N(\omega_f).$$

Hence, $x * (y * (y * x)) \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X by Lemma 1. □

Theorem 8. *Any commutative ideal of X can be realized as level commutative ideals of some neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. Let A be a commutative ideal of X and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X in which

$$T_N : X \to [-1,0],\ x \mapsto \begin{cases} \alpha & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases}$$
$$I_N : X \to [-1,0],\ x \mapsto \begin{cases} \beta & \text{if } x \in A, \\ -1 & \text{otherwise}, \end{cases}$$
$$F_N : X \to [-1,0],\ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Division into the following cases will verify that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.

If $(x * y) * z \in A$ and $z \in A$, then $x * (y * (y * x) \in A$. Thus,

$$T_N((x*y)*z) = T_N(z) = T_N(x*(y*(y*x))) = \alpha,$$
$$I_N((x*y)*z) = I_N(z) = I_N(x*(y*(y*x))) = \beta,$$
$$F_N((x*y)*z) = F_N(z) = F_N(x*(y*(y*x))) = \gamma,$$

and so (11) is clearly verified.

If $(x * y) * z \notin A$ and $z \notin A$, then $T_N((x * y) * z) = T_N(z) = 0$, $I_N((x * y) * z) = I_N(z) = -1$ and $F_N((x * y) * z) = F_N(z) = 0$. Hence

$$T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z),\ T_N(z)\},$$
$$I_N(x*(y*(y*x))) \geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\},$$
$$F_N(x*(y*(y*x))) \leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}.$$

If $(x * y) * z \in A$ and $z \notin A$, then $T_N((x * y) * z) = \alpha$, $T_N(z) = 0$, $I_N((x * y) * z) = \beta$, $I_N(z) = -1$, $F_N((x * y) * z) = \gamma$ and $F_N(z) = 0$. Therefore,

$$T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z),\ T_N(z)\},$$
$$I_N(x*(y*(y*x))) \geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\},$$
$$F_N(x*(y*(y*x))) \leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}.$$

Similarly, if $(x * y) * z \notin A$ and $z \in A$, then (11) is verified. Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. Obviously, $T_N^{\alpha} = A$, $I_N^{\beta} = A$ and $F_N^{\gamma} = A$. This completes the proof. □

4. Conclusions

In order to deal with the negative meaning of information, Jun et al. [2] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [9,10] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigated several properties. We have considered relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We have discussed characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

Acknowledgments: The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812). The authors wish to thank the anonymous reviewers for their valuable suggestions.

Author Contributions: Y.B. Jun initiated the main idea of the work and wrote the paper. S.Z. Song and Y.B. Jun conceived and designed the new definitions and results. F. Smarandache and S.Z. Song performed finding examples and checking contents. All authors have read and approved the final manuscript for submission.

Conflicts of Interest: The authors declare no conflict of interest.

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