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Abstract: A bipolar single-valued neutrosophic (BSVN) graph structure is a generalization of a bipolar fuzzy graph. In this research paper, we present certain concepts of BSVN graph structures. We describe some operations on BSVN graph structures and elaborate on these with examples. Moreover, we investigate some related properties of these operations.

Keywords: graph structure; bipolar single-valued neutrosophic (BSVN) graph structure

MSC: 03E72; 05C72; 05C78; 05C99

1. Introduction

Fuzzy graphs are mathematical models for dealing with combinatorial problems in different domains, including operations research, optimization, computer science and engineering. In 1965, Zadeh [1] proposed fuzzy set theory to deal with uncertainty in abundant meticulous real-life phenomena. Fuzzy set theory is affluently applicable in real-time systems consisting of information with different levels of precision. Subsequently, Atanassov [2] introduced the idea of intuitionistic fuzzy sets in 1986. However, for many real-life phenomena, it is necessary to deal with the implicit counter property of a particular event. Zhang [3] initiated the concept of bipolar fuzzy sets in 1994. Evidently bipolar fuzzy sets and intuitionistic fuzzy sets seem to be similar, but they are completely different sets. Bipolar fuzzy sets have large number of applications in image processing and spatial reasoning. Bipolar fuzzy sets are more practical, advantageous and applicable in real-life phenomena. However, both bipolar fuzzy sets and intuitionistic fuzzy sets cope with incomplete information, because they do not consider indeterminate or inconsistent information that clearly appears in many systems of different fields, including belief systems and decision-support systems. Smarandache [4] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set has three constituents: truth membership, indeterminacy membership and falsity membership, for which each membership value is a real standard or non-standard subset of the unit interval $[0^-, 1^+]$. In real-life problems, neutrosophic sets can be applied more appropriately by using the single-valued neutrosophic sets defined by Smarandache [4] and Wang et al. [5]. Deli et al. [6] considered bipolar neutrosophic sets as a generalization of bipolar fuzzy sets. They also studied some operations and applications in decision-making problems.

On the basis of Zadeh's fuzzy relations [7], Kauffman defined fuzzy graphs [8]. In 1975, Rosenfeld [9] discussed a fuzzy analogue of different graph-theoretic ideas. Later on, Bhattacharya [10] gave some remarks on fuzzy graphs in 1987. Akram [11] first introduced the notion of bipolar fuzzy graphs. In 2011, Dinesh and Ramakrishnan [12] studied fuzzy graph structures and discussed their properties. In 2016, Akram and Akmal [13] proposed the concept of bipolar fuzzy graph structures. Certain concepts on graphs have been discussed in [14–19]. Ye [20–22] considered several applications of single-valued neutrosophic sets. Inthis research paper, we present certain concepts of bipolar single-valued neutrosophic graph structures (BSVNGSs). We introduce some operations on BSVNGSs and elaborate on these with examples. Moreover, we investigate some relevant and remarkable properties of these operators.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [23–29].

2. Bipolar Single-Valued Neutrosophic Graph Structures

Definition 1. *[4] A neutrosophic set N on a non-empty set V is an object of the form*

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where $T_N, I_N, F_N : V \to [0^-, 1^+]$ *and there is no restriction on the sum of* $T_N(v)$, $I_N(v)$ *and* $F_N(v)$ *for all* $v \in V$.

Definition 2. *[5] A single-valued neutrosophic set N on a non-empty set V is an object of the form*

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where $T_N, I_N, F_N : V \to [0, 1]$ *and the sum of* $T_N(v)$, $I_N(v)$ *and* $F_N(v)$ *is confined between 0 and 3 for all* $v \in V$.

Definition 3. *[23] A BSVN set on a non-empty set V is an object of the form*

$$B = \{(v, T_B^P(v), I_B^P(v), F_B^P(v), T_B^N(v), I_B^N(v), F_B^N(v)) : v \in V\}$$

where $T_B^P, I_B^P, F_B^P : V \to [0, 1]$ *and* $T_B^N, I_B^N, F_B^N : V \to [-1, 0]$. *The positive values* $T_B^P(v), I_B^P(v)$ *and* $F_B^P(v)$ *denote the truth, indeterminacy and falsity membership values of an element* $v \in V$*, whereas negative values* $T_B^N(v), I_B^N(v)$ *and* $F_B^N(v)$ *indicate the implicit counter property of truth, indeterminacy and falsity membership values of an element* $v \in V$.

Definition 4. *[23] A BSVN graph on a non-empty set V is a pair* $G = (B, R)$*, where B is a BSVN set on V and R is a BSVN relation in V such that*

$$T_R^P(b,d) \le T_B^P(b) \wedge T_B^P(d), \quad I_R^P(b,d) \le I_B^P(b) \wedge I_B^P(d), \quad F_R^P(b,d) \le F_B^P(b) \vee F_B^P(d),$$
$$T_R^N(b,d) \ge T_B^N(b) \vee T_B^N(d), \quad I_R^N(b,d) \ge I_B^N(b) \vee I_B^N(d), \quad F_R^N(b,d) \ge F_B^N(b) \wedge F_B^N(d)$$

for all $b, d \in V$.

We now define the BSVNGS.

Definition 5. *[30] A BSVNGS of a graph structure* $\check{G}_s = (V, V_1, V_2, \ldots, V_m)$ *is denoted by* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$*, where* $B =< b, T^P(b), I^P(b), F^P(b), T^N(b), I^N(b), F^N(b) >$ *is a BSVN set on the set V and* $B_k =< (b,d), T_k^P(b,d), I_k^P(b,d), F_k^P(b,d), T_k^N(b,d), I_k^N(b,d), F_k^N(b,d) >$ *are the BSVN sets on* V_k *such that*

$$T_k^P(b,d) \le \min\{T^P(b), T^P(d)\}, \; I_k^P(b,d) \le \min\{I^P(b), I^P(d)\}, \; F_k^P(b,d) \le \max\{F^P(b), F^P(d)\},$$
$$T_k^N(b,d) \ge \max\{T^N(b), T^N(d)\}, \; I_k^N(b,d) \ge \max\{I^N(b), I^N(d)\}, \; F_k^N(b,d) \ge \min\{F^N(b), F^N(d)\}$$

for all $b, d \in V$. *Note that* $0 \le T_k^P(b,d) + I_k^P(b,d) + F_k^P(b,d) \le 3$, $-3 \le T_k^N(b,d) + I_k^N(b,d) + F_k^N(b,d) \le 0$ *for all* $(b,d) \in V_k$*, and* (b,d) *represents an edge between two vertices b and d. In this paper we use bd in place of* (b,d).

Example 1. *Consider a graph structure* $\check{G}_s = (V, V_1, V_2, V_3)$ *such that* $V = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$, $V_1 = \{b_1b_2, b_2b_7, b_4b_8, b_6b_8, b_5b_6, b_3b_4\}$, $V_2 = \{b_1b_5, b_5b_7, b_3b_6, b_7b_8\}$, *and* $V_3 = \{b_1b_3, b_2b_4\}$. *Let* B *be a BSVN set on* V *given in Table 1 and* B_1, B_2 *and* B_3 *be BSVN sets on* V_1, V_2 *and* V_3, *respectively, given in Table 2.*

Table 1. Bipolar single-valued neutrosophic (BSVN) set B on vertex set V.

B	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
T^P	0.5	0.4	0.4	0.5	0.3	0.4	0.5	0.3
I^P	0.4	0.3	0.4	0.4	0.2	0.4	0.5	0.4
F^P	0.6	0.5	0.4	0.6	0.4	0.7	0.4	0.5
T^N	−0.5	−0.4	−0.4	−0.5	−0.3	−0.4	−0.5	−0.3
I^N	−0.4	−0.3	−0.4	−0.4	−0.2	−0.4	−0.5	−0.4
F^N	−0.6	−0.5	−0.4	−0.6	−0.4	−0.7	−0.4	−0.5

Table 2. BSVN sets B_1, B_2 and B_3.

B_1	b_1b_2	b_2b_7	b_4b_8	b_6b_8	b_5b_6	b_3b_4	B_2	b_1b_5	b_5b_7	b_3b_6	b_7b_8	B_3	b_1b_3	b_2b_4
T^P	0.4	0.4	0.3	0.3	0.3	0.4	T^P	0.3	0.3	0.4	0.3	T^P	0.4	0.4
I^P	0.3	0.3	0.4	0.4	0.2	0.4	I^P	0.2	0.2	0.4	0.4	I^P	0.4	0.3
F^P	0.6	0.5	0.6	0.7	0.7	0.6	F^P	0.6	0.4	0.7	0.5	F^P	0.6	0.6
T^N	−0.4	−0.4	−0.3	−0.3	−0.3	−0.4	T^N	−0.3	−0.3	−0.4	−0.3	T^N	−0.4	−0.4
I^N	−0.3	−0.3	−0.4	−0.4	−0.2	−0.4	I^N	−0.2	−0.2	−0.4	−0.4	I^N	−0.4	−0.3
F^N	−0.6	−0.5	−0.6	−0.7	−0.7	−0.6	F^N	−0.6	−0.4	−0.7	−0.5	F^N	−0.6	−0.6

By direct calculations, it is easy to show that $\check{G}_{bn} = (B, B_1, B_2, B_3)$ *is a BSVNGS. This BSVNGS is shown in Figure 1. Generated with LaTeXDraw 2.0.8 on Saturday March 11 20:30:24 PKT 2017.*

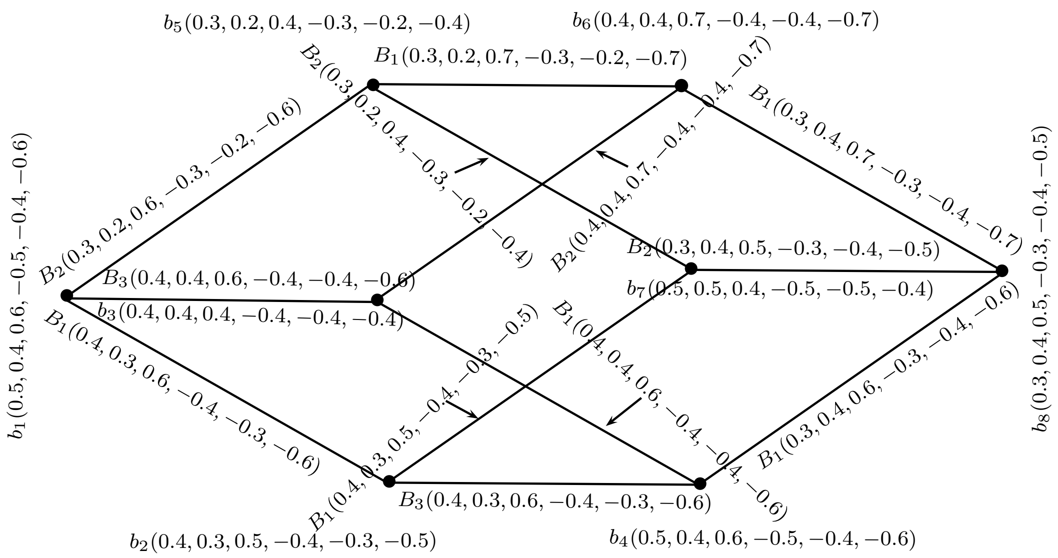

Figure 1. A BSVN graph structure.

Definition 6. *A BSVNGS* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ *is called a* B_k*-cycle if* $(supp(B), supp(B_1), supp(B_2), \ldots, supp(B_m))$ *is a* B_k*-cycle.*

Example 2. *Consider a BSVNGS* $\check{G}_{bn} = (B, B_1, B_2)$ *as shown in Figure 2.*

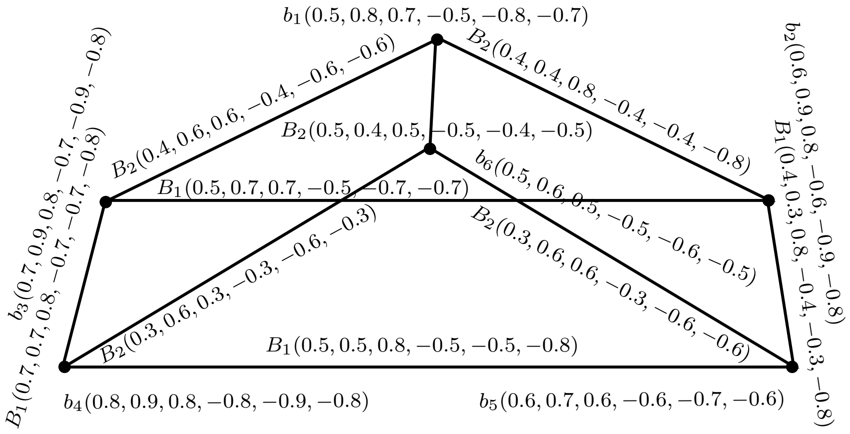

Figure 2. A BSVN B_1-cycle.

$\check{G}_{bn}$ is a B_1-cycle, as $(supp(B), supp(B_1), supp(B_2))$ is a B_1-cycle, that is, b_2-b_3-b_4-b_5-b_2.

Definition 7. *A BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is a BSVN fuzzy B_k-cycle (for any k) if $\check{G}_{bn}$ is a B_k-cycle and no unique B_k-edge bd exists in $\check{G}_{bn}$, such that $T^P_{B_k}(bd) = \min\{T^P_{B_k}(ef) : ef \in B_k = supp(B_k)\}$, $I^P_{B_k}(bd) = \min\{I^P_{B_k}(ef) : ef \in B_k = supp(B_k)\}$, $F^P_{B_k}(bd) = \max\{F^P_{B_k}(ef) : ef \in B_k = supp(B_k)\}$, $T^N_{B_k}(bd) = \max\{T^N_{B_k}(ef) : ef \in B_k = supp(B_k)\}$, $I^N_{B_k}(bd) = \max\{I^N_{B_k}(ef) : ef \in B_k = supp(B_k)\}$ or $F^N_{B_k}(bd) = \min\{F^N_{B_k}(ef) : ef \in B_k = supp(B_k)\}$.*

Example 3. *Consider a BSVNGS $\check{G}_{bn} = (B, B_1, B_2)$ as depicted in Figure 3.*

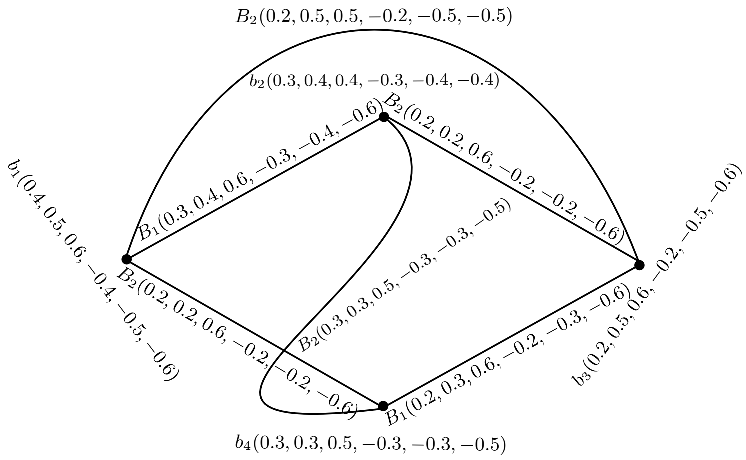

Figure 3. A BSVN fuzzy B_2-cycle.

$T^P_{B_2}(bd) = \min\{T^P_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$, $I^P_{B_2}(bd) = \min\{I^P_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$, $F^P_{B_2}(bd) = \max\{F^P_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$, $T^N_{B_2}(bd) = \max\{T^N_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$, $I^N_{B_2}(bd) = \max\{I^N_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$ *or* $F^N_{B_2}(bd) = \min\{F^N_{B_2}(ef) : ef \in B_2 = supp(B_2)\}$.

Definition 8. *A sequence of vertices (distinct) in a BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is called a B_k-path, that is, $b_1, b_2, \ldots, b_m$, such that $b_{k-1}b_k$ is a BSVN B_k-edge, for all $k = 2, \ldots, m$.*

Example 4. *Consider a BSVNGS $\check{G}_{bn} = (B, B_1, B_2)$ as represented in Figure 4.*

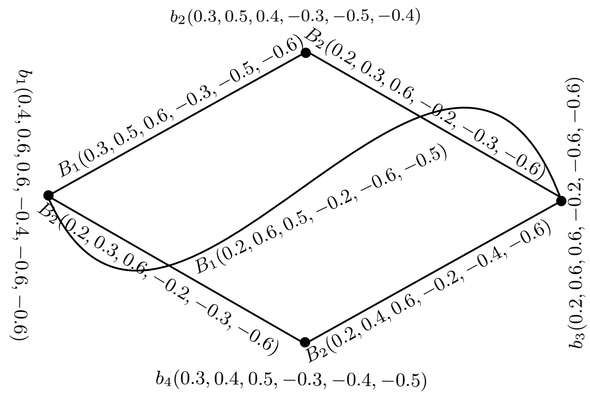

Figure 4. A BSVN B_2-path.

In this BSVNGS, the sequence of distinct vertices b_1, b_4, b_3, b_2 *is a BSVN* B_2*-path.*

Definition 9. *Let* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ *be a BSVNGS. The positive truth strength* $T^P.P_{B_k}$*, positive falsity strength* $F^P.P_{B_k}$*, and positive indeterminacy strength* $I^P.P_{B_k}$ *of a* B_k*-path,* $P_{B_k} = b_1, b_2, \ldots, b_n$*, are defined as*

$$T^P.P_{B_k} = \bigwedge_{h=2}^{n} [T^P_{B_k}(b_{h-1}b_h)],\ I^P.P_{B_k} = \bigwedge_{h=2}^{n} [I^P_{B_k}(b_{h-1}b_h)],\ F^P.P_{B_k} = \bigvee_{h=2}^{n} [F^P_{B_k}(b_{h-1}b_h)]$$

Similarly, the negative truth strength $T^N.P_{B_k}$*, negative falsity strength* $F^N.P_{B_k}$*, and negative indeterminacy strength* $I^N.P_{B_k}$ *of a* B_k*-path are defined as*

$$T^N.P_{B_k} = \bigvee_{h=2}^{n} [T^N_{B_k}(b_{h-1}b_h)],\ I^N.P_{B_k} = \bigvee_{h=2}^{n} [I^N_{B_k}(b_{h-1}b_h)],\ F^N.P_{B_k} = \bigwedge_{h=2}^{n} [F^N_{B_k}(b_{h-1}b_h)]$$

Example 5. *Consider a BSVNGS* $\check{G}_{bn} = (B, B_1, B_2, B_3)$ *as shown in Figure 5.*

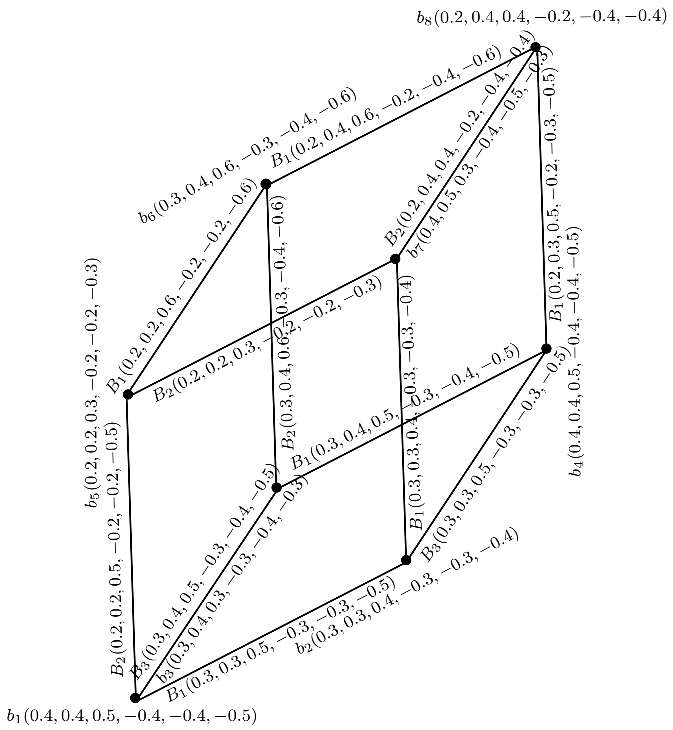

Figure 5. A bipolar single-valued neutrosophic graph structure (BSVNGS) $\check{G}_{bn} = (B, B_1, B_2, B_3)$.

In this BSVNGS, there is a B_1-path, that is, $P_{B_1} = b_5, b_6, b_8, b_4, b_3$. Thus, $T^N.P_{B_1} = -0.2$, $I^N.P_{B_1} = -0.2$, $F^N.P_{B_2} = -0.6$, $T^P.P_{B_1} = 0.2$, $I^P.P_{B_1} = 0.2$ and $F^P.P_{B_2} = 0.6$.

Definition 10. *Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS. Then*

- *The B_k-positive strength of connectedness of truth between two nodes b and d is defined by* $T^{P\infty}_{B_k}(bd) = \bigvee_{l\geq 1}\{T^{Pl}_{B_k}(bd)\}$, *such that* $T^{Pl}_{B_k}(bd) = (T^{Pl-1}_{B_k} \circ T^{P1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $T^{P2}_{B_k}(bd) = (T^{P1}_{B_k} \circ T^{P1}_{B_k})(bd) = \bigvee_y (T^{P1}_{B_k}(by) \wedge T^{P1}_{B_k}(yd))$.
- *The B_k-positive strength of connectedness of indeterminacy between two nodes b and d is defined by* $I^{P\infty}_{B_k}(bd) = \bigvee_{l\geq 1}\{I^{Pl}_{B_k}(bd)\}$, *such that* $I^{Pl}_{B_k}(bd) = (I^{Pl-1}_{B_k} \circ I^{P1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $I^{P2}_{B_k}(bd) = (I^{P1}_{B_k} \circ I^{P1}_{B_k})(bd) = \bigvee_y (I^{P1}_{B_k}(by) \wedge I^{P1}_{B_k}(yd))$.
- *The B_k-positive strength of connectedness of falsity between two nodes b and d is defined by* $F^{P\infty}_{B_k}(bd) = \bigwedge_{l\geq 1}\{F^{Pl}_{B_k}(bd)\}$, *such that* $F^{Pl}_{B_k}(bd) = (F^{Pl-1}_{B_k} \circ F^{P1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $F^{P2}_{B_k}(bd) = (F^{P1}_{B_k} \circ F^{P1}_{B_k})(bd) = \bigwedge_y (F^{P1}_{B_k}(by) \vee F^{P1}_{B_k}(yd))$.
- *The B_k-negative strength of connectedness of truth between two nodes b and d is defined by* $T^{N\infty}_{B_k}(bd) = \bigwedge_{l\geq 1}\{T^{Nl}_{B_k}(bd)\}$, *such that* $T^{Nl}_{B_k}(bd) = (T^{Nl-1}_{B_k} \circ T^{N1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $T^{N2}_{B_k}(bd) = (T^{N1}_{B_k} \circ T^{N1}_{B_k})(bd) = \bigwedge_y (T^{N1}_{B_k}(by) \vee T^{N1}_{B_k}(yd))$.
- *The B_k-negative strength of connectedness of indeterminacy between two nodes b and d is defined by* $I^{N\infty}_{B_k}(bd) = \bigwedge_{l\geq 1}\{I^{Nl}_{B_k}(bd)\}$, *such that* $I^{Nl}_{B_k}(bd) = (I^{Nl-1}_{B_k} \circ I^{N1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $I^{N2}_{B_k}(bd) = (I^{N1}_{B_k} \circ I^{N1}_{B_k})(bd) = \bigwedge_y (I^{N1}_{B_k}(by) \vee I^{N1}_{B_l}(yd))$.
- *The B_k-negative strength of connectedness of falsity between two nodes b and d is defined by* $F^{N\infty}_{B_k}(bd) = \bigvee_{l\geq 1}\{F^{Nl}_{B_k}(bd)\}$, *such that* $F^{Nl}_{B_k}(bd) = (F^{Nl-1}_{B_k} \circ F^{N1}_{B_k})(bd)$ *for* $l \geq 2$ *and* $F^{N2}_{B_k}(bd) = (F^{N1}_{B_k} \circ F^{N1}_{B_k})(bd) = \bigvee_y (F^{N1}_{B_k}(by) \wedge F^{N1}_{B_k}(yd))$.

Definition 11. *Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS and "b" be a node in $\check{G}_{bn}$. Let $(B', B'_1, B'_2, \ldots, B'_m)$ be a BSVN subgraph structure of $\check{G}_{bn}$ induced by $B \setminus \{b\}$ such that $\forall e \neq b, f \neq b$*

$$T^P_{B'}(b) = I^P_{B'}(b) = F^P_{B'}(b) = T^P_{B'_k}(be) = I^P_{B'_k}(be) = F^P_{B'_k}(be) = 0$$
$$T^N_{B'}(b) = I^N_{B'}(b) = F^N_{B'}(b) = T^N_{B'_k}(be) = I^N_{B'_k}(be) = F^N_{B'_k}(be) = 0$$
$$T^P_{B'}(e) = T^P_B(e),\ I^P_{B'}(e) = I^P_B(e),\ F^P_{B'}(e) = F^P_B(e),\ T^N_{B'}(e) = T^N_B(e),\ I^N_{B'}(e) = I^N_B(e),\ F^N_{B'}e) = F^N_B(e)$$
$$T^P_{B'_k}(ef) = T^P_{B_k}(ef),\ I^P_{B'_k}(ef) = I^P_{B_k}(ef),\ F^P_{B'_k}(ef) = F^P_{B_k}(ef),\ T^N_{B'_k}(ef) = T^N_{B_k}(ef),\ I^N_{B'_k}(ef) = I^N_{B_k}(ef)$$
$$F^N_{B'_k}(ef) = F^N_{B_k}(ef), \forall \text{ edges } be, ef \in \check{G}_{bn}$$

Then b is a BSVN fuzzy B_k cut-vertex if $T^{P\infty}_{B_k}(ef) > T^{P\infty}_{B'_k}(ef)$, $I^{P\infty}_{B_k}(ef) > I^{P\infty}_{B'_k}(ef)$, $F^{P\infty}_{B_k}(ef) > F^{P\infty}_{B'_k}(ef)$, $T^{N\infty}_{B_k}(ef) < T^{N\infty}_{B'_k}(ef)$, $I^{N\infty}_{B_k}(ef) < I^{N\infty}_{B'_k}(ef)$ and $F^{N\infty}_{B_k}(ef) < F^{N\infty}_{B'_k}(ef)$, for some $e, f \in B \setminus \{b\}$. Note that vertex b is a BSVN fuzzy $B_k - T^P$ cut-vertex if $T^{P\infty}_{B_k}(ef) > T^{P\infty}_{B'_k}(ef)$, it is a BSVN fuzzy $B_k - I^P$ cut-vertex if $I^{P\infty}_{B_k}(ef) > I^{P\infty}_{B'_k}(ef)$, and it is a BSVN fuzzy $B_k - F^P$ cut-vertex if $F^{P\infty}_{B_k}(ef) > F^{P\infty}_{B'_k}(ef)$. Moreover, vertex b is a BSVN fuzzy $B_k - T^N$ cut-vertex if $T^{N\infty}_{B_k}(ef) < T^{N\infty}_{B'_k}(ef)$, it is a BSVN fuzzy $B_k - I^N$ cut-vertex if $I^{N\infty}_{B_k}(ef) < I^{N\infty}_{B'_k}(ef)$ and it is a BSVN fuzzy $B_k - F^N$ cut-vertex if $F^{N\infty}_{B_k}(ef) < F^{N\infty}_{B'_k}(ef)$.

Example 6. *Consider a BSVNGS $\check{G}_{bn} = (B, B_1, B_2)$ as depicted in Figure 6, and let $\check{G}'_{bh} = (B', B'_1, B'_2)$ be a BSVN subgraph structure of the BSVNGS $\check{G}_{bn}$, which is obtained through deletion of vertex b_2.*

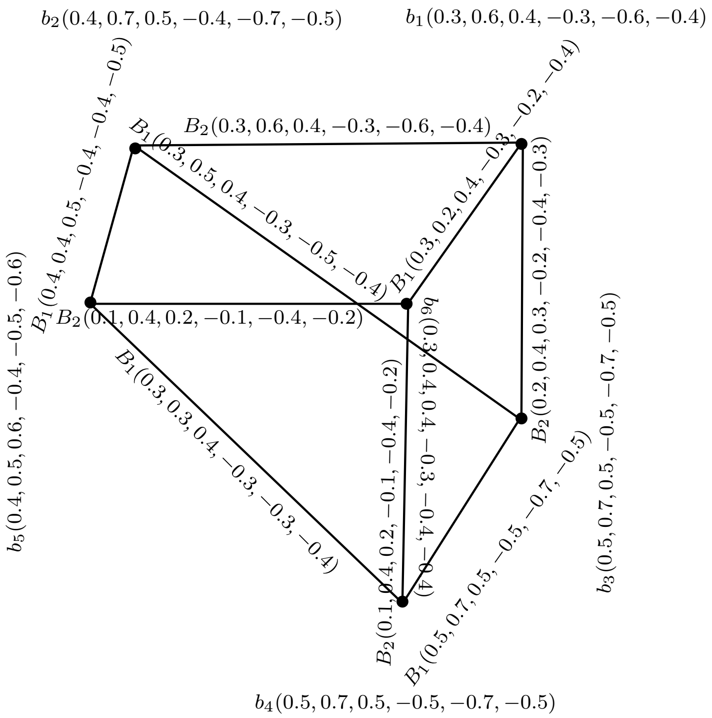

Figure 6. A BSVNGS $\check{G}_{bn} = (B, B_1, B_2)$.

The vertex b_2 *is a BSVN fuzzy* $B_1 - I^P$ *cut-vertex and a BSVN fuzzy* $B_1 - I^N$ *cut-vertex, because* $I^{P\infty}_{B'_1}(b_2b_5) = 0$, $I^{P\infty}_{B_1}(b_2b_5) = 0.5$, $I^{P\infty}_{B'_1}(b_4b_3) = 0.7$, $I^{P\infty}_{B_1}(b_4b_3) = 0.7$, $I^{P\infty}_{B'_1}(b_3b_5) = 0.3$, $I^{P\infty}_{B_1}(b_3b_5) = 0.4$. $I^{N\infty}_{B'_1}(b_2b_5) = 0 > -0.5 = I^{N\infty}_{B_1}(b_2b_5)$, $I^{N\infty}_{B'_1}(b_4b_3) = 0.7 = I^{N\infty}_{B_1}(b_4b_3)$ *and* $I^{N\infty}_{B'_1}(b_3b_5) = -0.3 > -0.4 = I^{N\infty}_{B_1}(b_3b_5)$.

Definition 12. *Suppose* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ *is a BSVNGS and bd is a* B_k*-edge. Let* $(B', B'_1, B'_2, \ldots, B'_m)$ *be a BSVN fuzzy spanning subgraph structure of* $\check{G}_{bn}$*, such that*

$$T^P_{B'_k}(bd) = 0 = I^P_{B'_k}(bd) = F^P_{B'_k}(bd),\ T^N_{B'_k}(bd) = 0 = I^N_{B'_k}(bd) = F^N_{B'_k}(bd)$$
$$T^P_{B'_k}(gh) = T^P_{B_k}(gh),\ I^P_{B'_k}(gh) = I^P_{B_k}(gh)$$
$$F^P_{B'_k}(bd) = F^P_{B_k}(bd),\ T^N_{B'_k}(gh) = T^N_{B_k}(gh),\ I^N_{B'_k}(gh) = I^N_{B_k}(gh),\ F^N_{B'_k}(bd) = F^N_{B_k}(bd),\ \forall \text{ edges } gh \neq bd$$

Then bd is a BSVN fuzzy B_k*-bridge if* $T^{P\infty}_{B_k}(ef) > T^{P\infty}_{B'_k}(ef)$, $I^{P\infty}_{B_k}(ef) > I^{P\infty}_{B'_k}(ef)$, $F^{P\infty}_{B_k}(ef) > F^{P\infty}_{B'_k}(ef)$, $T^{N\infty}_{B_k}(ef) < T^{N\infty}_{B'_k}(ef)$, $I^{N\infty}_{B_k}(ef) < I^{N\infty}_{B'_k}(ef)$ *and* $F^{N\infty}_{B_k}(ef) < F^{N\infty}_{B'_k}(ef)$, *for some* $e, f \in V$. *Note that bd is a BSVN fuzzy* $B_k - T^P$ *bridge if* $T^{P\infty}_{B_k}(ef) > T^{P\infty}_{B'_k}(ef)$, *it is a BSVN fuzzy* $B_k - I^P$ *bridge if* $I^{P\infty}_{B_k}(ef) > I^{P\infty}_{B'_k}(ef)$ *and it is a BSVN fuzzy* $B_k - F^P$ *bridge if* $F^{P\infty}_{B_k}(ef) > F^{P\infty}_{B'_k}(ef)$. *Moreover, bd is a BSVN fuzzy* $B_k - T^N$ *bridge if* $T^{N\infty}_{B_k}(ef) < T^{N\infty}_{B'_k}(ef)$, *it is a BSVN fuzzy* $B_k - I^N$ *bridge if* $I^{N\infty}_{B_k}(ef) < I^{N\infty}_{B'_k}(ef)$ *and it is a BSVN fuzzy* $B_k - F^N$ *bridge if* $F^{N\infty}_{B_k}(ef) < F^{N\infty}_{B'_k}(ef)$.

Example 7. *Consider a BSVNGS* $\check{G}_{bn} = (B, B_1, B_2)$ *as depicted in Figure 6 and* $\check{G}'_{bs} = (B', B'_1, B'_2)$*, a BSVN spanning subgraph structure of the BSVNGS* $\check{G}_{bn}$ *obtained by deleting* B_1*-edge* (b_2b_5) *and that is shown in Figure 7.*

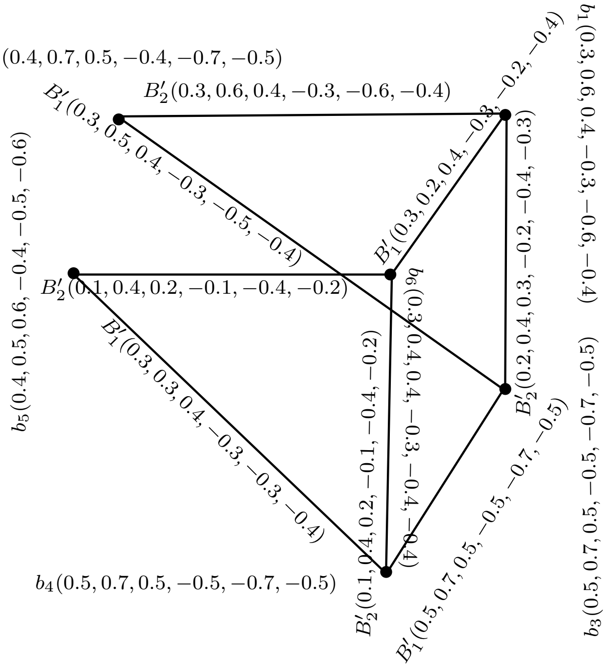

Figure 7. A BSVNGS $\check{G}'_{bs} = (B', B'_1, B'_2)$.

This edge (b_2b_5) *is a BSVN fuzzy* B_1*-bridge, as* $T^{P\infty}_{B'_1}(b_2b_5) = 0.3$, $T^{P\infty}_{B_1}(b_2b_5) = 0.4$, $I^{P\infty}_{B'_1}(b_2b_5) = 0.3$, $I^{P\infty}_{B_1}(b_2b_5) = 0.4$, $F^{P\infty}_{B'_1}(b_2b_5) = 0.4$, $F^{P\infty}_{B_1}(b_2b_5) = 0.5.$, $T^{N\infty}_{B'_1}(b_2b_5) = -0.3 > -0.4 = T^{N\infty}_{B_1}(b_2b_5)$, $I^{N\infty}_{B'_1}(b_2b_5) = -0.3 > -0.4 = I^{N\infty}_{B_1}(b_2b_5)$, *and* $F^{N\infty}_{B'_1}(b_2b_5) = -0.4 > -0.5 = F^{N\infty}_{B_1}(b_2b_5)$.

Definition 13. *A BSVNGS* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ *is a* B_k*-tree if* $(supp(B), supp(B_1), supp(B_2), \ldots, supp(B_m))$ *is a* B_k*-tree. Alternatively,* $\check{G}_{bn}$ *is a* B_k*-tree if* $\check{G}_{bn}$ *has a subgraph induced by* $supp(B_k)$ *that forms a tree.*

Example 8. *Consider the BSVNGS* $\check{G}_{bn} = (B, B_1, B_2)$ *as depicted in Figure 8.*

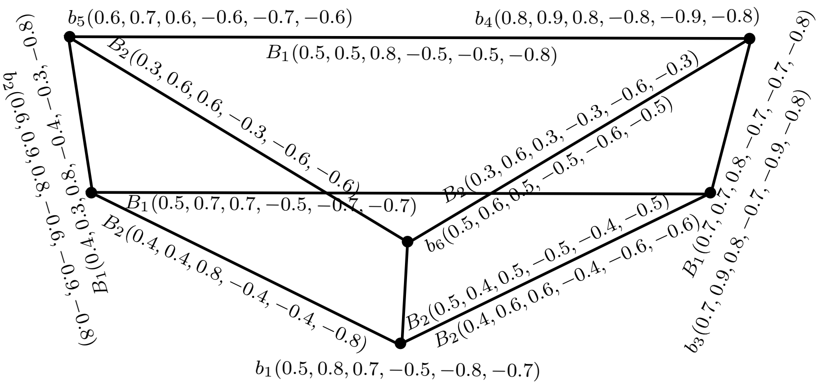

Figure 8. A BSVN B_2-tree.

This BSVNGS $\check{G}_{bn} = (B, B_1, B_2)$ *is a* B_2*-tree, as* $(supp(B), supp(B_1), supp(B_2))$ *is a* B_2*-tree.*

Definition 14. *A BSVNGS* $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ *is a BSVN fuzzy* B_k*-tree if* $\check{G}_{bn}$ *has a BSVN fuzzy spanning subgraph structure* $\check{H}_{bn} = (B', B'_1, B'_2, \ldots, B'_m)$ *such that for all* B_k*-edges, bd not in* $\check{H}_{bn}$*:*

1. $\check{H}_{bn}$ *is a* B'_k*-tree.*

2. $T^P_{B_k}(bd) < T^{P\infty}_{B'_k}(bd)$, $I^P_{B_k}(bd) < I^{P\infty}_{B'_k}(bd)$, $F^P_{B_k}(bd) < F^{P\infty}_{B'_k}(bd)$, $T^N_{B_k}(bd) > T^{N\infty}_{B'_k}(bd)$, $I^N_{B_k}(bd) > I^{N\infty}_{B'_k}(bd)$, *and* $F^N_{B_k}(bd) > F^{N\infty}_{B'_k}(bd)$.

In particular, $\check{G}_{bn}$ *is a BSVN fuzzy* $B_k - T^P$ *tree if* $T^P_{B_k}(bd) < T^{P\infty}_{B'_k}(bd)$*, it is a BSVN fuzzy* $B_k - I^P$ *tree if* $I^P_{B_k}(bd) < I^{P\infty}_{B'_k}(bd)$*, and it is a BSVN fuzzy* $B_k - F^P$ *tree if* $F^P_{B_k}(bd) > F^{P\infty}_{B'_k}(bd)$*. Moreover,* $\check{G}_{bn}$ *is a BSVN fuzzy* $B_k - T^N$ *tree if* $T^N_{B_k}(bd) > T^{N\infty}_{B'_k}(bd)$*, it is a BSVN fuzzy* $B_k - I^N$ *tree if* $I^N_{B_k}(bd) > I^{N\infty}_{B'_k}(bd)$*, and it is a BSVN fuzzy* $B_k - F^N$*tree if* $F^N_{B_k}(bd) < F^{N\infty}_{B'_k}(bd)$*.*

Example 9. *Consider the BSVNGS* $\check{G}_{bn} = (B, B_1, B_2)$ *as depicted in Figure 9.*

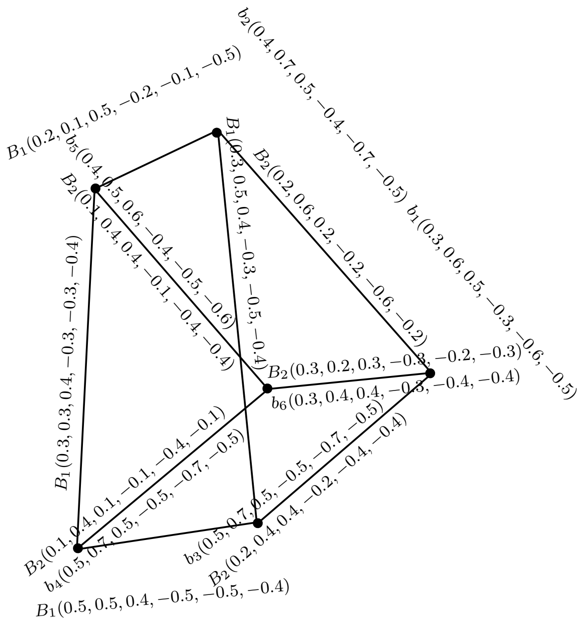

Figure 9. A BSVN fuzzy B_1-tree.

It is B_2*-tree, rather than a* B_1*-tree. However, it is a BSVN fuzzy* B_1*-tree, because it has a BSVN fuzzy spanning subgraph* (B', B'_1, B'_2) *as a* B'_1*-tree, which is obtained through the deletion of the* B_1*-edge* b_2b_5 *from* $\check{G}_{bn}$*. Moreover,* $T^{P\infty}_{B'_1}(b_2b_5) = 0.3$, $T^P_{B_1}(b_2b_5) = 0.2$, $I^{P\infty}_{B'_1}(b_2b_5) = 0.3$, $I^P_{B_1}(b_2b_5) = 0.1$, $F^{P\infty}_{B'_1}(b_2b_5) = 0.4$, $F_{B_1}(b_2b_5) = 0.5$. $T^{N\infty}_{B'_1}(b_2b_5) = -0.3 < -0.2 = T^N_{B_1}(b_2b_5)$, $I^{N\infty}_{B'_1}(b_2b_5) = -0.3 < -0.1 = I^N_{B_1}(b_2b_5)$ *and* $F^{N\infty}_{B'_1}(b_2b_5) = -0.4 > -0.5 = F_{B_1}(b_2b_5)$.

Now we define the operations on BSVNGSs.

Definition 15. *Let* $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ *and* $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ *be two BSVNGSs. The Cartesian product of* $\check{G}_{b1}$ *and* $\check{G}_{b2}$*, denoted by*

$$\check{G}_{b1} \times \check{G}_{b2} = (B_1 \times B_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1m} \times B_{2m})$$

is defined as

$$\text{(i)} \quad \begin{cases} T^P_{(B_1 \times B_2)}(bd) = (T^P_{B_1} \times T^P_{B_2})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_1 \times B_2)}(bd) = (I^P_{B_1} \times I^P_{B_2})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_1 \times B_2)}(bd) = (F^P_{B_1} \times F^P_{B_2})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$$

(ii) $\begin{cases} T^N_{(B_1\times B_2)}(bd) = (T^N_{B_1} \times T^N_{B_2})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_1\times B_2)}(bd) = (I^N_{B_1} \times I^N_{B_2})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_1\times B_2)}(bd) = (F^N_{B_1} \times F^N_{B_2})(bd) = F^N_{B_1}(b) \wedge F^P_{B_2}(d) \end{cases}$

for all $(bd) \in V_1 \times V_2$,

(iii) $\begin{cases} T^P_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}} \times T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}} \times I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}} \times F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$

(iv) $\begin{cases} T^N_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}} \times T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}} \times I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\times B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}} \times F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$

for all $b \in V_1$, $(d_1d_2) \in V_{2k}$, *and*

(v) $\begin{cases} T^P_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (T^P_{B_{1k}} \times T^P_{B_{2k}})(b_1d)(b_2d) = T^P_{B_2}(d) \wedge T^P_{B_{1k}}(b_1b_2) \\ I^P_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (I^P_{B_{1k}} \times I^P_{B_{2k}})(b_1d)(b_2d) = I^P_{B_2}(d) \wedge I^P_{B_{2k}}(b_1b_2) \\ F^P_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (F^P_{B_{1k}} \times F^P_{B_{2k}})(b_1d)(b_2d) = F^P_{B_2}(d) \vee F^P_{B_{2k}}(b_1b_2) \end{cases}$

(vi) $\begin{cases} T^N_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (T^N_{B_{1k}} \times T^N_{B_{2k}})(b_1d)(b_2d) = T^N_{B_2}(d) \vee T^N_{B_{1k}}(b_1b_2) \\ I^N_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (I^N_{B_{1k}} \times I^N_{B_{2k}})(b_1d)(b_2d) = I^N_{B_2}(d) \vee I^N_{B_{2k}}(b_1b_2) \\ F^N_{(B_{1k}\times B_{2k})}(b_1d)(b_2d) = (F^N_{B_{1k}} \times F^N_{B_{2k}})(b_1d)(b_2d) = F^N_{B_2}(d) \wedge F^N_{B_{2k}}(b_1b_2) \end{cases}$

for all $d \in V_2$, $(b_1b_2) \in V_{1k}$.

Example 10. *Consider* $\check{G}_{b1} = (B_1, B_{11}, B_{12})$ *and* $\check{G}_{b2} = (B_2, B_{21}, B_{22})$ *as BSVNGSs of GSs* $\check{G}_{s1} = (V_1, V_{11}, V_{12})$ *and* $\check{G}_{s2} = (V_2, V_{21}, V_{22})$*, respectively, as depicted in Figure 10, where* $V_{11} = \{b_1b_2\}$*,* $V_{12} = \{b_3b_4\}$*,* $V_{21} = \{d_1d_2\}$*, and* $V_{22} = \{d_2d_3\}$*.*

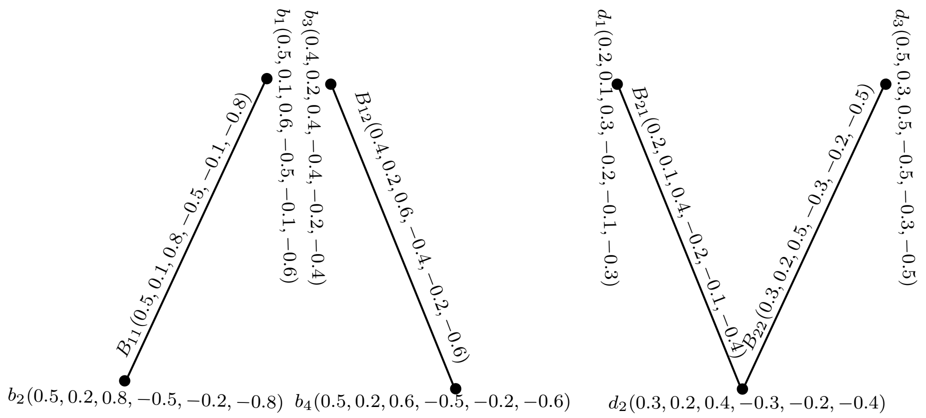

Figure 10. Two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$.

The Cartesian product of $\check{G}_{b1}$ *and* $\check{G}_{b2}$*, defined as* $\check{G}_{b1} \times \check{G}_{b2} = \{B_1 \times B_2, B_{11} \times B_{21}, B_{12} \times B_{22}\}$*, is depicted in Figures 11 and 12.*

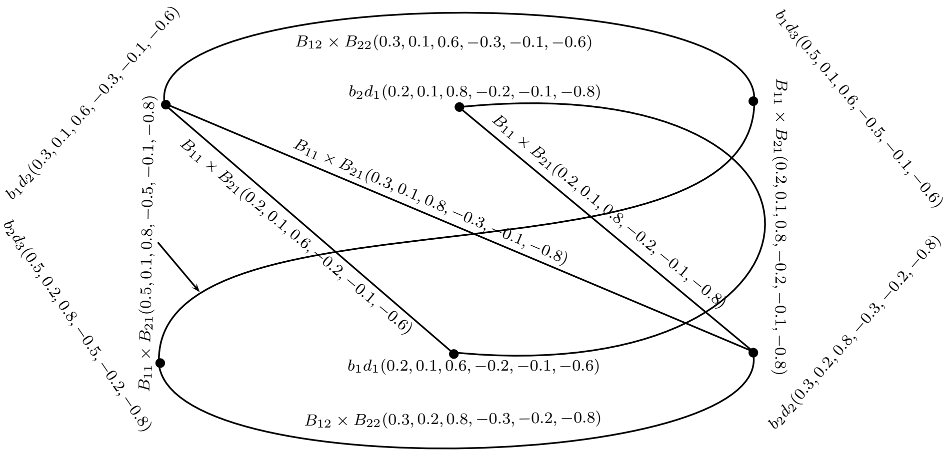

Figure 11. $\check{G}_{b1} \times \check{G}_{b2}$.

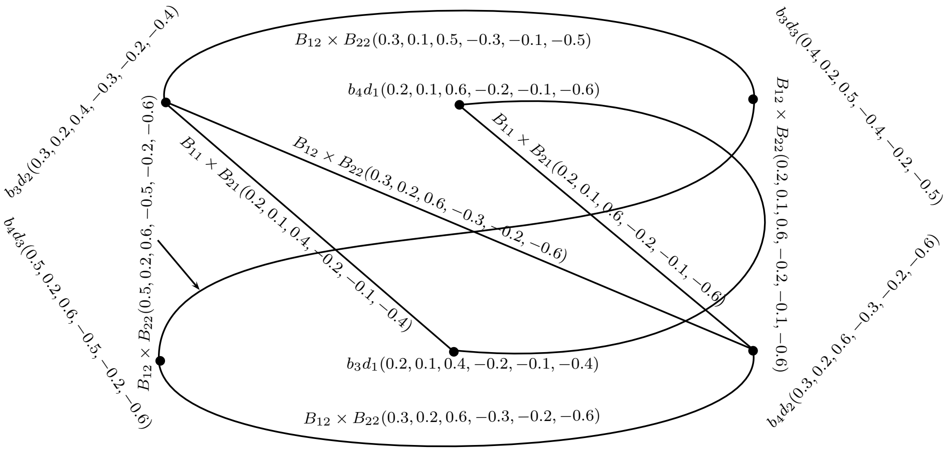

Figure 12. $\check{G}_{b1} \times \check{G}_{b2}$.

Theorem 1. *The Cartesian product* $\check{G}_{b1} \times \check{G}_{b2} = (B_1 \times B_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1m} \times B_{2m})$ *of two BSVNSGSs of GSs* $\check{G_{s1}}$ *and* $\check{G_{s2}}$ *is a BSVNGS of* $\check{G_{s1}} \times \check{G_{s2}}$.

Proof. Consider two cases:

Case 1. For $b \in V_1$, $d_1d_2 \in V_{2k}$,

$$\begin{aligned} T^P_{(B_{1k} \times B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ &\leq T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1 \times B_2)}(bd_1) \wedge T^P_{(B_1 \times B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k} \times B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ &\geq T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1 \times B_2)}(bd_1) \vee T^N_{(B_1 \times B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\times B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2)\\
&\leq I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\times B_2)}(bd_1) \wedge I^P_{(B_1\times B_2)}(bd_2)
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\times B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2)\\
&\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\times B_2)}(bd_1) \vee I^N_{(B_1\times B_2)}(bd_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\times B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2)\\
&\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\times B_2)}(bd_1) \vee F^P_{(B_1\times B_2)}(bd_2)
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\times B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\times B_2)}(bd_1) \wedge F^N_{(B_1\times B_2)}(bd_2)
\end{aligned}$$

for $bd_1, bd_2 \in V_1 \times V_2$.

Case 2. For $b \in V_2$, $d_1d_2 \in V_{1k}$,

$$\begin{aligned}
T^P_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= T^P_{B_2}(b) \wedge T^P_{B_{1k}}(d_1d_2)\\
&\leq T^P_{B_2}(b) \wedge [T^P_{B_1}(d_1) \wedge T^P_{B_1}(d_2)]\\
&= [T^P_{B_2}(b) \wedge T^P_{B_1}(d_1)] \wedge [T^P_{B_2}(b) \wedge T^P_{B_1}(d_2)]\\
&= T^P_{(B_1\times B_2)}(d_1b) \wedge T^P_{(B_1\times B_2)}(d_2b)
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= T^N_{B_2}(b) \vee T^N_{B_{1k}}(d_1d_2)\\
&\geq T^N_{B_2}(b) \vee [T^N_{B_1}(d_1) \vee T^N_{B_1}(d_2)]\\
&= [T^N_{B_2}(b) \vee T^N_{B_1}(d_1)] \vee [T^N_{B_2}(b) \vee T^N_{B_1}(d_2)]\\
&= T^N_{(B_1\times B_2)}(d_1b) \vee T^N_{(B_1\times B_2)}(d_2b)
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= I^P_{B_2}(b) \wedge I^P_{B_{1k}}(d_1d_2)\\
&\leq I^P_{B_2}(b) \wedge [I^P_{B_1}(d_1) \wedge I^P_{B_1}(d_2)]\\
&= [I^P_{B_2}(b) \wedge I^P_{B_1}(d_1)] \wedge [I^P_{B_2}(b) \wedge I^P_{B_1}(d_2)]\\
&= I^P_{(B_1\times B_2)}(d_1b) \wedge I^P_{(B_1\times B_2)}(d_2b)
\end{aligned}$$

$$\begin{aligned} I^N_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= I^N_{B_2}(b) \vee I^N_{B_{1k}}(d_1d_2) \\ &\geq I^N_{B_2}(b) \vee [I^N_{B_1}(d_1) \vee I^N_{B_1}(d_2)] \\ &= [I^N_{B_2}(b) \vee I^N_{B_1}(d_1)] \vee [I^N_{B_2}(b) \vee I^N_{B_1}(d_2)] \\ &= I^N_{(B_1\times B_2)}(d_1b) \vee I^N_{(B_1\times B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= F^P_{B_2}(b) \vee F^P_{B_{1k}}(d_1d_2) \\ &\leq F^P_{B_2}(b) \vee [F^P_{B_1}(d_1) \vee F^P_{B_1}(d_2)] \\ &= [F^P_{B_2}(b) \vee F^P_{B_1}(d_1)] \vee [F^P_{B_2}(b) \vee F^P_{B_1}(d_2)] \\ &= F^P_{(B_1\times B_2)}(d_1b) \vee F^P_{(B_1\times B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}\times B_{2k})}((d_1b)(d_2b)) &= F^N_{B_2}(b) \wedge F^N_{B_{1k}}(d_1d_2) \\ &\geq F^N_{B_2}(b) \wedge [F^N_{B_1}(d_1) \wedge F^N_{B_1}(d_2)] \\ &= [F^N_{B_2}(b) \wedge F^N_{B_1}(d_1)] \wedge [F^N_{B_2}(b) \wedge F^N_{B_1}(d_2)] \\ &= F^N_{(B_1\times B_2)}(d_1b) \wedge F^P_{(B_1\times B_2)}(d_2b) \end{aligned}$$

for $d_1b, d_2b \in V_1 \times V_2$.

Both cases hold for all $k \in \{1, 2, \ldots, m\}$. This completes the proof. □

Definition 16. *Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be two BSVNGSs. The cross product of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by*

$$\check{G}_{b1} * \check{G}_{b2} = (B_1 * B_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1m} * B_{2m})$$

is defined as

(i) $$\begin{cases} T^P_{(B_1*B_2)}(bd) = (T^P_{B_1} * T^P_{B_2})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_1*B_2)}(bd) = (I^P_{B_1} * I^P_{B_2})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_1*B_2)}(bd) = (F^P_{B_1} * F^P_{B_2})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$$

(ii) $$\begin{cases} T^N_{(B_1*B_2)}(bd) = (T^N_{B_1} * T^N_{B_2})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_1*B_2)}(bd) = (I^N_{B_1} * I^N_{B_2})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_1*B_2)}(bd) = (F^N_{B_1} * F^N_{B_2})(bd) = F^N_{B_1}(b) \wedge F^P_{B_2}(d) \end{cases}$$

for all $(bd) \in V_1 \times V_2$, and

(iii) $$\begin{cases} T^P_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} * T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} * I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} * F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$$

(iv) $$\begin{cases} T^N_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} * T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} * I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}*B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} * F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$$

for all $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$.

Example 11. *The cross product of BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Figure 10 is defined as $\check{G}_{b1} * \check{G}_{b2} = \{B_1 * B_2, B_{11} * B_{21}, B_{12} * B_{22}\}$ and is depicted in Figure 13.*

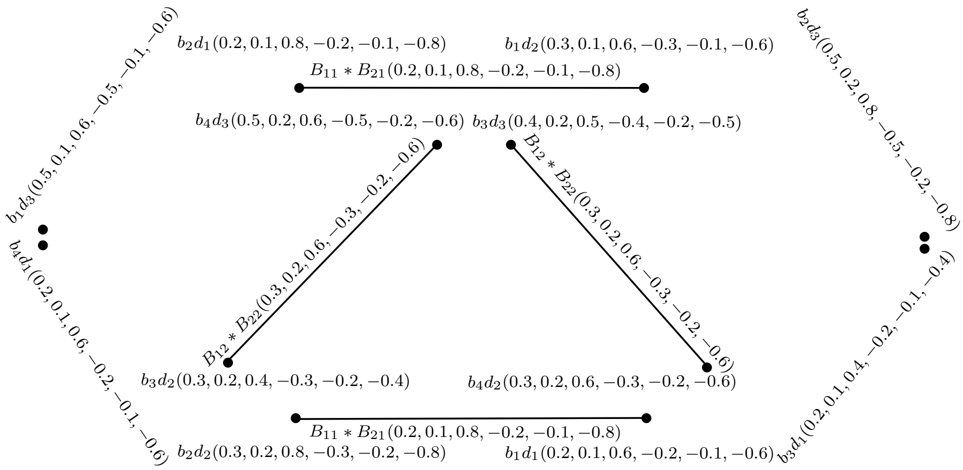

Figure 13. $\check{G}_{b1} \times \check{G}_{b2}$.

Theorem 2. *The cross product* $\check{G}_{b1} * \check{G}_{b2} = (B_1 * B_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1m} * B_{2m})$ *of two BSVNSGSs of GSs* $\check{G_{s1}}$ *and* $\check{G_{s2}}$ *is a BSVNGS of* $\check{G_{s1}} * \check{G_{s2}}$.

Proof. For $b_1b_2 \in V_{1k}$, $d_1d_2 \in V_{2k}$,

$$\begin{aligned}
T^P_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\
&\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2] \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\
&= [T^P_{B_1}(b_1) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b_2) \wedge T^P_{B_2}(d_2)] \\
&= T^P_{(B_1*B_2)}(b_1d_1) \wedge T^P_{(B_1*B_2)}(b_2d_2)
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\
&\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2] \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\
&= [T^N_{B_1}(b_1) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b_2) \vee T^N_{B_2}(d_2)] \\
&= T^N_{(B_1*B_2)}(b_1d_1) \vee T^N_{(B_1*B_2)}(b_2d_2)
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\
&\leq [I^P_{B_1}(b_1) \wedge I^P_{B_1}(b_2] \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)] \\
&= [I^P_{B_1}(b_1) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b_2) \wedge I^P_{B_2}(d_2)] \\
&= I^P_{(B_1*B_2)}(b_1d_1) \wedge I^P_{(B_1*B_2)}(b_2d_2)
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\
&\geq [I^N_{B_1}(b_1) \vee I^N_{B_1}(b_2] \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)] \\
&= [I^N_{B_1}(b_1) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b_2) \vee I^N_{B_2}(d_2)] \\
&= I^N_{(B_1*B_2)}(b_1d_1) \vee I^N_{(B_1*B_2)}(b_2d_2)
\end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \\ &\leq [F^P_{B_1}(b_1) \vee F^P_{B_1}(b_2] \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)] \\ &= [F^P_{B_1}(b_1) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b_2) \vee F^P_{B_2}(d_2)] \\ &= F^P_{(B_1*B_2)}(b_1d_1) \vee F^P_{(B_1*B_2)}(b_2d_2) \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}*B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \\ &\geq [F^N_{B_1}(b_1) \wedge F^N_{B_1}(b_2] \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)] \\ &= [F^N_{B_1}(b_1) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b_2) \wedge F^N_{B_2}(d_2)] \\ &= F^N_{(B_1*B_2)}(b_1d_1) \wedge F^N_{(B_1*B_2)}(b_2d_2) \end{aligned}$$

where $b_1d_1, b_2d_2 \in V_1 * V_2$ and $h \in \{1, 2, \ldots, m\}$. □

Definition 17. *Let* $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ *and* $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ *be two BSVNGSs. The composition of* $\check{G}_{b1}$ *and* $\check{G}_{b2}$*, denoted by*

$$\check{G}_{b1} \circ \check{G}_{b2} = (B_1 \circ B_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1m} \circ B_{2m})$$

is defined as

$$\text{(i)} \quad \begin{cases} T^P_{(B_1 \circ B_2)}(bd) = (T^P_{B_1} \circ T^P_{B_2})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_1 \circ B_2)}(bd) = (I^P_{B_1} \circ I^P_{B_2})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_1 \circ B_2)}(bd) = (F^P_{B_1} \circ F^P_{B_2})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$$

$$\text{(ii)} \quad \begin{cases} T^N_{(B_1 \circ B_2)}(bd) = (T^N_{B_1} \circ T^N_{B_2})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_1 \circ B_2)}(bd) = (I^N_{B_1} \circ I^N_{B_2})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_1 \circ B_2)}(bd) = (F^N_{B_1} \circ F^N_{B_2})(bd) = F^N_{B_1}(b) \wedge F^N_{B_2}(d) \end{cases}$$

for all $(bd) \in V_1 \times V_2$,

$$\text{(iii)} \quad \begin{cases} T^P_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}} \circ T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}} \circ I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}} \circ F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$$

$$\text{(iv)} \quad \begin{cases} T^N_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}} \circ T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}} \circ I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k} \circ B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}} \circ F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$$

for all $b \in V_1$, $(d_1d_2) \in V_{2k}$,

$$\text{(v)} \quad \begin{cases} T^P_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (T^P_{B_{1k}} \circ T^P_{B_{2k}})(b_1d)(b_2d) = T^P_{B_2}(d) \wedge T^P_{B_{1k}}(b_1b_2) \\ I^P_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (I^P_{B_{1k}} \circ I^P_{B_{2k}})(b_1d)(b_2d) = I^P_{B_2}(d) \wedge I^P_{B_{1k}}(b_1b_2) \\ F^P_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (F^P_{B_{1k}} \circ F^P_{B_{2k}})(b_1d)(b_2d) = F^P_{B_2}(d) \vee F^P_{B_{1k}}(b_1b_2) \end{cases}$$

$$\text{(vi)} \quad \begin{cases} T^N_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (T^N_{B_{1k}} \circ T^N_{B_{2k}})(b_1d)(b_2d) = T^N_{B_2}(d) \vee T^N_{B_{1k}}(b_1b_2) \\ I^N_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (I^N_{B_{1k}} \circ I^N_{B_{2k}})(b_1d)(b_2d) = I^N_{B_2}(d) \vee I^N_{B_{1k}}(b_1b_2) \\ F^N_{(B_{1k} \circ B_{2k})}(b_1d)(b_2d) = (F^N_{B_{1k}} \circ F^N_{B_{2k}})(b_1d)(b_2d) = F^N_{B_2}(d) \wedge F^N_{B_{1k}}(b_1b_2) \end{cases}$$

for all $d \in V_2$, $(b_1b_2) \in V_{1k}$*, and*

$$\text{(vii)} \quad \begin{cases} T^P_{(B_{1k} \circ B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} \circ T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2) \\ I^P_{(B_{1k} \circ B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} \circ I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2) \\ F^P_{(B_{1k} \circ B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} \circ F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2) \end{cases}$$

(viii) $\begin{cases} T^N_{(B_{1k}\circ B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} \circ T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2) \\ I^N_{(B_{1k}\circ B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} \circ I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2) \\ F^N_{(B_{1k}\circ B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} \circ F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2) \end{cases}$

for all $(b_1b_2) \in V_{1k}$ *,* $(d_1d_2) \in V_{2k}$ *such that* $d_1 \neq d_2$*.*

Example 12. *The composition of BSVNGSs* $\check{G}_{b1}$ *and* $\check{G}_{b2}$ *shown in Figure 10 is defined as* $\check{G}_{b1} \circ \check{G}_{b2} = \{B_1 \circ B_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}\}$ *and is depicted in Figures 14 and 15.*

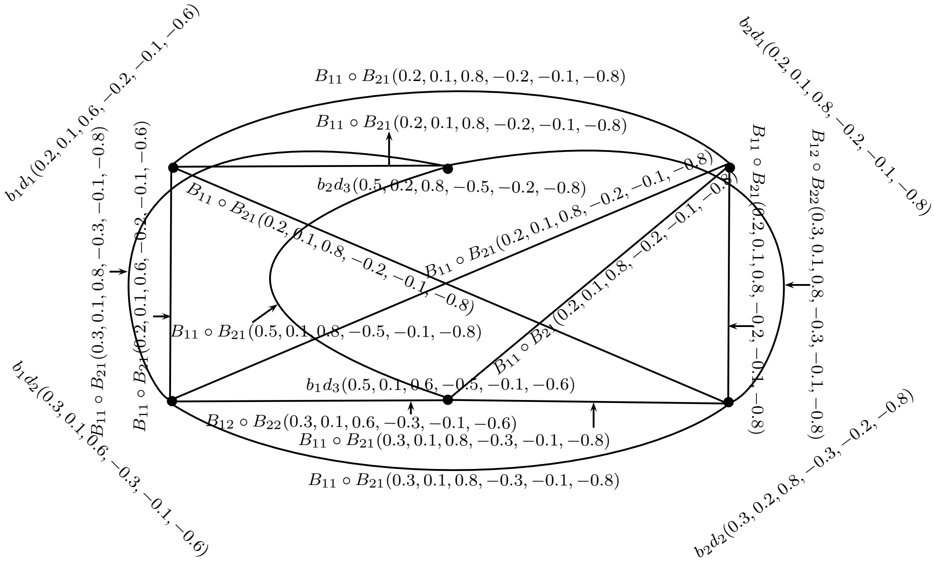

Figure 14. $\check{G}_{b1} \circ \check{G}_{b2}$.

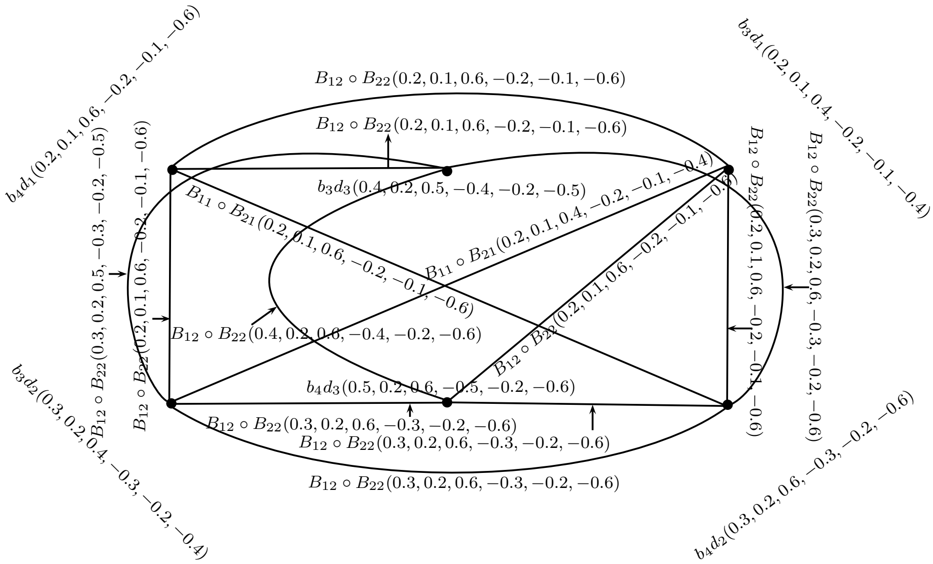

Figure 15. $\check{G}_{b1} \circ \check{G}_{b2}$.

Theorem 3. *The composition* $\check{G}_{b1} \circ \check{G}_{b2} = (B_1 \circ B_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1m} \circ B_{2m})$ *of two BSVNGSs of GSs* $\check{G}_{s1}$ *and* $\check{G}_{s2}$ *is a BSVNGS of* $\check{G}_{s1} \circ \check{G}_{s2}$*.*

Proof. Consider three cases:

Case 1. For $b \in V_1$, $d_1d_2 \in V_{2k}$,

$$\begin{aligned} T^P_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ &\leq T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1\circ B_2)}(bd_1) \wedge T^P_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ &\geq T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1\circ B_2)}(bd_1) \vee T^N_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} I^P_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ &\leq I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)] \\ &= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)] \\ &= I^P_{(B_1\circ B_2)}(bd_1) \wedge I^P_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} I^N_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ &\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)] \\ &= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)] \\ &= I^N_{(B_1\circ B_2)}(bd_1) \vee I^N_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \\ &\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)] \\ &= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)] \\ &= F^P_{(B_1\circ B_2)}(bd_1) \vee F^P_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}\circ B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \\ &\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)] \\ &= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)] \\ &= F^N_{(B_1\circ B_2)}(bd_1) \wedge F^N_{(B_1\circ B_2)}(bd_2) \end{aligned}$$

for $bd_1, bd_2 \in V_1 \circ V_2$.

Case 2. For $b \in V_2$, $d_1d_2 \in V_{1k}$,

$$\begin{aligned} T^P_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= T^P_{B_2}(b) \wedge T^P_{B_{1k}}(d_1d_2) \\ &\leq T^P_{B_2}(b) \wedge [T^P_{B_1}(d_1) \wedge T^P_{B_1}(d_2)] \\ &= [T^P_{B_2}(b) \wedge T^P_{B_1}(d_1)] \wedge [T^P_{B_2}(b) \wedge T^P_{B_1}(d_2)] \\ &= T^P_{(B_1\circ B_2)}(d_1b) \wedge T^P_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= T^N_{B_2}(b) \vee T^N_{B_{1k}}(d_1d_2) \\ &\geq T^N_{B_2}(b) \vee [T^N_{B_1}(d_1) \vee T^N_{B_1}(d_2)] \\ &= [T^N_{B_2}(b) \vee T^N_{B_1}(d_1)] \vee [T^N_{B_2}(b) \vee T^N_{B_1}(d_2)] \\ &= T^N_{(B_1\circ B_2)}(d_1b) \vee T^N_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} I^P_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= I^P_{B_2}(b) \wedge I^P_{B_{1k}}(d_1d_2) \\ &\leq I^P_{B_2}(b) \wedge [I^P_{B_1}(d_1) \wedge I^P_{B_1}(d_2)] \\ &= [I^P_{B_2}(b) \wedge I^P_{B_1}(d_1)] \wedge [I^P_{B_2}(b) \wedge I^P_{B_1}(d_2)] \\ &= I^P_{(B_1\circ B_2)}(d_1b) \wedge I^P_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} I^N_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= I^N_{B_2}(b) \vee I^N_{B_{1k}}(d_1d_2) \\ &\geq I^N_{B_2}(b) \vee [I^N_{B_1}(d_1) \vee I^N_{B_1}(d_2)] \\ &= [I^N_{B_2}(b) \vee I^N_{B_1}(d_1)] \vee [I^N_{B_2}(b) \vee I^N_{B_1}(d_2)] \\ &= I^N_{(B_1\circ B_2)}(d_1b) \vee I^N_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= F^P_{B_2}(b) \vee F^P_{B_{1k}}(d_1d_2) \\ &\leq F^P_{B_2}(b) \vee [F^P_{B_1}(d_1) \vee F^P_{B_1}(d_2)] \\ &= [F^P_{B_2}(b) \vee F^P_{B_1}(d_1)] \vee [F^P_{B_2}(b) \vee F^P_{B_1}(d_2)] \\ &= F^P_{(B_1\circ B_2)}(d_1b) \vee F^P_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}\circ B_{2k})}((d_1b)(d_2b)) &= F^N_{B_2}(b) \wedge F^N_{B_{1k}}(d_1d_2) \\ &\geq F^N_{B_2}(b) \wedge [F^N_{B_1}(d_1) \wedge F^N_{B_1}(d_2)] \\ &= [F^N_{B_2}(b) \wedge F^N_{B_1}(d_1)] \wedge [F^N_{B_2}(b) \wedge F^N_{B_1}(d_2)] \\ &= F^N_{(B_1\circ B_2)}(d_1b) \wedge F^N_{(B_1\circ B_2)}(d_2b) \end{aligned}$$

for $d_1b, d_2b \in V_1 \circ V_2$.

Case 3. For $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$ such that $d_1 \neq d_2$,

$$\begin{aligned} T^P_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2) \\ &\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2] \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b_1) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b_2) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1\circ B_2)}(b_1d_1) \wedge T^P_{(B_1\circ B_2)}(b_2d_2) \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2) \\ &\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2] \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b_1) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b_2) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1\circ B_2)}(b_1d_1) \vee T^N_{(B_1\circ B_2)}(b_2d_2) \end{aligned}$$

$$\begin{aligned}I^P_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2)\wedge I^P_{B_2}(d_1)\wedge I^P_{B_2}(d_2)\\ &\leq [I^P_{B_1}(b_1)\wedge I^P_{B_1}(b_2]\wedge[I^P_{B_2}(d_1)\wedge I^P_{B_2}(d_2)]\\ &= [I^P_{B_1}(b_1)\wedge I^P_{B_2}(d_1)]\wedge[I^P_{B_1}(b_2)\wedge I^P_{B_2}(d_2)]\\ &= I^P_{(B_1\circ B_2)}(b_1d_1)\wedge I^P_{(B_1\circ B_2)}(b_2d_2)\end{aligned}$$

$$\begin{aligned}I^N_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2)\vee I^N_{B_2}(d_1)\vee I^N_{B_2}(d_2)\\ &\geq [I^N_{B_1}(b_1)\vee I^N_{B_1}(b_2]\vee[I^N_{B_2}(d_1)\vee I^N_{B_2}(d_2)]\\ &= [I^N_{B_1}(b_1)\vee I^N_{B_2}(d_1)]\vee[I^N_{B_1}(b_2)\vee I^N_{B_2}(d_2)]\\ &= I^N_{(B_1\circ B_2)}(b_1d_1)\vee I^N_{(B_1\circ B_2)}(b_2d_2)\end{aligned}$$

$$\begin{aligned}F^P_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2)\vee F^P_{B_2}(d_1)\vee F^P_{B_2}(d_2)\\ &\leq [F^P_{B_1}(b_1)\vee F^P_{B_1}(b_2]\vee[F^P_{B_2}(d_1)\vee F^P_{B_2}(d_2)]\\ &= [F^P_{B_1}(b_1)\vee F^P_{B_2}(d_1)]\vee[F^P_{B_1}(b_2)\vee F^P_{B_2}(d_2)]\\ &= F^P_{(B_1\circ B_2)}(b_1d_1)\vee F^P_{(B_1\circ B_2)}(b_2d_2)\end{aligned}$$

$$\begin{aligned}F^N_{(B_{1k}\circ B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2)\wedge F^N_{B_2}(d_1)\wedge F^N_{B_2}(d_2)\\ &\geq [F^N_{B_1}(b_1)\wedge F^N_{B_1}(b_2]\wedge[F^N_{B_2}(d_1)\wedge F^N_{B_2}(d_2)]\\ &= [F^N_{B_1}(b_1)\wedge F^N_{B_2}(d_1)]\wedge[F^N_{B_1}(b_2)\wedge F^N_{B_2}(d_2)]\\ &= F^N_{(B_1\circ B_2)}(b_1d_1)\wedge F^N_{(B_1\circ B_2)}(b_2d_2)\end{aligned}$$

where $b_1d_1, b_2d_2 \in V_1 \circ V_2$.

All cases are satisfied for all $k \in \{1, 2, \ldots, m\}$. □

3. Conclusions

The notion of bipolar fuzzy graphs is applicable in several domains of engineering, expert systems, pattern recognition, signal processing, neural networks, medical diagnosis and decision-making. BSVNGSs show more flexibility, compatibility and precision for a system than single-valued neutrosophic graph structures. In this research paper, we introduced certain concepts of BSVNGSs and elaborated on them with suitable examples. Further, we defined some operations on BSVNGSs and investigated some relevant properties of these operations. We intend to generalize our research of fuzzification to (1) concepts of BSVN soft graph structures, (2) concepts of BSVN rough fuzzy graph structures, (3) concepts of BSVN fuzzy soft graph structures, and (4) concepts of BSVN rough fuzzy soft graph structures.

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