Existence and Characterizations of Weighted Chebyshev Polynomials

Notation and Definitions

Notation:
- \( E \subset \mathbb{C} \) a compact set with infinitely many points.
- \( w : E \to [0, \infty) \) an upper semi-continuous function.
- \( P_n = \{\text{monic polynomials of degree } n\} \)
- \( P_n = \{\text{polynomials of degree } n\} \)

Definitions:
- A weighted polynomial of degree \( n \) (on \( E \) with weight \( w \)) is a function of the form \( \wp \in P_n \) with \( w \in P_n \).
- For a fixed \( E \) as above, \( T_n \in P_n \) is called a weighted Chebyshev polynomial (w.r.t \( w \)) if it minimizes \( ||\wp||_E \) over all \( \wp \in P_n \). That is,
  \[ ||T_n||_E = \inf \{ ||\wp||_E : \wp \in P_n \} \]
- A point \( z_0 \in E \) is called a weighted extremal point of a function \( p_n \) (w.r.t \( w \)) when \( ||(wp)(z_0)||_E = ||wp||_E \).
- The set of all weighted extremal points of \( p_n \) (w.r.t \( w \)) will be denoted \( A_0 \).

Chebyshev Polynomials

It is hard to visualize graphs of complex valued functions (since the graph would have to be 4 dimensional). What we do instead is think of how the domain of a function changes.

\[ ||\wp||_E \text{ is the maximum distance from the origin to a point on } \wp(E), \text{ the Chebyshev Polynomial minimizes } ||\wp||_E. \]

Upper Semi-Continuous Functions

Definition:
- A function \( E \to [0, \infty) \) is said to be upper-semi-continuous iff \( w(z) \geq \limsup w(z) \) for each accumulation point \( z \in E \).
- Upper Semi-Continuous functions can have various types of discontinuities, the key point is that the function value is always \( \geq \) any limit values.

Weighted Chebyshev Polynomials Exist

Theorem:
- For each fixed pair \( w, E \) as above there is some \( T_n \in P_n \) such that
  \[ ||T_n||_E = \inf \{ ||\wp||_E : \wp \in P_n \} \]

Weighted Chebyshev Polynomials are Unique

Theorem:
- \( T_n \) is unique if \( w \) is non-zero at least \( n+1 \) points.

Weighted Chebyshev Polynomials Alternate

Theorem:
- If \( E \subset \mathbb{R} \) then \( \wp \) is a weighted Chebyshev polynomial of degree \( n \) iff it has extremal points \( x_0 < \ldots < x_n \) with \( \wp(x_i) = -\wp(x_{i+1}) \) for all \( i \in \{0, \ldots, n-1\} \).

Kolmogorov's Characterization

Definitions:
- A weighted continuous function on \( E \) is a function of the form \( \wp \) where \( w : E \to [0, \infty) \) is an upper semi continuous weight function and \( f \in C(E) \).
- We say a weighted polynomial \( \wp \), \( p \in P_n \) is a best approximation to the weighted continuous function \( \wp \) on the compact set \( E \subset \mathbb{C} \) if
  \[ ||f - \wp||_E = \inf_{\wp \in P_n} ||f - \wp||_E \]

Theorem:
- A weighted polynomial \( \wp \), \( p \in P_n \) is a best approximation to the weighted continuous function \( \wp \) iff for all \( q \in P_{n-1} \)
  \[ \max_{z \in A_0} \Re ((\wp f(z) - (wp)(z))wq(z)) \geq 0 \]
  where
  \[ A_0 = \{z \in E : (\wp f(z)) - (wp)(z) = ||f - \wp||_E \} \]

Rivlin's Characterization

Theorem:
- A monic polynomial \( p \in P_n \) is a weighted Chebyshev polynomial on \( E \) iff there exist \( m \leq 2n+1 \) extremal points \( z_1, \ldots, z_m \) and positive numbers \( a_1, \ldots, a_m \) such that
  \[ \sum_{j=1}^{\infty} a_j = 1 \]
  and for each \( q \in P_{n-1} \),
  \[ \sum_{j=1}^{\infty} a_j p(z_j) q(z_j) = 0. \]