Spring 5-16-2020

Elementary Math Teachers' Narrated Noticing: What Occurs Between Teacher Listening And Teacher Response

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By

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DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Multicultural Teacher and Childhood Education

The University of New Mexico
Albuquerque, New Mexico

May 2020

DEDICATION

"Tell me to what you pay attention and I will tell you who you are."
- Josè Ortega y Gasset

Dedicated to Kenny, Gray, Sutton, and Holden.

ACKNOWLEDGMENTS

Thank you to my dissertation chair, Dr. Laura Haniford, for the many hours spent in her office discussing noticing, listening, processes, and math.

Thank you to my dissertation committee, Dr. Laura Haniford, Dr. Peele-Eady, Dr. Kersti Tyson, and Dr. Kathryn Watkins, for the years of support and encouragement that helped me get through my PhD program.

Thank you to my family, for their patience and belief in me.

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ABSTRACT

Teacher noticing requires listening to student sense-making, interpreting the mathematical understandings teachers hear, and deciding how to respond based on what they notice in a specific interaction. Teachers manage what they notice about students in instructional interactions to both make key teaching decisions and adjust their interactions with specific learners. Elementary teachers narrate their procedural and declarative knowledge in the moment between listening and deciding how to interact. The data is generated through qualitative interviewing, observation, and documentation. There is a cyclical process of listening to student sense-making then recalling the student criteria that triggered an interaction, as well as a balance between impulsivity and overthinking (in which the tension between a teacher's best knowledge and the gathering of more student information is considered), before a teacher commits to a specific response. Teacher listening processes, context, and ability to narrate interactions are considered to describe how student sense-making can drive the opportunities to learn that teachers provide.

Keywords: knowledge of content and students, pedagogical content knowledge, elementary mathematics, narrated noticing, listening, student sense-making

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CHAPTER ONE

Introduction

The science of education (Thorndike, 1913–1914) was based on the observation and numerical representation of teacher actions during instruction. This modern research framework, termed the "grand narrative," used a taxonomy of educational objectives built on observable, classifiable teacher behaviors. The universal case was of prime interest, because teaching was viewed as a set of transmittable actions. This predominant view of teachers and teaching still informs a significant strand of teacher knowledge research (Lampert, 2001; Shulman, 2005). This viewpoint is problematic, because it simplifies teaching to a linear "one size fits all" presentation of content to what is most likely a diverse classroom of students.

Teaching is more than simply providing someone with useful knowledge, more than the description of steps in a procedure. What students learn depends upon teachers' and students' interactions with the curriculum (Ball & Forzani, 2011), students' interactions with the teacher, and the particular context in which these interactions occur (Dreier, 1993; 2009). The quality of the interactions that teachers have with students' interactions with content depend, in part, upon a teacher's ability to listen to students' thinking when they encounter content (Ball & Forzani, 2011). Listening to students making sense of content enables teachers to think (content knowledge) and act (pedagogical knowledge) flexibly during instructional interactions.

The primary function of teachers' interactions with students and content is to provide learning opportunities that support students' developing understanding (Charalambos, Hill & Ball, 2011). Teachers must listen to students' sense-making to develop processes that assist them in the provision of opportunities for learning to all children in complex and diverse contexts. The grand narrative research framework fails to account for all of the aforementioned complexities that teachers' listening and interpreting student sense-making adds to each interaction.

We require more studies attuned to what types student sense-making teachers listen to, as well as how teachers interpret what they hear, to adjust instruction to meet the needs of specific learners in specific contexts (Thames & Ball, 2013; Ball, Thames & Phelps, 2008; Clandinin & Connelly, 2000; Empson & Jacobs, 2008; Hill, Rowan & Ball, 2005; Lampert, 2001; Shulman, 2005; Dreier, 1993; 2009). The results of these types of studies, while complex and nuanced, are useful for both teachers and those trying to understand the ways teachers listen to students to notice student sense-making. Teachers must notice student sense-making, attend to it, and interpret it in a fast-paced, complex environment filled with many opportunities for interaction. Listening is a part of noticing, because what a teacher listens to is as important as how they apply the knowledge gained via hearing (Sherin & van Es, 2009). Teachers can use what they hear in student sense-making to inform their choices as they interact with students (Hintz & Tyson, 2015). Liping Ma (1999) refers to teachers' abilities to categorize student sense-making as "packaging," and Rodriguez and Fitzpatrick (2014) call teachers' thinking about students' verbal and nonverbal communication "processing." I combine these two ideas of categorizing and processing students' sense-making to highlight the pedagogical and content knowledge embedded in teacher narratives of listening to and interpreting student sense-making (Hintz & Tyson, 2015) in the specific moment between a teacher's listening to students' sense-making and deciding how to interact with them.

Elementary teachers teach children; thus, it makes sense that teachers aim to interpret instructional interactions through the sense-making of the child, situated in a particular context (Dreier, 1993), where the optimal interaction is the "moment when what is said is what is heard"

(Tyson, 2011, p. 5). This means that the teacher would encounter student sense-making in the way the child intended their thinking to be interpreted (Dreier, 1993). Teachers who listen – and attempt to listen – to students' sense-making in this manner contribute to the re-humanization of education (Gutierrez, 2015), which requires that teachers challenge the view of students as deficient. Additionally, teachers must also challenge the belief that student deficiency can be remedied purely through increased teacher pedagogical and content knowledge (Gutierrez, 2015). Teachers – and those who teach them how to teach – must know what teachers notice and, more specifically, what teachers listen for in their students' thinking, what they do with what they notice in successive interactions (Empson & Jacobs, 2008), and how they use this information to tailor instruction.

Additional studies seeking to understand what happens in a specific moment between when a teacher listens to student sense-making and how they decide to interact with that student are necessary. Teacher narratives of the context-dependent interactions involved in noticing (which I contend includes listening, interpreting, and responding) students' sense-making are needed, so that all teachers can be prepared with sufficient resources to foster student learning.

The Nature of the Problem

The purpose of my research is to learn more about the process that occurs between the moments when a teacher listens to student sense-making and chooses to respond during instructional interactions. Teacher listening is a process, not a static category of teacher knowledge. Listening requires the teacher to attend, interpret, and respond – virtually simultaneously – to students' sense-making strategies. My conceptualization of teacher noticing depends upon a focus on the teacher's process of interpretation – beginning with their narrative of what student sense-making they are listening to (or for), followed by how they interpret what they hear, and, finally, how they determine a response – all within their specific instructional context.

I have chosen mathematics instruction as the lens through which to describe this process, because of the prevailing perspective that in “no other living science is the part of the presentation, of the transformation of disciplinary knowledge to knowledge as it is to be taught so important” and in “no other discipline is the distance between the taught and the new so large” (Bass, 2005, p. 417). It is true that teaching math involves the ability to unpack something one knows well to make it accessible to and learnable by someone else (Ball & Forzani, 2010a). This perspective can also cause teachers – and those learning to become teachers – to be guided by the anticipation of student misconceptions, which comprises a deficit view of learners as needing to conform to teachers’ ways of thinking (Gutierrez, 2015). Teaching is more than providing and consequently supporting the use of specific mathematical procedures or resources to solve a problem. Children reasonably – and developmentally – hold their own mathematical ideas and strategies, which can differ from teachers’ own thinking about mathematics. For this reason, learning to listen and respond to student sense-making during a period of mathematics re-humanization (Gutierrez, 2015) is particularly relevant.

Ideally, teachers can consider ideas and skills from their students’ perspectives. Teachers’ ideal interactions with students would involve their listening to students’ sense-making and then utilizing the noticed strategies to provide students with appropriate learning opportunities as they emerge through an interaction (Empson & Jacobs, 2008; Gutierrez, 2015). The rapid process of teachers’ interpretation of the student sense-making they listen to – and their chosen response to that sense-making – is unobservable. We need teachers to narrate this process so that other teachers, as well as teacher educators, can learn not only from the dominant research axis of methodology and core practices (grand narrative research) but also from the context-dependent components of identity and power (the re-humanization perspective's critical critique of grand narrative research) to fine tune their interactions with students to appreciate the sense-making of all learners involved.

Teachers can listen to student thinking and interpret what they hear to adjust instruction; such adjustments will, in turn, affect students and interactions. Teacher listening is based on the extent to which the teacher notices the sense-making they hear. Teacher narratives about when they notice the student sense-making they respond to could occur at the strategic site where the student's interaction with the content intersects with the teacher's interaction with the student's practice space (Lampert, 2001); I provide more detail on this topic in the conceptual framework, described in Chapter Two. To understand how a teacher participates (i.e., thinks and acts given the student sense-making they listen to) in an instructional interaction, we must study how they participate in that social practice (i.e., their narratives of their processes of noticing the student sense-making they listen to). Herein lies a monumental opportunity. Teacher narratives of listening to student sense-making are key to developing pedagogically appropriate interactions that provide opportunities for all students to learn content (Gutierrez, 2015). The teacher impacts the teaching interaction.

Teachers' noticing based on children's mathematical thinking has recently gained ground in primary mathematics research. My research positions the teacher as a learner, as well: a learner who must move from a possessing general, whole class vision of instruction to becoming a teacher who has both a fine-grained focus on the individual student and the content knowledge needed to address a mathematics curriculum with great depth and breadth, so that all students have opportunities to learn.

I contend that further research is needed to critically examine the moment between teacher listening and response, and specifically what a teacher notices about students' mathematical sense-making; how a teacher elicits and interprets an individual student's thinking; and why a teacher implements a particular instructional strategy in response to student sense-making. Then, we must consider how these pedagogical processes can be shared with researchers, those preparing to become teachers, and practitioners.

Researchers, teacher educators, and teachers must engage in understanding the processes teachers undergo to notice (listen, interpret, and respond to) student mathematical thinking and choose a pedagogically appropriate interaction. Teacher narratives of how they personally interact with students to extend student thinking, in the midst of whole class math instruction (Hill, 2010; Lampert, 2001), are needed. Teachers' decision making is shaped by what they notice (Schoenfeld, 2011); however, what they notice – and how they act on it – is a function of their content knowledge, pedagogical knowledge, goals (Schoenfeld, 2011), and context (Gutierrez, 2015). This dissertation seeks to increase our understanding of teacher noticing by describing how teachers narrate between the moment of listening to student sense-making and their instructional response.

Problem Statement

What teachers do in their classrooms affects student achievement (Gage, 1989; Good & Brophy, 2000; Hill, Rowan & Ball, 2005). Widespread agreement exists that teachers possess unique knowledge of students' mathematical ideas and thinking (Ball, Hill & Shilling, 2008; Boaler, 2013; Lampert, 2001, 2002; Lampert & Cobb, 2003). Teachers' capacity to consider content from another's perspective and to understand what another person is doing requires mathematical reasoning and skill (Ball & Hill, 2009; Baumert et al, 2010; Lampert & Cobb, 2003). However, simply telling teachers to listen "better" will not solve our educational problems or address achievement gaps among students.

As educators, we hold a position of power. We shape our interactions with our students. We create learning opportunities where, whether or not we are cognizant of it, we are responsible for eliciting the sense-making of certain students and not others, and consequences exist for the interactions we participate in (Gutierrez, 2015). We must prepare – and be – teachers who critically consider the learning opportunities we provide to students; which should provide teachers with moments to interact with students' sense-making, as well as to advocate for children doing math in ways that challenge hegemony in education.

Researchers, teacher educators, and teachers need narratives of teachers' processes of listening to their own students' sense-making, as well as their organization and interpretation of their understanding of what they noticed (Ball, 2002; Clandinin & Connelly, 2000; Davis, 1997; Lampert, 2001, 2002; Pinar, 2004; Shulman, 2005), so that we can consider how to appropriately utilize our pedagogical and content expertise to extend students' understandings within the learning opportunities we provide.

Teachers' narratives of listening to student sense-making can help us understand how noticing the details of students' thinking and exploring the underlying principles apparent in this sense-making can help us develop pedagogically appropriate interactions with all learners. This dissertation adds to our understanding of teacher listening in instructional contexts. These narratives of teachers listening to and interpreting student sense-making, paired with pedagogy and content knowledge, document how teachers can provide equitable learning opportunities for all students. Before we can utilize our pedagogical and content knowledge, we must learn to listen to and learn from the sense-making of our students.

Research Questions

My research question is "How do teachers narrate their process of listening to, interpreting, and responding to elementary student sense-making?" My sub-questions are "What student mathematical sense-making do teachers listen to?", "How do teachers elicit and interpret individual students thinking?", and "How do teachers identify and implement an instructional strategy/intervention in response to student sense-making?"

Significance

Teachers' narratives of listening to their students' mathematical sense-making will enable us to better understand the complexities of practice and develop a means to impart this process to teachers, so they can better provide learning opportunities for all students. This is not a new argument. Lee Shulman (1987, 1996, 2005), Magdalene Lampert (2001, 2002), and Deborah Ball (2005, 2008, 2009) have validated the need for research focusing on the particulars of teachers' educational contexts and instructional interactions. We must learn how teachers notice students' sense-making, so teachers can do more than transform their subject matter expertise into a form that students can comprehend.

The contextualized interactions teachers that are able to narrate are patterns that might be useful in future similar interactions, not formulas. The creation of another list of "core practices" would not be useful and is not the aim of my work. These approaches can be overgeneralized and could lead to scripted instruction and a technical view of teaching (Hayden & Baird, 2018). Teachers' narratives of how they use, blend, and choose from the multiple types of strategies they possess to craft and deliver effective instruction in situated practice (Hayden & Baird, 2018) may help all of us consider how noticing could impact our own students and classrooms.

History of Research Interest

My interest in teacher noticing, listening, and knowledge of content and students stems from both a qualitative research class project I completed in 2014 and my son's 2016 speech delay diagnosis and label. I begin with the project, which focused on identifying which factors mediate the expression of pedagogical content knowledge among elementary math teachers.

I acquired much knowledge during this learning experience, and perhaps the most fruitful moment was when we were encouraged to write analytic memos. I wrote several additional questions during the class research project, most of which were left unresolved, since the scope of the project did not lend itself to finding answers. My questions revolved around teacher experience and pedagogical content knowledge expression and how observing teacher instruction, focusing particularly on how teachers respond to students, could help me better understand students' expressions of mathematical knowledge of content and students, a sub-unit of pedagogical content knowledge. My interpretation extended to describing and naming interactions, as well as what was learned by teachers through participating in those interactions.

My deep interest in teachers' mathematical knowledge of content and students also evolved in response to my son's public-school system label. Much concern existed among myself, speech therapists, and teachers about how my son's differences (he was nonverbal through age three) would emerge in instructional activities. The imposed label placed on him had implications upon teaching and the grouping structures used with him during instruction. I voiced my desire for his teachers to use approaches that valued the varied abilities of my son and all students; however, even after subsequent testing revealed that my son's misdiagnosis was a stage of his development and not a symptom of a more serious medical condition, I felt that his teachers and school constantly communicated messages to him about his ability and learning through their interactions with him. I would describe my first interactions with the local public school system via a narrative entitled "curriculum and instruction is on point; the problem is obviously with the child." Instead of teachers considering instructional interactions with my son (who scored far above grade level in mathematics and logical reasoning during special education assessments), practitioners and teachers jumped to a diagnosis. In my opinion, despite their knowledge of curriculum standards and benchmarks, teachers failed to recognize the range and pace of child development if that development happened to be slightly off course. I could not see how teachers' content knowledge alone was sufficient for interacting with my son. My research can demonstrate that teachers are capable of noticing student sense-making, interpreting student sense-making, and adapting instruction for children like my son and all students.

Students use knowledge differently in different situations, and that knowledge is co-constructed with other people with whom they are interacting and aspects of the situation in which they are working (Boaler, 2002). To put it simply, I expected teachers to notice my son's sense-making to interpret interactions that occurred with him. This expectation arose from my own personal goals as a teacher.

I taught fourth- and fifth-grade reading, English, and language arts for one year and then fourth- and fifth-grade math and science for six years, until the birth of my son. I then tutored children in math for two years, in kindergarten through fifth grade, through an after-school program funded by the public school system. I concluded this segment of my professional teaching career by teaching eighth-grade US history for a year, before celebrating the birth of my daughter. I now work with pre-service teachers in an alternative teacher licensure program during their first semesters of student teaching. Although teaching has changed in its scope and capacity for me, the essence of teaching bears the same nuances. I passionately believe that teachers' narratives of their processes – of listening, noting the procedural and declarative knowledge they utilize to hear student sense-making, and deciding how to interact in the moment – will lead to a more thorough conceptualization of the knowledge resources teachers need to have more effective, and equitable, interactions with their students.

I assume that teachers' explanations come from somewhere and that teachers decide what to teach, how to represent it, how to question students about it, and how to cope when the best laid plans for instruction go awry. I believe that sources of teacher knowledge exist, that a teacher knows when and why certain practices should be implemented, and if they do not know, that they can be taught.

For me, being a teacher entails constantly negotiating the worlds of students, staff, and families and constantly checking and reconstructing lessons and lesson sequences to meet the needs of diverse learners (while heeding curricular demands from the district and state). I assume that I can observe and ask teachers to reflect on and narrate how they use what they notice about their students to respond to their sense-making. I hold these assumptions because I am a teacher.

Teacher noticing is an important teaching process but a complicated one that may be challenging to learn. We must support teacher reflection on recent student–teacher–content interactions at an appropriate level of challenge and provide informative feedback and opportunities to repeat, reflect, and correct. In summary, I turn now to why I have chosen math as the lens through which I examine the teacher process of listening to student sense-making.

Significance of the Current Study

Teachers' narratives of their processes for attending to children's strategies, interpreting children's understandings, and deciding how to respond based on children's sense-making could serve as dynamic models for how teachers can finetune their interactions to how students learn difficult content in any classroom setting.

A growing body of knowledge exists specifically related to children's mathematical sense-making. The United States Department of Education's National Mathematics Advisory Panel (2008) suggested that a major goal for K–8 mathematics education should be proficiency with fractions, precisely because such proficiency is foundational for algebra. The panel noted that early elementary school children possess a rudimentary understanding of simple fraction relations. Student development of formal mathematical fraction concepts and procedures is not well understood and is an area in critical need of further study (Final Report of the National Mathematics Advisory Panel, 2008). My research into teachers' narratives of their listening to students' sense-making during instruction in the domains of algebraic reasoning and number and operations regarding fractions is timely and critical. Educators – and those who prepare educators – have the chance to address issues of student access and achievement by providing teachers with more than just subject related content and pedagogy. We can rewrite the narrative about who is good, not only at math but all school subjects, by valuing the sense-making of students, by learning from those who can tell us how to notice and listen to it.

In this chapter, I have begun to describe how teachers' narratives of their listening and interacting processes with student mathematical sense-making may help researchers describe – and teachers learn – how to provide learning opportunities that extend students' sense-making. In Chapter Two, I review the literature regarding the importance of teacher noticing. I present my methodology in Chapter Three. I introduce the reader to my participants individually in Chapters Four through Six and provide a cross-case analysis and conclusions in Chapters Seven and Eight, respectively. Next, in Chapter Two, I discuss pertinent initial and current investigations into teacher knowledge and present the conceptual framework for my research.

CHAPTER TWO

The Importance of Teacher Noticing – A Review of the Literature

Introduction

I described in Chapter One how this study seeks to better describe the process of teacher listening to student sense-making from the point of hearing to the point of interacting. Teachers' narratives of their interactions with individuals, small groups, and their whole class provide insight into how noticed student sense-making can be paired with content knowledge and pedagogical knowledge to extend each child's learning opportunities.

Mathematics instruction is the lens through which I describe teachers' narratives of their interactions with students and content. A teacher should connect age appropriate pedagogy and math content to craft high-quality interactions that provide opportunities for all students to learn. A teacher's mathematical pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers (Shulman, 1986), is a necessary component of teachers noticing what students think and recognizing what is important about that thinking.

Deborah Ball and colleagues (1990) were among the first single-subject researchers to dig into pedagogical content knowledge in math. They have completed extensive research with mathematics teachers, focusing on the mathematical knowledge needed to teach math. Ball's analyses are ongoing, with a focus on content knowledge-related teaching practices that are essential for effective math teachers. She sees persuasive evidence that the knowledge needed for teaching math is multidimensional. Ball describes two types of teacher knowledge – pedagogical content knowledge and knowledge of content and students – that she has found to impact how teachers transform instruction for their students (Ball & Bass, 2000; Ball & Hill, 2009; Ball, Thames & Phelps, 2008; Boaler, 2013; Lampert, 2001; Mason & Davis, 2013). In fact, teaching standards documents from the National Council of the Teachers of Mathematics (NCTM) and the National Board for Professional Teaching Standards (NBPTS) require that teachers hold knowledge of students as learners (NCTM, 2013) and that teachers have the ability to recognize the preconceptions and background knowledge that students typically bring to each subject (NBPTS, 2016). Precisely what student mathematical sense-making the teacher is responsible for noticing – and how they are to notice it – is yet to be mapped precisely (Ball, Hill & Schilling, 2008), despite these concrete teaching requirements.

Ball (2013) identified 19 high leverage teaching practices used as part of her Teacher Education Initiative; she utilizes these practices in training teachers-to-be across all subject areas. I focus on three of these practices in my research of teacher mathematical narratives: recognizing and identifying common patterns of student thinking in math, identifying and implementing an instructional strategy or intervention in response to common patterns of student thinking, and eliciting and interpreting individual students' thinking (Teaching Works, 2013).

I have chosen these high leverage practices because each plays a critical role in student–teacher–content interactions, but the process for meeting each of these ends is unobservable and, perhaps, un-narratable by those who teach. A teacher must first create a context where noticing students' mathematical sense-making matters. A teacher should provide opportunities that encourage students to share their thoughts; then, a teacher can notice (elicit and interpret) student mathematical sense-making. Teachers must have an idea of what kinds of student sense-making they may focus on (recognizing and identifying common patterns of student thinking in math). Finally, teachers must combine all of the information noticed in the moment to uniquely interact with the student (identifying and implementing an instructional strategy or intervention in response to common patterns of student thinking) (Jacob, Lamb, & Philipp, 2010).

I argue that a lack of detailed understanding regarding these processes exists. Teachers' narratives of their determination of the course of an interaction has not been completely understood, due to the process' implicit nature. More nuanced constructions of the teacher process of listening to students' mathematical sense-making, in authentic classroom interactions, are needed to understand why and how teachers adjust their interactions.

This chapter provides an overview of the transition from teaching being viewed as the collection and representation of teachers' generalizable instructional practices to it being considered an organization of complex processes that requires teacher awareness to narrate (Ball, 2002; Empson & Jacobs, 2008; Rodriguez & Fitzpatrick, 2014). I first provide a summary of the progression of teacher knowledge research and then provide a review of contemporary research into teacher pedagogical content knowledge in mathematics, making the case for the following: all learners possess unique mathematical thinking that impacts their learning; interactions with students require a teacher to notice students' sense-making to respond appropriately; and teachers can notice student sense-making by listening. I provide this discussion to situate this study within the larger field of teacher knowledge, teacher pedagogical content knowledge, and teacher noticing research within mathematics. I summarize by arguing that we need to understand the process of how teachers listen and respond to student sense-making through teacher narratives of their individual math instruction contexts. I end this chapter with an explanation of the conceptual framework I use to describe how teachers narrate their processes of listening to students' mathematical sense-making.

Initial Investigations into Teacher Awareness of Student Learning

In the early years of the twentieth century, teachers were encouraged to develop their ability to observe student learning (Erikson, 2011). Presumably, teachers could look at the children they were to teach and “see” the students were learning. Dewey (1904) distinguished two types of student behaviors that teachers could observe to determine that children were learning: children’s outer attention, or their surface appearance of attending to what the teacher was teaching, which was relatively easy to see, and inner attention, or the interest of the child, which may or may not be displayed to the teacher through the child’s behavior (Dewey, 1904). Dewey argued that it was of fundamental pedagogical importance for teachers to be able to distinguish between the two; however, it was a common error of novice teachers to confuse the two attention types. For example, a student could demonstrate – and the teacher observe – the behavior of looking at the board as the teacher wrote. However, the teacher had no way of observing if the student’s looking at the board was focused on the content or if the child was daydreaming. This was superficial teacher noticing. Pedagogical noticing would require teachers to consider what they were noticing and why this mattered in a child’s educational progress.

Systematic research on teacher behaviors developed after World War II, in contrast to student behaviors that teachers might notice as they taught (Erikson, 2011). Researchers were primarily interested in teachers’ thinking about planning curricula (Borg, 2009; Calderhead, 1996; Carter, 1990). Researchers’ curriculum studies aimed to connect what was taught by teachers and what was learned by students. Thorndike’s (1913) work attempted to make sense of teacher instruction by recording teacher behaviors in the classroom that were directly related to the teachers’ desired student outcomes.

Science of education. Thorndike was a measurement-oriented psychologist who further popularized the idea that teaching should be evaluated through the observation and numerical representation of teacher behaviors (1914), termed the “grand narrative.” The grand narrative research framework became the way teacher curricular knowledge and teacher student learning knowledge was researched. Different groups of researchers focused on various aspects of teacher behaviors, but the primary research technique remained classifying teacher actions in the classroom. However, Dewey was about to change that.

Science of education aimed at teacher enrichment. Dewey (1929) suggested that the purpose of a science of education should be the enrichment of the teacher's capacity for heightened understanding and intelligent decision making, rather than to control the behavior of teachers. Currently, many researchers use studies of curriculum to limit teachers' judgments and constrain classroom practices to set routines (Darling-Hammond & Synder, 1996). Thus, curriculum research sought, on one hand, uniformly applicable rules for teaching practice, and on the other, understanding of content dependent interactions and outcomes. A "cognitive revolution" (Erickson, 2011) in teacher thinking and student learning developed in the early 1960s and 1970s, and research attention returned to the study of teacher thinking.

Context-dependent interactions and outcomes. Dewey (1956) extended curriculum research by re-focusing on the learner. He suggested that a child's interests and activities ought to be noticed, because these interests could point teachers toward subjects of study. Teachers must notice student interests for their curricular significance and use them for educational experiences. Teachers could not wait for these learning opportunities but needed to arrange for such experiences to occur; they became planners and managers of educational experiences.

Bobbitt (1918, 1972) also extended the definition of "curriculum" by emphasizing the social conditions outside the school. He made room for out-of-school experiences, or students' informal knowledge. Bobbitt termed in-school experiences directed and curriculum that operated outside of school undirected. He suggested that life needed to be studied carefully to identify the skills students needed. These skills should be divided into specific units, the units organized into experiences, and the experiences provided to students. He listed nine areas in which educational objectives should be specified containing 160 educational objectives. Dewey and Bobbitt seemed to recognize that planned experiences of any kind are all a teacher can work with, no matter the age of the students or the subject matter taught. It was possible for teachers to plan and set instructional goals based on those objectives. Tyler's (1949) planning model utilized components of Thorndike's grand narrative and Bobbitt's educational objective development to do just that.

Tyler's planning model. Tyler's (1949) planning model was often used to guide researcher observations of teacher knowledge. The model asked curriculum planners to consider the following: what educational purposes should the school seek to attain? What educational experiences can be provided that are likely to attain these purposes? How can these educational experiences be effectively organized? And how can we determine whether these purposes are being attained? Tyler offered a prototype of curriculum development utilizing the scientific approach. This model recommended four essential steps for effective planning: specifying objectives, selecting learning activities, organizing learning activities, and specifying evaluation procedures. Teachers were expected to determine whether students were meeting the established learning outcomes. Educators were then expected to return to their lesson plans to make the adjustments necessary to ensure students achieved the learning outcomes.

Both Bobbitt and Tyler were sensitive to the particularized quality of curricular plans. Curricula had to fit a setting, school, and district, as well as respond to local conditions (Jackson, 1986). Both Bobbitt and Tyler defined educational objectives and devised learning experiences; Tyler further organized and evaluated learning experiences.

Tyler's model began to challenge the instructional relevancy and applicability of transmittable blanket practices and notice the particular contexts of students. How practitioners saw – and see – and thought about their world surely shaped on what teachers did as much as any number of explicit rules or principles they may attempt to follow (Jackson, 1986). Tyler's planning principles were considered by Bloom in his development of a taxonomy of educational objectives; Bloom's educational objectives also provided a frame for how teachers' instructional objectives should consider learner needs.

*Bloom's taxonomy***.** Bloom (1956), with his collaborators Englehart, Furst, Hill and Krathwohl, developed a taxonomy of educational objectives built on the observable and classifiable behaviors of teachers. This framework consisted of six major categories: knowledge, comprehension, application, analysis, synthesis, and evaluation. Bloom's taxonomy was intended to help teachers organize a set of objectives to evaluate the types of learning opportunities they provided for students. Teachers could check whether they were providing opportunities for all students. Teacher instruction that aligned with the objectives would establish learning goals that helped teachers and students understand the purposes of their interactions and varied ways for students to demonstrate the accomplishment of the objectives. Eisner's (1967) integrated ends–means model used the same framework: the idea that educational objectives needed to be clearly specified, because they provided goals, facilitated the selection and organization of content, and made it possible to equitably evaluate the outcomes of instruction.

Eisner's integrated ends–means model utilizing Bloom's taxonomy. Eisner (1967) suggested that educational objectives could help – and hamper – the ends of instruction. Researchers used Bloom's taxonomy and the integrated ends–means model to document teachers' effective teaching behaviors. These types of investigations into teacher planning aimed to standardize the behaviors of effective teachers to ensure positive outcomes (defined as higher grades) for learners (Clark & Joyce, 1975; Clark & Yinger, 1977; Peterson, Marx & Clark, 1978). This line of teacher knowledge research was designed to identify specific teacher behaviors accounting for improved academic performance among students (Shulman, 1986).

These studies often used teachers' experience and education, which have since been shown to bear little relationship to student outcomes (Hanushek, 2008), to stand in for direct measures of teacher knowledge (Hill et al, 2008). The difficulty with such a measure was its assumption that a year of schooling produced the same amount of student achievement, or skills, over time and in every classroom for every student. Additionally, the outcomes of instruction are more complex than a few educational objectives can encompass; the process of instruction has many outcomes that cannot be specified in advance (Eisner, 1967). Teacher knowledge measured simply as experience or courses completed counts time spent in schools, without understanding what processes occur there. It does not provide a complete or accurate picture of what teachers know or do.

This line of research acknowledged that teaching required skilled technique and action on the part of the teacher. Eisner (1967) claimed that teachers additionally needed to employ a variety of processes to teach students and that this planning should be evident in a teacher's instructional planning. Jackson's (1986) research built on Eisner's claims and began to investigate teachers' interactions with students in the classroom.

Jackson. Jackson's quantitative research numerically represented and diagrammed teacher movements during classroom instruction. Jackson observed that elementary school teachers often moved around their classrooms and that each teacher had a surprisingly high number of interactions with students within a short period. He suggested, based on his observations, that these seemingly common and ordinary interactions of classroom life were precisely the parts calling most urgently for renewed vision (Jackson, 1986). He chose to modify his quantitative investigation in teacher planning behaviors and incorporate the qualitative technique of interviewing teachers he had observed. He asked the teachers he observed why they did what they did in the classroom and concluded that teachers liked to be in control of their classrooms and hated being given specific tasks, textbooks, or plans they had to follow. Teachers wanted the freedom to choose what, when, and how to teach, because they desired increased self-governance over their students (above curricular demands) (Jackson, 1986). The teachers he interviewed, considered the highest performing by their administrators, appreciated students' differences and felt that tailoring their approaches to various students was crucial. This was particularly enlightening given that teaching was still mostly viewed as a set of transmittable actions that could and should be passed from one generation of instructors to the next. Doyle (1986) built on Jackson's conclusion that teacher and student interactions of school life needed more researcher attention.

Doyle. Doyle (1986) listed five key dimensions that framed teaching and learning in classrooms: multidimensionality, simultaneity, immediacy, unpredictability, classroom climate, and history. Doyle's work elaborated upon the complexity of teaching: classroom interactions involve fast and complex communications; different life experiences lead teachers to notice some students or issues more than others; teachers lack training on how to monitor and study their instructional behavior; teachers rarely receive systematic or useful feedback about their behavior (Good & Brophy, 2000). Teachers had many interactions with many students each day, and in each, they were expected to instantaneously interpret and respond to student behavior. Good and Brophy conducted a range of studies focusing on teacher awareness of their own behaviors.

Good and Brophy. Good and Brophy researched and presented classroom problems that could occur in part because of lack of teacher awareness and information. The authors (1974) noted teachers' dominance of classroom communication, that their instruction lacked an emphasis on meaning and intrinsic student motivation, and a teacher overreliance on repetitive seatwork as possible consequences of a lack of teacher development of noticing. Good and Brophy noted that teachers' lack of awareness could affect their decisions during instruction.

Good and Brophy (1970) also reported that while the frequency of teacher contact with students of different achievement levels was similar, there were important variations in the quality of the student–teacher–content interactions in terms of praise and working with students in moments of struggle. Additionally, boys had more interactions with teachers than girls (Good & Brophy, 1974). Brophy (1982) also argued that students' opportunities to learn would affect the types of interactions teachers could have with their students. In a nutshell, teachers played a role in determining the instruction their students received, both consciously and unconsciously.

Teacher awareness. Researchers gave increasing attention to what teachers paid attention to, both while they taught and what they saw other teachers noticing in videotapes of their instruction. Erickson (2011) provided a series of propositions about teacher noticing in his research "Teachers' Practical Ways of Seeing and Making Sense," based upon his experiences in observing and working with early-grade teachers.

Erickson's propositions concerning teacher noticing. Erickson (1986) suggested that what teachers paid attention to is necessarily selective, highly influenced by the teacher's prior experiences in teaching, and highly variable across individual teachers. A new branch of teacher knowledge research, teacher effects research, assessed relationships between teachers' and students' classroom actions and changes in students' knowledge, skills, values, or dispositions.

Teacher effects research. Brophy and Good (1986) noted the characteristics of teachers who elicited strong achievement test score gains for students: a belief that students are capable of learning and that teachers are capable of teaching successfully; students being provided opportunities to learn; classrooms being organized and effective; curricula being covered rapidly; teachers actively instructing; and teachers monitoring students' progress by providing feedback and remedial instruction, as needed.

Despite Jackson's, Doyle's, and Good and Brophy's conclusions that interactions in schools needed more researcher attention, teacher effects research studies in teachers' classrooms still tended to document general teaching behaviors (Clark & Joyce, 1975; Clark & Yinger, 1977; Peterson, Marx & Clark, 1978). Today, teacher knowledge is still investigated in the context of the grand narrative, with an emphasis on how teachers manage their classrooms, organize activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of questions, plan lessons, and judge general student understanding (Shulman, 1986). Researchers have shown that teachers do pay attention to their students and choose to make instructional adaptations based on what they see, yet this noticing is not highlighted consistently, nor does it consider that not all learning is visible.

Summary of Initial Studies into Teacher Awareness

Grand narrative research contributed a great deal of insight into how teachers behaved in the classroom, as well as what they paid attention to in order to discern if students were learning. Researchers often depended upon teachers' self-reports of their practices and behaviors, obtained through researcher-led questionnaires, interviews (researchers audio taped teachers outlining their lesson plans), and teachers' viewing of other teachers' videos of instructional segments (Carpenter et al, 1989; Clark & Yinger, 1977). These investigations contributed to our understanding of teachers in classrooms and the personal and professional contexts that influence their actions in the classroom. Yet, sequences of teacher action, such as teaching context, including student characteristics, classroom setting, time of day, term, and subject matter, continued to be overlooked. Most significantly, what teachers noticed about their students' sense-making and how they used that information to teach was not reported.

Research emphasis on generalizable teaching behaviors was logical; teachers had to rely on methods allowing them to manage student behavior and learning while attempting to meet more of the needs of more of their students (Brophy, 1986). Teachers are charged with instructing classes, but classes comprise individuals. Effective teaching involves orchestrating a vast array of student learning opportunities suited to each student–teacher–content interaction rather than the continued performance of a few, presumably generic, teaching behaviors.

The groundwork had been laid to attempt to explain the complex interplay of what teachers notice about their students and how this underlies teachers' chosen pedagogical interactions. Early research into teacher knowledge concentrated mostly on quantity of behaviors rather than quality of interactions and did not sufficiently consider contextual concerns specific to the content being taught. Shulman's (1986) work began to address this needed change.

Shulman's Theoretical Framework of Teacher Knowledge

Shulman's (1986) work significantly changed the larger educational discourse regarding teacher knowledge. He proposed a theoretical framework for considering how knowledge is organized in the minds of teachers, with a special emphasis on content. Shulman (1987) believed that the way to understand and document teacher knowledge is to focus on how teachers transform their subject matter expertise into a form students can comprehend. He further suggested that to understand how teacher knowledge develops, researchers must consider how knowledge is organized in teachers' minds. Shulman (1986) suggested this could be achieved by distinguishing three categories of teacher content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge.

Shulman's domains of teacher knowledge. Content knowledge, or subject matter knowledge, refers to content knowledge already in the mind of the teacher. Teacher knowledge is distinguished as knowledge that something is so and why (Shulman, 1986). Pedagogical content knowledge is subject matter knowledge for teaching (Shulman, 1986) and is conceptualized as the ways of representing and formulating a subject that makes it comprehensible to others. Shulman suggested that many alternative forms of representation should be available to a teacher, some of which would derive from research, and others originating in the wisdom of practice (Shulman, 1986). Shulman argued that pedagogical content knowledge would include an understanding of what makes the learning of specific topics easy or difficult for students and the strategies most likely to be fruitful in reorganizing the understanding of learners.

Curricular knowledge is knowledge of the full range of programs designed for the teaching of content topics at a given level, knowing the variety of instructional materials available, and knowing when to use them (Shulman, 1986). Shulman suggested that the mature teacher should be expected to possess such understanding about the alternatives available for instruction. Each type of teacher knowledge (content, pedagogical, and curricular) can be organized in three forms: propositional, case, and strategic knowledge.

Shulman's organization of teacher knowledge domains. Propositional knowledge, which is knowledge of what teachers do, can be broken down into three components: principles arise from a teacher's disciplined empirical or philosophical inquiry, maxims arise from practical experience, and norms consist of moral or ethical reasoning. Propositional knowledge contains and simplifies much complexity. Teachers must also know when to apply each knowledge type.

Shulman (1986) suggested that teachers can also organize their knowledge into case knowledge, or knowledge of specific, well documented, and richly described events. This form of knowledge organization consists of specific instances of practice and can assist teachers in their use of propositional knowledge. Case knowledge can be broken down into three components: prototypes exemplify theoretical principles (from propositional knowledge), precedents communicate principles of practice (maxims from propositional knowledge), and parables convey norms or values (also a form of propositional knowledge). Finally, the third form of teacher knowledge organization is strategic knowledge, which arises as teachers confront situations or problems and use organized knowledge for instructional decisions.

Summary of Shulman's knowledge domains. In summary, Shulman describes a teacher who is a master of procedure, content, and rationale. He suggests teachers to be capable of reflection and hypothesizes that such reflection would allow the narration of how their knowledge is organized. This sort of reflective awareness of how and why one performs further complicates, rather than simplifies, teaching (Shulman, 1986). Shulman's teacher knowledge – the idea that teachers employ content expertise to generate new explanations, representations, and clarifications based on what they know about their unique group of students – suggests that the process of teaching can be fully understood only when observable teaching actions and unobservable knowledge processes are brought together and examined in relation to one another.

Shulman's Impact on Teacher Knowledge Research

Many scholars, inspired by Shulman's formulation of knowledge for teaching, began more descriptive studies of teacher knowledge (Hill et al, 2008). A central contribution of Shulman and his colleagues was to reframe the study of teacher knowledge to attend to the role of content in teaching (Ball, Thames & Phelps, 2008), in contrast to research focused on general aspects of teaching (the grand narrative and universal teaching practices). A second contribution was to represent teacher content understanding as crucial to the profession of teaching (Ball, Thames & Phelps, 2008). Shulman's pedagogical content knowledge, with its focus on teachers' repertoire of students' representations, conceptions, and misconceptions, broadened conceptions of how teacher knowledge may affect teaching, by suggesting that a mix of knowledge of content and pedagogy is central to the knowledge needed for teaching (Ball, Thames & Phelps 2008).

However, Shulman's pedagogical content knowledge only provided a conceptual orientation (Ball, Thames & Phelps, 2008). The first strand of work that followed Shulman's proposal showed how teachers' orientations to their subject matters influenced how content was taught. Grossman (1990) articulated how teachers' orientations to literature shaped how they approached texts with students. Wilson and Wineburg (1988) illuminate how social studies teachers' disciplinary backgrounds affect how they represent historical knowledge for high school students. The research conducted in the field of mathematics differs markedly from social studies and language arts.

Ball (1990) showed that what teachers must understand to teach math differs greatly from what would suffice for non-teachers. She described differences between teachers' abilities to accurately represent mathematical operations for students. Studies such as Ball's demonstrated that teaching demands an integration of content ideas with how students learn content.

The second and third strands of work following Shulman's proposal contributed to understanding the knowledge teachers need about common student conceptions and misconceptions by documenting teachers' pedagogical knowledge. These studies can be classified into two groups: deficit and affordance approaches (Hill et al, 2008).

In deficit studies in mathematics, researchers have linked a teacher's lack of knowledge and its impact on the instruction provided to students. Deficit studies (e.g., Cohen [1990], Borko et al [1992], and Thompson [1994]) suggest teachers might know math but struggle to unpack it for students. Ma (1999) compared US and Chinese elementary teachers' understandings of fundamental (primary) mathematics, noting that many US teachers (ranging from student to experienced teachers) lack a deep conceptual understanding of many areas covered in elementary mathematics. Ma hypothesizes that elementary teachers in the two countries possess different pockets, or knowledge packages, which are structured bodies of mathematical knowledge that she described as central to teacher pedagogical content knowledge related to math. Teacher pockets of knowledge are classified by the connectedness of teachers' instruction, the basic ideas presented in mathematics lessons, and the longitudinal coherence occurring during teaching.

Affordance studies highlighted the benefits that strong teacher knowledge created for classroom instruction (Hill et al, 2008). Researchers focused on the practice of teachers and examined what teachers with more college mathematics classes (defined as teacher knowledge in affordance studies) could do with students (Hill et al, 2008). Teachers with more knowledge differed in the richness of the mathematics available for student learning (Fenemma & Franke, 1992). Carpenter, Franke, and Levi (2003) investigated instructional strategies that improved student learning and provided descriptions of mathematical problems and forms of questioning found useful for fostering growth in student mathematical understanding.

Deficient and affordance studies suggest elements of mathematics instruction that may be impacted by teacher knowledge, including the following: teacher knowledge of the mistakes they make while teaching; how teachers interpret student utterances; how teachers use representations and explanations; and how teachers use language to clearly convey mathematical ideas (Hill et al, 2008). Research on teacher knowledge, then, needed to focus on student mathematical sense-making. Teachers needed to first pay attention to how students made sense of math and, second, consider how they manage to adjust instruction for specific students and groups of students based on student understanding.

Theoretical Perspectives that Impact Current Teacher Knowledge Research

Each teacher's responses are unique to their students and classroom contexts (Ball, 2002; Lampert, 2001; Shulman, 1987). Researchers must strive to understand teachers' teaching processes through each teacher's unique pedagogical interactions (Baumert et al, 2010; Ball, Thames & Phelps, 2008; Ball, Hill & Schilling, 2008; Shulman, 2005). Shulman observed, "The character of the narrative form may be particularly well suited to the situated-ness of the teacher teaching process" (1987, p. 144). Researchers can use the narrative form to understand why teachers do what they do during instructional interactions. The researcher using the narrative form would first ask a teacher to focus on a specific interaction that occurred, second, ask the teacher to explain what they noticed about their students, and third, ask the teacher to narrate how this awareness impacted their instruction. A researcher using narrative inquiry values the prior experiences of teachers and aims to use these experiences to generate understanding (Clandinin & Connelly, 2000; Elbaz, 1991; Shulman, 1987).

It is crucial to note that research in other fields has identified a phenomenon in which skilled professionals struggle to verbalize (narrate) memories of perceptual experiences (Flegal & Anderson, 2008). This means that a professional of a high level of expertise may face challenges in reflecting consciously on what they know about a process or skill. Difficulties may arise in their ability to interpret or become aware of interactions they had, and describe the procedural or declarative knowledge used in those interactions (Flegal & Anderson, 2008), especially as it relates to recalling a memory (an experience or interaction that has already occurred). If this concept is applied to the teaching profession, it is possible that elite teachers may not be able to narrate the relationships they know exist between their procedural/content knowledge and their declarative/pedagogical knowledge. While it is true that a teacher's narrative of their interactional processes is the paramount data from which to generate understandings of teacher listening and interacting, current researchers using narrative inquiry to understand teachers' listening processes must recognize that a teacher's inability to verbalize what they know does not mean that they do not know it.

Many researchers now recognize that teacher narratives are an essential part of understanding and developing nuanced cases of teachers' noticing (Borg, 2006; Elbaz, 1991; Shulman, 1986, 1987). This highlights a coinciding shift in curriculum theory. Teachers must undergo a conceptual change, which requires some serious reflection and thinking, to notice their students' mathematical thinking. Pinar's curriculum theory suggests an autobiographical examination of self, which would help teachers do just that.

Pinar's curriculum theory. Pinar (2004) proposed a shift from considering "curriculum" as a noun to seeing it as a verb. His post-modern research deviated from the grand narrative framework. The grand narrative's curriculum theory framework often depended upon social efficiency, meaning that curriculum had to directly and specifically prepare students for tasks in the adult world (Bobbitt, 1918). A curriculum was a set of experiential objectives that developed habits for student activities in the grand narrative perspective. Progressive reform called for child-driven curriculum at the child's present capacity. This view entailed recognizing the significance of previous experiences to guide future ones (Dewey, 1899).

Pinar's unique perspective of "curriculum" as a verb required educators to undertake an autobiographical examination of self. Educators were presented with the opportunity for a complicated conversation with themselves, as an ongoing project for self-understanding. Pinar suggested this self-understanding would contribute to a teacher's mobilization toward engaged pedagogical action. This examination required four steps: first, the teacher remembered their educational experiences of the past and how these past experiences guided them in the development of their own personal attitudes and beliefs about education. Next, the teacher considered the future and what they wanted instruction to look like and be for themselves and their students, based on their experiences with instruction. Then, the teacher analyzed their present teaching circumstances. Finally, the teacher considered how to use their insights from the past, present, and future to create transformed educational environments. Pinar's idea behind such an investigation into oneself was guided by the thought that educators who choose to facilitate their students' personalized journeys of understanding cannot do so without undertaking this similar journey of understanding.

Pinar's curriculum theory could provide a method to help teachers access their implicit teaching and listening processes. Teachers' autobiographical examinations of self could possibly assist them in accessing their reflections on their own self-evaluations, skills, experiences, and motivations, as these relate to their teaching practice. The four aforementioned components may help researchers better understand and describe a teacher's instincts in context.

Teaching context. Most teachers understand their teaching contexts and are adapt at translating their knowledge to students. However, as teachers increasingly improve at what they do, their ability to communicate their understanding to help others (for example, researchers and student teachers) learn a skill or process may continually suffer (Scotti, 2016). We can ask a teacher for specific information that could help us describe their teaching context or about how they evaluate their own teaching, skills, experiences, and motivations.

Self-evaluation. Every teacher has a particular approach to teaching (Lampert, 2001). We must know more about the elements of a teacher's work to learn from it. In self-evaluation, we ask a teacher to make thoughtful judgments about how successful they think their approach to curriculum and instruction is.

Skills. Teachers have many strategies to cover their content. We must ask what they believe was right to do in their interactions with students to be effective and responsible teachers. In skills, we ask teachers to consider the implications of their teaching actions.

Experiences. Teachers hold perspectives on what and how they should be teaching (Lampert, 2001). Past instructional experiences may impact whether students have opportunities to memorize and apply rules, reason from assumptions to conclusions, or compute using basic content procedures. In experiences, we ask teachers to consider the learning opportunities they provide to students by examining where, the subject matter, and the grades they have taught.

Motivations. Our descriptions of teaching often capture little of the work teachers experience from inside the role (Lampert, 2001). We must describe the processes teachers consider to create interactions likely to result in student learning. The challenge is in asking teachers to explicate why they communicate content to students as they do.

Teachers bring personal understandings, conceptions, and orientations to subjects they teach. Researcher focus on teacher context (self-evaluation, skills, experiences, and motivations) can better describe teachers' relationships with content and students' sense-making of content.

Shulman's impact on contemporary teacher knowledge research. Researchers must understand teachers' processes of listening, beginning with the classroom context described by individual teachers' interactions in it. An interpretive pathway exists between teacher response and teacher knowledge (Clandinin, 2013; Clandinin & Connelly, 2000). It is necessary for researchers to listen to teachers' narratives of what they notice about students' mathematical sense-making; this is the only way researchers can understand the complex ways teachers provide students with opportunities to learn. Without understanding the processes involved in teacher listening, a part of the intention of teacher interactions will remain unknown.

Shulman (1986) argues that strategic sites exist where learning for teaching occurs. These could also be sites where teachers can narrate what they notice about students' sense-making. Locating these sites and specifying what teachers notice about their students has proven difficult.

Contemporary Focus on Teacher Knowledge Use in Math

Shulman – and researchers utilizing his teacher knowledge framework – affirm that teacher noticing requires teacher awareness of student sense-making, a teacher's narrative of how they interpret what they understand about their students' sense-making, and a teacher's narrative of how and why they choose to adjust their instruction based on what they noticed. It is known that mathematical knowledge matters for teaching math, but research into teacher pedagogical content knowledge has shown that conventional content knowledge is insufficient for skillfully handling the mathematical tasks of teaching (Ball & Cohen, 1996). Thus, studies are need that specify how teachers are aware of student mathematical sense-making. The work of Ball and colleagues expanded on Shulman's landmark pedagogical content knowledge research by identifying and defining specialized mathematical knowledge crucial to teachers of math.

Ball's mathematical knowledge for teaching. Teachers require specialized mathematical knowledge, termed mathematical knowledge for teaching (Ball & Hill, 2009), which Ball and Forzani (2009) have found is a significant predictor of student outcomes and is also strongly related to the mathematical quality of teacher instruction. Mathematical knowledge for teaching includes the use of mathematical explanations and representations, responsiveness to students' mathematical ideas, and the ability to avoid mathematical imprecisions and errors.

Thames and Ball (2010) found the most frequent tasks of teaching were posing mathematical questions, giving and appraising explanations, choosing or designing tasks, using and choosing representations, recording mathematical work on the board, selecting and sequencing examples, analyzing students' errors, appraising students' unconventional ideas, mediating discussions, attending to and using math language, defining terms mathematically and accessibly, and choosing or using math notation. These frequent tasks of teaching fall into the category of subject matter knowledge (Thames & Ball, 2010). Subject matter knowledge (Ball & Hill, 2009) is knowing when an answer is correct, choosing definitions, and knowing how to conduct a procedure. Specialized mathematical knowledge is mathematical knowledge and skill unique to teaching (Ball, Thames & Phelps, 2008; Thames & Ball, 2010), such as discussing how math language is used, how to choose, make, and use mathematical representations effectively, and how to explain and justify one's mathematical ideas (Ball, Thames & Phelps, 2008; Thames & Ball, 2010). Teachers' crucial, specialized knowledge enables them to interpret and analyze student work, provide mathematical explanations intelligible to young learners, and forge links between mathematical symbols and representations (Ball & Hill, 2009).

Ball (2008) and colleagues partitioned Shulman's pedagogical content knowledge into the domains of knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. They devote most of their research to the development of the domain of knowledge of content and students, and as such, the remainder of this review focuses on the development of this domain, since it is the one most relevant for my research.

Ball's knowledge of content and students. Knowledge of content and students combines knowing about students and math. Interactions occur between specific mathematical understanding and familiarity with students, both as specific groups of learners with unique sense-making and individuals with unique sense-making (Ball, Thames & Phelps, 2008). Teachers' knowledge of common student conceptions and misconceptions of particular math topics, how to sequence particular content, and how to use various instructional models to develop topics for students (Ball, Thames & Phelps, 2008) are central processes to the domain.

Ball and colleagues identify the "high leverage" practices that underlie effective teaching (Ball & Forzani, 2011). These are defined by Ball and Forzani (2009) as the ability to recognize key patterns of thinking, ideas, and misconceptions that students in a specific grade level typically have when they encounter a given idea. The associated high leverage tasks are analyzing student errors, encountering unconventional solutions, choosing examples, and assessing the mathematical integrity of a representation (Ball & Hill, 2009).

The most deeply developed high leverage tasks are associated with analyzing student errors (Ball, Hill & Schilling, 2008). Teacher analysis of student errors involves four categories of teacher knowledge. The first is teacher knowledge of common student errors; a teacher must identify and provide explanations for student errors, as well as be aware of what errors tend to arise with what content (Ball, Hill & Shilling, 2008). The second category of teacher knowledge is knowledge of how students understand the content; teachers must interpret student productions as sufficient to show understanding and decide which student productions indicate better understanding (Ball, Hill & Shillings, 2008). The third category of teacher knowledge is the analysis of student errors; teachers must know students' developmental sequences to identify problem types, topics, and mathematical activities that are easier or more difficult at specific ages, as well as determine what students typically learn first and have an awareness for what students might be able to do (Ball, Hill & Shilling, 2008). The final knowledge category is teacher knowledge of common student computational strategies; a teacher must be familiar with landmark number and fact families (Ball, Hill & Shilling, 2008). Ball and colleagues summarize knowledge of content and students as teachers knowing their students well – not only their personalities and preferences, but also their ideas, ways of thinking, misconceptions, and interests. This is not to say that students "are" their misconceptions; Gutierrez's (2015) re-humanization of mathematics instruction helps us – as teachers – consider how we can position ourselves and our students as authors and drivers of mathematics, instead of simply as individuals with misconceptions.

Re-humanization of mathematics instruction. Teaching is a profession largely about making classroom interactions positive and rigorous for students (Gutierrez, 2012). Teachers should be aware of possible student content misconceptions; however, the goal of rigorous mathematics should be to "prepare students to think more critically about mathematics" (Gutierrez, 2009, p. 11), so they can be critical citizens instead of "mere consumers." Gutierrez (2018) suggests that mathematical interactions ought to provide students with culturally and linguistically meaningful activities, where teachers see themselves as advocates. A teacher who re-humanizes mathematics instruction understands the roles of power, privilege, and oppression in the history of mathematics education (Gutierrez, 2018) and listens to student sense-making to ensure that opportunities for all students to learn are provided. Additionally, these opportunities should value students' home communities, other ways of viewing the world, and doing math in creatively insubordinate ways (Gutierrez, 2015). A teacher's ability to respond to student sense-making with appropriate instructional interactions requires skillful – and thoughtful – processes of interpretation (Ball & Forzani, 2011; Empson & Jacobs, 2008). Ideally, all interactions with students would begin by recognizing the sense-making that students bring to the interactions.

Teacher awareness of student sense-making during interactions. The primary function of teachers' interactions should be to support learners' understanding (Charalambous, Hill & Ball, 2011). Teachers must learn to unpack their knowledge of content and students (Charalambous, Hill & Ball, 2011) to articulate how they listen to student sense-making. Teachers' instructional explanations are interactions. Teachers' articulations, demonstrations, or arrangements of experiences that communicate some portion of math to students are instructional explanations (Charalambous, Hill & Ball, 2011); their good instructional explanations establish what students need to know, build on students' existing concepts and skills, and consider what students need to learn (Charalambous, Hill & Ball, 2011). Teachers' instructional explanations consider student thinking, clarify what is to be learned, chunk and sequence information, and monitor sequential student progress toward learning objectives (Charalambous, Hill, & Ball, 2011). Charalambous, Hill, and Ball's (2011) research compiled criteria for evaluating teachers' instructional explanations, including the pedagogical interaction being meaningful and easy to understand; the appropriate definition of key terms and concepts; the expansion and highlighting of mathematical ideas; the explanation of process without skipping steps; the clarification of transitions between successive steps; the awareness of the audience; the appropriate use of suitable examples and representations; and the clarification of the considered question, as well as how it is answered. As described above, teachers must know, or learn, how to listen to students' mathematical sense-making to provide pedagogically rich learning opportunities. Ball, Hintz, and Kazmi, Tyson and Stein, Engle, and Smith and Hughes suggest the listening process begins with talk.

Math talk. Teachers must notice how students make sense of the math they respond to during instruction, as well as begin to align the many sense-making approaches students use with canonical understandings of the nature of mathematics (Stein, Engle, Smith & Hughes, 2008). Ball (1993) suggests that it is the responsibility of expert practitioners and researchers to help novice and current teachers learn how to continually size up, step in, and redirect instruction when important mathematical ideas are – or are not – being developed during math instruction.

Student sense-making should be used as the starting point from which the teacher's and students' ideas can interact to form a more powerful and efficient mathematical interaction (Stein, Engle, Smith & Hughes, 2008). A teacher's orchestration of mathematical talk during instruction is a contemporary instructional method for eliciting student responses, supporting students, and extending student mathematical thinking.

Teachers' anticipation of likely student responses to mathematics instruction requires that teachers, at a minimum, actually do the mathematical tasks they are asking students to consider (Stein, Engle, Smith & Hughes, 2008). Teachers should devise and work through many solution strategies, so they can anticipate student sense-making. Next, teachers should monitor students' responses to tasks, identify strategies or representations that are used by students to make sense of the math they are working on, make sense of the students' sense-making, and learn how students' mathematical thinking relates to canonical mathematical ideas (Stein, Engle, Smith & Hughes, 2008) by responding to students in an interaction that provides an opportunity to learn the content. Teachers could select specific students to present their mathematical thinking and sequence the selected responses to help individual students – and the class – make connections between different ways of making sense of key canonical math ideas. This process depends completely upon teachers listening to students' mathematical thinking and instructional contexts.

Not all mathematical discussions have the same aim or should be led in the same way (Hintz & Kazemi, 2014). Discussions should achieve a mathematical goal, and students should know what and how to share, so their ideas can be heard and used by others (Hintz & Kazemi, 2014). Most importantly, teachers must orient students to mathematical ideas so everyone has the opportunity to engage in mathematical thinking and sense-making. Mathematical discussions are rich sites for comparing similarities and differences; generating justifications; determining efficient strategies; defining and discussing appropriate uses for mathematical models, tools, vocabulary, and notation; and reasoning through strategies (Kazmi & Hintz, 2014).

Teachers must notice their students' mathematical thinking; what is less clear is how teachers listen to their students' mathematical thinking and how what teachers notice about their students is used to appropriately respond to students' mathematical thinking in these complex, contextual, pedagogical interactions. A contemporary colleague of Ball's, Magdelene Lampert (2001), further develops our understanding of what teachers must notice about students' mathematical thinking by explicating how she locates sites of pedagogical interaction (Shulman, 1986) and adjusts her instruction based on her knowledge of her students. She examines what she does during mathematics instruction and why.

Lampert's Research into Teacher Knowledge of Content and Students

Lampert explicates Ball's knowledge of content and students, an important part of Shulman's pedagogical content knowledge. Lampert's (2001) practitioner research study investigates her own insight into her mathematics instruction. Lampert recognizes that her practice requires knowledge about her students and that what she needs to notice occurs during student–teacher–content exchanges. Lampert describes how she discerns her students' sense-making, examines this information critically from her perspective, and creates learning opportunities that are faithful to the subject matter and constructive to the understandings of her students. It is essential for researchers and teachers to understand why these instructional adjustments occur. The important conceptual differences from the literature I review above is her focus on relationships – while content still matters, she turns her attention more specifically to the people interacting and, most importantly, their interacting.

Lampert's conceptualization of the knowledge of content and students' domain. Lampert (2001), like Shulman and Ball, asserted that the teacher narrative is the most comprehensive representation of the work of teaching. Lampert's analytic frame is based on her structuring of activities in relation to content and her knowledge about her students. Lampert advocates for capturing the complexity of individual practice; she strives to specifically narrate how her knowledge of her students and the content are revealed in her pedagogical responses to student mathematical thinking so others may consider how they too can practice the same level of awareness in their instruction. She captures and narrates the complexity of her instruction by identifying practice spaces within her classroom relationships.

The first relationship – and practice space – is between the teacher and the student. The second is between the teacher and the content. Students also have a relationship with the content, so the third relationship and practice space is between the teacher and the students' practice with the content. Although Lampert claims that this three-pronged model is only a start to portraying the complexities of teacher pedagogical interactions, this frame provides an opportunity for practitioners and researchers to represent a deep and extensive record of teaching that allows others to consider the information teachers listen for in pedagogical interactions. We must understand how a teacher listens to students' mathematical sense-making.

Contemporary Research into Teacher Noticing of Student Mathematical Sense-making

Professional noticing of children's mathematical thinking requires that a teacher attend to children's mathematical strategies, interpret children's understandings, and decide how to respond based on those understandings (Jacobs, Lamb & Phillip, 2010). Each teacher will make their own sense of children's mathematical ideas and strategies, which can differ from the teacher's own thinking about mathematics (Empson & Jacobs, 2008). This means that different teachers can examine the same student mathematical thinking yet see it differently. Teachers' organized ways of seeing and understanding student mathematical thinking have been referred to as professional vision (Goodwin, 1994). Professional noticing of children's mathematical thinking requires not only attention to children's strategies but also an interpretation of the mathematics reflected in those strategies. How a teacher makes specific aspects of students' mathematical thinking salient by marking, or using them in some fashion, has been termed "highlighting" (Goodwin, 1994). Ball's most recent research attends to these details of students' thinking and explores the underlying mathematical principles present in it.

Ball's (2013) 19 high leverage practices help teachers learn how to make inferences about what students understand, as well as consider how to extend that understanding. Three of her 19 high leverage practices pertain directly to my research: recognizing and identifying common patterns of student thinking; identifying and implementing an instructional strategy or intervention in response to common patterns of student thinking; and eliciting and interpreting individual students' thinking (Thames & Ball, 2013). Ball's high leverage practices view the teacher as a resource who shares context, clarifies the processes involved in identifying students' mathematical thinking details, interprets children's understandings, and describes deciding how to support or extend children's thinking. Teachers' narratives of what they notice about their students' sense-making are as important as how they interpret what they notice.

A teacher's reasoning about a student's mathematical thinking should be consistent with research on math development, as well as the details of the child's strategy. Jacobs, Empson, Krause, and Pynes (2015) applied a multiple case study approach to investigate how teachers support and extend students' mathematical thinking in both one-on-one interactions and classroom settings. The teachers who participated in this research ensured children made sense of the problem, explored details of a child's existing strategy, encouraged the child to consider other strategies, invited the child to generate symbolic notation, and adjusted the problem to match the child's understandings (Jacobs et al, 2015). Notably, teachers who explored the details of a child's existing strategy also posed starter questions to the child, pressed the child for a detailed explanation of his or her problem-solving process, probed the child's representation to highlight connections to the problem context, and invited the child to think ahead before executing a problem-solving step. Jacobs, Empson, Krause, and Pyne (2015) observe that teacher noticing of children's mathematical thinking underlies all of the aforementioned supporting and extending moves; however, it is not a focus of their research. It is precisely the process of teacher listening to student mathematical sense-making that I focus my research on.

We have yet to understand the process of teacher listening to student mathematical sense-making. We can begin to conceive the listening process by considering what we can learn from cognitively guided instruction (CGI), which is a research-based model of children's sense-making that teachers can use to interpret, transform, and reframe their listening.

Cognitively guided instruction. Children can construct for themselves strategies that model actions or relationships in mathematical tasks when the opportunities for learning occur in the right context for them (Carpenter & Fennema, 1992). The idea behind CGI is that children enter school with a great deal of informal knowledge, which can serve as the basis of their developing mathematical sense-making. A teacher who listens and makes sense of students' mathematical sense-making has a context in which they can interpret and provide appropriate opportunities for students to learn.

Students tend to begin by using direct modeling strategies, which give way to more efficient counting strategies, which are generally more abstract ways of modeling a problem (Carpenter & Fennema, 1992). Finally, students come to rely on number facts (Carpenter & Fennema, 1992). Student problem-solving strategies come naturally in the perspective of CGI; students need not be taught a strategy for a problem. Children can construct strategies for themselves in the appropriate learning context. Teachers may use CGI's conception of problem-solving as a basis for making sense of children's strategies. The emphasis of CGI is on what children can do when given the chance to reason things out for themselves and the kinds of math sense-making that occurs when they do.

Summary

Student sense-making that teachers listen and respond to is a crucial and powerful aspect of teaching. Each learner uses a unique pattern of mathematical sense-making to respond, and teachers can listen to and interpret this sense-making to provide all with opportunities to learn.

Teacher knowledge was, and still sometimes is, represented by what teachers do during instruction. The aim of these types of investigations into teacher planning was to standardize the behaviors of effective teachers. This line of teacher knowledge research was designed to identify specific teacher behaviors that accounted for improved academic performance.

Why a teacher did certain things was insignificant, because research was, and is, interested in transmittable, replicable practices. These studies often used teacher experience and teacher education, which have since been shown to bear little relationship to student outcomes, to determine which teaching practices were “best.” Transmittable teaching practices do not re-humanize mathematics instruction, nor do they respect the ability of teachers to listen to and understand student sense-making. While researchers have documented practices that are essential for effective math teachers, I am interested in documenting not the practice but the process by which teachers listen, notice, interpret, and respond to student mathematical thinking. In short, I am interested in how teachers determine the practice, not in the practice itself. To develop sensible recommendations for what teachers must know in the classroom, it is necessary to consider the ranges of variation in observed teacher pedagogical interactions in various contexts, by asking teachers to narrate the learning opportunities they provide for their students, as well as why they have chosen them. Next, I describe the conceptual framework guiding my research into teacher listening to student mathematical sense-making.

Conceptual Framework for my Research: Introduction

I have argued that teachers can listen to student mathematical sense-making, interpret what they notice, and respond through providing appropriate opportunities to learn. Instructional interactions like these require teachers to fuse student sense-making, pedagogical knowledge, and content knowledge. I wish to understand what occurs in the mind of the teacher in the specific moment between teacher listening to student sense-making and the culminating instructional interaction. My conceptual framework centers on interactions.

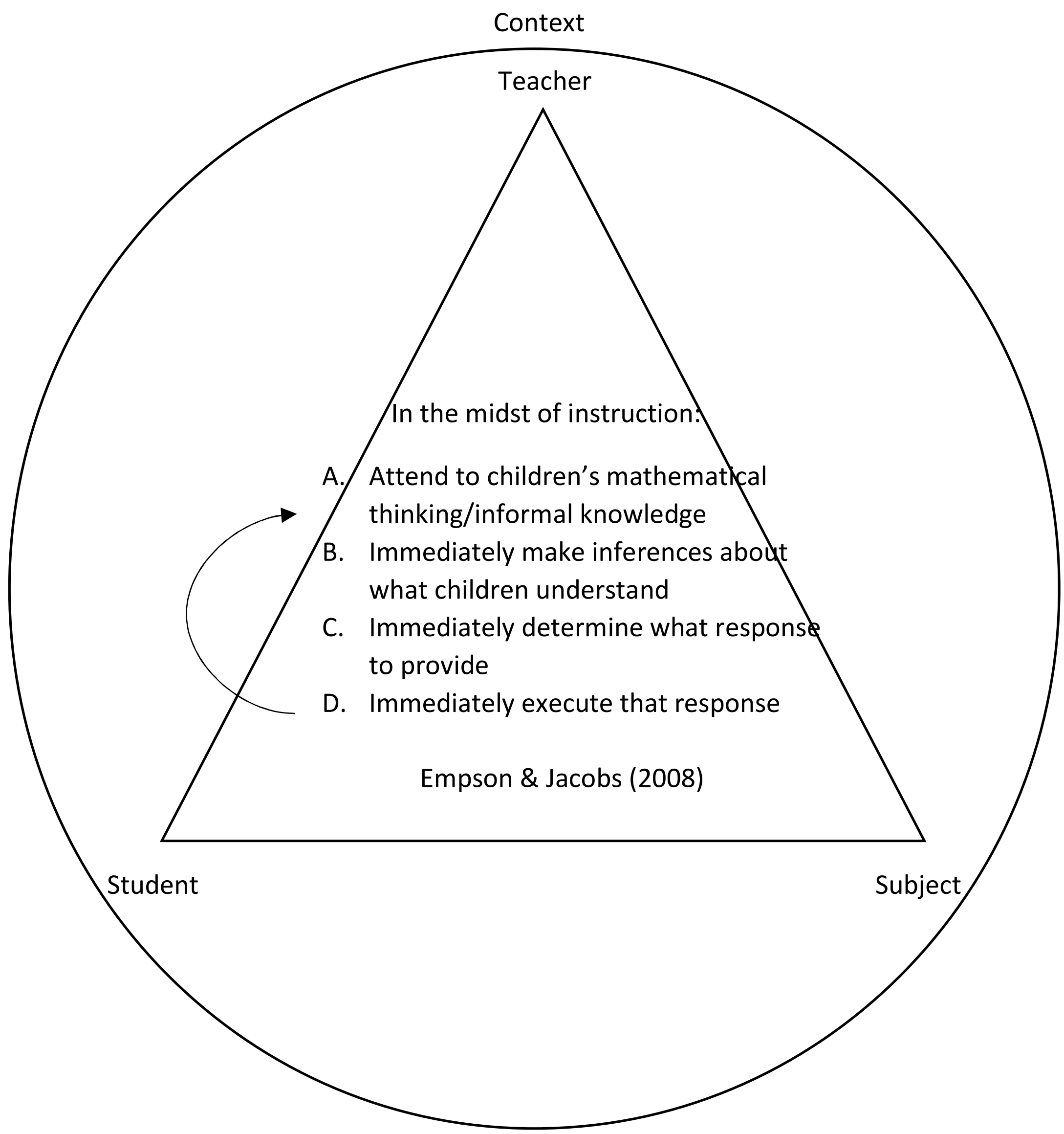


Figure 1. Teacher process. This figure illustrates what teacher participation in in-the-moment interactions could look like.

My research focuses on analyzing teachers' narratives of strategic sites (Lampert, 2001). Figure 2 depicts the strategic sites embedded within the context, the relationships between teacher, students, and subject, and the process of teacher listening to student sense-making.

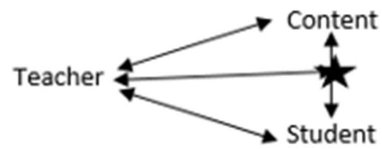


Figure 2. Lampert's organization of the strategic site of teachers' interaction with students' interaction with the content, which are Parts A and B of the teacher noticing process in Figure 1.

Concept of strategic sites. Strategic sites (Lampert, 2001; Shulman, 1986), or points of interaction, exist where teacher listening can be recognized and narrated (see Figure 2). Unpacking what is occurring in these interactions – and helping teachers unpack them – has been under-researched. Locating this type of strategic site requires zooming in on a lesson and noticing student sense-making, both verbal and written, of the content. Zooming in further requires observing and documenting how teachers' responses change when they listen to student mathematical thinking. Teachers may need intentional assistance in the reconstruction of these interactions to be able to narrate the processes that occurred in the moment.

I next describe the conceptual resources I draw upon to frame my investigation and analysis of vital components of interactive strategic sites. My conceptual resources include teachers' teaching context, students' mathematical sense-making, and the culminating process by which teachers respond to students. I begin with teachers' autobiographical examination of self.

Teacher teaching context. Teachers' personal narratives of their interactions with students' sense-making provided the orientation (Clandinin & Connelly, 2000), or the context, for me to make sense of their thinking and acting while embedded in their classroom (Sherin & Star, 2011). The teachers' perceptions and descriptions of their context is necessary for making sense of their pedagogical interactions.

The teaching context comprises four components: teacher self-evaluation, teacher skills, teacher experiences, and teacher motivation. These components are ever present in a teacher's context. They may change, and they include – but are not limited to – the following: their day-by-day experiences that are shaped within a longer term historical context; their personal and social interactions and the place or situation in which they occur; and their personal feelings, hopes, reactions, and dispositions, paired with the feelings, hopes, reactions, and dispositions of those around them (Clandinin & Connelly, 2000). It is through the teacher's description of their context that they, as well as other teachers, may be able to narrate what they recognize and understand about their students and the sense-making they bring to an interaction.

Concepts of student mathematical sense-making. Student mathematical sense-making is the ways children solve problems, create representations, and make arguments (Empson & Jacobs, 2008). Students have their own interactions with content by making their own meaning of what the teacher says and does. Teachers' pedagogical choices occur in response to students' moves, and students' moves happen in response to the teacher.

Teachers have knowledge that they draw upon in their pedagogical interactions with students. This knowledge is useful when used in response to listening to children's sense-making. Teachers' content knowledge, while an important component of teacher ability, is useless if the opportunities a teacher presents to students to learn the content are not in response to the student sense-making they hear. A teacher can listen to a child's thinking by attending to the child's strategies, interpreting the child's understandings, and deciding how to respond, based on the child's understandings (Jacobs, Lamb & Phillip, 2010). Good teachers employ a variety of information about students to judge the appropriate ways to interact with them. All teachers must be aware of the different experiences, backgrounds, and needs of each child they instruct (Boaler, 2009), since teaching mathematics does not involve only the precise presentation of knowledge.

Teacher process of listening to student sense-making. The teacher process of listening begins when teachers selectively attend to students' mathematical sense-making. They must draw on their existing content knowledge to interpret the student mathematical thinking they hear and determine how to provide the appropriate opportunity to learn. I need my participants to narrate themselves in context within the moment of listening to student sense-making and determining how to respond. I want teachers to provide the most robust descriptions of how and why they adapt their instruction based on their learners' sense-making that they can.

I must also be cognizant that some teachers may not be able to explain why they do what they do. Teachers, as they continuously improve at teaching, may find it challenging to communicate their understanding of student sense-making or help others learn specific skills related to noticing student thinking. I am asking teachers to consciously reflect on their instructional skills, by recalling their procedural and declarative knowledge in specific moments in a process of interaction. Sometimes, the procedural knowledge underlying the skills I observe and ask them to narrate will exceed what they can express verbally (Flegal & Anderson, 2008). Additionally, I am asking participants to describe something that, at the time of interview, may be nothing more than a memory. I must account for my participants' evaluation of their selves as educators and their motivation, skills, and experiences as primary components of their instructional context because these elements may impact the ability to narrate (Flegal & Anderson, 2008) what occurs between the moment of listening to student sense-making and determining how to respond.

My research focuses on a very specific moment of teacher listening. A teacher must know the concepts and procedures contained in a subject area, as well as how to teach said content, to help all students learn. We know that teacher content knowledge alone is insufficient.

I believe the narratives teachers can provide of their process of noticing will help us consider how to teach a process for listening to student sense-making to pre-service and current teachers.

Summary

A real need exists for research seeking to unpack how teachers listen to students' sense-making. Teachers' articulation of their processes for listening to student sense-making is key to describing a part of the complexity of teaching and understanding the kinds of content and pedagogical knowledge necessary for teachers to be able to provide learning opportunities for all students. Teachers may face challenges in narrating their processes; we will learn from these interactions by thoroughly describing the context of each teacher's instruction. Now that I have situated the need for my study and described my conceptual framework, in the next chapter, I describe the data I collected, as well as the tools used in analysis, to add to our understanding of teacher listening to student mathematical sense-making.

CHAPTER THREE

Introduction

Chapters One and Two describe my study, which began with the premise that teaching is complex because it involves myriad interactions (Lampert, 2001, 2002; Lampert & Cobb, 2003; Shulman, 1987, 2005). I describe in Chapter Two (see conceptual framework, Figure 1) that teachers must listen to their students' mathematical sense-making to provide opportunities for all students to learn. Teachers have a process of interpretation to understand what they hear in their students' sense-making and make key teaching decisions to adjust their interactions with specific learners. As I argue above, we must understand what occurs between the moments of teacher listening and teacher interaction. This process (see conceptual framework in Chapter Two) occurs in the strategic site where the teacher interacts with the student's content interaction. My research focuses on teachers' narratives of processes for listening, interpreting, and responding to student sense-making. Next, I describe my research design.

Design of the Study

Teachers gain knowledge about their students' sense-making when students interact with content and respond to teacher instruction. What teachers hear in students' sense-making can help them adapt interactions with students. My research occurred in a Southwestern city during math instruction addressing algebraic reasoning (ethno-), focusing on the processes by which teachers listened and responded to students' mathematical sense-making (methods), and listening to the teachers' narratives, in which they described their processes for interacting with students' practice with the content (-ology). Thus, this research is an ethnomethodology (Garfinkel, 1967).

My ethnomethodological study assumed that members of specific socio-cultural groups (elementary teachers) had some shared methods (of listening to students' mathematical sense-making) that they may use to construct the meaning of their social interaction (processes of listening to and interpreting student sense-making to adapt instruction) (Rawls, 2003). Additionally, a key core concept for ethnomethodology, as well as for my research, is indexicality (Garfinkel, 1967): my understanding of teachers' descriptions of their interactions depend upon the context in which it was embedded (Garfinkel, 1967). A common approach for studying what people notice while performing an activity is to ask them to verbalize what they are seeing and thinking while it occurs (Ericsson & Simon, 1993). Asking teachers to verbalize their thinking during instruction was unfeasible, given the ongoing nature of instruction. Therefore, I asked participants to narrate their listening after instruction had occurred.

Methodological Framework

The teacher in context was of prime interest in my research. People are embodiments of lived interactions, and these interactions can best be understood temporally, personally, socially, and in specific places (Clandinin & Connelly, 2000). My participants and I generated a text that offered readers a place to imagine their own applications of the narrated processes provided by my participants. I described – and my participants member-checked (described later in this chapter) – how they interpreted their interactions with students' mathematical sense-making and how they chose to respond. One can learn vicariously from each teacher's interactions through our co-constructed descriptions (Stake, 2005). What one learns from a particular case could be transferred to similar situations; one determines what can be applied to one's own context.

Multi-case study framework. I derived an up-close, in-depth understanding of three cases of teacher-narrated student–teacher–content interactions in classroom contexts. Instructional interactions require in-depth description. As defined in the conceptual framework, I investigated complex interactions with many variables important in understanding teacher-narrated processes.

The purpose of my research was to access my participants' interpretations of their listening processes in interactions with specific students in specific contexts (Glesne, 2011). We constructed a member-checked narrative based on observations, interviews, and artifacts, and I wrote member-checked case studies of each participant based on these narratives. Finally, I conducted a cross-case analysis. I sought similarities and differences between cases that helped me richly describe teachers' processes based on their self-reported data.

The primary concern of this multi-case study was to articulate teachers' processes for listening to student mathematical sense-making and those processes used to interact appropriately with students, given the student sense-making that was listened to. I observed and video recorded, interviewed, and examined artifacts with three public elementary teachers in their classrooms during mathematics instruction over up to two weeks during one trimester in a Southwestern city. I next describe my observations, interviews, and artifact examination.

Methods

Data Generation

I asked teachers for accounts of their pedagogical interactions. Teacher narratives, while self-reported, are beneficial because they emphasize the nuance of each student interaction. Next, I address qualitative interviewing and how Seidman's (2006) three-part interview sequence and my triangulation of generated data address concerns relating to self-reported data.

Qualitative interviewing. I was interested in teachers' narratives of their interactions with students' mathematical sense-making. Interviewing helped me understand the experiences of my participants and the meaning they made of their experiences or, in the case of this research, the meaning my participants made of their interactions. Interviewing provided access to the context framing my participants' behavior and allowed me to understand the meaning of that behavior (Seidman, 2006). I used a three-interview sequence, with each interview a maximum of 90 minutes, spaced 3–7 days apart (Seidman, 2006). The interview structure provided internal consistency, because my transcripts show how much I kept quiet and that the thoughts caught on paper seem to be the participants' and not mine, and the consistency of participant responses over the one to two weeks again indicate the participants' responses are theirs.

The first interview was a focused life history. I began the first interview with Pinar's modified autobiographical examination of self (Appendix A). The purpose of the initial interview was to understand my participants' teaching contexts. My second interview focused on the narrator describing the details of specific interactions. (Appendix B, Question 1a). I asked my participants to concentrate on the concrete details of their interactions (the focuses of the second interview are described and identified in the observation and video recording section). I asked participants to reflect on the meanings of their interactions in the third interview (Appendix B, Questions 1b, 1c-6 as follow-up questions as needed), which began with a review of the teacher's videotaped interaction and sometimes included a re-reading of several interview transcripts to assist the narrator in describing their noticing of student mathematical thinking. Each teacher and I spent between 90 to 180 minutes (1.5–3 hours) in interview locations of their choosing.

Observation. I observed teachers' pedagogical interactions in their classrooms. My observations were important to my research, because teachers could not recall the purpose of their interactions in the interviews alone (Mason, 2005). Teachers provided access to the contexts that framed their behaviors, which gave me a way to understand the meaning of their behavior. I used the video and field notes of teachers' classroom instruction to review and replay what occurred during certain student–teacher–content interactions (see Appendix C for my Observation Protocol) as needed.

I video recorded teacher mathematics instruction between Interviews 1 and 2 and between Interviews 2 and 3. All observations and video recordings occurred in teachers' classrooms during mathematics instruction, which occurred approximately two hours a day.

Field notes for observations and interviews. My observations and interviews with my participants included audio and video recording, and as well as field note writing. I made a record of my observations, interpretations, and experiences of the interviews and observations. I needed to be aware of the details of place and time and of the complex shifts between teacher noticing and interpretation. Interviews required a collaboration between me and my participants; it was important to me that they had the opportunity to validate the narrative I wrote.

Additionally, field notes helped me fill in the richness, nuance, and complexity of the teacher responses to student feedback that were recorded. I knew that my observations would be selective and based on my observational perspective (Mason, 2005; Merriam, 1999), so I completed a planning and preparation for observing and interviewing procedure to link my research questions to observations I might make in the field (Appendices B and C). I kept a separate column in my field notes for my thoughts and questions while in the field to consider how my observer (and teacher) status might shape my data (Mason, 2005). My field notes helped me reflect on the interactions I observed and listened to in a richer, more complex way than the audio and video alone could help me re-construct.

I referenced my field notes throughout my research. They began as two columns. I recorded the interactions as I saw them on the right-hand side; this was also what the audio and video captured. I recorded my inner responses about the interactions I observed on the left-hand side of my field notes. I used both columns to prepare for interviews and write analytic memos. These memos assisted me in iterative processes of analyses.

I transcribed audio from interviews, video from observations, and my field notes as soon after an observation/interview period as possible, which made me very familiar with the data. There were a few documents that the teachers and I used to triangulate the data generated from observations and interviews. Next, I share why artifacts are an important form of data generation.

Documentation. My purpose for using documentation as a data generation technique was to assist teachers in their consideration and narration of what they noticed when they planned and adjusted mathematics instruction. I used teacher lesson plans, students' in-class products and homework, and teacher–student mathematical representations to help me describe classroom practice (content, math processes, language, expectations). My participants and I looked at the aforementioned documents on the days that I observed in their classrooms. My participants used these classroom documents to remind themselves of their lesson objectives, as well as to remember what they expected children to make sense of. The data that we were able to generate together was dependent upon our research relationship. Next, I discuss my role as researcher.

Researcher Position. My narrative inquiry involved entering and exiting participants' contexts during their interactions, which did not begin when I entered the classroom nor end when I exited. My participants and I were changed simply because we interacted. Participants' experiences of their interactions and their perspectives on what occurred determined how they told their story. I was a participant observer; simply being in teachers' classrooms made me a participant in their teaching contexts.

Each of these data generation techniques contributed to our rich description of teaching interactions in all their complexity. I now describe what the interviews, observations, and artifact analysis comprised at each of my research sites.

Observations, Interviews, and Documentation in the Field

Our period of data collection was October through December 2018. I developed my cases by first negotiating access to school sites.

Negotiating Access

A series of people needed to give their consent before I could contact teachers at public and charter schools and ask them to be a part of my research. First, I obtained approval to research in the public school system from the Research Review Board (RRB) of the local school district. The RRB required that I submit their application form after I obtained university institutional review board approval, with copies of all data collection instruments, informed consent forms for all adult participant groups, assent forms for all student participants, and translations of all forms for parents and students speaking languages other than English. The RRB reviewed my research application against the following criteria: interest to the local school district (hereafter referred to as a large Southwestern urban school district, per RRB guidelines), relevance, technical adequacy, minimal intrusiveness on instructional and employee time, and protection of student privacy. The review process took one month; I received my RRB approval letter in the summer of 2018.

I decided to obtain the consent of the teachers, and then the principals, after receiving district approval. I chose this order of negotiating access because my research focus was on teachers. Teachers should have first say about their willingness to participate. This order of negotiating access proved to be much more difficult and time consuming than I had anticipated.

Public school teachers. I used Foster's (1994) idea of community nomination: a process that should have enabled me to study teachers whose practice could typify what the public school community thinks about its best teachers. I sent out a recruitment email (Appendix D) to all elementary public school teachers in the district, to be equitable in my selection of participants. The recruitment email requested that possible participants complete a survey. Respondents were self-selecting, but I could then identify focal teachers of varied backgrounds whose perspectives represented most of the pool. This recruitment process would have provided an opportunity for teachers to choose whether to participate, without their choice being apparent to other teachers.

I created the survey using Google Documents. The first five questions, related directly to Ball's (2013) 19 high leverage practices, were on a scale of 0–5 (0 meaning completely disagree, 5 meaning completely agree with the statements). The five statements were 1) "The goal of classroom teaching is to help student learn worthwhile knowledge and skills and develop the ability to use what they learn for their own purposes"; 2) "All students deserve the opportunity to learn at high levels"; 3) "Learning is an active sense-making process"; 4) "Teaching is interactive with and constructed together with students"; and finally, 5) "The contexts of classroom teaching matter, and teachers must manage and use them well." Teachers who answered with more 5s or toward the 5 end of the scale self-reported to attend to the complexity of teaching, which my research aimed to narrate and describe. My next three questions were short answer and pertained directly to the three high leverage practices (Teaching Works, 2018) that I asked my participants to narrate.

The three short-answer question prompts were the following: "Provide an example of when you asked a student to explain their thinking. What did you learn from your student's explanation?", "Have you ever noticed a pattern in how students' thinking in math develops?

Please provide an example," and "Provide an example of a time you adjusted math instruction to meet the needs of your learners." The final questions asked possible participants to self-report what grade they currently taught, at which elementary school, how long they had been teaching, what other grade levels they had taught, and if they felt comfortable, gender and ethnicity. Lastly, I asked teachers if they were willing to participate in my research, and if so, to provide a phone number or email where I could reach them to begin the formal informed consent process.

Unfortunately, trying to gain access to teachers first proved challenging. Most possible participants did not respond at all. My email went out over the summer, which was bad timing on my part. I did receive a few email responses. They were from principals (two) who indicated that a teacher at their school forwarded them my email and told me I was not to contact teachers prior to gaining principal consent to conduct my research at their school site (even though the RRB approved my research design). A handful (five) of teachers from two schools completed the survey, and I arranged to meet with their respective school principals to gain access to their school sites. The principals thanked me for asking, but I was denied the opportunity to work with their teachers for various reasons (i.e., the level of instructional supports from the Public Education Department, level of staff turnover, too many other commitments at this time). So, I decided to begin by setting up appointments for August with elementary school principals around the city to share to my research.

Public school principals. I initially contacted three public school principals via their public school phone numbers. I contacted their school offices to set up times to meet face to face at their convenience. Two of these principals maintained their appointments with me. I used the opportunity to discuss my research in detail. I used my conceptual framework (see Chapter Two, Figure 1), my recruitment flier (Appendix E), and my recruitment narrative (Appendix F) to situate my research and my conceptual framework to respond to questions. I made clear that the data belonged to the participants and myself and stated my dedication to meeting each participant's expectations, such as respecting instructional time, timeliness in my arrivals and departures, working around teachers' schedules, and providing teachers with drafts of analysis that they could review and critique. One principal immediately signed a letter of principal support and said, "Send out the emails! I'm sure we can get a group to help you out." The other principal said that my research would need to be considered and reviewed by his instructional council. If the instructional council approved my research, then he would get back in touch with me. I asked if there was a way that I could attend the meeting to speak on my behalf. My request was denied, and despite multiple attempts to reconvene, I never again heard from this principal. I immediately sent emails to the teachers at the first public school site; 10 teachers responded affirmatively. The public school system was one possible pool of teachers for my research; I concurrently sought possible participants in charter schools.

Charter school principals. Charter schools within the district were their own entities, so there was no independent review board (like the RRB of the public schools) that made an additional judgment on the applicability of my research to their educational settings. I had to contact charter school principals individually to ask for their consent to conduct research at their sites. There were 13 elementary charter schools within the district. I contacted each principal by email and used my recruitment script as a guide to negotiate access to the teachers. Five principals thanked me for my interest in their schools but denied me access to their schools as research sites. A seventh was a new principal who wanted time to learn about the school community before agreeing to participate. The principal contacted me in December saying that they could not participate. I decided to move forward with the 10 participants that I was able to negotiate access to at the first public school site.

Participant selection. My ideal research sample was five participants, and 10 possible participants responded to my survey. I considered how I might select among the possible participants, should they all confirm their participation after the informed consent process. First, I identified the teachers with the highest average scores on the first five questions (the possible informants whose scores are closer to 5 have a self-reported perspective toward teaching that I want to capture in my research) of my screening survey. Additionally, the possible participants' written narratives assisted me in the identification of teachers whose practice was self-reportedly representative of the pool. Seven of these possible participants opted not to participate in my research during the face-to-face informed consent process.

Informed Consent of Participants

I contacted all 10 teachers who completed the survey. I read my recruitment script over the phone or sent it via email (Appendix F), per the teacher's desired mode of contact. It took about two weeks to make contact with each possible participant. As mentioned above, at this point seven of the teachers chose not to participate. One discovered she was pregnant and stated that she did not "have the energy to be mom, teacher, and researcher." The second experienced a prolonged stay in the hospital for a blood infection that began in his leg and spread throughout his body. I understood when he said, "I need to dedicate my time to my students," upon his return. The third became my son's teacher when the school lost a teacher. We mutually decided that research in her classroom would be challenging given the change in context. The fourth was a gifted teacher whose hours were cut to part time. She communicated that she simply did not "want to participate anymore." The sixth did not see how she could have me and a student teacher in the classroom at the same time. The seventh did not respond until December that she was ready to consider giving me access to her classroom in February; I had completed my observations and interviews with the three remaining participants in December. I thanked her for her interest in my research and her dedication to teaching. I shredded all seven of those teachers' screening surveys and focused on the three who had listened to my informed consent presentation and still wished to meet with me.

Each teacher asked me to meet them in their classrooms during their preparation periods. I had an informed consent document that provided specific information about the study to my participants in a manner understandable to them (Appendix G). This document described that the study involved research; explained the purposes of the research and the expected duration of the teacher's participation; described the procedures we would follow, the foreseeable risks/discomforts to the participant, and the benefits (what I hoped to learn and how that knowledge would contribute to our understanding of teaching); and finally, described how I would maintain data confidentiality. I reiterated that participation was voluntary and that they could discontinue participation at any time.

I answered questions from the possible participants; most revolved around their opportunities to negotiate observation times and structure the timing of interviews, to ensure all research was conducted within their contracted school day. All three participants agreed to voluntarily participate in the study. They signed their consent forms, and I signed at the same time. The signed consent forms were stored in a locked file cabinet in my home office, apart from the second locked file cabinet at my work office, which housed our generated data. I then asked participants to choose pseudonyms; Mrs. BB, Mrs. Branches, and Mrs. Kapeesh were referred to as such throughout all phases of our data generation and analysis.

Informed consent of students. My research was focused on teachers and their narratives of noticing their students' mathematical sense-making. However, students' responses (both verbal and written) both precede and follow teacher responses. Students were not the primary focus of my study, although they played a vital role; I focused on how teachers respond to students during mathematics instruction. I wanted to include observations of students' voice and homework/in-class work, to use this data with the teacher during interviews (to assist teachers in recalling the student interactions I asked them about).

Children are a protected population; thus, it was essential to notify parents/guardians of my research in the classroom so that their consent, or dissent, could be obtained, as well. Parents/guardians provided permission for their children to participate in the research. I created parent/guardian consent (Appendix H) and student assent (Appendix I) forms containing the same components as the teacher informed consent document. I abided by each participating teacher's prescribed method for distributing these forms. I did not use the verbal responses or work of students whose parent(s)/guardian(s) did not provide assent. I reviewed the verbal and nonverbal feedback of students whose parent(s)/guardian(s) provided assent with participating teachers to assist with narratives. I only noted students' verbal and nonverbal feedback by numbers (related to their desk position in the classroom) to be used as prompts in teacher interviews. The teachers identified other student variables that we chose to describe as part of what they generally notice about students' sense-making, ensuring the descriptors could not be linked to any individual student. I used member checking to help ensure student anonymity. Teachers read and critiqued my data analysis to ensure that no student was identifiable. I also employed member checking to determine whether I accurately represented the students. Following this description of methodology and methods, I next describe the data analysis.

Data Analysis

My analysis was iterative; I attended to all evidence, addressed the most significant aspects of my cases, and appropriately used my prior, expert knowledge. My analysis emphasized sharing narratives of noticing and the collaborative co-construction of meaning. My explanation building process, which was consistent with a multi-case approach, began with a ground-up case description.

Codes, Coding, and Analysis

I used codes to categorize similar data for further analyzing and drawing conclusions. My field notes began with two columns: one for my observations and the other for my feelings and thoughts. I transcribed my own field notes and added my code assignments and jottings (Emerson, et al, 2011). I chose to transcribe and code manually, because I wanted a systematic overview of the data to elicit a clear idea of its scope. My decisions helped me to build my arguments, as well as to recognize that my argument may not be the only possible version of reading the data (Mason, 2005). My analysis was concurrent with data generation.

Open coding. I began open coding as I listened to and read each participant's interview, since this was the first data generated. I began with a narrative approach to analysis that did not work because my participants did not use personal narratives to describe their noticing. Narrative codes were codes such as Ex (explicative) – explain why something happened and c (comparator) – the participant compares what did occur to what did not. The pattern emerged of teachers wanting to narrate for specific reasons, and identifying these moments of interaction lead to the generation of codes such as CSC&M (common student conception or misconception) – a teacher can narrate what they know or noticed about what students may or may not understand about math content –and SC (sequence content) – a teacher narrates why they teach the content in the order they do. I then read each participant's respective classroom observations and interviews in chronological order, so I could begin to assign descriptive codes (short words or phrases assigned to chunks of data). My observation codes began with the identification of types of strategic site interactions: TIC (teacher's interaction with the content), TIS (teacher's interaction with the student), SIC (student's interaction with the content), and TISIC (teacher's interaction with students' interactions with the content). These became more specific when I added codes for how a teacher interpreted student interactions, including PMI (how the teacher presents mathematical ideas), RSQIU (how the teacher chooses to engage the student with the content), EMP (the teacher finding an example to make a specific mathematical point), and EPSC (the teacher asking the student to evaluate the plausibility of their own claim). I repeated this cycle with each participant's data and generated a list of 44 codes.

Axial coding. I then grouped related codes into categories. I chose one category as the core category during selective coding and related all other categories to this one. The axial codes began with noticing – NN (not noticing), BN (barely noticing), M (marking), and R (recording) – and were expanded to include codes corresponding to the theme of what teachers do when noticing – ACS (attending to children's strategies), ICS (interpreting children's understandings), and DCS (deciding how to respond based on children's understandings). Finally, I based my assertions on the interrelationships between categories and wrote analytic memos expanding upon the significance of codes and patterns (Miles, Huberman & Saldana, 2014).

Analytic memos. I wrote analytic memos as I transcribed our generated data to "begin to develop tentative ideas about categories and relationships" (Maxwell, 2013, p. 105). I used my memo writing as an opportunity to consider, reflexively, what I thought I might be understanding about the data. I used my memos to reflect on my establishment of relationships and to note points of elaboration for future interviews and observations.

Code book. I created and maintained a code book to organize the codes emerging during data analysis. I finetuned the codes and their definitions as my study progressed and relationships between cases and narrative descriptions became more specific with narrated examples.

I shared my data analysis with respondents for participant feedback. I wrote memos frequently as I reviewed old generated data and as new data was added, to become as familiar as possible with the data. I paid attention to the words, representations used, and sequences of pedagogical interactions to construct a version of what the data meant to me. I focused on my participants' narratives of how they made sense of their students' mathematical thinking, as well as sought the feedback of my participants.

Participant feedback. I brought my data analysis to participants (two to three times per participant) to read and critique until they were satisfied with the draft. We discussed discrepancies between my interpretation of the participants' narratives and interpretations of their interactions. Ultimately, only the participants themselves could help me interpret the data. I respected the complexity of practice. When participants added to or modified my interpretation of the data, their interpretations were used in the final iteration of analysis.

Case Development

I described my participants' narratives of their interactions with students' mathematical sense-making. I created an analytical template, which helped me follow an explanation building strategy (Yin, 2009) as I revisited the generated data. First, I examined each teacher's individual set of generated data. Then, I turned to my additional cases looking for underlying similarities, differences, and connections to begin to form an explanation (Miles, Huberman & Saldana, 2014). This is the case comparison approach (Yin, 1981).

Case comparison approach. I cited evidence as I shifted from data collection and within-case analysis to cross-case analysis and my overall findings and conclusions (Yin, 1981). My within-case analysis with my first participant provided tentative descriptions, explanations, and connections for the narrated interactions of Participant 2. My within-case analysis with Participant 2 had some relevant conditions that appeared to resemble Participant 1, and so on with each successive case (explanation building). I constructed an adequate explanation for each case individually, with participant feedback. Then I completed the cross-case analysis, using the aforementioned coding strategies. Next, I discuss potential challenges with this type of research.

Limitations

My research focused on three participants, and most of our generated data came from narrated, self-reported data; thus, I understand why perceived issues of validity could exist. I next discuss how I addressed the threats of generalizability, validity, and reliability.

Generalizability. Readers can learn vicariously from an encounter with a case through our co-constructed narrative description (Stake, 2005); what our audience learns from a case could be transferred to their self-described similar situations. The reader – not me – determines what can apply to his or her context.

Validity. Validity refers to research findings that truly represent the phenomena one claims to measure. I addressed construct validity by defining related pedagogical interactional terms in my conceptual framework. I asked participants to generate data from multiple sources of evidence, since I used primarily teacher narratives as descriptions of these concepts. I asked informants to review drafts of their personal case study reports (participant checking). I addressed internal validity by using explanation building (Maxwell, 2013), which is primarily used with the narrative form, hence, its applicability. I made my initial conceptual framework and began the explanation building process by comparing my initial case against my framework. I then considered what revisions could be made to my initial thoughts and compared my revisions to a second, and a third, case. Although my subset of teachers is self-selected, it represents a range of years teaching, ethnicities, educational backgrounds, and teaching techniques. Their responses may reflect the response of the entire pool, due to unique self-descriptors.

Reliability. I developed a case study database, which is a separate and orderly compilation of all data from all case studies. My file for each case included all data generated in the field. I developed it so that other persons (dissertation committee) could inspect the entire database, apart from reading my case analysis and drawing their own conclusion. I reduced my bias by frequently reviewing my research questions and using member checking to examine alternative explanations. Finally, I created a case study protocol as my interview, observation, and documentation patterns developed in the field.

Methodological Summary

This part of Chapter Three has described my methodology, the specific data I collected, and my analysis for this study. The aim of my research was to document how teachers narrated their listening and interactional processes. My objective was to provide a description of teacher-narrated pedagogical interactions during math instruction. Teachers shared narratives of their interactions so we could learn more about what teachers listened to – and how they utilized what they heard – to interact with students. Next, I describe the research site and how we chose the interactions to share in our final narrative.

Research Site Description

Tuff Elementary is a two-story brown brick public school established in a Southwestern city in 1939. It houses kindergarten through fifth grade. Each teacher in kindergarten through third grade, music, and special education has a classroom inside, or attached to, the main building. The fourth- and fifth-grade, gifted, gym, and art classrooms are housed in portables. Student enrollment has remained consistent for around five years. However, the school experienced its first student population dip in the 2016–2017 academic year and its first teacher loss (one third-grade teacher and two gifted teachers) due to the continuing student population decrease during the 2017–2018 and 2018–2019 academic years. Some teachers suggested that this trend in enrollment originated with a massive administrative upheaval and coinciding teacher loss following the 2016–2017 school year. Highly respected teachers left Tuff Elementary following that school year, and many students followed these teachers to nearby charter and magnet schools. Additionally, fewer school-aged children live in the vicinity of Tuff Elementary than did a few years ago.

Tuff Elementary was once known for being a true neighborhood community school. Teachers used to live in proximity to the school, and administration, teachers, and parents/guardians worked hand in hand. The school's parent teacher association (PTA) used to provide a strong communication link between the school and families. Teachers, administration, and staff were constantly present at after-school and weekend events that built Tuff's renowned community. Presently, a tension exists between staff, administration, and families, which has led to fractures in their partnerships.

Tuff experiences about 25% teacher turnover annually today (compared to nearly 100% consistent teacher retention, up to 2016). Student enrollment has dropped by about 100 students, so more student transfers are being admitted. Of five nearby public schools, four have received failing grades for the past three or more years, and students and their families are granted student transfers to schools such as Tuff with passing school grades.

Test scores at this school are above the state average, but some students are still not performing at grade level. More than half of Tuff's students are proficient in reading (37% state average), about one-third are proficient in math (20% state average), and over half are proficient in science (40% state average). Students who identify as "white" are significantly above the state averages across the assessed subjects. Students who identify as "Hispanic" are also above the state averages, but not by as large of a margin. Students who identify as "low-income" are also above the state averages, but by a very slim margin.

The changes in student population and family dynamics, due to new families moving into the neighborhood and new families transferring into Tuff, challenge teachers at Tuff. Many teachers at Tuff do not feel prepared, or supported, to teach groups of students with such diverse backgrounds and ranges of academic ability and need. The school currently employs a full-time principal, a part-time vice principal (who works at two elementary schools), and 28 grade-level teachers (four teachers per grade, with the exception of third, where there are only three). There is a full-time annual PE teacher and a full-time art or music teacher every other year. There are five special education teachers, four gifted teachers, two certified occupational therapist assistants, a school nurse, a counselor, an instructional coach, a librarian, an occupational therapist, a speech language pathologist, a physical therapist, a social worker, and a technology coordinator. A safety team of four specially trained educational assistants are present daily to assist with classroom disruptions that sometimes relate to students with individual behavioral educational plans.

Three teachers from Tuff Elementary, Mrs. BB (a first-grade teacher), Mrs. Branches, and Mrs. Kapeesh (fifth-grade teachers) agreed to participate in my research. Each teacher narrated how their teaching at Tuff Elementary was impacted by their perceptions of this instructional context. The reader will notice that each teacher refers to their teaching contexts by first describing their classroom composition via labeling students as on, above, or below grade level. Student academic diversity significantly affects how these teachers interact with their students, the content, and their students' interactions with the content.

Choosing Interactions

My research hinged on the recognition that interactions (between teachers and students, students and the content, and the teacher and the content) provide the context for discussing what student mathematical sense-making teachers listen to; how teachers make connections between the particulars of a specific interaction emerging during classroom instruction and what they know about how their students learn math; and how the teacher describes their listening, interpretation, and response processes.

My participants and I discussed what it means to notice students' mathematical sense-making and created with four categories of noticing, which closely relate to Mason's (2002): NN, BN, M, and R. Our definitions of each category, which are reminiscent of Mason's, helped us decide which interactions we would include in the final narrative of noticing: NN describes teachers' inability to narrate how students' sense-making contributes to a viewed instructional interaction; BN means that teachers can recall a viewed interaction and how a student(s) responded but cannot narrate their interpretations of student sense-making; M means that teachers have something to say about the interactions they reviewed without prompts or questions from me; and R means that teachers have kept student work, written a personal note, or so forth, so they can return to an interaction without reviewing video recordings of their instructional interactions. All participants agreed that not noticed interactions would not be included in their chapter narratives; this decision involved how they felt they would be perceived by readers. I was permitted to use these episodes in my cross-case analysis (Chapter Seven) to show similarities and differences in noticing. The teachers had various reasons for wanting to include what they noticed from the remaining three categories. Generally, the teachers thought they could improve their noticing by highlighting how they might make specific aspects of what they noticed about their students' mathematical thinking salient by narrating it, even if what they noticed was small. We decided, then, to include interactions in which the teachers self-identified as BN, M, and R.

Some participants watched their instructional videos as an aid in recalling moments when they listened, interpreted, and responded to their students, while others had specific interactions they wished to discuss. Finally, one struggled to narrate any moments of interaction. I used Leatham, Peterson, Stockero, and Zoest's (2015) conceptual framework for identifying a unit of analysis – an observable student action or small collection of connected actions – to assist my participants, if needed, to select interactions for their narration. I watched participants' instructional videos for in-the-moment interactions (teacher–student, student–content, or teacher–content) that began with teacher noticing of student mathematical sense-making, may have been mathematically significant, and may have been a pedagogical opportunity.

In Chapters Four, Five, and Six, the reader is introduced to Mrs. BB, Mrs. Branches, and Mrs. Kapeesh, respectively. They share their narratives of the student sense-making they listened to. The reader should remember that the goal of this research was not to identify specific pedagogical interactions that teachers should copy. Instead, we worked together to determine what we could understand about how teachers listen to student sense-making. This matters because the narrative reasons my participants provided for noticing what they did about students' sense-making and interacting how they did, given what they thought they understood about that sense-making, had implications for the types of learning opportunities students had in their classrooms. We begin with Mrs. BB.

CHAPTER FOUR

"You Never Know What the Kids Might Need or When It will Make Sense": Mrs. BB's Narrated Noticing of her First-grade Students' Mathematical Sense-making

My participants and I lead the reader through each case analysis by sharing several interactions with the reader (the process for choosing these interactions and narratives is presented at the end of Chapter Three). I describe each interaction, framed by my three research sub-questions. I guide the reader through the mathematical sense-making the teacher noticed by providing a context for the interaction the reader is about to participate in; the teacher's elicitation and interpretation of student sense-making through the teacher's and students' pedagogical interactions; and the teacher's narrative of how they chose to respond to their student's sense-making, given what they heard (Research Sub-questions are outlined in Chapter One, p. 9). Finally, I focus the reader's attention on an analytic point. I illustrate and explore my claims through each participant's narratives of noticing grounded in the details of their specific listening (Emerson, Fretz & Shaw, 2011).

The Student Sense-making Mrs. BB Noticed: Her Context for Interactions

Mrs. BB had been teaching 38 years at the time I undertook this research. Mrs. BB always "wanted her parents to be proud of her." This backdrop provides the initial description of the context for Mrs. BB's classroom interactions.

Self-evaluation. Mrs. BB's years and variety of teaching experiences have led to her to feel that she "knew many of the best techniques for teaching most subjects." She considers herself a "life-long learner." Her description of self is reflected in the many educational experiences she has participated in prior to, and during, her teaching career.

Experiences. Mrs. BB said she was "always a good student." Her elementary and middle school educative experiences were filled with extra academic opportunities, mostly through workbooks her parents bought her, because school was "easy," she was an "over-achiever," and she "always got the answers right." She had a waitress job during high school to "make good tips and save for college" and was also a volunteer candy striper at a local hospital. Mrs. BB's freshman year of college was spent studying nursing. She did not care for it, so she transferred to an Eastern university and changed her major to elementary education, with a minor in special education. One day, Mrs. BB's "crazy roommate, Kim" came home and said that she had signed them up to learn about a special student teaching project on the Navajo Reservation. She took two graduate classes to "learn about teaching in a different culture" and spent three years teaching fourth grade on reservations in the southwestern United States. She earned her master's degree in behavior disorders and became dually certified in both regular and special education. She felt limited by the size of her current classroom, how many students she had in her class, and the mandated curriculum that she taught.

Motivation. Mrs. BB was always interested in learning instructional strategies that helped students with "impairments" and also recognized the importance of finding ways to "help students with their behavior." She dreamed of having students who were "very creative and would constantly ask to either extend lessons or ask for lessons following their own interests."

Mrs. BB was one of four first-grade teachers at Tuff Elementary. Her classroom marked the beginning of the first-grade hallway. A soft hum of children's chatter can usually be heard through the classroom door's pane of glass, which is covered with emerald green construction paper. Her classroom is

> …a small old room with a wonderful wall of windows and built in bookshelves underneath. I am lucky to have a Promethean Board to use with math, science, and

> English Language Arts lessons, and to give the students Brain Breaks with the site gonoodle.com. I have a meeting area made of my own rugs on the tile. There are also five tables with four desks in each group. There are two tables for small group work. There are six computers against the back wall to use for a group of students. There are 20 students in my class, 10 girls and 10 boys. There are five English language learners, and four students have been retained either in kindergarten or in first grade. Academically, six of the students are below grade level in a least one area, seven are on grade level, and seven are above grade level. The students really like to do reading and math rotations, so they get to move three or four stations during a lesson.

Skills. At the time of research, Mrs. BB used the district adopted curriculum, Stepping Stones Math, which had lessons to differentiate between students below, on, and above grade level. She also used some Everyday Math (a different math curriculum) games and strategies that she learned when teaching with that series many years prior. Additionally, Mrs. BB supplemented these curricula with her knowledge of Math Recovery Strategies (a curriculum that assists teachers in identifying how students' number sense is developing) from five years as a math recovery teacher. Mrs. BB had a repertoire of instructional techniques, because she had "taught kindergarten through fourth grade, special education, and gifted education." She had "taken many classes and participated in professional development opportunities."

The Kinds of Student Sense-making Mrs. BB Listens for

There are three categories of student mathematical sense-making that Mrs. BB listens to most frequently: student number sense, student communication conveyed by visual representations, and children's generation and defense of responses. The following vignette shows one of Mrs. BB's daily routines emphasizing student number sense and algebraic thinking – students' opportunities to notice, record, and build patterns with Unifix cubes (colorful linking blocks with a single peg on top; students connect the blocks, top to bottom, to create chains or towers that allow them to investigate counting and one-to-one correspondence, as well as explore pattern formation, repetition, and description). We begin by providing the context for interactions emphasizing student number sense with daily routines.

Teacher Noticing that Emphasizes Number Sense with Daily Routines

Near the front of the classroom, a dilapidated, but refurbished, cardboard chest sits, noticeably void of drawers. The students had the overflowing drawers, containing Unifix cubes, at their seats. The students' objective was to design a pattern out of the linking cubes, organized by color, and then use mathematically appropriate terminology to describe the pattern sequence to a fellow classmate. Students in first grade were expected (by grade-level common core standards) to identify their constructed pattern (for example, a student could link red cube, orange cube, red cube, orange cube) and assign each color a corresponding letter of the alphabet to describe the pattern mathematically (for example, red cube=A and orange cube=B; the pattern the child created with the red and orange cubes would be named ABAB).

Context for student sense-making noticed in Interaction 1. Mrs. BB interacted with a student's interaction with the content presented above. Mrs. BB noticed that this student

> was having a hard time just writing numbers. And I think it's just a lack of practice. Some classrooms, they do a lot of counting and a lot of different things with math, patterns and that, and but, she didn't know patterns. And so, she was just doing different colors. So then, we had been working on making AB patterns. Make an ABC pattern. And um so, we had gone further, like the calendar is AABB for October, and so they know that's the pattern.

Mrs. BB narrated that she noticed this student struggled to write numbers and linked Unifix cubes together randomly. Mrs. BB decided to use a large classroom calendar to build on the student's oral counting skills and introduce pattern design and identification. For example, in August, she used an alternating sun, cloud, sun, cloud, or ABAB, pattern for all students to identify and describe. Mrs. BB used a triangle, square, circle (all the same color) pattern for the month of September, so the student could identify and describe an ABC pattern. Mrs. BB incorporated what she noticed about her student ("She is developing skills writing numbers in standard mathematical notation and developing skills in identifying and articulating patterns") into the interaction that follows.

Mrs. BB's elicitation and interpretation of student sense-making in Interaction 1. Eighteen students are snugly situated in clusters of four desks. Pairs of student heads are leaned toward each other, sharing the space over a box wrapped in floral contact paper. A gentle patter of plastic against plastic and cardboard can be heard, as children dig through boldly colored Unifix cubes, searching for the block that will contribute to their 12-train design (a connected stack of Unifix cubes, one on top of the other, that contains a total of 12 cubes). A soft murmur fills the air as the students excitedly whisper their color combinations. A spattering of students utter alphabetic sequences while walking their fingers like feet up coinciding Unifix cubes in their trains. A student springs from her seat and gallops to Mrs. BB, who is completing her final preparations for the day's math lesson, her brow furrowed as she taps fervently on her laptop, connected to the Promethean Board.

Student: Look at my colors! Look at my colors!

Mrs. BB: Does it repeat?

Student: Yeah. Mine is white, light blue, white, dark blue.
The student points to each colored Unifix cube as she names them.

Mrs. BB: White, light blue, white, dark blue. *Mrs. BB mimics the student behavior of pointing to each color of the pattern train while the student says the color aloud.* And it repeats. And look – *She then points to the colors in the same pattern, but alternatively gives each color a letter name, instead of a color name.* A (*points to white)* B (*points to light blue)* A *(points to white)* C *(points to dark blue)* A *(points to white) – She stops in the middle of the pattern.*

Student: B!

Mrs. BB's response, given what she noticed. In this teacher interaction with the student (see Lampert's strategic sites of interactions, Chapter Two), Mrs. BB noticed that her student had created a color pattern that they had not covered in class with the calendar sequence or named alphabetically. Mrs. BB made a pedagogical choice to stop her recitation of the letter pattern sequence for two reasons: "to provide the child with the opportunity to make sense of the alphabetic pattern organization" and "to determine if the child was able to continue making sense of the pattern independently." The child leaped away triumphantly. Mrs. BB continued to observe the ensuing student–student interaction that occurred, as her student interrupted her elbow partner to describe her newly discovered pattern description correctly (ABACAB) from the end of the Unifix train.

Listening process claim for Interaction 1. Mrs. BB balanced a tension between her best content and pedagogical knowledge and gathering more information about her students' sense-making. She considered how an instructional strategy was performed before and how it could perform in the present, based on the student sense-making she had gathered and her knowledge of how to use that sense-making to elicit the best result. She used what she noticed about student mathematical sense-making in the past to consider the upcoming instructional interactions with an individual student. In this interaction, she listened to the student dictate a common pattern of student development in student number sense and algebraic thinking. The student's sense-making used physical objects (the Unifix cubes) to directly model the relationship (the Unifix cube color pattern) she created with the Unifix cubes. Mrs. BB knew that

> I can expect [the student's] strategies to become more efficient soon, like, more efficient counting strategies. For example, instead of starting at the edge of the train and naming the pattern repeatedly, [the student] could just say ABAC is the pattern.

Her student's communication was moving from being conveyed by primarily visual representations to the oral articulation of identified patterns. Mrs. BB considered how to alter her interaction with this student based on what she initially noticed about her student's sense-making and the sense-making the student was demonstrating in the moment. Initially, Mrs. BB had not planned on interacting with the students at all during this activity. She needed the time to continue preparing for her lesson. Instead, she built on what she noticed ("the students' building, recording, and articulation of a pattern") and used this strategic pedagogical moment to provide a pivotal opportunity for the student to notice, record, and build patterns using Unifix cubes and then relay this new knowledge to another student.

The opening vignette of this chapter shows one of Mrs. BB's daily routines that emphasizes noticing student number sense and algebraic thinking; the students' opportunity to notice, record, and build patterns using Unifix cubes. I was able to see another variation of pattern building, choral counting (Interaction 2: emphasizing number sense with daily routines). The next teacher interaction with the content focuses on choral counting.

Context for sense-making noticed in Interaction 2. Mrs. BB was aware that her instruction was about subtraction. In this teacher interaction with the content, she

> knew that we needed to start doing counting back, and because that's something we didn't actually have to do today but is something that's coming. Plus, oral counting at the beginning of a lesson is something we do.

The following interaction is one that Mrs. BB marked, meaning that this is an interaction she thought to remark on without a prompt from me. She knew from prior noticing during subtraction unit introductions (gathering information on student sense-making) that students would need opportunities to count back to make the connection to subtraction (her consideration of how an interaction has performed before in combination with using the student sense-making she noticed, to use her best knowledge and get the best student result). She considered strategies used in her prior teaching experiences and applied them to a new interaction with her entire group of students when she introduced their subtraction unit.

Mrs. BB's elicitation and interpretation of student sense-making in Interaction 2. Mrs. BB and her students stand in an oval at the front of the classroom on her rugs. The lesson begins with students counting forward from one to 10, while passing a bean bag gently into the hands of the child responsible for the next number in the count.

> Mrs. BB: Okay, now, that was forward. Now we're going to go from 10 down to one. That's backward. Ten.
>
> Student 2: Nine.
>
> Student 3: Eight.
>
> Student 4: Seven.
>
> Student 5: Six.
>
> Student 6: Five.
>
> Student 7: Four.
>
> Student 8: Three.
>
> Student 9: Two.
>
> Student 10: One.
>
> Mrs. BB: Right. That was backward. Now we're going forward again. Now you start the sequence.

Mrs. BB's response, given what she noticed. Mrs. BB looked at each speaking student and nodded as they counted. She mouthed the numbers with her students' oral counting. She explains,

> We always do oral choral counting first as a group and then we move to counting individually. Sometimes they just need a reminder. If we do it again the second time

> individually, the student usually gets the sequence of the numbers. I also try to switch it up a little bit every day.

Listening process claim for Interaction 2. Mrs. BB used what she already knew about students' sense-making to select an interaction based on her best knowledge. Her chosen interaction had succeeded in the past, and she thought it would yield similar student results in the present context. Mrs. BB noticed how children think about concepts (counting forward is the same is adding, and counting backward is like subtracting) and interacted with her students in a scaffolding manner (first, chorally to learn and practice a counting sequence and then individually to determine student level of mastery) to help all students attain fluency with relating counting to addition and subtraction (CCSS, Operations and Algebraic Thinking, Standard 1.OA.C5, 2018). She emphasized student number sense (regularity in counting sequence and pattern of counting backward) and algebraic thinking in early mathematics with daily routines.

Mrs. BB was aware of the sameness of this daily routine with her prior first-grade classrooms and chose to enact the math content (counting backward) in this specific student–teacher–content interaction (altogether chorally, and then individually with teacher support as needed), because she was aware that this particular daily interaction could, and does, hold in her current classroom. She explains,

> The purpose of this pattern routine is to emphasize number sense and algebraic thinking. I am looking for students to express regularity in repeated reasoning by counting backward from 10 to one. Additionally, I am encouraging attention to generalization, from the specific observation of counting backward to the broader insight that counting backward is related to subtracting.

Mrs. BB was aware that her students needed to tackle counting back from 20 next, and due to her familiarity with children's common misconceptions about math ("those tricky teens and then 12, 11") she scaffolded instruction (her lesson the next day presented counting backward from 20) for herself and her students, by providing an authentic interaction for clarifying backward counting mistakes, should they arise. Next, in Interaction 3, we focus on Mrs. BB's second noticing category, students' sense-making conveyed by visual representations.

Teacher Noticing of Students' Communication Conveyed by Visual Representations

Mrs. BB's narratives of her noticing students' communication conveyed by visual representations were a combination of "never give a kid an answer" and "encouraging children to generate their own questions and then develop, explain, and defend their responses." She helped students see the relationship among numbers in subtraction equations (which sets the stage for algebra in later school years) through her balance of patterned questioning of students' sense-making and introduction of formal math vocabulary. Mrs. BB explains,

> We don't ever want to tell answers. We want them to explain their thinking. And if they can't explain their thinking in words, figure out how else – I approach each student with a different way of attending to the task, whether it's go back and draw it, show me on your fingers – I reiterate, show me, tell me lots of times! Eventually, kids get there.

Mrs. BB begins her introduction to subtraction by asking (a teacher initiated productive mathematical question that may assist her in recognizing, identifying, and interpreting patterns in students' sense-making) students to share their language for describing the action of subtraction.

Mrs. BB: Has anybody ever heard that word before? Sub-trac-tion? Does anybody know what it means and like to share with us?

Student: Minus or subtract means take away.

Mrs. BB: Alright. Awesome. Anyone else like to share?

Student 1: Um, subtraction…is, um…

Mrs. BB: Have you heard the word before?

Student 1: Plusses?

Mrs. BB: Addition is plusses. And subtraction is…
Keep thinking. If you come up with it, raise your hand, okay?

Student 2: To me, subtraction means take away.

Mrs. BB: Okay. How many people have heard that word before? Subtraction? Okay. We need to be able to count, so I'd like – we're going to pass this around we're just going to go up to 10 forward 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and then whoever has 10, the next person says 10 again and 10, 9, 8, 7, 6, 4, 5, 3, 2, 1 (sic) and go backward. Let's see if we can do that, okay?

Mrs. BB knew "the connection between counting forward and addition and subtraction and counting backward." She provided the opportunity for students to build their number sense with the daily choral count routine and assisted children in clarifying what subtraction is. Finally, she ensured that all of her students began with the same set of prior knowledge ("understanding addition as putting together and adding to, understanding subtraction as taking apart and taking from" (CCSS K.OA.A1-A5) and "relating counting to addition and subtraction" (CCSS 1.OA.C5).

Context for sense-making Interaction 3. In this interaction, Mrs. BB assigns students to homogenous groups to work on a Step Ahead problem, the final problem in the Stepping Stones daily workbook sequence, which asks students to take what they learned from the days instruction and apply it to an authentic problem, unlike the drill work completed in the opening segments of a lesson. Students are required to generate seven different subtraction equations that all begin with six as the total. Mrs. BB moves to a pair of boys working on the Step Ahead. She kneels, at face level with her students. One of the children writes "6–7=1." Mrs. BB uses her purple glitter pen to tap on that equation.

Mrs. BB's elicitation and interpretation of student sense-making in Interaction 3.

Mrs. BB: Now this is where it gets tricky. You have to go through and see what numbers did you use to take away. So show me that one.

Student: Show six minus seven?

Mrs. BB: Show me six fingers. Take away seven. Can you take away seven? Will that work? Can you do it?

Student: (Gasp!) I found it Mrs. BB!

Mrs. BB's response, given what she noticed. Mrs. BB chose to use a prompt (a strategically chosen question) to help her, as well as the child, recognize how his thinking about subtraction is organized. She asked the student to visually represent with his fingers, so he can reflect upon his response, as well as try to defend it. She gave the student an opportunity to evaluate the plausibility of his claim. Mrs. BB chose to structure the modeling of finger counts using the method prescribed by the Stepping Stones curriculum. She says,

> The most important part is to show that if I want to show seven, I show one hand with the four fingers then the thumb, then the other thumb and then the pointer, so that the seven fingers are together and then the other three fingers make 10. The formation of raising the fingers together shows combinations of 10.

Mrs. BB has her palms facing out to the students with her thumbs together, counting left to right, like reading a book. "So when we're counting, they want us to try and go like this, one, two, three, four, five, six, seven, eight, nine, 10."

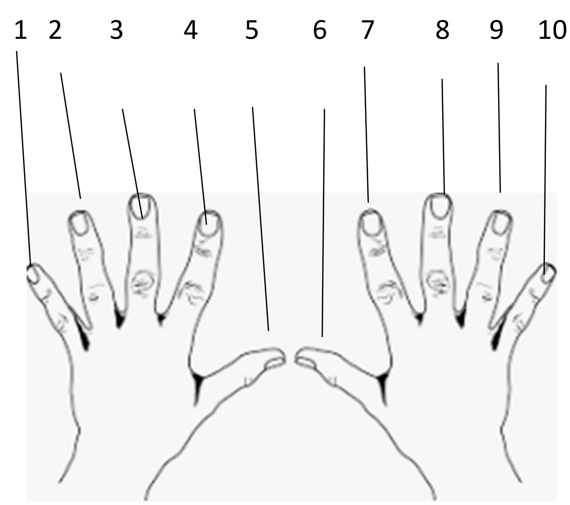


Figure 3. How kids count on fingers.

> And the reason is, and after you hear the reason, it's like, oh, this is probably – I thought it's a good thing. Because then when you say, okay, show me nine, and then one more

makes 10. So, what is it? Nine plus one. Following across. If you say seven, and you have to do it so you're looking – so you're looking left to right – so then I have seven here plus three equals 10. So that in a month, when we get to combinations of 10, they're going to be so on it.

Listening process claim for Interaction 3. Mrs. BB tried a new-er to her strategy to respond to student sense-making, instead of relying on what she already knew. Mrs. BB noticed the roots of students' algebraic thinking in the ways they organize information (building on students knowing what makes 10) and provided an opportunity for students to model their thinking with their fingers that allowed for multiple methods of representation of 10 (direct modeling or counting strategies). She drew a model, hands out-stretched (see above) with a count to mark specific aspects of the student's thinking. She narrates,

> This technique of modeling counting helped me form an appropriate interaction with him [the student] so he can become fluent in counting on, making 10, and adding and subtracting within 10.

Teacher Noticing Children's Generating Own Responses and Beginning to Defend Them

Mrs. BB used her knowledge of children's thinking to interpret how students were organizing their new information about subtraction. The new information Mrs. BB had provided them with was "how to calculate the unknown amount in subtraction equations, rote count backward from 10, and solve subtraction word problems" and cross referenced that with her awareness of hallmarks of students' mathematical competence, "particularly, students using an appropriate count-back strategy in one-digit number subtraction problems." Mrs. BB's subtraction instruction provided an opportunity for students to tell their own versions of their subtraction stories, using different words that involve taking away. Mrs. BB knew that "It is important for her students to have many opportunities to make the distinction between addition and subtraction" and often shares a story with her students that she chose to share with me:

> And the other thing about subtraction is I wanna teach them, you know – I found subtraction a little difficult because I wasn't great with counting back. Ya know? It's like, I wasn't sure of those numbers, and so what I did when I was a waitress, I counted up to subtract. If you bought a drink for 65 cents and you paid a dollar for it, I would need to give you some change. The drink was 65 cents, so instead of subtracting, I say, 65 plus what gives me a dollar? Then I say, 60 and then how many tens, 70, 80, 90, so that's 30 cents, and then five more, so that's 35. That's why I'm doing this. Math is so abstract sometimes. And so, I try to tell them that and then it's funny because there are always some kids that will come back and say, I add to subtract too. And I'm like, you know what? There's so many different ways to figure out math problems.

Mrs. BB had high expectations for students to demonstrate fluency during the learning opportunities she provided. Her most important observation was, "They [students] should know about subtraction situations is the total and one part are known, and the answer is the unknown part of the total." Mrs. BB used a pen that is an inspiring shade of purple to remind herself – and convey to her students – that they are on their way to attaining fluency by marking purple stars in students' workbooks when they explained their sense-making in a way that Mrs. BB determined demonstrated fluency (the student could name the total and both parts of subtraction equations).

Mrs. BB organized the students in homogenous groups during their middle of the lesson rotations and adapted instruction to address what she noticed about specific students' number sense and algebraic reasoning. We now participate with a homogenous group, as they attempt to defend their answers to a subtraction problem. We see how sometimes, Mrs. BB does "what's in the plans," and sometimes she does not, depending upon what she notices about her students.

Mrs. BB: What's happening in this picture? How many hens? You know what? Draw a circle on the left-hand side. Left-hand side. Draw a circle. *She holds up the workbook. She points to the workbook picture of the hens in and outside the pen. She then uses her left hand to draw a circle on the left-hand side of the workbook's hen diagram. The hen diagram has a total of nine white, brown, and black hens.*

Mrs. BB: Okay. So, all of these little guys, the white and the brown and the black, are all hens. Count the hens and put the total in the circle. Go. Put your thumb up when you have it done. What did you get, S?

Student 1: Um, six. Six. *The student has counted only the hens that are inside the hen pen.*

Mrs. BB: Okay. All of them. All of them. Try counting one more time. *She uses her pencil to draw a circle around all the hens in the picture.*

Student 1: Nine.

Mrs. BB: Good. Could, could you do something with me? Put your pencils down, and remember how we talked this morning about if we just count all over? *Mrs. BB points with her left index finger at any chicken while she counts, without marking which chickens are counted.* Are we keeping track of it? No. So pick up your pencil, and can we make a little, like, "X" above each chicken as we count 'em? *Mrs. BB picks up her pencil and demonstrates making an "X" above a counted chicken.* Wait, wait! Before we do it, hang on; look up here; can you figure out how you're gonna do it? I see three, and then I see four, and then I see these two. *Mrs. BB demonstrates a way to group the chickens into parallel lines of four, three, and two. She draws a line with her pencil across each group of chickens.* Does that make sense?

Student 2: No.

Mrs. BB: Can you see those three? You don't see a line across here? *The child who said "no" now shakes her head. Mrs. BB points to the chicken "groups" in the student's workbook.* And then another line and another line? That's kinda like how I organize my thinking. If you see it a different way, that's fine, but as you count, put an "X" on it. Let's try. Ready? Go. One, two, three, four, five, six, seven, eight, nine. *Mrs. BB counts and puts Xs over the chickens using her four, three, and two groups.* How many of you got nine in your circle? Good. Okay, let's go down here. Here's our equation. What is the total number of hens? Write that in the first space.

Student 3: Nine.

Mrs. BB: Nine. That's our total. Nine. How many are running away? Look and see the little ones running away. *Mrs. BB uses her pencil to lightly circle the two hens that are running out of the hen pen in the picture.*

Student 1: Two.

Mrs. BB: Okay. Write it down. Now you count. How many are still in the hen pen? Write down your answer. The difference. *Mrs. BB uses her pencil to draw a circle in the air around the chickens that remain in the hen pen. All children look to see which part of the hen pen picture they should be* *paying attention to – which part of the picture corresponds to which part of the fill-in-the-blank equation.*

Student 5: Seven.

Mrs. BB: S, could you read your equation?

Student 4: Nine chickens in the hen pen. Two ran away, and there are seven left.

Mrs. BB: Does everybody agree? *Mrs. BB looks at the equations in everyone's workbooks and marks each correct response with a sparkly purple star.*

Mrs. BB narrates this interaction during filming. She instructs the children to begin working independently on the next subtraction problem, saying, "If you need help, let me know." She then walks toward me and steps out of camera frame. She narrates,

> I realized we had trouble counting this morning. There were lots of "ah-has" when I was seeing who was having trouble and who was getting it very quickly and who was having struggles with it. When they were counting the fish. That's where I got the big ah-ha. Cause they were going one, two, three, four, five, six, seven – no order at all. And students didn't get seven [the total]. They got eight or nine. Students were just going around, and um, and then I said, do you all see there's kinda like a row here and kinda like a row here. And then they could see that and I said, well, then, when we count, why don't we make an "X" or put a dot, make some kind of mark as we go across when we're counting so that we can keep track, so that we're not counting one fish three times. And then we get the right number. So then they all had to mark every single one. And that was okay because then they were saying, oh, all of those, when we mark 'em, that's a total. YES. THAT'S A TOTAL. Just getting that vocabulary in as many times as possible was good.

Mrs. BB's lesson plans did not include an objective for students to understand the relationship between numbers and quantities. She noticed, as she stated above, that she saw students moving their pencils in a haphazard manner (movements that indicated one-to-one correspondence was not being maintained). She decided that her students needed the opportunity to review the number names in the standard order, pairing each object with one, and only one, number name and each number name with one, and only one, object. Mrs. BB immediately provided a scaffolded interaction wherein students used drawings to represent a "taking from" word problem (CCSS, 1.OA.1). She used her interaction from the morning as a scaffold. She no longer counted aloud with students or directed the one-to-one correspondence in counting. She reminded students to "organize and mark their thinking" and "see groups," further encouraging students to demonstrate hallmarks of mathematical competence, by suggesting a "circle in the left-hand corner to write the total in," thus providing students the opportunity to show how they could represent and solve problems involving subtraction. Interaction 4 demonstrates how Mrs. BB was aware of how kids think about math, used multiple methods of representation, and presented informal and formal math language to be able to scaffold her students through the notation, generation, and defense of their mathematical sense-making.

Context for sense-making Interactions 4a and 4b. Mrs. BB told us earlier in the paper that part of her teaching context was four students who had been retained in either kindergarten or first grade. She narrated that her students' developing number sense and algebraic reasoning concepts impacted the types of interactions she orchestrated for learning. She knew that to build on the skills students developed in kindergarten, she needed to continue to focus on "representing and relating operations on whole numbers, initially with sets of objects." She discerned that she could provide two opportunities for her students to learn: first, to provide the opportunity to work on counting a set of a given size (CCSS, K.1) and second, to build on their knowledge of numbers to count the objects that remained in a set after some were taken away. She knew that by the end of first grade, all of her students needed to be able to relate counting to addition and subtraction and use the relationship between addition and subtraction (e.g., knowing that 8+4=12, one knows that 12–8=4). Mrs. BB had her students form homogenous groups and instructed her "lower level of number sense group" to "go ahead and color some fish" in response to the following prompt: There are seven fish in the tank. Color some of the fish orange. Complete an equation to show the number of fish that are not orange.

Mrs. BB's elicitation and interpretation of student sense-making in Interaction 4a. Mrs. BB gives her students time to color some of the fish orange, reminding them that, "Some doesn't mean all."

Mrs. BB: So, how many fish did we start with? Show me with your fingers. How many fish altogether did we start with? *Mrs. BB places her palm on the picture of the seven fish in the workbook and makes a circular motion. She scans the group for finger responses. S1 displays seven. S2 displays three. S3 displays eight and is counting his fingers. S4 is showing no response, and the remaining two students are showing seven, opposite of the way S1 s.*

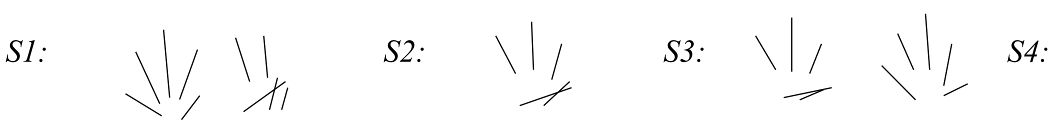


Figure 4. Student finger count representations.

Student 5: Seven.

Mrs. BB: That's our total, right? Show me how many you colored. *Mrs. BB scans the group of student numbers of finger for fish colored orange corresponding to the number actually colored orange.* Okay, so, is that how many, like, you took away? Okay, so then, how many are left? Could you fill in the equation here showing me your subtraction equation? *Mrs. BB points to the blank ___-___=___ in the workbook. This is the same model presented by Mrs. BB on the board at the beginning of the lesson.* What was your total? So, what number should we start with? How many did we have all-to-geth-er?

Student 4: Seven.

Mrs. BB: Yes. That's right. So we start here with seven. *Mrs. BB points to the total, the first blank in the subtraction equation in the workbook.* So, then, how many did we take away? You're going to start with seven. How many you colored is how many you took away. Then, how many do you have left? What's your equation, S1?

Student 1: Seven minus three equals four. *Mrs. BB writes "7–3=4" on her dry erase board after the student verbalizes each part of the equation.*

Mrs. BB: So, how does each number in the equation match the picture? And where did the three come from?

Student 1: I colored in three fish. *Mrs. BB points to the "3" in the equation.*

Mrs. BB: Okay, so those were the orange fish? *Mrs. BB changes the student response to "orange fish" and writes the words under the "3" in the equation.*

Mrs. BB: And then where were the four? Where did those come from? *Mrs. BB taps under the "4" in the equation three times with her left pointer finger.*

Student 1: The white ones that were left.

Mrs. BB: Okay. Good. So, where did the seven come from?

Student 1: Because altogether there were seven. *Mrs. BB writes "total" under the "7" in the equation on the dry erase board following the student's response.*

Mrs. BB: Right, because altogether, or we can say, the total, that's what we started with. Awesome. Who can tell me another equation that is different?

Mrs. BB chooses to interact differently with another homogenous group of students and alter her lesson plans to address what she notices about a particular group of students that have "greater number sense."

Mrs. BB's elicitation and interpretation of student sense-making in Interaction 4b. She presents this additional interaction, framed with the following problem: Write some numbers to show a subtraction fact. Draw a picture to match.

Mrs. BB: Thank you for working so hard. You guys are gonna get to the next part. So let's go ahead. You're going to write a number. Let's go between 6–12. Put your number in the box. So, what numbers can you choose? Who can tell us: what are the numbers?

Student 1: Six, seven, eight, nine, 10, 11, 12.

Mrs. BB: Yes. So, any of those numbers. Did everybody pick a number? That's your total. Pretend I picked 10. I'm going to put circles for my animals because we don't have to draw real animals. You could draw circles, Xs, squares, triangles. Put a "t" above that. That's your total. So, you're going to have some inside the pen and some outside getting some exercise. But when you add them together, they're going to come to that number. K? So, go ahead and put some inside, some outside, and make sure it's your total. Then go ahead and write your equation.

I like the way people are writing different things. *One student has chosen 10 for their total. Another chose six. Another, 11.*

Student 5: Do you write 11 in here and out here? *The student points to both inside the hen pen and outside the hen pen.*

Mrs. BB: You have to (pauses). Eleven would be your whole total.

Student 5: Some in here and some out here.

Mrs. BB: Yup. Try not to use doubles. Five over here, that's good. Five went away. Put a five there. Now, how many are left?

Student 5: Three!

Mrs. BB: You chose 11. So you need to have 11 in total. Then, how many went away? How many are left? *The student erased her answer. She looked at the total in her standard notation equation (11). She drew five circles outside the pen and wrote "5" in the part of the equation to be subtracted. She touched her pencil tip to the 11, then back to the 5. She said, "five," and then raised one finger for each count six, seven, eight, nine, 10, 11. She ended with six fingers raised. She wrote "6" after the equals sign and raised her eyes to Mrs. BB's face. Mrs. BB drew a purple star on the child's workbook, and the child proceeded to draw six circles inside the pen.*

So, let me show you something. I'm going to extend this. *Mrs. BB returned to her example of a total of 10 animals.* I have six in here, and four in here. So how many altogether?

Students: Ten.

Mrs. BB: How many are left in their exercise box?

Student 4: There's six.

Mrs. BB: So you know what, can I write an addition equation that would go with this? How many are inside and then how many are outside, and then how many altogether? Can you make an addition equation that goes with your subtraction equation?

Are you going to use the same numbers?

This is kinda like a family. I had 10, I took away four, and I got six. I wanna see if I can do an addition equation to go with it. Six plus four and I get 10. So what do you think? What's your total? Your total is, for subtraction, what you start with. Your total with addition is what you end with. The sum. S, can you tell us what you got? What's your addition equation?

Student 3: Three plus seven equals 10.

Mrs. BB: And now what's your subtraction equation?

Student 3: Ten minus three equals seven.

Mrs. BB: That's a family. It goes together.
That's a seven, right? You're on the right track. So here, 11 minus two is nine. So what's the addition to go with that? What plus two?

Student 2: Nine.

Mrs. BB: Yes. So write it down. Nine plus two equals – what did you get? What's this?

Student 2: Ten.

Mrs. BB: Hmm. Did you make 11 all together? Something isn't making sense.

Student 2: Because I need 11 and nine.

Mrs. BB: Show me your 11. *The student draws 11 circles inside the pen.* What's your take-away? *The student drew a circle around nine of the circles in the pen and drew a line indicating that these circles were moving as a group out of the pen.* What do you have left? *The student tapped the two un-circled circles. He then filled in his equation as "11–9=2."* Good. So, now, if you're adding it, you have this *(Mrs. BB points to the "2" in the equation)* plus what? *The boy points to the "9" in his equation.* Yes, write it down.

Mrs. BB's response, given what she noticed. Mrs. BB's process of noticing included an awareness of prior student mathematical conceptions and the strategies she used to remediate and extend children's thinking. She applied these strategies to current math interactions, where students' sense-making was similar to what she had experienced in other instructional contexts. Her noticing "developed during successive years of teaching first grade and cumulative years of teaching across grade levels." She discerned and noticed details in recurring interactions that could hold in other situations and assembled those noticed things to apply them to new, current, similar interactions. Mrs. BB's ability to structure various learning opportunities in the moment begins with knowing the content of the mathematical strands and the structures and interconnections of mathematical topics. I ask Mrs. BB to tell me more about this interaction. She says,

> So, I want them not only to be able to look at the picture and write it, but I want them to be able to tell, to say it, too. And then, telling the story of what happens, because that's a big thing with the math story problems and here, they want them [the students] to start making their own stories. That's why, normally, I wouldn't say you have to use, for example, four. I do that for a reason. Because they kinda need that scaffolding. That help. But I also don't want to stifle them, either.

This is why Mrs. BB varied the numbers given the homogenous groups in the final pen problem.

> With that group, I really wanted to open it up and I was going to say between 10 and 20, but then I backed up cause I was like, you know, then we're going to have problems with, oh, I meant to do 20 and I only drew 19, and you know, it's like, it can get too big. The representation. So I could've gone a little bit higher, but I think that worked, and I'm glad that I kept it to 12. Because all but one [student] got the fact family…We're going to get to fact families in like a month or so, and so, it's like, if I can teach these guys that there's a relationship, then, um, then the other ones [homogenous groups] are going to hear about that and then we're gonna be ready for it before we get it.

Listening process claims for Interactions 4a and 4b. Mrs. BB used what she already knew and her prior understanding of noticed student sense-making to provide students with opportunities to learn. Mrs. BB noticed the content of the mathematical strands and the structures/interconnections of mathematical topics to organize her instruction and gave students the opportunity to make sense of the problem. She gave students the opportunity to reason, construct arguments, and model with mathematics. We share our thoughts on each claim below.

Content of the mathematical strands and the structures and interconnections of mathematical topics. The kindergarten domain of operations and algebraic thinking contains the standard that students will understand addition as putting together and adding to and subtraction as taking apart and taking from. The cluster subsumed under this standard includes representing addition and subtraction, solving addition and subtraction word problems within 10, decomposing numbers less than or equal to 10 into pairs, finding the number that makes 10 for any number one through nine, and fluently adding and subtracting within five.

The first-grade domain of operations and algebraic thinking contains the standards that students will represent and solve problems involving addition and subtraction, understand and apply properties of operations and the relationship between addition and subtraction, add and subtract within 20, and work with addition and subtraction equations. The clusters subsumed under these standards include using addition and subtraction within 20 to solve word problems with unknowns in all positions, solving word problems that call for addition of three whole numbers, applying commutative and associative property of addition, understanding subtraction as an unknown–addend problem, relating counting to addition and subtraction, demonstrating fluency for addition and subtraction within 10, understanding the meaning of the equal sign, and determining whether equations involving addition and subtraction are true or false.

Mrs. BB utilized this background knowledge to slow the pace of her instruction and attend closely to what her students said and did to freshly determine (although keeping in mind what she may be able to generalize from her prior instructional experiences) how to create opportunities for students to make sense of the problem and persevere in solving it, reason abstractly and quantitatively, construct viable arguments, model with mathematics, and seek and express regularity in repeated reasoning. I present a more comprehensive list of interactions that led up to the story Mrs. BB chose to narrate above. Let us examine each of these opportunities, created by Mrs. BB, for students to interact with her and the content and for Mrs. BB to interact with students' interactions with the content.

Opportunity to make sense of the problem. Mrs. BB's introduction to subtraction began by having students recall what they already knew about addition. Mrs. BB guided this student-led, but pre-planned, discussion using an addition poster, of her own creation, which ensured that the most vital elements of addition were reviewed concretely. Mrs. BB's poster was divided into thirds. The top third had the word "addition." The middle third had "___+___=___," and the bottom third had the words "join" and "count on." She pointed to the first two blanks of the equation, referenced them as addends and named the third blank the sum. She related addition to subtraction.

Mrs. BB showed a similar subtraction poster on blue printer paper. The top third was labeled "subtraction." The middle third had the equation "___–___=___." She pointed to the first blank in the equation and named it "total." The second blank was pointed to for "take away." The third blank was named "difference." The bottom third had the word "take away." Her scaffolded interaction gave students the opportunity to begin to organize their ideas about addition and subtraction; she then provided a reason for students to know about addition and subtraction.

Opportunity to reason. Mrs. BB told students how they were going to utilize their knowledge about counting, adding, and subtracting. She announced the math objective: "I can do a take-away story using an equation with a minus sign." She again referenced the blue subtraction poster, saying, "I made an example right up here." The homogenous group of students were given the range of "6–12" to write their own subtraction story. The prior two groups (who were also guided to prominently note both the total and mark their one-to-one correspondence counts) were given ranges of "up to five" and "up to eight," respectively. She chose low numbers for their subtraction problems, because "We had just started this unit on subtraction, and I noticed from the Pre-test that some of the students were not familiar with this concept." Mrs. BB provided formal math language and a verbal example to provide an opportunity for students to build on her patterns. She chose to omit the other scaffolds. They (the circle with the total and marks for counting the total) were only used if she noticed a child was struggling. Students began their work on the problem independently.

Opportunity to construct an argument. Mrs. BB's choice of the total range from 6–12 for the students to begin with provided an appropriate scaffold from kindergarten counting and cardinality development including representing addition and subtraction with objects and decomposing numbers less than or equal to 10 into pairs, and addition work the students had already completed in first grade including demonstrating fluency for addition and subtraction within 10.

Opportunity to model with mathematics. Mrs. BB provided multiple ways for students to represent their sense-making. Many chose to convey their thinking using visual representations. Some put their total number of animals inside and outside the pen. Others drew all their animals inside the pen and circled those that ran away, while others modeled the entire subtraction story with their fingers. Mrs. BB could assist all students' use of notational norms that would serve them well in later learning. Every student was given the opportunity to tell their subtraction number story and encouraged to identify where the numbers came from and how they related to the story and subtraction equation.

Opportunity to look for and express regularity in repeated reasoning. Mrs. BB provided an opportunity for children to further defend their subtraction equations and develop their algebraic reasoning, by introducing the concept of fact families. This particular “fact family” interaction also began to encourage students’ attention to generalization. When students saw that 10–6=4 and 6+4=10, Mrs. BB helped them understand that this pattern is not unique but rather represents a mathematical property (the relationship between addition and subtraction).

Conclusion

Mrs. BB’s listening process involved first, recalling her best knowledge about the content and her associated strategies; second, gathering student sense-making; and third, considering if she had a stored interaction that had performed well in the past that could be repeated or whether she needed to gather more student sense-making and learn to interact in a new way. She favored using what she already knew and used her best knowledge to get the best student results (which would eventually be students’ automaticity with addition and subtraction facts). Most of the student sense-making she listened to had been listened to in the past and applied to present whole and small group instruction. She was cognizant of her opportunity to act differently, given what she heard in the moment with individual students, and usually chose to alter her interactions with individuals according to her perception of their number sense prior to beginning instruction. Her students’ developing number sense and algebraic reasoning were pivotal forces in what she noticed about her students’ mathematical sense-making.

Mrs. BB could narrate some of the steps she conducted in instruction (procedural knowledge) and some of her statements of fact regarding the relationships between what she knew about her students and why she interacted the way she did (declarative knowledge), to an extent. For an example of her declarative knowledge, I ask Mrs. BB how she could place students into homogeneous groups. She responds,

> I have them count forward, count backward, count a collection, um, and then, and then you just show them like simple addition, where you show them some – ah – this is all math recovery stuff, like, five cubes are here four cubes are here and then you cover them and say how many altogether. And see if they have that visual memory, or, you know with their counting by ones. Or, if they can say five, six, seven, eight, nine. You know. So, and then, and then, just I, you know, I don't have a real recording sheet. I just write down what I see. And then the ones that can't count to 30, you know, it's like, okay, they are really going to need a lot of help. And the ones who have counted, written, to a hundred are there in September, it's like, oh, that's telling me, and, and, then I say, circle 85 and they know. You know. So.

Mrs. BB had methods she frequently utilized for eliciting students' sense-making regarding their number sense. She used the student sense-making she heard for initial homogeneous grouping at the beginning of the school year. Mrs. BB could narrate how student sense-making could be used in isolated interactions but had a harder time narrating how similar heard student sense-making could be used in the moment, unless she had contemplated it before her lesson was taught.

Mrs. BB could narrate some steps of her listening process – that she listened to her children's sense-making in the roots of their algebraic thinking beginning with their number sense (counting backward and forward, counting a group, identifying a number) and then provided opportunities for students to model their thinking. Her opportunities for students to learn were framed by the mandated curriculum she taught. Mrs. BB knew the content of the mathematical strands and the structures/interconnections of mathematical topics in that curriculum, as well as children's developmental strategies, as categorized by CGI. She gave students ability appropriate opportunities to make sense of the content through the curriculum's pre-determined opportunities to reason, construct an argument, and model with mathematics, after she listened to their sense-making to determine their level of ability as she evaluated it – measured by the number sense she heard them describe.

Next, in Chapter Five, we meet Mrs. Branches, a fifth-grade teacher, whose narrative theme of noticing student mathematical sense-making is deeply embedded in her “moral obligations.”

CHAPTER FIVE

"It's all about them. That's my whole why": Mrs. Branches' Narrated Noticing of her Fifth-grade Students' Mathematical Sense-making

In Chapter Five, I guide the reader through the mathematical sense-making Mrs. Branches noticed, her elicitation and interpretation of student sense-making, and her narratives of listening and responding to her students' sense-making. I then focus the reader's attention on an analytic point by highlighting specific parts of her narratives of noticing.

The Student Sense-making Mrs. Branches Noticed: Her Context for Interactions

> It's who the kids are. I teach them what they need. I'm not going to teach fifth-grade standards when they don't understand third-grade standards. I have a moral obligation. I am forming the future. They need to be able to be functioning citizens. They do. And I can't, if they can't do one plus one, we have to build that for them. There's a place. There's a role for them. The student drives my teaching. So that's that.

Mrs. Branches self-identifies as the teacher who pushes for acceptance for each child, and because of that, she feels she is often labeled "the bad guy."

Self-evaluation. She regards herself as "*the* teacher who brought student concerns to their families" and ultimately, the one who bears the blame for those identified student learning issues. She describes "just trying to survive" and "do her best," while "working alone and feeling alone." At past school sites, she viewed herself as a teacher who "changed lives"; she "shaped the whole child," and the educative experiences she provided were "valued." This is not her perception of self at Tuff Elementary. At the time of research, she felt her job was becoming "harder and harder."

Experiences. Mrs. Branches had "taught in lower socioeconomic schools" where "I was students' whole world." She focuses on building relationships with students at the beginning of the year to gain knowledge "about them, from them." She regards her students as a "family" and part of a larger "community." She draws upon her various past teaching contexts to determine that "thematic daily units" may be key to inspiring her current students to "think outside the box."

Skills. When this research occurred, Mrs. Branches was working toward obtaining her master's degree in special education, with a focus on gifted instructional methods from an online university. She utilizes formative and summative assessments written herself by hand, because she "thinks better when she writes things out." She believes multiple ways to arrive at a solution exist and so expects students to explain their whys for doing things, as well – "But they can't."

Motivation. She is cognizant that her classroom context required some

> hand-holding and step-by-step instruction, giving students opportunities to learn and helping them understand that what I have them do is life changing and important, and showing them that everything we do within the four walls of our classroom can be applied to their community, their future.

Mrs. Branches' students drive what she does in her fifth-grade classroom. She describes the inspiring force behind her instruction as her "moral responsibility" to prepare her students to be "contributing citizens." Mrs. Branches acknowledges that she must ensure her students' proficiency with mathematical concepts presented in prior years before she can address current grade-level mathematics concept standards.

Re-humanization of instruction. Mrs. Branches' instruction is based upon relationships she builds with – and among – her students, which reinforce respect for individuals (Gutierrez, 2015). Her instruction requires a classroom context that "meets the needs of every single student and challenges them to push boundaries." Her students must be "able to think critically through problems without panicking – recognizing that every idea and attempt at a solution will lead them closer to the correct answer."

Mrs. Branches has instituted a classroom context for mathematics instruction that allows students to be the "drivers of their learning." She has "shaped the whole child by providing them with experiences to better their math skills, social skills, language skills, and creativity skills." She describes her instruction as the "opposite of a canned one size fits all curriculum that required step-by-step teaching and the use of workbooks and worksheets." Instead, Mrs. Branches hopes to be seen as a key component to the overall well-being of the children.

> Students see me as a learner with them. They feel comfortable asking questions. They feel comfortable taking risks. They understand that life in not defined by a standardized test or the standard algorithm.

Perhaps her greatest challenge in creating such a classroom context is that each class differs every year – "I would like to say that life could be easy, and I could go year to year and have the same, the same lessons if you will. Every class is different." She describes a prior teaching site, where the population she worked with was very much like her current class. She explains,

> I was teaching letters and letter sounds, and phonics. Now, I have four English Language Learner students, so that kinda drives what I do. I also have three that I have to teach at a third-grade level, so that drives what I do. In reading, I teach first-grade reading. So again, they drive what I do.

She optimistically generalizes, "So, I think drawing upon your different experiences – but each class is different. So, for this group, I really had to pull away from whole group instruction." She explains how a vital and continuing part of the school year is to

> really make a point to talk to my kids. Just to know who they are. Build those relationships. So, that's where I get most of my knowledge from, is from them. Just talking to them. I never look at the incoming teacher cards. Because I want to form my own opinions. This class brings a lot of trauma and different socioeconomic statuses that wasn't here in the class last year. I knew I had to start [instruction] from scratch.

Mrs. Branches quickly explains, "But, it took me three weeks of pulling out my hair to kinda figure out what worked for them." She asks, "How do we take all of this [student trauma, family/school priorities, parents/guardians] into account? How do we accommodate 26 different learning styles when we have 26 different state mandates?"

The year the research occurred, her math class had "only a two-grade level, 2.5-grade level span. Which isn't terrible." Mrs. Branches knew "exactly" where her students were at in their mathematical thinking, at very specific moments for specific content, either through formative assessments or summative assessments. She writes her own math assessments.

> I make mine and hand write them because I'm old school and I don't – I think better when I write things out, and so my stuff is written out by hand, but, it's, I can't type it. For some reason. Because then, it doesn't stay with me. That [writing formative and summative assessments] requires pulling from so many different resources and there's the time. It's an issue. But – they [the kids' responses to the assessments] drive my lessons. They drive what I do. They drive small groups. It's how it has to happen due to the span of [mathematical] knowledge. So, I use that as my starting point, and from there, it's basically a web of where everybody plugs into that. So that's my groupings. And it's reworked constantly.

Mrs. Branches addressed earlier in this paper that she cannot teach fifth-grade standards until she addresses concepts in which students lack proficiency. Mrs. Branches noticed at the beginning of the school year that whole group instruction was not working. She narrates,

> I had a lot of kids that would sleep, um, they just couldn't focus, a lot of fidgeting – it just wasn't working. So that's when I came up with the idea of this, so that they [the kids] could kinda drive. What were their passions. And I think we are in a really nice groove now.

The "this" (line 2 of the above narration) that Mrs. Branches refers to is her choice to "practice the use of her own autonomy to teach what she wants."

There were several different categories of student mathematical sense-making that Mrs. Branches listened to most frequently during this research: student sense-making with major concepts and procedures that define number (including operations and the problems students solve), student sense-making with algebraic thinking topics that work with equations, and student sense-making with problem-solving (making sense of the problems and persevering in solving them). Please note that while Mrs. Branches teaches fifth grade, she consistently reminds herself of her students who perform below grade level. As a consequence, what she notices – and how she responds to what she noticed – typically matches the student sense-making that would be noticed in primary (kindergarten through third grade) content standards.

Teacher Noticing of Major Concepts and Procedures that Define Number

Mrs. Branches shared that until last year (2018), the fifth-grade team had been using the Stepping Stones curriculum as a framework for their instruction. This year, 2019, three of the four fifth-grade teachers determined that Engage New York was the curriculum that would provide the backbone of their instruction due to the "inherently higher level of rigor demanded of teachers and students." Mrs. Branches concedes,

> I don't even have the math workbooks. I left them in the book room. Because that's not life. The approach I've always taken is I need to prepare them for life, and life is not a workbook. You can't, when you're solving a life problem, it's not linear. And that's what worksheets and canned curriculum is, its linear. And it teaches them not to be able to pull from their English Language Arts background for math, and not to pull from science. [The students] are really used to having the answer provided to them. They're like deathly afraid to make mistakes. So, my own open-ended questions kinda fosters the mistake process also. I allow them to work through their own thoughts.

Mrs. Branches is "not a huge fan of leveling students, or grouping them according to ability, but at some point, it becomes necessary. My small groups are very fluid, though." I asked Mrs. Branches to explain how she formed her groups, because I only saw her utilize one small group for computational instruction during our observations; all other instruction was either whole group or brainstorming to make anchor charts (posters that students made to demonstrate their current level of knowledge about content and added to throughout the year), and group selection was random. Mrs. Branches explained how her groups were formed:

> So, I group my small groups according to common levels, common concepts that they have mastered, and that they're ready to go to the next step. But, they change, and they almost change weekly. It's all driven by what they're showing me, what they're

> mastering, what they are demonstrating. What I'm looking for is pretty much a hundred percent proficiency on a performance assessment – I'm walking around and looking at things before switching [groups] around. I know that's high. A lot of people look for 80, maybe 85, but I really want it to be independent and I want the why behind it. For them. So, I want them to be able to tell me what they're doing – so I listen to them speaking to themselves working through procedures. That's how I pull my groups.

Mrs. Branches tunes her brain to her context; her interactions with students focus on their civic development and providing equitable opportunities to learn for all students. She listens to student sense-making; however, the learning gaps she attempts to close are so vast, and appear so vividly and momentarily, that she often does not have time or know how to address what she hears. But this is not due to a lack of preparation. The sense-making to which she listens to plan her instruction is often verbalized by students in the moment through Mrs. Branches questioning.

She uses the sense-making she hears to prepare her sequence of lessons. Her plans are bulleted and succinct. Her bullet-pointed lesson plans include the words "anchor chart," "pattern block fractions," "number stories," and "pizza problems," but no student learning objectives or outcomes or lists of supplies needed for lesson preparation; she believes that open-ended questions are the only tool teachers should use in interactions with students. She notes, "It's how you get them [students] engaged. It's how you get them to think. And they need to think."

Mrs. Branches often struggles to formulate open-ended questions, however, because she often attempts to adapt her instruction – to meet students at the level of sense-making she hears – so rapidly that she almost never has enough time to deeply consider her repertoire of content strategies and pedagogical knowledge to her desired level of specificity. She has a seemingly infinite number of ideas for how she could walk students through their sense-making in dialogue but a finite amount of time to search for the best interaction. Reflection and reflexivity are important components (and characteristic of elite teachers) of Mrs. Branches' interactions with students. While her listening process does not ensure that Mrs. Branches can provide her students with answers, it does impact her ability to provide valuable learning opportunities that extend her students' sense-making.

Context for student sense-making noticed in Interaction 1. Mrs. Branches is aware of the core content standard for students to continuously develop their number sense, even though the grade-level "standards don't drive" her instruction, because the standards do not yet provide opportunities for all her students to learn. This trajectory begins with counting and advances to include place value and operations, which are all directed toward computational fluency with whole numbers and fractions. Mrs. Branches' instruction emphasizes her use of open-ended questions, not providing direct instruction, and not providing answers. She wants to discover what her students can do when they have the chance to reason things out for themselves and the kinds of sense-making that emerge when they do.

During this research, Mrs. Branches chose to re-introduce her students to working with fractions by returning to the third-grade standards in this domain, specifically addressing that students develop an understanding of fractions as numbers (i.e., understanding the fraction 1/b as the quantity formed by one part when a whole is partitions into b equal parts and the fraction a/b as the quantity formed by a parts of the size 1/b) (CCSS, 2018, CCSS.MATH.CONTENT.3.NFA.A.1). Mrs. Branches noticed while planning her instruction that a 2.5-grade-level span of mathematical ability existed among her students. Mrs. Branches structured her fraction instruction given what she noticed, meaning that her instruction needed to have equitable learning opportunities for third- through mid-fifth-grade sense-making. Mrs. Branches could not verbalize her knowledge of content standards, but due to her re-humanization stance on mathematics instruction, she knew she needed to provide opportunities to learn that respected her students' sense-making, as well as various opportunities for them to learn the content. She chose to help students rebuild their meaning for fractions through solving and discussing a word problem. She presented a problem that asked the children to identify a fraction, based on a given number of parts. The story was concrete, meaning that children saw and manipulated the shapes.

The opportunity to learn that Mrs. Branches created came from the *Teaching Channel.* She chose to let her students use manipulatives to “help students understand that a fraction names a quantity that has been formed when you partition a whole into equal sized pieces.” Mrs. Branches gives students time and organized a discussion to explore and discover the relationships between the pattern blocks. The *Teaching Channel* video suggests the teacher ask students guiding questions such as, “Do you see any other relationships to the (hexagon, trapezoid, rhombus)?, prove the relationship between (hexagon) and (triangle), and what shape is found in all other shapes?” The final prompt reads, “The hexagon represents our whole. Partition the whole in equal parts as many ways as possible.”

The students are supposed to be able to describe and demonstrate that the hexagon can be partitioned into two trapezoids (one-half), three rhombuses (one-third), or six triangles (one-sixth). Children are supposed to take this knowledge and extend it to ensure that they understand that all of these particular pattern blocks can be portioned into triangles (a rhombus comprises two triangles, a trapezoid three triangles, and a hexagon six triangles).

Then, the teacher can ask, “What fraction does one triangle represent…one trapezoid represent…one rhombus represent?” Finally, students create their own whole and then have to identify a pattern block part. Mrs. Branches orchestrates a dialogue, which incorporates a few of the guiding elements from the *Teaching Channel’s* lesson and then has the students begin with the aforementioned extension.

Mrs. Branches' elicitation and interpretation of student sense-making in Interaction 1.

B: *Mrs. Branches picks up a hexagon, a trapezoid, a rhombus, and a triangle pattern block. She is holding them in her hand in the order: green (triangle), blue (rhombus), red (trapezoid) and then yellow (hexagon). She holds up the blocks so students can see the layers.* Alright. X, what do these four shapes have to do with our fraction conversation?

S: You can break the hexagon into triangles.

B: How many triangles in my hexagon, X?

S: Six.

B: What are you thinking, X?

S: So, I think they all sorta fit into each other. This hexagon has six triangles. But one of the red squares (sic) can fit into there. And that's like three triangles. They all fit in and make a bunch of fractions.

B: Okay, what shape specifically?

S: The hexagon.

B: So this would be your what? *Mrs. Branches holds up the hexagon.*

Ss: Whole!

B: And these would be your…*Mrs. Branches holds up the trapezoid, rhombus, and triangle.*

S: Fractions!

S: Numerators!

B: Numerators. Each shape can be made by a certain number of triangles. So, what do you have to have, X, in order to be able to divide up your whole?

S: Well, you have to have triangles.

Mrs. Branches then explains to the students that they are going to create a design, or a picture, of their choosing. The only requirement is that the students trace the pattern block using its representative color, so that all students can identify the pattern block shapes. After 10 minutes of creative time…

B: K, so here's what we're doing, X. you're going to trade papers with somebody. And you're going to figure out the fraction of red on the picture.

S: What if there isn't any red?

B: The fraction of red on their picture.

S: Can I explain it to the class?

B: No, you cannot. X has a brain.

S: *This student has received a picture with no red pattern blocks.* Zero.

B: Zero out of? What's going to be your denominator?

S: How do you know what numbers to put in the numerator and denominator?

B: What's the definition of denominator? What does denominator represent? What does it mean?

S: Denominator is what the top number is divided by. I know, the whole?

B: The whole picture.

S: So you have to count up every little square (sic). Is it how many triangles are on it?

S: I was thinking about red. Like, these two are red. Out of what though? Mrs. Branches, like the entire picture?

S: I think I get it. How many times this, goes into this. *The student points to a red trapezoid and then points to the entire picture.*

S: This is impossible.

B: It's not impossible. You just have to figure out the denominator. So what's the issue in this?

S: *The student has a picture that uses four red trapezoids and one yellow hexagon. The student has written "4 or 12" on the bottom of his drawing.* If the red is four or 12.

B: So, what are you basing this on? Are you basing this on, like, this is one piece, this is two, this is three, this is four? Or what are you thinking? *Mrs. Branches draws an imaginary circle around each of the trapezoids as she counts.*

S: That. But then, there's 12.

B: Alright. *Mrs. Branches looks at her classroom clock and directs the children to come back together. She identifies a student (a benchmark response) to bring up the picture he was working with.*

B: Okay, so, what you were supposed to look at, X, was the fraction of red. X, what was your process?

S: I decided to find how many singular red triangles there were. I counted 15. Because there were 15 red triangles on it. Then, 65. I counted all these squares (sic) and I got 65. So then, I said that would be 15 sixty-fifths.

B: So what he did, X, was a broke it down into what, for his denominator?

S: Ah, singular triangles.

B: So he broke it down into individual triangles. Why though?

S: Because everything's made of triangles. All the pattern blocks. All the shapes.

B: What can all the shapes be divided into?

Ss: Triangles.

B: So what's the common denominator?

Ss: Triangles.

B: So, X counted all the triangles and he got 65. Then he counted red. Fifteen are red.

Mrs. Branches' response, given what she noticed. Mrs. Branches encouraged students to explain their thinking by asking open-ended questions. The discussion that Mrs. Branches orchestrated encouraged students to begin constructing viable arguments as they interacted with the content and each other. I asked Mrs. Branches why she ultimately chose this method of questioning and why she only had one student share his sense-making. She replied,

> Because as much as I would like to let every child get the answer, the wait time – and as much as I want to sit there and force them to just say anything, at some point – this sounds terrible – but time is not on my side. So, at some point, I have to call on the students that I know are going to provide a correct response so that we can keep the process moving. Um, it's not ideal, but there are times when I know, okay, this student does have the correct answer, they can model it, they can show what they can, they can use their words to add to the conversation and we'll move along quickly. Um, it's not perfect. It's not the best answer that I could give you, and I wish I had a better one, but I'm short on time sometimes. So.

Listening process claim for Interaction 1. Mrs. Branches tuned her brain, her content knowledge, and pedagogical knowledge, to her teaching context, and that context revolved around her students' sense-making. The student sense-making Mrs. Branches listened to helped her choose which interactions to keep stored for immediate use. Mrs. Branches led a discussion designed to help her notice students' abilities to partition a variety of pattern block shapes. We know this because she opens her lesson by having students define fractions using physical pattern block models – partitioning a whole into equal parts (lines 1–28). When students respond that the triangle, trapezoid, and rhombus were "fractions" (line 30), she favors the response "numerators" (line 32) without explaining why. She later addresses this opportunity to learn with interactions that have implications for the remainder of her lesson. For example, the student who is working with four red trapezoids and one yellow hexagon (line 82) is trying to determine the numerator of her fraction. The student identifies the denominator as five (four trapezoids plus one hexagon equals five pattern blocks). Mrs. Branches asks a clarifying question to determine how the student got to the denominator of five (lines 86–88). The student's response indicates that she counted the total number of pattern blocks to get her denominator (line 90) but also that she was struggling to make sense of whether to count the total number of trapezoids for the numerator or to partition the trapezoids into triangles for the numerator. Mrs. Branches responds with "alright" (line 92). This student has above grade-level mathematical ability, and Mrs.

Branches knows that the student just needs more time to grapple with the content. Mrs. Branches knows the best interaction to have is to not have one at all, although she cannot verbalize why she knows the sense-making she hears from this student is best resolved by the student's independent sense-making.

She noticed overall that some of her students had knowledge of geometry ("like partitioning the hexagon and rhombus into triangles") and also that "Many students struggle with the number – the fractional parts that the geometric pieces represented." She did not indicate what she would do with that awareness.

Mrs. Branches chose a student benchmark example to lead and close the learning opportunity, because she noticed that continuing to devote a great deal of time to solving and discussing this task, especially at the onset of fraction instruction, was not going to allow her to complete her lesson plans. A portion of her students were able to learn (or recall) how to partition a whole unit into x equal parts, with the size of a part 1/x that resulted (relational understanding of unit fractions). The closing interaction in this lesson indicates that Mrs. Branches listens to her students' sense-making to determine who can explain their thinking to others. Perhaps this is because she recognizes that among her repertoire of strategies that must be sorted through to select the specific single interaction that will provide learning opportunities for all students, student-led dialogue is the most important contextual piece she needs.

Teacher Noticing of Student Sense-making with Problem-solving

Mrs. Branches narrated that she related to her students in a variety of ways. She said she "appreciated the similarities and differences that characterized each child in her classroom." She tried to learn what her students knew, how they thought, who they were, and what motivated them. She claimed to achieve these ends by asking her students to share their prior knowledge when new content areas were introduced. This was an important theme in the narratives Mrs. Branches' told: that she spent a lot of time getting to know her students.

Context for sense-making Interaction 2. Mrs. Branches began every domain of study in math with an anchor poster. She said,

> I like to have these anchor charts for them [students] to be able to refer back to. And then, I like to make these working posters so that the students can contribute. We can cross things out, they are a part of the anchor chart. I like it to all be hands on workable. And I like them to come back to it. Anytime they need to. For my poster, I just want them to go up to things, but they're not there yet. By the end of the year, I hope that they feel extremely confident going up to anything and adding to it. Um, but, it's new and so – hopefully. I model crossing out, adding to, developing thoughts, keeping the conversations going through my interactive posters if you will.

Mrs. Branches had her students recall what they knew about fractions. She asked the children to extend what they already learned about fractions and draw on those relationships to help them understand the fundamental properties of operations with fractions.

Mrs. Branches' elicitation and interpretation of student sense-making in Interaction 2. Mrs. Branches works with her students to create a fraction anchor chart. She uses the anchor chart activity as an opportunity to develop students' understanding by eliciting their ideas, and later, relating their ideas to the content she chooses to teach.

B: Alright. Ready? We're creating posters! OOOO! AHHHH!

Ss: Yes!

B: On your poster, X…

S: What do you know about…

S: Triangles?

S: Pringles?

S: *Mrs. Branches writes "fractions" on her poster board.* Fractions. Yay.

B: What do you already know about fractions? Oh, I love the hands. You're gonna have an opportunity, X, to document your knows, your knowings, your knowledge, X, on your poster. Mmm K? I just want to know what you already know about fractions. What could go on this poster? What do you think?

S: Fractions are everywhere?

B: Fractions are everywhere? What else?

S: Fractions have numerators and de-monators (sic) Ack! De-men-ators (sic).

B: Okay, so vocabulary. Fraction vocab.

S: Example problems.

B: Okay, could we give real world examples? X, if we know 'em? What else could we do?

S: If you have pieces of a cake and you give them away, five, how many are, how many fraction representations do we have left?

B: Okay, so you're going to create a word problem with fractions.

S: You could use a visual.

B: Ooo. Music to my ears.

S: You could do different ways of teaching people about decimals.
B: Oh. He said decimals. But I was talking about fractions.

S: But aren't they the same? The same thing?

B: Decimals and fractions…

S: Because of one hundredth, is like, out of one hundred. Or a fraction of a piece of pizza.

B: So maybe X is feeling extra extraordinary and she's going to put how decimals and fractions…

S: Are the same!

Mrs. Branches passes out poster paper to student-selected groups and rotates around the classroom four times, while students make sense of what they know about fractions.

B: Alright, so, I want a working definition. What is a fraction?

S: It's like splitting things into smaller pieces.

B: X, what would you add or take away from X's definition?

S: Um, I would add that fractions can be everywhere.

B: X, what would you add to or take off? What are your thoughts for definition of a fraction?

S: Um, there's a numerator and a denominator. Oh, but we are still on the definition part, aren't we though?

B: I think that that could fit in, X. So would you like to give me a fraction example so that I could define vocabulary?

S: Like a fraction number? Like five-sixths?

S: But there's more to the numerator and the denominator.

B: Well, let's define it. X, which one's the numerator?

S: The top one is.

B: Okay. And any idea what the numerator tells you, X?

S: I was going to say something, then may I answer this question?

B: Absolutely. What would you like to add?

S: I would change "something" to "wholes."

B: So, instead of going with "something," you're saying split wholes.

S: Also, the numerator is how much of the whole is put…

S: Can we add there's so many shapes to explain a fraction? And what I've learned is that shapes, like when you cut them into portions, like the fraction has to be equal. Like, if it's going to be a circle and you split it in half, it can't be one-half is just a little bit and the other half is really big. It has to be equal.

B: What do you want to do, X?

S: Denominator. The denominator, um, would be the whole, or, um, how much you have? Of that one thing?

S: The numerator is the fraction of the whole. Of the denominator.

B: So this is the fraction of the denominator? *Mrs. Branches points to the "5" in "5/6."*

S: It's how much is taken away from the denominator.

B: What are you thinking, X? What could you add to our poster, here?

S: What you could add is that fractions, um, sometimes could be used as, um, decimals?

B: Can you talk to me more about that?

S: Like, if you use it as money? Like five dollars and 36 cents. That's kinda like a fraction because it could be one out of 63?

B: What is one out of 63? Here's your example. *Mrs. Branches draws an imaginary circle with her left index finger around the "$5.36" on the poster.* What part of this is the fraction, X?

S: The hundreds?

B: So, the six in in what place value, X?

S: The six?

B: What place value position?

S: Uh…

B: And you have lots of friends in here that would be more than willing to support you.

S: I think that six is the denominator.

B: Okay, six is the denominator. X, what do you think?

S: Just the six by itself?

B: Whatever you want to say.

S: Well, I think the 36 together, like, both numbers are the numerator because 36 of 100 cents in a dollar. So if we take away 36, we have, uh, 63 cents left.

B: So, X, what she's saying is there's 100 total cents to make up a whole dollar, and in your problem, you used 36 out of…

S: A hundred.

B: I love it. I still didn't get place value, but we're going to continue. To continue to add to our working definition poster. Just like we always do, right? We're always adding to our poster.

Mrs. Branches' response, given what she noticed. The completed anchor chart looked like this:

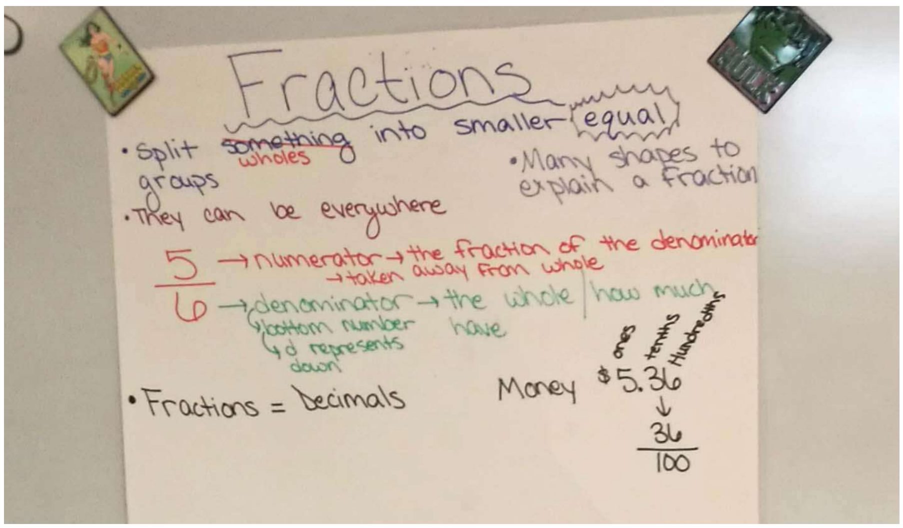


Figure 5. Fraction anchor poster.

The final poster read, "Fractions: Split ~~something~~ wholes into smaller equal groups. Many shapes can explain a fraction. They can be everywhere. 5/6: 5 is the numerator – the fraction of the denominator taken from whole. 6: is the denominator: The whole/how much have, bottom number, d represents down. Fractions=decimals=money. $5.36=5 ones, 3 tenths, 6 hundredths. 36/100."

Mrs. Branches listened to students' sense-making. She used an anchor chart so she could be aware of what her students knew – or what they thought they already knew – about fractions. She heard (as indicated by her writing the students' verbal responses on the anchor paper with marker, beginning at line 53) that students knew "numerator" and "denominator" and that somehow fractions, decimals, and money were equal. Mrs. Branches' succeeding narrative again focused on the re-humanization of mathematics by not only focusing on the math content but on additional social behaviors. When I asked Mrs. Branches to elaborate on why she had utilized an anchor chart to review prior fraction content, she narrated,

> I ask people to build upon it. Um, and having them write in pen and things like that, we celebrate mistakes, and it's important that we're seeing the mistake and building upon it and working on it, and it's the same thing in a conversation. Okay, he said this, what does any, what are some other approaches, thoughts, about that response. And they're, we've gotten really good about being really respectful when we are disagreeing which I think is a global problem. So, even something as small as an incorrect math response and how you approach that is a teachable moment. So, I let them build upon that.

Mrs. Branches' response focused on celebrating mistakes, building upon sense-making, and being respectful during a disagreement. I asked Mrs. Branches to elaborate on what she heard about the student's sense-making of the content, and she said,

> I like to all be hands on workable. And I like them to come back to it. Anytime they need to. For my poster, I just want them to go up to things, but they're not there yet. By the end of the year, I hope that they feel extremely confident going up to anything and adding to it. Um, but it's new and so – hopefully. I model crossing out, adding to, developing thought, keeping the conversations going through my interactive posters if you will.

Listening process claim for Interaction 2. Mrs. Branches again tuned her listening to her context, this time to consider how and whether she ought to arrange her interactions. Mrs. Branches could validate what she thought she already knew about her students' sense-making through the use of the anchor charts and allow students to share their own sense-making in a way that celebrates their sense-making, instead of classifying students by their misconceptions. Mrs.

Branches determined that some students' sense-making was tied to physical or pictorial models of fractions that show a whole broken into equal parts (review lines 90, 97, and 104 for examples). She described her declarative knowledge, that this "kind of student sense-making is a good start, but not where I'd like them to end up." Mrs. Branches then showed me a fraction rap video she was going to have students watch which clearly explicated the relationship between numerators and denominators. I heard students singing

> I only want a part of it, that's a fraction, that's a fraction. The numerator's way up top, and that's just how much we've got. Denominator's down under the line, equal parts, we can find. I only want a part of it, that's a fraction, that's a fraction.

through the portable window the next day.

Context for sense-making Interaction 3. Mrs. Branches tries to have her math discussions revolve around critical thinking skills. She reports, "I expect that we have perseverance when we are attacking a problem. So, open-ended questions make them start to, you know, develop that perseverance. So, it's critical that they work through it themselves." She routinely stresses the importance of problem-solving and encourages her students to make sense of the problems they solve and to persevere in solving them. During my observation, Mrs. Branches chose four random fraction number problems and presented them to her students. Each group of students was given a different problem to solve collaboratively. We focus on the fourth question in the student–teacher–content interaction that follows.

Mrs. Branches' elicitation and interpretation of student sense-making in Interaction 3. Mrs. Branches begins by telling the students they are going to do standing word problems today. She says, "These are word problems that you do all the time, X. How are we seeing them in a school setting?"

S: Scary.

S: We can't do it.

S: We easily give up.

S: Our brains go numb.

B: But, question. Question for you. When X is calculating how many hours until he's free from school, is that scary?

S: No.

B: And does he do it?

S: Yeah.

B: Alright, X, when we get to a word problem, what's the first step, X?

S: It's reading the problem.

B: Reading comprehension. We have to understand what it is asking us to do.

S: Is it always necessary to do a character analysis?

B: It is not always necessary to do a character analysis, X. But, is that one strategy?

S: Yes. It gets you into the problem.

B: What's another strategy for helping with the reading comprehension component?

S: Like reading it again?

B: Okay. Re-reading. X, what else could you do to understand that text, X?

S: You can find the key words.

S: Sometimes I try to figure out if we're doing division, multiplication…

B: So you try to figure out the operation.

S: You can circle the numbers and key words you need. Highlight important words.

S: You could create a visual.

B: Right. Today you're working with fractions. We're going to divide up. X, we're not sitting down. Why are we standing? What's the thought behind this?

S: It helps us process our brains (sic).

S: Blood flow.

S: You're in it.

B: I want to talk about one more thing. Collaboration.

S: You put your heads together.

Students count off and moved to their respective heterogeneous groups. The group that we follow is assigned a number story about Tony, a clown who is driving to the circus. The problem states that Tony drove a total of 112.5 miles and that he changed lanes every 12.5 miles. The students are supposed to determine how many times Tony changed lanes on his trip.

The student–teacher–content interaction is as follows:

B: What's the situation over here? *Students have written the words "character analysis" on their poster.*

S: He's a taxi driver.

B: Who's a taxi driver? Are we talking about…

S: He changes lanes every 12.5 miles. So, it looks like he's all by himself…

S: So, what do we know about taxis…*After Mrs. Branches walks away, the students have a three minute conversation about the color of taxis, and a debate over whether the taxi is an Uber or not.*

B: *Mrs. Branches returns on her second rotation around the classroom.* I see Tony looks very fantastic. And your taxi looks great. *The students have drawn a taxi and a clown character on their poster.* What's the next step here?

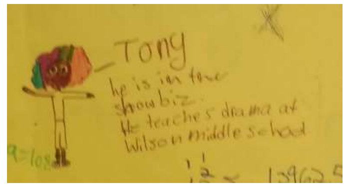


Figure 6. Student story problem character analysis.

S: Well, I solved the problem.

S: This part is a very important part. Everything else isn't very important. *A student points to the sections of the number story that are underlined in red pen. The student points to "12.5" and "112.5" on the poster.* I mean, the question is there. The rest is just a statement. But this is the important part that we need to figure out. I wrote down our numbers that we're going to use. And we've done our character analysis. So, now what's left is we have to find out how to solve it, and then we have to solve it.

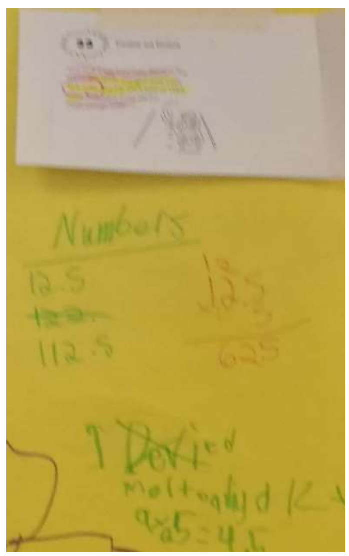


Figure 7. Student standing story problem work.

B: So, how are you going to solve it? What are you thinking?

S1: Uh. Nine times 12. Remainder 4.5.

B: Isn't that what these remainders are? The decimals? You're thinking division. What are you going to divide?

S1: One hundred twelve divided by 12.

S2: I mean, that's a really good way to figure that out. But I think that we're not dividing. Because the driver changed lanes every 12.5 miles.

S3: Oh, wait, I got it!

S2: You did? What's the answer?

B: X, she's multiplying. What do you think?

S1: Divide.

B: But wait, you didn't divide. You multiplied.

S1: Kinda.

B: Talk to her.

S1: I multiplied, um, I multiplied nine times half.

S2: Where did you get nine and a half?

S1: Uh, from the five. And then, nine times 12 equals 108, which is the closest.

B: Why don't you write this. Because I'm wondering if X will be able to get what you're saying if she sees it. And then, X, you and X can see what X is thinking.

B: *Mrs. Branches is starting her third rotation around the classroom. The students in this group have not started any computation since Mrs. Branches was last there. They intercept Mrs. Branches for assistance.* And my friends. What are we doing here?

S2: Well, we tried to figure out, like, but we each got…

S3: Different answers. So we're a little…

S1: It's 9.5.

B: So, what X is saying, he says, 12 times nine in 108. So he felt that was as close as he could come to 112.5.

S2: Okay.

B: Then I told X, well, what about this point five? So, if he takes a half, plus a half, that's a…

S1: Whole.

B: So, half, nine times, what does he get?

S1: Forty-five.

B: So, a half, plus a half, plus a half, plus a half, plus a half, plus a half, plus a half, plus a half, plus a half.

S1: It's 9.5.

S2: Why nine?

S1: Because 9.5, uh, that's, that equals…

B: So now you're saying 9.5 times 12.5. What do you think?

S2: So, 9.5 times 12.

B: So here's his journey. *Mrs. Branches draws a number line on the yellow poster. The number line starts at zero and ends at 112.5.* He started at zero and he ended at 112.5.

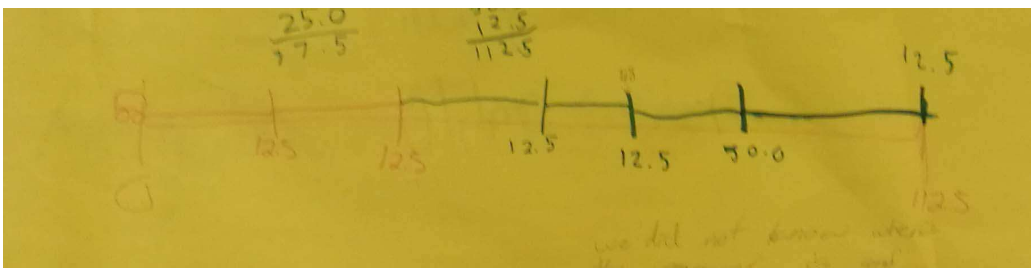


Figure 8. Student problem-solving for standing story problems.

S2: How often does he change lanes. I'm thinking.

B: Why are you thinking? Does it say? Does it tell you?

S2: Well, I'm just taking a guess.

B: *Mrs. Branches points to the words in the number story as she read them.* The driver changed lanes every 12.5 miles. So when he starts here, here's his car. He drives 12.5 miles. He changes lanes. *Mrs. Branches makes a dot, marks and writes "12.5 miles," makes a line above the number line to show the distance the car has traveled and labels this line with a one.* How many more miles before he changes lanes again?

S2: Twelve point five. I get it now. I was confused earlier but now I see that you need to get this number to see how many times it goes into this number.

B: What are you going to do next?

S2: I'm going to do 12.5 times five. Yeah. Five. Just try numbers. *Student writes "12.5×5" on the poster. She gets 625, then looks to the teacher and her peers. The same student moves to another part of the poster and writes "12.5×8."*

B: You know the answer. Do you want to help your group out? *About five minutes later, Mrs. Branches calls time, and groups rotate counter-clockwise to critique the work of a different group.*

B: *To new group:* You guys have to solve this. They did not solve it. They started this visual here. The driver started at zero, driving this total distance, and then every time 12.5 hit, he changed lanes. They didn't know what to do.

S4: What about multiplication?

B: How would multiplication fit it?

S4: Cause they could try like 12.5 times 10 or so…

B: And see where it ends? Okay. And what would the 10 represent?

S4: Uh, that would represent how many times he switched lanes.

B: I think you're thinking. I think you should build upon that though. Maybe X can do that math.

Mrs. Branches' response, given what she noticed. Mrs. Branches listened to all the children's sense-making. She chose a student who solved the problem to describe his thinking and then provided an opportunity for the children to come up with their own strategies for sharing their thinking and solving the problem. After several rotations around the classroom, Mrs. Branches noticed no progress had been made on the required computation. She drew a number line for the students and demonstrated how to "chunk" the line into measured 12.5 unit parts, while keeping a running total. "It was unclear if the students did not understand the model, or were frustrated by the problem, given that their choice of problem-solving was to finally guess numbers, even though a member of the group *had* a strategy for solving the number story." She later narrates,

> The group that followed, or checked, this groups' number story solution may have been a more suitable group for [the student who was able to solve the problem]. This group could use multiplication as the basis to make sense of multiplying fractions.

Mrs. Branches seemed to notice that students in separate groups were using similar computational strategies and consider how students' sense-making could provide a learning opportunity for students who were struggling with how to begin the computation.

Listening process claim for Interaction 3. Mrs. Branches listened to collect information about her students' sense-making for a specified amount of time (a total of 15 minutes in three five-minute rotations). What is interesting and consistent about Mrs. Branches' listening process is that she does not instantly commit to a specific structured interaction. Mrs. Branches used more than one in-the-moment observation of student sense-making and trial and error to help herself, and other students, listen to and understand each-others' sense-making.

The student who tried to explain his sense-making was beginning to exhibit relational understanding of fractions as composite (multiplicative relationship between the numerator and the denominator). The student who responded with 9.5 was using an algebraic relationship in his reasoning about fractions; he was using a relational thinking strategy, because he implemented it on the basis of his understanding of mathematical relationships. One of the relationships embedded in this student's thinking was the distributive property of multiplication over addition. Although the student knew how to solve the problem, he struggled to explain his thinking to his peers, and his peers struggled to make sense of his explanation. The remaining members of his group were challenged by the request to partition 112.5 into 12.5 equal parts (partitive division) using a number line and needed the direct instruction and modeling of their teacher to start computing. Mrs. Branches knew these strategies were on opposite ends of the spectrum of student fraction conceptualization, yet she persisted in her attempts to have the students explain their sense-making and only intervened when she perceived the access to knowledge gap too large for the material to continue to be discussed by all students in the manner it was. That was when she chose to introduce the model.

Teacher Noticing of Student Sense-making with Algebraic Thinking: Patterns, Expressions, and Equations

Mrs. Branches narrated for us that many of her students were not yet performing at grade level. Her instruction "adapted to the needs, backgrounds, and experiences of her students." Mrs. Branches determined the progress and mastery of students working on double digit multiplication and division by working with them in small groups.

Context for sense-making Interaction 4. Mrs. Branches described how most of her students "know the traditional algorithms" without necessarily understanding why the algorithm worked.

> They're [students] stuck in just standard algorithm. "And my teacher told me to carry the one, so I carry the one." So, it's really for me about the why behind it. Why are we regrouping? Why are we borrowing? What does this mean? Is there a different approach? Same thing with the worksheets and life. There's multiple ways to arrive at a solution. And they [students] are really stuck that this is the only, only way. And even though they don't even understand the steps, um, it is an absolute refusal for some of them to branch out into a different strategy. So, that is a huge obstacle for me right now. "And it's just carry the one." Well, what does that mean, and why, and they can't tell me. You'll notice, a lot of my day was conversations. Let's talk. And they haven't had that. They've had somebody stand up and say carry the one, and so it's just ingrained. And it's hard to break habits in fifth grade.

Mrs. Branches' elicitation and interpretation of student sense-making in Interaction 4.

B: *Mrs. Branches clears off three desks to make a tabletop. She has seven students. Six students are seated at the table, and one has chosen to sit alone at her desk.* Alright, friends. I want to review division. I just want to make sure we are on the same page. That and multiplication. Ready, X? Let's go. *Mrs. Branches writes "217×25=" on the dry erase board.*

S: *Students have dry erase markers and are writing on their desks. Four of the students are using what would be considered a traditional algorithm when multiplying with a two-digit number. One student is using partial products by place values. Another student is using partial sums.*

B: We fly on this now.

S: I'm finished. *Two students next to each other show their answers to each other. Their answers are not the same.*

B: Comparing?

S: Oh wait, never mind.

B: Ah, you see it. X and X, keep going. *The students using partial products and partial sums have not yet finished the computation.* Let's flip it up. *Mrs. Branches writes "217 divided by 35." Mrs. Branches watches the students compute. The student using partial sums has two columns of addends and is staring into the distance. The same four students who used the traditional algorithm to compute the double digit multiplication problem use the traditional algorithm to solve the division problem. Mrs. Branches moves back to the student using partial sums.* What could you do? Cause that one was right. *Mrs. Branches identified where the student misaligned her places values when adding which has thrown off her sum.*

S: Oh.

B: That's okay. Sometimes that happens. There's a multiplication problem on the board. *The student erases all her work and does not work the problem to a solution.* Okay, ladies and gentlemen, what was the answer to this? *She points to "217×35.*

S: Seven thousand, five hundred ninety-five.

B: K. And the quotient.

S: It was six remainder seven.

B: Okay. So, instead of having remainder, what can we do?

S: They are extra.

B: You had 217 suckers. Put, um, to 35 classes. Each class got six. So, you've got seven suckers left. Those are your…

S: Remainders.

B: They are remainders. Are they your whole? We have whole suckers left. Right? What can we do with those suckers?

S: You can share them with the classes. By breaking them into smaller pieces.

B: So, we're going to take our whole and what?

S: Divide it.

B: Into?

S: Decimals?

B: I hear decimals, but I need more. We're dividing our whole into…

S: Half.

S: Into fifths! Into fifths!

B: Into fifths? Why fifths?

S: Because if you did seven times five it would equal 35. So each class would get one-fifth of the lollipop.

Mrs. Branches' response, given what she noticed. Mrs. Branches' interactions included two that she wanted to talk about: the student using partial sums to multiply and the division problem that turned into an opportunity to talk about equally sharing lollipops. We begin with the multiplication.

Mrs. Branches had two students that complete the double digit multiplication problem using the traditional algorithm, got the same answer, and were done. A third compared her work, re-did some computation, and ended up with the same result. The fourth student used a partial products and partial sums strategy.

> I worked with her specifically to try and get her to see where she was using partial products, to use more partial products rely less on the repeated adding. She was losing track of where she was. But I also know what, why, I know the why behind the choice to work alone and the reluctancy to move past that strategy. So, it's just something that I just – kinda something I have to work with. This student is actually an amazing mathematician but doesn't have the confidence.

Mrs. Branches continued to "emphasize making mistakes and revising thinking" with this student and determined she was going to ensure that the student "saw math as a subject that needed to make sense to her, instead of a set of procedures to follow just because someone told her to." This interaction again demonstrates Mrs. Branches' stance on re-humanizing mathematics – every student deserves the opportunity to learn and make sense of their mathematics. Everyone in her class is a mathematician.

Finally, the division problem that turned into an equal sharing problem. Mrs. Branches concocted a connection between her prior unit of study on division computation and the current unit on fractions. She took the division problem, 217/35, and changed it into a number story. Mrs. Branches knows that "my students needed more opportunities to connect fractions to prior work we had done in math class." However, she cannot describe why what she noticed and how she chose to modify her interaction with this particular group of students is meaningful (characteristic of an elite teacher).

Listening process claim for Interaction 4. Again, Mrs. Branches did not instantly commit to a specific structured interaction. She used more than one in-the-moment observation of student sense-making and her arranged interactions to listen to and understand student sense-making. Mrs. Branches listened to her students' sense-making based on her teaching preferences ("autonomy") to meet the needs of her students (to be "critical thinking productive citizens"). She knew that most of her students needed her support to be able to begin to use relational thinking – and use it consistently – but she still struggled to narrate why her modified interaction was an appropriate pedagogical choice (characteristic of an elite teacher). Her choice gave students the opportunity to interpret a problem that included a set of objects that they could count and individually split into parts. Students had the opportunity to make sense of the idea that a countable set of objects could also include fractions of an object.

Summary

Mrs. Branches' focus on re-humanizing mathematics and preparing her students to be contributing critical citizens consistently guided her listening. She was tuned to her teaching context and this resulted in many interactions that led to sequenced interactions that were sensitive to her students' developing sense-making. Although Mrs. Branches was frequently unable to narrate the procedural and declarative knowledge she utilized in her listening process, her consistent brain tuning to her context and her consistent process of listening to student sense-making to choose which interactions to arrange and to keep stored for use demonstrate her commitment to the tenants of mathematics re-humanization (Gutierrez, 2018). Next, in Chapter Six, we learn what student sense-making Ms. Kapeesh listened for and how it was impacted by her focus on a "growth mindset."

CHAPTER SIX

"I love a diverse group of students; however, I would love for the growth mindset to have been part of their education from kindergarten, so challenges felt exciting and not daunting": Mrs. Kapeesh's Narrated Noticing of her Fifth-grade Students' Mathematical Sense-making

Mrs. Kapeesh notices her students' mathematical sense-making. Her noticing is driven by her desire to instill and cultivate their growth mindset. Her teaching context is "a classroom full of student work, references for learning and growing, charts celebrating growth, and learning targets up for everything we learn." She teaches a diverse group of students, which she describes as "one of the most challenging groups she has taught in many years."

> The needs are so great, mainly socially, that cause learning to get off track daily. Academically, I have a majority on or above reading level with three to four largely below grade level. In math, a majority of students are slightly below or on grade level. I have two much higher than grade level, and three that are greatly below grade level. I have one social emotional student that joins our class for the entirety of the day.

The Student Sense-making Mrs. Kapeesh Noticed: Her Context for Interactions

Mrs. Kapeesh's philosophy of teaching drives her instructional style. She embodies the growth mindset. She tells students the truth about their current achievement. She expects to work with her students to learn a repertoire of approaches to assist them as they work through challenges and setbacks on their way to mastering grade-level concepts.

Self-evaluation. Mrs. Kapeesh values her positive relationships with her families and students. She believes she has "open communication, it is clear that I value their children as a whole child, and I value family input greatly." She finds students are "most successful when there is a team behind them, and this is always my goal."

Skills. Mrs. Kapeesh recognizes that “her most successful math lessons in the last 10 years involved a seminar-type setup where discussion and learning was student led.” At the time of research, she used the Engage New York curriculum as her guiding framework for math instruction.

> It took me a long time to learn how to do this [seminar-type setup] well, and it is a hard skill to know how to adapt so quickly to student responses and learning, but I believe it’s the most beneficial. I believe that by me planting a seed and having students talk with each other and justify their answer, or teach someone else, they arrive to a clearer understanding of math.

Mrs. Kapeesh aims to maintain a student–teacher talking ratio of about 80–20 and attempts to stick to it.

> There are many lessons I feel as though I need to control more, and they don’t go as well. Students are bored and lose focus more. They don’t feel accountable or engaged and I think both of those are key to pushing students to own their own learning.

She bookends her lessons with “a clear direction or answer and one strategy that students could work.” Overall, she wants her students to explain their thinking. Some questions or phrases she uses to promote student-led learning include “What do you already know? What do you think? Why do you think that? Do you agree or disagree? Why? Do you want to add anything? Why is that important?” She believes these questions push her students’ critical thinking.

Experiences. Mrs. Kapeesh struggled with math in elementary school but was eventually recommended for advanced math classes in middle school. She remembers being very intimidated in the advanced math class and “not getting much help so I requested to be put back in grade-level math.” She felt more confident in earning failing grades in high school. She recalls that she “didn’t have strong teachers necessarily; however, I felt more comfortable making mistakes and asking for help.”

Motivation. Mrs. Kapeesh wishes she “was more confident in the growth mindset approach and felt stronger about my ability to excel” looking back on her formative math years. Her instructional goal this year (2018) was to "focus on things that I can control." She believes that developing a child's growth mindset is one of those things.

Next, I narrate and describe Mrs. Kapeesh's noticing of student sense-making in four different interactions: an assessment review, pattern block snowflakes, tallied desserts, and fraction hamburgers. I describe each interaction, framed by our three research sub-questions. I guide the reader through the mathematical sense-making the teacher noticed by providing a context for the interaction the reader is about to participate in; the teacher's elicitation and interpretation of student sense-making through the teacher's and students' pedagogical interactions; and the teacher's narrative of how she chose to respond to her students' sense-making, given what she listened to. Finally, I focus the reader's attention on an analytic point and illustrate and explore our claims through Mrs. Kapeesh's narratives of noticing grounded in the details of her specific interactions (Emerson, Fretz & Shaw, 2011).

The Student Sense-making Mrs. Kapeesh Noticed: The Setup for Interactions

Mrs. Kapeesh narrates that her interactions focus on her students' growth mindset: "Students need to try new strategies and seek input from others when they're stuck. They need a bunch of strategies not just 'trying their best' to learn and get smarter." She designs discussions that attempt to achieve mathematical goals, as well as communicate that all students' ideas are valued and contribute to the overall conceptual understanding of the group.

Teacher Noticing that Highlighted Students' Growth Mindsets

Mrs. Kapeesh initially prepared two fraction lessons for me to observe. A day before my scheduled observation, Mrs. Kapeesh told me she needed to change her lesson plans. Her students had taken an assessment covering their prior unit on decimals and types of division;

there was a range of student scores, and Mrs. Kapeesh described the importance of reviewing the work with her students:

> It is important for students to understand their mistakes in order to connect to future units and have a larger conceptual understanding. In this case, the past unit was decimals and the present unit is fractions. They are connected and students should be able to make that connection by understanding their mistakes. It's important for students to be held accountable for their learning and circling back to their own choices helps them understand they are held to fixing those mistakes in the future.

Context for student sense-making noticed in Interaction 1. Mrs. Kapeesh told me I was more than welcome to reschedule my visit, but I requested that I still observe as planned. Mrs. Kapeesh modified her instruction because she had noticed something about her students, namely that they could naturally extend what they knew about decimals and division to fractions by making sense of our base-10 number system and introducing a new notation for familiar numbers. However, the strategies her students used on the test indicated a good deal of variety in their understanding of decimals and division. Mrs. Kapeesh needed to review the concepts assessed, so she could ensure that her students would make sense of the connections between decimals, division, and equal sharing, to be able to extend this understanding to fractions.

Mrs. Kapeesh's elicitation and interpretation of student sense-making in Interaction 1.

> K: What this score says is very important because it shows me what you know, and what you might be struggling with. And that is okay. What I care more about in this conversation, is that you can reflect on your own paper, you can find your own mistakes, and that we can have a conversation about how we're going to get better. I'm looking for reflective conversations. Definitely not negative comments. This is a time that you say, oh, I found this mistake. Here's what I did wrong. Here's what I would do differently. *Mrs. Kapeesh hands back the tests and has every student think of one thing they are proud of. Students share their responses with peers. She begins her review of concepts with question four, the first question, which a significant number of students missed.* When it asks you for decimal division, what does that mean?

S: Decimal division means that instead of having a remainder, here you have a decimal point and a number.

K: Yes. Let's do eight into 26. *Mrs. Kapeesh writes "26/8" on the dry erase board with a pink marker. She computes using the traditional algorithm. The 3 goes over the 6 in 26. 3×8=24. The 24 is subtracted from 26, which leaves a remainder of 2.* Instead of having three remainder two, what would I do?

S: You put a decimal and add a zero, to make the two, 20.

K: *Mrs. Kapeesh puts a decimal point after the 26 to make 26.0. She brings down the zero to make the reminder 20. The decimal point is moved to after the 3 in the quotient.* This is decimal division. So then we say, eight goes into 20 how many times?

S: Oh, um, twice. Twice.

K: That was way too long. Let's try again. It goes into 20 how many times?

Ss: Twice!

K: *Mrs. Kapeesh writes "2" after the decimal point behind the 3. She writes "16" under 20 and subtracts, leaving 4. She then writes another zero in the hundredths place for 26 and draws an arrow down to the 4.* Why are the answers using remainder division and decimal division different?

S: Um, decimal division will give you a different answer cause you would keep adding a zero and you would get a bigger number than the actual number.

K: K. Let's add to that.

S: Um, so, with remainder division, you just keep the number you end with. I the number that you divide, like 26, it would be three remainder two, but if you were doing decimal division, eight doesn't go into two, so you'd have a decimal point and bring down zero where it would – you would get a different number each time.

K: Anyone want to add to that?

S: Bottom line. Decimal division is more precise.

Mrs. Kapeesh's response, given what she noticed. Mrs. Kapeesh believes it is important for her students to understand their mistakes to connect to future units and have a larger conceptual understanding, in this case, between decimals, division, and fractions. Mrs. Kapeesh stated, “They are connected, and students should be able to make that connection by understanding their mistakes.” She continued,

> I love lightbulb moments and I think this is one of them. It’s important to connect in all ways because they are the same representation but displayed differently. I also want students to feel confident in their learning, and they feel much more comfortable with decimals, so I want them to feel they can take on fractions as well.

Mrs. Kapeesh chose to review the standard algorithm with her students (lines 16–37). The review was procedural (meaning first we do this, then we do this). In lines 16–18, Mrs. Kapeesh completes the computation on her own. She asks her students, “What do I do next?” (line 19). and “Why are the answers using remainder division and decimal division different?” (lines 36–37). She encourages children to explain their thinking, and their explanations rely on describing the procedure they used to compute (lines 39–40 and 44–48).

Listening process claim for Interaction 1. Mrs. Kapeesh demonstrated a particular balance between impulsivity and spending too much time considering alternative strategies and collecting more student sense-making. She knew what student sense-making to consider, their procedural knowledge, and had an interaction in mind to assist student sense-making. This was a cyclical process of trying and rejecting interactions over several years. Mrs. Kapeesh notices her students’ mathematical sense-making. She notices students’ procedural knowledge (lines 16–36). She notices and responds to her students’ written, and then verbal, strategy explanations. The interaction that ensues needed to help students make sense of what to do with what was left over. Since her students’ explanations of division were procedurally based, it made sense to Mrs. Kapeesh that their response to this question would also be based on the procedure, which it was (lines 39–40 and 44–48).

Context for student sense-making noticed in Interaction 2. Mrs. Kapeesh's fraction teaching and learning objectives are as follows: students will understand that a fraction is part of a whole; students will be able to name ways we use fractions in our real lives; and students will make equivalent fractions with the number line, the area model, and numbers. Mrs. Kapeesh opens her unit on fractions with a "Snowflake Pattern Block Activity" to gather information about her students' current understanding of fractions.

Mrs. Kapeesh's elicitation and interpretation of student sense-making in Interaction 1.

K: You're going to get a piece of printer paper. You're going to fold it in half hamburger style. Then, I'm going to give you pattern blocks, and you are going to create a snowflake. You can make that snowflake look however you want, but you only have four minutes to do so. When I call time, you will draw your snowflake on the top half of your paper. And I'm going to ask you to color it. Cause colors are important. Step one, fold paper. Step two, make snowflake. Step three, draw snowflake. *Mrs. Kapeesh walks around her classroom once to observe student snowflakes. She then works on getting a series of questions listed on the board. She ends up writing them by hand because the Promethean Board is not working. She directs the students to use their own snowflake designs to respond to the following questions:*

K: Number one. How many total blocks did you use?

S: Do I need to count these squares (sic)?

K: Yes. All the pattern blocks. Number two. What fraction of the blocks are triangles? Number three, what fraction of the blocks are blue? Number four. What fraction of the blocks are hexagons?

S: What do hexagons look like again?

K: How many sides?

Ss: Six.

K: Question five. What fraction of the blocks are pattern blocks?

Ss: What?

K: Think about it. *Mrs. Kapeesh then collects the students' work.*

Mrs. Kapeesh's response, given what she noticed. Mrs. Kapeesh wishes that she had the time to have her students discuss their answers and show their snowflakes. She believes this would have "connected understandings even more," but she struggles to elaborate upon why this would be so. Mrs. Kapeesh was unable to dedicate as much time to this activity as she initially anticipated, because many of her students were unable to demonstrate mastery of decimal and division content on an assessment the day prior (Interaction 1). Mrs. Kapeesh identified a need for her students to review the division material. She opted to shorten the time sequences on this introductory fraction activity to ensure that she was still able to collect enough information on her students' fraction sense-making to design discussions that would allow students to discover "why things about fractions make sense," while noting she had not "maxed the potential" of the exercise.

Mrs. Kapeesh utilizes evidence of her students' interaction with this content to plan her instruction for the following lesson. She collects the students' pattern block snowflakes and responses to the questions listed above (lines 66–80) as "pre-test data." She notices that "all students were able to use the *a/b* fraction notation, but had various issues with identifying the whole, numerators, and denominators." Mrs. Kapeesh decides that she will need more discussion to understand what her students already know about fractions. She determines that the "Tally Desserts" activity (Interaction 3) that follows will be her next pedagogical move.

Listening process claim for Interaction 2. Mrs. Kapeesh listens to her students' sense-making for a specified amount of time and then instantly commits to a specific interaction. Mrs. Kapeesh notices students' procedural knowledge of fractions.

Mrs. Kapeesh's questions (lines 66–80) are used to learn about her students' fraction sense-making. Mrs. Kapeesh notices students' "mastery of fractions," as the students' ability to note the whole, a part, and a part of the whole of a model of a unit.

Context for sense-making noticed in Interaction 3. Mrs. Kapeesh continues to introduce fractions to her students with a dessert selection tally mark chart activity, a pedagogical strategy to understand more about what the students knew about fractions:

> This was an idea to hook students into recalling there are fractions in our everyday lives. I don't love this beginning. I would have chosen a more concrete, clear connection so students understood numerator and denominator. I would have included other vocabulary such as fraction, what part of a whole means – maybe I would start with a real-life problem that involved fractions and we would break it apart.

I was intrigued by Mrs. Kapeesh's desire to try something new, to start with a real-life problem that involved fractions and break it apart. However, Mrs. Kapeesh still opted to respond to her students' interaction with the snowflake pattern block fraction activity with her "very fictional dessert story." She used an equal sharing division problem to connect division to fractions, just as she suggests here.

Mrs. Kapeesh's elicitation and interpretation of student sense-making in Interaction 3.

K: From your experience so far, who loves fractions? *(About one-third of the class raises their hands.)* Who hates fractions? Who has a complicated relationship with fractions? Here it is, loves. I love fractions. They are fun, they are in our real lives, I actually know how to do this, and what my goal is. I have two goals. This will definitely go to winter break. That we work on fractions. Parts of it a little later. I want you to love them and I want you to understand them. Most of the time, once you understand them, ya love them. And I think sometimes we hate them because they're confusing and I was always getting them wrong, and I'm messing them up. But we are going to do a lot of really engaging things to get you used to fractions. So, I'm going to tell you a very fictional story.

S: How fake?

K: Very fake. I would love to buy every one of my students in this math class their favorite dessert.

Ss: Really?

K: Very fictional. Now, if I wanted to know what your favorite dessert was, any idea on how I could find out?

S: You could ask us what our favorite dessert is.

K: How? How would I ask you? *She is now holding two large colored posters, one blue and one orange.*

S: You ask every student in the class what they like. Then they mark it down. Then you make a pie chart which is a type of fraction anyway.

K: I'm going to ask you to come up here, and you'll draw a tally on the one that you like. *(All students come to the poster and mark their choice with a tally.)* Okay. Thank you. So, how many people from this chart chose cookies? *(There is one tally mark next to the row "cookies.")*

Ss: One.

K: How many people chose ice cream?

Ss: Two.

K: How many people chose none?

Ss: Zero.

K: How many people chose other?

Ss: Twelve.

K: So, how many people chose cookies again?

Ss: One.

K: Out of how many?

Ss: Seventeen.

K: So what I'm going to write here is one out of 17.

S: Fractions!

S: Next is two out of 16.

K: Nope.

S: But…

K: It's still going to be out of the total. So, X made an interesting comment. Which actually, I think, happens quite often. She said for this one we put one-sixteenth, because we already had one person. Why would that not be correct? You can answer that, X.

S: If you made a circle and you drew like 17 pieces of pie, or whatever, and then, um, you color one, I know what I mean, but…

K: It's alright. Call on somebody to put it into words for you.

S: So, the reason why you have 17 pieces is that you have 17 total people in the class. Not 16.

S: But why don't you minus the one that got the cookie? Because you colored in one square (sic) of the circle out of 17. So there would be 16 squares left.

S: That's not the whole number though.

S: There's still 17 people. One just got taken out.

K: So, does anyone know what this number represents? *Mrs. Kapeesh circles the number that is in the top part of the fraction.*

S: It represents how many pieces were taken. Like, say it was a piece of cake, or a cookie, or a bowl of ice cream. And you cut it in slices. And since it's like two seventeenths of a bowl of ice cream, you take two slices of ice cream and that would be like why you put two seventeenths.

K: Call on someone else to add on to that.

S: It's called the dividend. No, wait, no – it's called the numerator! It's what's taken out of the whole. Or, say, the special pieces.

K: Copy me. Part of a whole. *Mrs. Kapeesh writes "numerator" next to the 1 in "1/7," as she says "part of a whole."*

Ss: Part of a whole.

S: The denominator is the whole.

K: So that number is going to tell you how many you have in total.

Mrs. Kapeesh's response, given what she noticed. Mrs. Kapeesh decides to lead a targeted discussion about fractions and fraction equivalency using fraction hamburgers (which follows after our claim). She does not choose to continue to assist the student with her fractional part–whole misconception (lines 61–65) directly. Mrs. Kapeesh does provide her definition for numerator, "part of the whole," and denominator, "D, down for denominator – and it represents the whole."

Listening process claim for Interaction 3. Mrs. Kapeesh listens to her students' sense-making and goes through a process of acknowledgment (lines 61–107) when an interaction assists in promoting student sense-making, and rejection, when an interaction does not assist in student sense-making. Mrs. Kapeesh wants to adapt to student responses and learning, but she does not know how to altogether respond to students' sense-making in the moment. She tries to clear up the student's misconception (line 61) by introducing "part of the whole" vocabulary (lines 103–107). She needs to provide an opportunity for her student (and others in the class, as well, as their explanations do not indicate conceptual understanding of "part of the whole" either) to understand the relationship between fractional quantities.

Mrs. Kapeesh previously taught sixth-grade math, where a rule was commonly used to "make sense" of computation. She noticed that her past students did not have a conceptual understanding of why that rule worked, and they struggled to use the rule with new problems. She described that her students felt they needed to have the right "tools" to begin problem-solving. She explained,

> Students were taught procedures on how to solve a computation problem – just find the key words, use the rule. But they didn't understand how to solve a problem and struggled with explaining their own work. I wanted to change a couple of things. How students talk

about math should be rooted from conceptual understanding and also how students feel about math. I want them to love making mistakes and finding ways to fix it. I want it to feel exciting to take risks in thinking.

Mrs. Kapeesh understands that fractions can be encountered as a natural outcome of whole number division; students can begin to see fractions as numbers that follow some of the same basic principles as whole numbers. Mrs. Kapeesh provides the students an equal sharing problem to solve (one that could be modeled pictorially if the students choose to make sense of it that way). Her students can solve the problem using a strategy they had at hand (division with remainders and decimal division was Mrs. Kapeesh's previous unit of study).

Mrs. Kapeesh extends the problem to request a written fraction built on her students' understanding of equal sharing and fraction and decimal equivalence. Her interaction with the content and her students intentionally varies from the script provided by Engage New York.

problem. You need to glue it in, and you need to solve it.

S: Explain it, too?

K: Not explain, just solve. We are going to explain verbally. X, will you read that please?

S: Yes. If 15 kilograms of rice are separated equally among four containers, how many kilograms are in each container? Express your answer as a decimal.

K: Thank you. *Mrs. Kapeesh begins walking around the classroom as students independently approach the problem using their own strategies. Mrs. Kapeesh stops to watch a student demonstrate putting in his decimal point after his whole number quotient and then putting in a zero, but he had not brought down the zero to initiate his decimal division.* Fifteen divided by four is three. You said four times three is 12. Fifteen minus 12 is three. That's fine. But four doesn't go into 30 three times. So, where is this three coming from?

S: *The student looks at his computation for a moment, scowls, then says,* Oh! *The student smiles, erases the three, brings down the zero and writes seven after the decimal point in the quotient.*

K: *Mrs. Kapeesh pats the students on the shoulder and walks away.* Almost everyone, unless you weren't done, got this right. Which, big whoop, you got it right, but you have to be able to explain it. So, can you read it again, X?

S: Fifteen kilograms of rice are separated into four containers. How many kilograms of rice are in each container? Express your answer as a decimal.

K: What do we know? What do you we know, X?

S: We know 15 kilograms of rice are spreaded (sic) into four containers equally.

K: That's awesome. I don't know if I could have said that better. That's awesome. So what did you do for this problem, X?

S: I did four divided by 15 (sic).

K: *Mrs. Kapeesh writes "4/15" on the dry erase board.* When you say four divided by 15…

S: Aw, no, the other way.

K: K. So, how do you say it? *Mrs. Kapeesh erases "4/15" from the board.*

S: Four divided by 15.

K: *Mrs. Kapeesh writes "4/15" on the board again.* Four divided by 15. You got this.

S: Fifteen divided by four.

K: There you go. You could also say four divided into 15. Okay, so when you, I'm going to do this part quick because you guys got this. What I want to do is more with the answer. *Mrs. Kapeesh computes using traditional algorithm.* Okay, four goes into 15…

Ss: Three times.

K: That's 12. I subtract and get three. Why did you not do three remainder three? X?

S: Because a remainder is not a valid answer and you're putting things into groups, so what we need to do, you need to put a decimal point behind the 15 and the top three, put a zero after the decimal point, after the 15, and then bring it down.

K: Awesome. Thank you. Four goes into 30…

Ss: Seven…

Ss: Twenty-eight. Two. Bring down the zero. Twenty. Five.

Ss: Kilograms!

K: What does this number represent?

S: All the kilograms of rice there is.

K: Who wants to add, or change, or agree? X?

S: I want to add and change because it's not every kilogram of rice. It's how many kilograms of rice are in each container.

K: Okay. Somebody tell me this number. How do we say this number? *Mrs. Kapeesh circles the "3.75" quotient on the board.*

S: Three and seventy-five-hundredths kilograms.

K: Do you agree with her?

Ss: Yes.

K: Tell somebody, each of you, how you say this number. Then, on your paper, I want you to attempt to write this as a fraction. We are not making this up, guys. Decimals and fractions are the same. They show a part of the whole. Or in this case, a whole *(Mrs. Kapeesh circles the 3 in the quotient)* and a part *(Mrs. Kapeesh circles the .75 in the quotient).* X, how did you write this?

S: Uh, I had no clue how to write it.

K: We're going to help you. So we have three and. What is three? What is three, X?

S: Ah, it represents three. It's a whole. *The student at the board writes "3/3."*

K: It's the whole number. So, how would you write three as a whole number?

S: By itself?

K: By itself. *(The student writes just "3.")* Yes ma'am. What is seventy-five-hundredths? It's not a whole number; I will tell you it's part of a whole. Write it exactly as it sounds. Seventy-five hundredths. *(The student writes "75/100.")* Did she do it? This is what I was looking for. Now, if you were to simplify this, the hundredths, you can think about quarters. X, what did you put?

S: Three and three-fourths.

> K: I need you to make connections from decimals, which you're so good at, add it to fractions. You know how to do this, so today we're going to talk about equivalent fractions. Fractions that might look a little bit different, but they are actually equal.

Mrs. Kapeesh leads a discussion that asks her students to define and clarify (Kazemi & Hintz, 2014) what they understand about the relationships between division, decimals, and fractions (lines 202–203). She notices the procedural representations some students are using (lines 205–217). Mrs. Kapeesh listens closely to student sense-making and is more explicit in how she interacts about converting between decimals and fractions, because this is the goal of her interaction: for her students to explain their thinking and develop a deeper conceptual sense of the relationship between decimals and fractions.

Mrs. Kapeesh has all students attempt to convert the decimal to a fraction (lines 288–292) so she can identify what she thinks her students understand. Her choice to call on a student who struggles with writing 3.75 as a fraction is intentional (lines 292–307). Even though this student–teacher–content interaction focuses on how the fraction "sounded" (lines 305–307), Mrs. Kapeesh wants to make this sense-making public, so more students might learn to be "okay with the struggle."

Context for sense-making noticed in Interaction 4. Mrs. Kapeesh decides to lead a targeted justification discussion about fractions and fraction equivalency using fraction hamburgers. Students are expected to develop, use, and justify their explanations about why one-half and six-twelfths are equivalent:

> Fraction hamburgers are so fun, and students have been eyeing them since the beginning of the year. Half is a safe fraction to work with, given that students can truly understand half of everything. What is trickier is four-eighths or six-twelfths, but when actually putting them on top of each other, they can see it is equivalent.

Mrs. Kapeesh's elicitation and interpretation of student sense-making in Interaction 4. Mrs. Kapeesh's expects her students to develop, use, and justify their explanations about why one-half and six-twelfths are equivalent, given their prior experiences with fraction hamburgers.

K: These are called fraction hamburgers. *Mrs. Kapeesh holds up a Styrofoam hamburger in a plastic container.* These are super fun and today, they're going to be a learning tool. *Mrs. Kapeesh puts the students into partner groups and distributes a hamburger to each pair.* Take out a whole. Go.

S: *Students open the fraction hamburger containers.*

K: If you have a whole, hold it up.

S: *Almost all partner groups immediately hold up a peach colored circle, which represents the hamburger bun.*

K: How do you know it's a whole? You can raise your hand for this. How do you know it's a whole?

S: It's not broken into teeny little pieces.

K: Does anyone want to add to that?

S: Cause there's no missing pieces.

K: That's awesome.

S: Um, because it's got no cuts through the middle meaning it is whole.

K: Perfect. Makes sense, right? Now, I want you to find the halves. The one halves. *Mrs. Kapeesh writes "½" on the dry erase board.*

S: *Some students begin holding up a one-half circle, which is a brown hamburger patty.*

K: I need all the halves. *Mrs. Kapeesh models holding up both of her hands with a half hamburger patty in each hand. She looks around the room to ensure each group is holding up the two halves of brown hamburger patty.* How do you know it's one-half? And you may not answer this question by saying, because I saw the label "one-half" on the parts. How do you know it's one-half?

S: Do you see how the bun's one whole? *Student holds up the bun.* So, it's one whole. It's not going to split in half. And when you see these, it's not one whole. *Student holds up both one-half hamburger patties.*

K: Call on someone else to add to that.

S: It's a half because, you can tell it's a half because half means two, I'm pretty sure. And there's two pieces. That makes this piece one-half. *Student holds up the one-half hamburger patty.*

K: What do you notice in the fraction one-half? What do we know about the bottom number?

S: Um, it's always going to be the same as the top number, or bigger than the top number.

K: Let me clarify my question. What do we know about the denominator here? *Mrs. Kapeesh circles the "2" in the denominator of the fraction "½" on the board.*

S: The denominator tells us that is the full number. So if you add one more then that would be a whole.

K: Yes. I'm going to share with you the way we say that in math. This is our total. *Mrs. Kapeesh points to the "2" again.* When you show me two of the brown, which is the hamburger patty in this case, you should have two of them which is my total. And my numerator tells me what?

S: Um, it tells the fraction.

K: The part of it, right? So, when you have two, this is only one of the two. One-half. *Mrs. Kapeesh holds up one of the one-half hamburger patties.* If you have two out of the two total pieces, what does it become? *Mrs. Kapeesh holds up both one-half hamburger patties.*

Ss: One whole.

K: One whole. So, when you have – if the number at the bottom is the same as at the top, you have a whole there. So when we talk about fractions, we are talking about a part of that whole. Kapeesh?

Ss: Kaposh.

Mrs. Kapeesh's response, given what she noticed. Each of Mrs. Kapeesh's students are engaged in solving and discussing the fraction hamburger representations at an appropriate level, given what she currently knows about their fraction sense-making. She has a problem for her students to solve.

K: Now, here's my challenge for you and your partner. You're going to take one-half. And I want you to find any other fractional size but also has to be equal to this same amount. It has to fit perfectly on this half. *(Students work independently to find a solution.)*

S: I used twelfths.

K: How many?

S: Six.

K: What's another idea?

S: Um, I chose one-third and one-sixth.

K: The fractions had to be the same size.

S: Oh, then one-fourth. Two-fourths.

S: I have another. Three-sixths.

S: Five-tenths.

S: Four-eighths.

K: Talk with your partner. Something you notice. Something that is happening with all these fractions. So, what do you notice?

S: They're all split in half. Like, eight divided by two equals four. And 10 divided by two equals five. And so forth and so on.

S: The student goes up to the board. So basically, with this one, well, with all of them, this is multiplied by a certain number. This one (the student points to one-half) is multiplied by two to get this one (student points to two-fourths). This one (one-half) is multiplied by four to equal this (four-eighths). And yeah, for all those a different number.

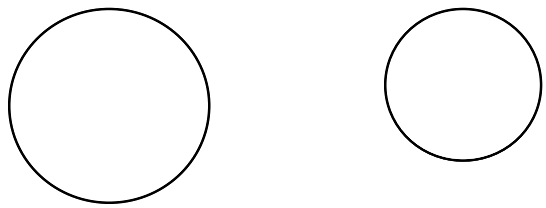
S: All of these fractions right here are equivalent to one-half. Since you put all of these little pieces on just one-half.

K: Equivalent. What does that mean?

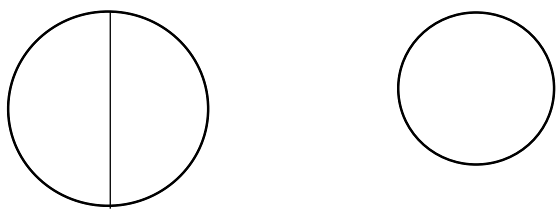
S: Equivalent means that they are all equal.

K: So explain to me why one-half and six-twelfths are equivalent. Talk in your group. Be ready for a debate. *(The students work in groups to come up with their sense-making strategies. Three children use a similar circle strategy to demonstrate equivalence which follows.)*

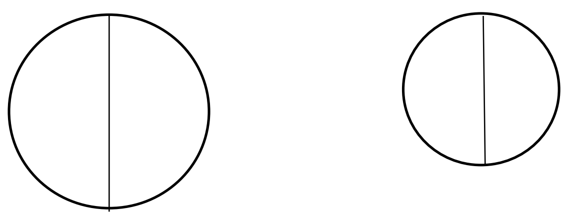
S: *The student draws two similar circles on the board.* Okay, so these are not perfect circles.



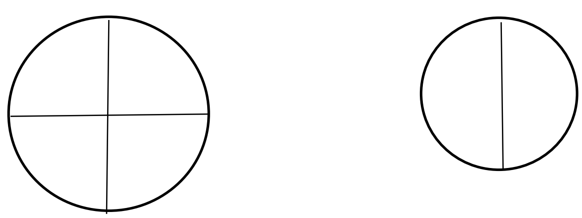
She splits one in half with a single line from top to bottom. So this is one-half.



She does the same with the other circle. And then this is still one-half.

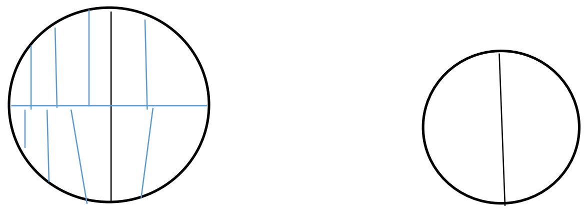


Compared to this. *She draws a line from left to right on one circle, creating fourths.* So, I split this in half, and this in half. And then, that is fourths.



She then attempts to break the fourths into twelfths but is not sure how to divide the fourths into equal parts. She draws a line, perpendicular to the edges of the slice, not crossing the vertex of the circle and counts "1," another line and counts

"2." Those are not equal. *This continues all the way around the circle until she says "12." She picks up a red dry erase marker and shades one of the two halves on the first circle and six of the unequal slices from the second circle.* This is equal to this because they take up the same amount of space in the circular formation.



K: I don't agree with that because those circles look different.

(The fourth student uses a similar rectangle strategy which follows.)

S: Okay, so, these are two triangles (sic). Rectangles. *The student draws a rectangle on the dry erase board. She then draws a similar rectangle underneath. She draws a line through the center of both rectangles. She shades the entire top half of both rectangles.* Half. Shade this in.

She makes the bottom half of the second rectangle have six parts. She then splits each of the sixths in two. It's still half of the whole. It's the same whole. It's the same size. It's just split into different amounts of parts. Or parts.

Figure 9. Student justifications of fraction equivalency.

Mrs. Kapeesh's response, given what she noticed. Mrs. Kapeesh sees and hears students' emerging ideas about how one-half and six-twelfths are related. Mrs. Kapeesh's questions and statements (lines 391, 395, 450) continue to press the students for why the two fractions are equivalent.

Listening process claim for Interaction 4. Mrs. Kapeesh listened to her students' sense-making and went through a process of acknowledgment when an interaction assisted in promoting student sense-making, and rejection, when an interaction did not assist in student sense-making. However, in this interaction, Mrs. Kapeesh tried to recall what student sense-making triggered each successive interaction. She allowed the students to guide the conversation and the opportunities to learn that would assist the students in extending their thinking; however, she was the one who chose the student examples that were presented on the board and the order in which they would be presented. Most of the responding students base their explanations on a relationship between fraction hamburger pieces (lines 399–427). The final student response attempts to explain the concept of equivalent fractions – what remains the same when one fraction is transformed into another (lines 459–471). Additionally, Mrs. Kapeesh has a student who indicates understanding the multiplicative relationship between the numerator and denominator (lines 379–385).

Conclusion

Mrs. Kapeesh knows that her students need hands-on engaging material to be focused. She taught an academically diverse group, including both struggling and highly performing students, and determined that pruning a student's growth mindset was instrumental in the students' development of perseverance in working through challenging mathematical concepts.

She facilitated seminar-type math lessons that were student led. At the end of our time together, Mrs. Kapeesh reported that she knew

> …that three out of the 18 students in class are confused over the conceptual understanding of fractions, like, they can't describe the relationship or the meaning between the numerator and the denominator. The part of the whole. Seven out of the 18 cannot explain equivalent fractions mathematically without visuals.

I asked Mrs. Kapeesh what she was going to do with what she noticed about her students' fraction sense-making. She said that she was going to have her students "represent fractions on a number line next."

K: So here is a number line. Write it. *Mrs. Kapeesh draws an open number line on the dry erase board. Students write directly on their desks with their dry erase markers. Mrs. Kapeesh writes "½" on the upper right-hand corner of the dry erase board.* And I am going to represent halves. *She draws a short perpendicular line at the left-hand side of her number line and labels it "0."* So, what is my whole? I start at zero. Make sure your number line is starting at zero. And if you look at the denominator, that tells me our total. So, where do you think my number line ends?

S: Two.

K: Two?

Ss: Yes.

Ss: No.

S: Or one?

K: Why one?

S: Because it's one whole.

K: My wording could have confused you because we're talking about halves, but we're talking about a whole in this case. One whole. So, right in-between will give me what?

Ss: One-half.

S: If you have a fraction, there's always going to be a half, no matter what.

K: It might just look different right? Three-sixths, six-twelfths…

S: But you can't represent one-third.

S: Yes you can.

K: Out of thirds. We'll talk about it, but not quite yet. Let's understand halves first. So, instead of putting the whole, what else could I put here? *Mrs. Kapeesh points to the "1" at the end of the number line.*

S: Uh, I don't know how to say it. Two over two.

K: *Mrs. Kapeesh writes "2/2" on the dry erase board under the "1."* We talked about this before. This is my amount over my total, and when they match, I have my total. Right? I have two out of my total two. So that's how you would write that as a number line.

S: Or two-hundredths?

K: Two-halves.

S: Two and two-tenths?

K: Now, I want you to write a number line for thirds.

Mrs. Kapeesh attempts "to keep a neutral tone while hearing answers so students can come to their own conclusions about their and their peers' justifications" until she solidified the correct answers. She wants to give students time to think, provide opportunities to share, restate student sense-making using mathematical terminology, sequence student responses, and encourage students to engage with the content by encouraging them to explain what they understood. Mrs. Kapeesh listens to her students' sense-making. She demonstrates a particular balance between impulsivity and overthinking. She knows what student sense-making to consider and usually has an interaction in mind to assist student sense-making. She listens to her students' sense-making for a specified amount of time and then instantly commits to a specific interaction. She can do this because she has used a cyclical process of trying and rejecting interactions over several years. Mrs. Kapeesh listens to her students' sense-making and goes through a process of acknowledgment, when an interaction assists in promoting student sense-making, and rejection, when an interaction does not assist in student sense-making.

Next, in Chapter Seven, I share my cross-case analysis. I draw the reader's attention to the similarities and differences across Mrs. BB's, Mrs. Branches', and Mrs. Kapeesh's narrated noticing of their students' mathematical sense-making.

CHAPTER SEVEN

CROSS-CASE ANALYSIS

Chapters Four, Five, and Six introduced the reader to Mrs. BB, Mrs. Branches, and Mrs. Kapeesh. Each teacher narrated what they noticed about, as well as their processes for listening to, student mathematical sense-making. In this chapter, I share the similarities and differences I noticed across these three cases. My cross-case analysis covers the themes of contexts, interactions, and processes of listening; these themes helped me best categorize how participating teachers' noticing afforded learning opportunities to their students.

Organization of Teachers' Narrated Noticing

In Chapter Two, I described the possibility for teachers to consider their processes for listening to student sense-making and narrate said processes and knowledge for use in interactions with their students, the content, and students' interactions with the content. My participants narrated knowledge of what teachers do, or procedural knowledge (Klegal & Anderson, 2008), as well as specific instances of practice dependent upon noticed relationships between students and content, or declarative knowledge (Klegal & Anderson, 2008). I noted in Chapter Two that teachers' narratives of their interactions and procedural/declarative knowledge may be challenging, if not impossible, for them to verbalize (Klegal and Anderson, 2008). In the following sections, I describe the similarities and differences between my participants' contexts, interactions, and listening processes.

How I Compared the Interactions

My research hinged on teachers' ability to identify and narrate contextual interactions that might assist them in explaining their process of listening to student sense-making (see Chapter Two). My participants chose to include interactions they believed they could discuss.

Before beginning the discussion of the similarities and differences across interactions in these cases, I reemphasize the diverse instructional contexts of these teachers. Each teacher's practice space was multi-faceted and varied, and each of their contexts raised personal challenges and practical resources for their teaching (Drier, 2008). Each teacher who participated in my research worked hard at their craft of teaching.

We must recall that teaching is not a set of homogeneous interactions. Each teacher participated in their context by either interacting to maintain the status quo or contributing in a way to change it (Drier, 2008). My participants realized that they exerted some influence on their teaching contexts and possessed a particular ability to interact in, and knowledge about, that context. My participants all listened and interacted differently, depending partially upon the teaching context they identified. So, as we review similarities and differences, it is crucial to consider which contexts these teachers were a part of and how they chose to take part (Drier, 2008). I use my conceptual framework to guide us through this process.

How My Conceptual Framework Assisted in my Analysis

I wanted to understand what occurred in the minds of teachers in the specific moment between teacher listening to student sense-making and the culminating instructional interaction. My theoretical frame relied on the Hawkins Triangle (1974) to depict the relationships that exist between teacher, student, and subject. The circle represents the context, or social practice, in which the teachers' interactions and thinking processes occurred (Gutierrez, 2015; Drier 2008). The flow chart inside the triangle (Empson & Jacobs, 2008) depicts how students and teachers, through their listening to student sense-making, were continuous drivers of instructional interactions. Finally, we needed to capture a specific momentary process that teachers believed they could narrate from a specific interaction, which we attempted via identifying strategic sites

(Lampert, 2001). I use each of these tenets of my conceptual framework to describe similarities and differences across my three participants.

Identification of strategic interaction sites. I began my cross-case analysis by counting each coded TISIC that the teachers agreed to discuss (see Lampert strategic sites diagram in Chapter Two). I categorized the TISIC codes into how teachers noticed and the types of interactions teachers had and tallied the codes I used to describe each TISIC interaction. I then focused on the TISIC interactions where the teacher could recall the interaction and their culminating RSQIU. This code is directly tied to my research questions. Finally, I returned to the interviews to code each teacher narrative tied to the TISIC–RSQIU code for moments when teachers could or could not narrate their processes for listening to the student sense-making involved in those interactions. I begin by providing a summary of each teacher's case, embedding their listening processes within their interactions and context. I then compare and contrast across cases.

Participant Summaries

This section presents a three-part summary of each participant – their context, interactions, and listening processes – beginning with Mrs. BB.

Mrs. BB's context. Mrs. BB's context comprises her experiences, skills, self-evaluation, and motivation.

Mrs. BB's experiences. Mrs. BB's principled inquiries into teaching elementary math included earning a master's degree (in behavioral disorders) and an endorsement in special education. Mrs. BB "reads for information," and this is how she stumbled upon CGI, math recovery, and appropriate scaffolding. She has continually attended professional development events to be “up to date on the multitude of strategies available for teaching math.” Her instruction relies on “writing down what I see.”

Mrs. BB’s self-evaluation. Her practical experiences over the past 38 years have led her to believe that she “knows many of the best techniques” for getting students to the point of using “mental math”; yet, her work with Title I schools, gifted programs, math games (Everyday Mathematics), and Math Recovery led her to recognize she must always “provide different ways of attending to the task,” whether it be “draw, show me fingers, show me, tell me.” She elicits “ah-ha” moments by observing who struggles and who understands the content. She always begins with her recognizing each student’s number sense and then building upon it. Her noticing often begins with choral oral counting (forward and backward, to see who may have misconceptions about number order).

Mrs. BB’s skills. She holds high expectations for students, in that they “will be able to do” the content presented to them. She knows all her students have the capacity “to try to figure it out.” She often supplements her curriculum using Math Recovery Strategies, such as “five frames” and “scaffolding,” since she has seen the limitations of consistently teaching directly to the entire group.

Mrs. BB’s motivation. Mrs. BB believes a teacher must “feel good about where they are with teaching, and their philosophies” to teach how she does. Play and practice rank high on her to-do list.

Mrs. BB’s interactions. The majority of Mrs. BB’s narratives arise from knowing what teachers do, or procedural knowledge. She tends to use one source of declarative knowledge – how students develop number sense – to narrate the relationship between her listening to student sense-making and how she chooses to respond to it.

Mrs. BB's ability to narrate. Mrs. BB would be considered an elite teacher by the teacher domain evaluation system used by the state. At the time of study, she was a Level III teacher nearing retirement. She could narrate almost all of her interactions by tying her knowledge of students' number sense to the strategies employed while working with student(s) and content.

Mrs. BB's procedural knowledge in interactions. Mrs. BB's current, and recent, instructional contexts have given her experiences with students with differing impairments and behavioral disorders, including second language learners, grade retention, and grade-level ability (above, at, or below). These contextual obligations, paired with her lack of curricular choice, require that Mrs. BB employ her own method of "math records" (like reading running records) that utilize student data from I-Ready Testing and modified Math Recovery sessions to determine patterns in students thinking so they can be grouped homogeneously. These groups are formed around her conception of students' number sense. Mrs. BB's instruction with her current group of students focus on providing multiple opportunities to solve subtraction problems "using manipulatives, fingers, ten-frames, counting back, getting the total, and drawing a picture."

Mrs. BB's declarative knowledge in interactions. Mrs. BB chooses to use homogeneous small groups, sequencing, and alternative resources to help "all her students learn from mistakes." Whether or not Mrs. BB follows curricular plans, she and her student groups begin and end together, practicing number sense-related drills daily. The interactions in the homogeneous groups are what vary. Students in each group are provided with opportunities and the freedom to see connections between past and current content. This begins with addition ("counting on, doubles, combinations to get to 10") and extends to subtraction ("fact families, turnaround facts, identifying the total, one-to-one correspondence"). These conversations depend upon "child modeling and children's explanations so that their thinking can be scaffolded and not stifled." Mrs. BB achieves these ends by providing the same, or similar, problems to every leveled math group and varying the numbers students have to work with, as well as the type of connecting dialogue she expects students to demonstrate.

Mrs. BB's listening processes. Mrs. BB's listening process involves first recalling her best knowledge about the content and her associated strategies; second, gathering student sense-making; and third, considering whether she has a stored interaction that had performed well in the past that could be repeated or whether she needed to gather more student sense-making and learn to interact in a new way. She tends to initiate this process prior to the start of instruction. She favors using what she already knows and applies her best knowledge to elicit the best student results (which eventually comprises students' automaticity with addition and subtraction facts). Most of the student sense-making she hears was heard in the past and applied to present whole and small group instruction. She is cognizant of her opportunity to act differently given what she hears in the moment with individual students and usually chooses to alter her interactions with individuals according to her perception of their number sense prior to beginning instruction.

Mrs. BB's summary. Mrs. BB is a veteran teacher with many years of experience, confidence in her abilities, and an awareness of how she can recall past interactions and experiences to do her best work, within the confines of her current teaching context. Mrs. BB can narrate most of her interactions. She uses students' number sense as the relational tie between listening to student sense-making and determining a response. Her listening process involves recalling instructional strategies, gathering student sense-making, and considering which prior interactions work, all before initiating instruction. Next, we review Mrs. Branches' case.

Mrs. Branches' context. Mrs. Branches re-humanizes mathematics instruction for her students.

Mrs. Branches' experiences. Mrs. Branches has "taught in lower socioeconomic schools," where "I was students' whole world." She has focused on building relationships with her students at the beginning of each year to gain knowledge "about them, from them." She regards her students as a "family" and part of a larger "community."

Mrs. Branches' self-evaluation. She sees herself as "*the* teacher who brought student concerns to their families" and, ultimately, the one who bears the blame for these identified student learning issues. She describes "just trying to survive" and "do her best" while "working alone and feeling alone." At past school sites, she viewed herself as a teacher who "changed lives." She "shaped the whole child" and provided "valued" educative experiences.

Mrs. Branches' skills. When this research occurred, Mrs. Branches was working toward obtaining her master's degree in special education with a focus on gifted instructional methods from an online university. She utilizes formative and summative assessments written herself by hand, because she "thinks better when she writes things out." She believes multiple ways exist to arrive at a solution and so expects students to explain their whys for doing things, as well – "But they can't."

Mrs. Branches' motivation. Mrs. Branches' students drive what she does in her fifth-grade classroom. She describes the inspiring force behind her instruction as her "moral responsibility" to prepare her students to be "contributing citizens." Mrs. Branches acknowledges that she must ensure her students' proficiency with concepts presented in prior years before she can address current grade-level mathematics concept standards.

Mrs. Branches' interactions. Mrs. Branches' struggles daily with knowledge of what teachers do; "How do we accommodate 26 different learning styles when we have 26 different state mandates?" She recognizes that every class is different and requires new lessons yearly and believes that teachers should "draw upon their different experiences."

Mrs. Branches' ability to narrate. Mrs. Branches listens to her students' sense-making based on her teaching preferences ("autonomy") to meet the needs of her students (to be "critical thinking productive citizens"). She knows that most of her students need her support to be able to begin using relational thinking – and to use it consistently – but she still struggles to narrate why her modified interaction is an appropriate pedagogical choice. This is characteristic of an elite teacher (Flegal & Anderson, 2008).

Mrs. Branches' procedural knowledge. Mrs. Branches knows she has "the autonomy to teach what she wants," because her teaching contract states as much, but quickly clarifies, "But if you don't know how, then that word [autonomy] doesn't work for you." Mrs. Branches takes her autonomy to the extreme and constantly attempts to make her present classroom context her ideal one. The reality of her current teaching context constantly leaves her disgruntled. She explains,

> The pressure is on for teachers. Our profession is currently frowned upon. We aren't valued. Society's thoughts on teachers are seeping into classrooms. Who do we please? The state? The district? The parents? The students? I have a moral responsibility to teach the student. We need to prepare that student to be a contributing member of society. That student needs basic skills for their life.

Mrs. Branches aims to "provide learning that meets the needs of every single student and challenge them to push boundaries." She wishes to achieve these ends without a specific curriculum and believes students should be "confident risk takers." Yet, she is aware that "we adopt for all without realizing that each school has a different population with different needs. We see this isn't the right approach, but we do nothing to change it."

Mrs. Branches' declarative knowledge. Mrs. Branches attempts to develop her contributing citizens by requiring "100% proficiency and the why behind it – for them [her students]." She claims her students drive what she does. She is not a fan of leveling her students but recognizes that "time is not on her side." She does not employ math workbooks or practice worksheets, "since students drive her instruction." When I asked her about her preference of using no textbooks or workbooks, she said,

> Because that's not life. I need to prepare them for life. And life is not a workbook. There are multiple ways to arrive at a solution. My students are stuck. They won't get out of the box. I ask them, what does that mean and why? They can't tell me. The standards don't drive my teaching. The student drives my teaching. Life problems aren't linear like workbooks and canned curriculum is.

Mrs. Branches greatly desires being "a learner and an observer with her students," not an "authoritative figure." She believes the easiest way to achieve this is by allowing students to lead conversations and asking them open-ended questions that push them to think critically through problems, "hopefully without panicking." The questions Mrs. Branches ask encourage students to explain their thinking. Mrs. Branches does not "tell kids they are incorrect; I tell them to build on it. We celebrate mistakes."

Mrs. Branches' listening processes. Mrs. Branches tunes her brain, content knowledge, and pedagogical knowledge to her teaching context, a context that revolves around her students' sense-making. The student sense-making Mrs. Branches listens to help her choose which interactions to store for immediate use. Mrs. Branches listens to collect information about her students' sense-making for a specified amount of time. What is interesting and consistent about Mrs. Branches' listening process is that she does not instantly commit to a specific structured interaction. Mrs. Branches uses more than one in-the-moment observation of student sense-making, as well as trial and error, to help herself – and other students – listen to and understand each-others' sense-making.

Mrs. Branches' summary. Mrs. Branches re-humanizes mathematics instruction for her students. She is mostly unable to narrate her listening processes and noticing. She tunes her brain to her context and listens to students' sense-making multiple times before deciding how to respond. We next review Mrs. Kapeesh's case.

Ms. Kapeesh's context. Mrs. Kapeesh's noticing narratives consistently arise from her ethical reasoning.

Mrs. Kapeesh's experiences. She teaches academically diverse groups of students and knows they relate well to "hands-on engaging material." Mrs. Kapeesh focuses on what she can control, like her relationships with colleagues, her classroom environment, and her relationships with families and students. She wishes to build a "support team" for her students.

Mrs. Kapeesh's self-evaluation. Mrs. Kapeesh feels she ensures that "all students are held accountable." It is her job to "solidify answers"; however, students must "circle back to their own choices" so they make "connections" across content and "understand" their mistakes.

Mrs. Kapeesh's skills. She knows that students, especially struggling students, relate well with visuals and hands-on materials and that "everyone achieves more understanding by explaining their thinking for others." She knows how to identify "first-talkers" by knowing and calling on her struggling learners and having them talk first.

Mrs. Kapeesh's motivation. Mrs. Kapeesh struggled in primary and secondary school mathematics. She feels she struggled because she was too afraid to get the wrong answer. She believes in the growth mindset, children's abilities to excel, and that all students should have confidence in failing. She feels that "student justification leads to clearer [student] understanding of math." So, her instruction is "student led." She believes students should be presented with math opportunities that are "safe" and "comfortable" so they can understand their mistakes. She "knows" her students need mainly visual and hands-on instruction and says she uses these resources in strategic heterogeneous groups.

Mrs. Kapeesh's interactions. Mrs. Kapeesh's narratives of her interactions with students and content are tied to her expectation of students' development of a growth mindset.

Mrs. Kapeesh's ability to narrate. Mrs. Kapeesh can state what portion of her students have mastered content, as measured by formative and summative assessments aligned with the common core state standards. Her knowledge of student sense-making is very specific and "in the moment." Her connection between her knowledge of student mastery and chosen interactions with students' and traditional computation algorithms is her belief in students' growth mindset.

Mrs. Kapeesh's procedural knowledge. Mrs. Kapeesh's instructional goals are for her students to "use fractions to compute correctly" and "represent fractions multiple ways, including visually and on the number line." She expects lessons to be 80% student explanation and talking, because she wants "students to explain their work." At the time of this research, she knew that her students were not performing at grade level and used this knowledge to adapt her instruction by providing instruction focused on traditional computation algorithms.

Mrs. Kapeesh's declarative knowledge. Mrs. Kapeesh focuses on "creativity and math that doesn't feel like math." She knows the needs of unique students (such as who was on, above, or below grade level and who had an individualized education plan). During this research, she knew that "three-eighths of her students were confused over the conceptual understanding of fractions – they can't describe the relationship or meaning between the numerator and the denominator." Additionally, she knew that seven out of 18 could not "explain equivalent fractions mathematically with visuals." Mrs. Kapeesh attempts to "incorporate" the growth mindset and "mathematical talk" but recognizes the challenges in quickly adapting to students' thinking.

Mrs. Kapeesh's listening processes. Mrs. Kapeesh demonstrates a balance between impulsivity and overthinking. She knows that she wants to develop students' growth mindset and usually has an interaction in mind, such as the introduction of a traditional computation algorithm to assist student sense-making. She listens to her students' sense-making for a specified amount of time and then instantly commits to that specific computational interaction. She can do this because she has used a cyclical process of trying and rejecting interactions over several years. Mrs. Kapeesh listens to her students' sense-making and goes through a process of acknowledgment, when an interaction assists in promoting student sense-making during dialogue, and rejection, when an interaction does not do so.

Mrs. Kapeesh's summary. Mrs. Kapeesh narrates a connection between her knowledge of students' specific in-the-moment mastery and how she would close the knowledge gap through the use of a traditional computation algorithm and students' belief in their ability to do it. She knows what she wants her students to know and listens for this sense-making through student-led dialogue. She provides an algorithm to help students reach her desired level of sense-making. I next discuss the similarities and differences between my three cases.

Cross-case Analysis: Similarities and Differences Across Cases

I initially sought similarities and differences across these cases using pre-defined categories of teacher knowledge we have learned from in the past, including Shulman's (1986) propositional and case knowledge and Ball's (2008) teacher knowledge of content and students, as I document above in the literature review. I considered three less researched and more current components of teacher noticing that could contribute to our understanding of teacher listening and help us harness its power, to extend the work of Shulman, Ball, and Lampert: teacher context, teacher listening processes, and teacher ability to narrate.

Context

When I first introduced teacher context in my methodology, I assumed that it would best applied in the form of the teacher's autobiographical examination of self. I thought I would present teachers' contexts through their stories of their past, present, and future educative experiences. In fact, the teachers did not utilize their autobiographical narratives in this fashion; they did not utilize these narratives at all in recalling their interactions. I actually removed the autobiographical narratives from my data set, until I realized the narratives were the interpretive pathway for beginning to describe the intent behind teachers' interactions, as they were revealed through their experiences, skills, motivations, and self-evaluations. We learned that context is more than the number of students in class, the mandated curriculum, or the state of one's relationships with colleagues. Each of my participants described an ideal classroom context, and it was through these conceptualizations – paired with a recognition of why their current teaching context was not their ideal context – that the aforementioned categories of experiences, skills, self-evaluation, and motivation became key parts of describing each participant's instructional context. When we grasped how each participant framed their context, we could follow that interpretive pathway to consider how and why teachers listened to their students' sense-making.

Listening Processes

Each of my participants has a process for listening to their students' sense-making and deciding how to respond. I must admit, this was pleasing to note, given current reform-oriented perspectives in mathematics instruction, such as the re-humanization of mathematics (Gutierrez, 2015), as discussed in Mrs. Branches' narrative. Teacher listening to student sense-making could be vital in reform-oriented math; however, it is an understudied component of instructional interactions (Hintz & Tyson, 2015). We know that student–teacher–content interactions provide opportunities for students to learn (Kazemi & Stipek, 2001; Lampert, 2001). We are still working to understand how teachers use the student sense-making they hear to decide how to respond (Jacobs, Lamb & Philipp, 2010). Better understanding teacher listening could teach us more about engaging in interactions with students that provide opportunities to learn.

We learn from Brent Davis (1997) that teachers must listen to students to better understand their teaching practices. Davis identifies three types of teacher listening that occur during mathematical talk: evaluative listening (Davis, 1997), listening for a particular answer; interpretive listening (Davis, 1997), asking for more information to be able to respond to what is heard; and hermeneutic listening (Davis, 1997), sharing responsibility for learning between teacher and students. Evaluative and interpretive listening reproduce students' sense-making, and teachers demonstrate understanding of student sense-making when they can reproduce students' thinking (Hintz & Tyson, 2015). Hermeneutic listening is productive (Hintz & Tyson, 2015); teachers and students create new understandings in unique interactions. Hintz and Tyson (2015) build upon Davis' (1997) listening categories to provide interactive themes that support a framework for complex listening: directing, purposefully amplifying, and being curious (Hintz & Tyson, 2015, p. 315). Next, I use the categories of listening (Davis, 1997) and themes of complex listening (Hintz & Tyson, 2015) to describe the listening processes of my participants.

Mrs. BB employs a sort of search-and-proceed listening process; she gathers information (searches) about her students' sense-making at a specific moment, usually prior to the initiation of instruction (versus in the moment of instruction, although these kinds of interactions also occur, just not as frequently). She then uses the student sense-making she has heard to consider her best strategies for helping students learn the content. I classified the majority of Mrs. BB's interactions as interpretive listening (Davis, 1997). She determines which interactions might yield the best results – or the best opportunity for students to learn – and teaches it (proceeds). This listening process builds upon Hintz and Tyson's purposefully amplifying. Mrs. BB acknowledges student sense-making to provide learning opportunities. A tension here is whether Mrs. BB should use an interaction that she has tried before and knows to work or should listen to more student sense-making and try something new. This is a challenging consideration for an elite/veteran teacher, especially when she believes she already possesses the best strategies for teaching most content.

Mrs. Branches' listening process requires adjusting her mind and her collection of similar instructional interactions to her teaching context. Mrs. Branches takes her time with this process and often attempts to listen to student sense-making on several occasions before sorting through her strategies and, finally, deciding how to respond. I think her listening process includes evaluative, interpretive and hermeneutic listening (Davis, 1997) and fits into the interactive theme of being curious (Hintz & Tyson, 2015), because she asks genuine questions about her students' sense-making .

Mrs. Kapeesh's process of listening and responding involves knowing what student sense-making she will listen for, listening for it, and instantly committing to a specific strategy that she selects prior to the interaction occurring in the moment. The majority of her listening is evaluative (Davis, 1997). Her process requires listening to student sense-making and directing students to pay attention to the sense-making she directs them to. She initially incorporates a cycle of noting when a strategy works and does not work, which is how she can determine what sense-making she listens for and how she will respond. I think her listening process builds on the interactive theme of directing students' listening (Hintz & Tyson, 2015).

When teachers listen to students as they work, what they hear can be used to inform the interactions they choose to have with their students (Hintz & Tyson, 2015). In each of these three case studies, the process of listening begins with the teacher noticing a student sense-making "trigger." My participants could not always narrate what that trigger was without reviewing transcripts or video of their instruction. However, they could begin to narrate their structure of decision making among their repertoire of stored strategies. Narrating the relationship between what sense-making they hear and their culminating response proved to be a challenge.

Ability to Narrate Interactions

Those who are considered the elite in their occupation may face challenges in sharing their instincts in a teachable way. Above, we consider how an individual at the top of their field may notice a diminishing ability to communicate what they understand to help others learn. Mrs. BB could narrate how she uses her general knowledge of how a student's number sense typically develops to determine how she interacts with that student. She was able to tell me about demonstrable student skills that she observed and grouped into categories of developing number sense to pre-determine the type of interactions she would have with students.

Mrs. Kapeesh could tell me what student sense-making she wanted to hear. What was more challenging to articulate was why her chosen interaction – providing direct instruction using traditional computational algorithms – was her interaction of choice. Mrs. Kapeesh could tell me about her listening process and the acknowledgment and rejection cycle but not why she knew it was the best learning opportunity to provide her students.

Mrs. Branches struggled to narrate why she did what she did. She told me about her moral responsibility to prepare contributing citizens. She was able to label her students as above, below, or on grade level. She did not name her work the re-humanization of mathematics, but she worked daily to end the hegemony she perceived in the educational system; she did so through listening to her students' sense-making and engaging in appropriate interactions based on what she heard.

The field of education may need to rely on narratives about context and processes of listening from teachers like those in this study to adequately prepare all teachers with the resources they need to be successful. How my findings among these three themes may contribute to our ability to address the challenges of listening to student sense-making well is the focus of my final chapter.

CHAPTER EIGHT

CONCLUSIONS

I spent six months in the field with my three participants. It is never easy to open oneself and one's classroom doors to anticipated scrutiny, and it was incredibly inspiring to work with individuals who were willing to narrate their work, review my writing and ensure it was their narrative, and confirm that the interpretations we made were true to their teaching contexts, even when what we uncovered was less than pleasant to work through. I am extremely grateful for the opportunity to have worked with individuals so dedicated to students and teaching; the parents, guardians, and students who allowed me to be a part of their educational journeys; and the school administration who supported me and the teachers who willingly gave their time to benefit education for all children. In my final chapter, I provide a brief summary of my key findings and share implications for scholarship in teaching, learning, and teacher education.

Key Findings

Teachers narrate instructional contexts in a variety of ways. In my research, teachers' autobiographical examinations provided an interpretive pathway for connecting their experiences, skills, self-evaluations, and motivations to the intent behind their interactions and listening.

Teachers could begin to narrate the steps in their listening processes, although the student sense-making that triggered these processes and the connection between the chosen interaction and the sense-making listened to was harder to narrate. Teachers described a search-and-proceed listening process, a contextual tuning and sorting listening process, and a commitment to specific interactions, based on pre-determined student sense-making criteria.

Finally, I learned that communicating an instinct for learning by another is less straightforward than it seems. Not every teacher can narrate the student sense-making they listen to or the relationships between the sense-making they hear and the interactions they choose.

Implications for Scholarship in Teaching, Learning, and Teacher Education

I return to my researcher positionality and a perspective of reflexivity to discuss the implications of my research. This research was, and continues to be, important to me, because of my status as a stakeholder in public schools as a parent and preparer of teachers. We need teachers who notice students' sense-making, instead of using categories of student descriptors to classify deficiencies. We also need teachers who can utilize methodology in pedagogically appropriate ways to adapt canned curriculum to the varied sense-making of our students. Our students are more than empty vessels with misconceptions to be identified and righted; we need teachers who offer learning opportunities to all students based on the student sense-making they listen to, interpret, and respond to.

I learned from my participating teachers that they first tune their brains to their teaching contexts. They recognize that they have a finite amount of time to recall strategies and content from the moment they listen to student sense-making. A cyclical process occurs between collecting student sense-making and recalling the student criteria that triggered an interaction, and a balance exists between impulsivity and overthinking (in which the tension between their best knowledge and gathering more student information is considered) before (sometimes instantly) committing to a specific response.

Teachers, researchers, policy makers, and other educational stakeholders could draw on my findings to reconsider the opportunities afforded to pre-service and current teachers to prepare for success with all students. My perspective is that teachers' opportunities to learn ought to be developmentally grounded, personalized, contextualized, and equitable. Nowhere is this more integral than in lesson planning. My new take on an old idea has the potential to prepare and develop teachers who enter the classroom ready to listen to students' sense-making.

Pre-service Teachers' Lesson Planning

Teachers must be given the opportunity to become masters of their craft. Preparing a lesson plan requires content and pedagogical knowledge. We expect students who are beginning to student teach to have some content and methodological knowledge, and we assist them in building a repertoire of pedagogical knowledge that helps them recognize why certain content, presented in a specific way, is appropriate at a particular moment. This begins with learning how to teach independently from a curriculum, which means a teacher must know how to lesson plan. A novice teacher's methodological preparation probably includes a focus on "core practices": practices that occur with high frequency in teaching and can be enacted in classrooms across curricula or instructional approaches (Grossman, Hammerness & McDonald, 2009, p. 277). While I disagree with providing teachers with a set of core strategies, because I consider this a crutch used in place of teaching student teachers and teachers how to listen to students' sense-making (bypassing the development of content and pedagogical knowledge), I contend that if we do teach a core practice, it ought to be knowing how to engage students in discussion. Teachers can learn how to support students to reason about central ideas, thereby deepening their own understanding of students' ideas (Franke, Kazemi & Battey, 2007). Teachers' facilitation of discussions that provide opportunities to learn is difficult work and varies across content and context. That said, some features of planning for and facilitating discussions are consistent with robust lesson planning. Identifying goals for a discussion (Kazemi & Hintz, 2014) can be aligned with selecting appropriate standards or benchmarks for a lesson and writing active student objectives to measure those goals. Anticipating student thinking, identifying key questions to pose connecting goals and knowledge of students' sense-making, and determining ways of representing sense-making (Kazemi & Hintz, 2014) align lesson preparation and instructional procedures.

Although my time working with pre-service teachers has been brief and limited to those in their first semester of student teaching in an elementary setting, I have learned that students have often failed to learn how to lesson plan and struggle to utilize standards to describe with clarity and specificity what student learning they are attending to. Student teachers are challenged by requests to lesson plan beyond the documentation of page numbers to be covered in the text, as scripted by a teacher's curriculum edition. Lesson planning is an opportunity for pre-service teachers to build their pedagogy. Instructors leading pre-service teacher fieldwork seminars have a duty to connect university coursework to clinical field service; nowhere is this more integral than in lesson planning.

University instructors must model powerful, substantial professional discourse that ties theory to practice. Lesson planning must be explicitly and concretely modeled and practiced. All teachers need to see and demonstrate how methodology, content, and pedagogy can and should be intertwined. Teaching requires teachers deep content knowledge, specialized content knowledge for teaching, and practical training to engage and empower all students. Teachers' initial field placements and seminars must focus explicitly on lesson planning that models how a merging of content, methodology, and pedagogy is possible and expected.

I suggest that a lesson plan template based on evaluative state teaching domains be designed and color coded by domain reference, so pre-service teachers can visualize how the lesson plan is a thorough representation of what is expected for each instructional interaction

Each lesson plan ought to be guided by a framework for planning and facilitating purposeful dialogue that enriches and deepens student and teacher learning. Pre-service teacher educators also have the responsibility to observe student teachers' actual instruction using these lesson plans and provide timely and robust feedback, with the opportunity for student teachers to reflect upon and then edit those original plans. Student teachers should focus on what they learned about their students' sense-making strategies in their reflection, as well as how they can utilize and build upon what they learned for use in future contexts. This is how pre-service teachers begin building the pedagogical content knowledge necessary for listening to students' sense-making: by purposefully aligning content with dialogue so they are forced to anticipate and consider how students might make sense of the content and how they can respond to it. Once lesson planning considers the sense-making of the learner, a teacher is cognizant of student sense-making. A teacher can listen to student sense-making and practice using it to provide equitable opportunities to learn.

Teaching

Teachers need to listen to *how* students understand content. I have argued that teachers must listen to students' sense-making; however, listening alone is not sufficient. Teachers must be prepared to teach their content, so when they hear students' thinking, they can determine when it is appropriate to use manipulatives, introduce a formula, or engage the class in student-led dialogue. Teachers may pair their knowledge of content and students with a repertoire of strategies to address the standards and assist them in responding to student sense-making (which may, or may not, follow developmental patterns). I believe it is fair to state that every teacher, particularly as it relates to the teachers in my research, notices (possibly including, but not limited to, listening to) students' sense-making. The challenge (particularly when teachers become cooperating or master teachers for student teachers) is that the process of listening, interpreting, and responding becomes so ingrained, so contextual, that the teacher need no longer write out the extensive narrative – nor can they recall the moment – in which that seamless process occurred. How can we take what I have found here – that teachers tune their brains to their teaching contexts; that they recognize the finite amount of time they possess to recall strategies and content, from the moment they listen to student sense-making; that a cyclical process of collecting student sense-making and recalling the student criteria that triggered an interaction exists, as well as a balance between impulsivity and overthinking before instantly committing to a specific response – and articulate that moment in a manner that student teachers can learn from and is recognizable for teachers to narrate. This is a significant barrier to overcome, and part of the solution may be embedded in the selection of master/cooperating teachers.

What we can learn from teachers' narrative abilities. The teachers in my research had varying levels of ability in recounting their procedural knowledge (the steps they conducted in instruction), as well as varying levels of success in narrating their declarative knowledge (statements of fact regarding the relationships between what they knew about their students and why they acted as they did in the moment). I think the variability of their narratives was impacted considerably by their contexts. I have carefully described each participant's self-identified context (self-evaluation, skills, experiences, and motivation) so we can consider how this insight could impact who universities select to be master and cooperating teachers. Next, I explain how teachers' experiences, skills, self-evaluation, and motivations should factor into our mentor teacher selection.

Experiences. The licensure system in the Southwestern state where my research occurred dictates that a Level III teacher (one who has earned a master's degree or National Board Certification and demonstrated that they meet the increased teaching competencies by submitting a Professional Development Dossier to the Public Education Department) is a master teacher who not only possesses a robust number of years (up to five years as a Level I, at least three as a Level II) in a teaching context but additionally has learned instructional techniques, strategies, and assessments appropriate for all students in each of the content areas taught in elementary, or in the main content area in secondary, school classrooms. Our state has identified a subset of teachers (elite Level III teachers) who can be called upon to mentor. Among these elite, we must select teachers who may be able to narrate their teaching interactions (although I must interject and suggest that there are probably Level I and Level II teachers who could narrate their procedural and declarative knowledge as well, if not better; I return to this shortly). We could utilize evaluations of these elite teachers to determine their range of skills.

Skills. Teachers are evaluated through a four-tiered system of domains: planning and preparation (Domain One), creating an environment for learning (Domain Two), teaching for learning (Domain Three), and professionalism (Domain Four). Subsumed in each domain are elements that ought to be observable in teachers' instruction. These criteria are objectively scored in evaluative categories: ineffective, minimally effective, effective, highly effective, and exemplary. We assume that a Level III teacher would typically score in the highly effective or exemplary range for basic teaching skills, such as including all students in learning; modifying instruction according to individual educational plans; adjusting instruction for interest, needs, language demands, and appropriate grade-level content; and adjusting lessons when checking for understanding, among a multitude of others. Yet, it is important to note that we still request that the elite – those teachers who may struggle the most to dictate their practice – share their insight with the newest of the teaching profession, those who have the most to learn. These student teachers might be better prepared for their teaching contexts if paired with teachers (perhaps a subsection of Level I and II teachers who are still deep in the process of honing their craft) who can disseminate the thinking processes they employ to utilize their procedural and declarative knowledge to inform their instruction. I do not mean to suggest that no Level III teachers can explain their thinking or that all Level I or II teachers are intrinsically better at narrating their listening. However, an opportunity exists here to consider why we use only content knowledge and experience to identify our master mentor teachers. We must utilize teachers' own evaluations of self – the third of the four components that comprise a teacher's context – as a nomination criterium to adopt the critical role of narrator.

Teacher self-evaluation. I had initially planned to use a peer-nomination approach to recruit participants who embodied the traits of a master teacher, as perceived by other teachers (see Chapter Two). Fortunately for my research, the impasse that occurred with my original plan led to the opportunity for teachers to self-nominate. Self-evaluation is difficult. At the very least, we undervalue – and at the most, over-estimate – our abilities. In either case, teachers willing to open the classroom to the purview of others probably realize that they have practitioner-related insights that could contribute to the development of a novice teacher. That offer should not be overlooked simply due to a licensure level. Perhaps licensure level should be one method to identify master teachers, in conjunction with peer nominations, administrative evaluation review, and university observation. I also wish emphasize the importance of teacher–university faculty liaison. Teachers and university faculty aspire to provide the best possible learning opportunities for our student teachers; our motivations are similar, which leads to my final point.

Motivation. Each of my participants had a reason for teaching how they did: to prepare contributing citizens, develop students who can use mental math to compute, and encourage students to adopt a growth mindset. More broadly, I believe all teachers have driving forces behind their instructional techniques, whether we can verbalize them or not. We are well aware that our students and student teachers vary in experiences, skills, evaluation of self, and motivations, all of which impact interactions. We have those students who will, students who cannot, and students who will not. Yet, we persevere, because listening helps us understand something about our learners, and because when we listen, we all gain new opportunities to make sense of what we hear, utilize our content knowledge, and develop pedagogical skill.

Summary

Teachers – both student and practicing – must learn how to facilitate interactions that build upon students' sense-making and canonical content concepts. Teachers need opportunities to learn – and practice – both listening to and being responsive to student sense-making and using student sense-making to plan and enact a lesson. Universities must focus on explicitly modeling lesson planning strategies to demonstrate how listening to student sense-making, content knowledge, and pedagogical knowledge is deeply intertwined in fieldwork and seminar.

Closure

I am excited to see where this conversation leads. I am sure we do not have all the pieces identified yet, but the fact that we are reflecting together and documenting authentic classroom instruction truly based on teacher listening is invigorating. We still have much to learn from teachers' narratives of listening to student sense-making. I am all ears.

Appendices

Appendix A

Modified Autobiographical Examination of Self

Research Questions

1. How do teachers adjust their instruction in response to student variables and feedback in elementary mathematics?
2. What student variables and feedback do teachers find most relevant?
3. How do teachers organize and bring to bear multiple types of knowledge in their responses to students?

I. Regressive Step Questions: The object of this part of the method is to observe oneself functioning in the past. The focus is educational experience.

1. Tell me about your elementary school experiences. Middle school experiences? High school experiences? College experiences?

   Follow-up: What memories do you have?

   Is there is teacher you recall?

   Is there a particular class/room you recall? What did you do?

   What kind of student do you think you were?

   How would you describe yourself as a student looking back?

II. Progressive Step Questions: Think about the future, of tomorrow, next week, the next few months, of the next academic year, of the next three years…bring attention back to intellectual interests, career.

1. Describe your ideal teaching context. Imagine an ideal teaching experience.

   Follow-up: What does the classroom look like?

   Describe the students in your classroom.

What curriculum do you use? Why?

What is your relationship with your colleagues? Administration? Parents? Students?

III. Analytic Step Questions: The present.

1. Describe your present teaching context. Describe your teaching reality.

Follow-up: What does your classroom look like?

Describe the students in your classroom.

What curriculum do you use? Why?

What is your relationship with your colleagues? Administration? Parents? Students?

IV. Synthetic Step Questions:

1. Tell me why you think your ideal teaching experience and your teaching reality are different? The same? (based on prior responses)

Appendix B

Interview Protocol

Review research questions (to align observations with questions):

1. How do teachers narrate their organization and appropriation of student mathematical thinking and informal math knowledge in their interactions with elementary math students?
2. What student mathematical thinking and informal math knowledge do teachers notice?
3. How do teachers recognize and identify common patterns of student thinking in math?
4. How do teachers elicit and interpret individual students thinking?
5. How do teachers identify and implement an instructional strategy/intervention in response to common patterns of student thinking?

Interaction-focused Questions:

1a. Tell me about the concrete details of this student interaction (show video clip and/or transcripts).

1b. Tell me about the meaning of this interaction (show video clip and/or transcripts, as well as Interview 2 transcripts as needed for Interview 3).

1c. Tell me about this student interaction? Why did you do X? Have you had other experiences with X? Did you do the same or different things? Why?

1d. What do you know about this student? what do you know about the content? Why did you say/do X? Have you had other experiences with X, or what lead up to X?

1e. Why did you ask this question/say this? What student feedback are you listening for? What do you do with that knowledge? Why? Tell me about experiences that have led you to use these questions/responses instead of others.

1f. What information do you need to have about your students to respond to (refer to clip)? What student feedback do you use to determine how to respond?

Follow-up Questions

1. What prior knowledge do you have about your student academically? Socially?
2. What have you learned about teaching this individual/this group of students from past or present teaching experiences?
3. Have you identified any areas of student confusion?
4. What have you learned, do you know, or hope to learn from in-class work/homework?
5. Ask teacher to tell stories of their teaching experiences, past and present. Are there recurring themes/patterns among students?
6. What types of decisions have you made based on prior knowledge and teaching experiences?

Appendix C

Observation Protocol

Review research questions (to align observations with questions):

1. How do teachers narrate their organization and appropriation of student mathematical thinking and informal math knowledge in their interactions with elementary math students?
2. What student mathematical thinking and informal math knowledge do teachers notice?
3. How do teachers recognize and identify common patterns of student thinking in math?
4. How do teachers elicit and interpret individual students thinking?
5. How do teachers identify and implement an instructional strategy/intervention in response to common patterns of student thinking?

What to observe:

1. **Video tape the entirety of the lesson**
2. **Capture student thinking (verbal or nonverbal) that precedes teacher response**
3. **Capture teacher response**
4. **Complete 1–3 as many times as possible during the math lesson**

A. Notice rhythm and cycles of math class:
   1. How students indicate the desire to participate
   2. Classroom management style/technique
   3. How instruction is implemented
      i. Size of instructional group
      ii. Seating arrangements
      iii. Classroom setup
      iv. Note pacing of lesson (time of lesson segments)
   4. Teacher and student movement around the classroom
   5. Classroom network model that depicts number/type of interactions teacher has with classroom students

B. The fraction content being presented:
   1. Teacher use of lesson plans and/or teacher's edition (from Appendix E), coherence to or deviation from
   2. Use Lampert's (2001) strategic site three-pronged model to represent the interactions
      i. Note time for video
      ii. Desk # location(s) for student(s) and student(s) response(s) (verbal/nonverbal/representation)
      iii. Content covered (common code standard #)
      iv. Teacher response
      v. Note use of gesture or representation
   3. Concept map of mathematical content being communicated by the teacher
      i. Diagram how teacher moves through the content with different student

Follow-up:

1. **Note the time of the initiation of the interaction and the duration/conclusion**
2. **Note the desk location of the student in the interaction, and whether the response was verbal or nonverbal (so that student representation/work can be referenced with the teacher in Interviews 2 and 3)**
3. **Note if diagrams, manipulatives, or bulletin boards around the room were referenced in the interaction (they may be off camera)**
4. **Diagram teacher generated representations if they are used**
5. **Note teacher pauses, postures, nodding, smiling, frowning – these may be moments when the teacher is thinking about something before their response Note the time of these gestures/movements**
6. **Note the curriculum/lesson plans used or references in the interaction**

Appendix D

How do teachers notice students' mathematical thinking?

Hi! My name is Allison Stansbury. I am a doctoral candidate in UNM's Teacher Education, Educational Leadership and Policy Program in the College of Education. I am studying what teachers notice about their students' mathematical thinking during fraction instruction. I am looking for elementary teachers who are willing to participate in an initial interview (90 minutes), the observation of 2-4 mathematics instruction periods (when instruction is focused on fractions), and two follow up interviews during which I will ask you to narrate the interactions you have with your students (each interview is 90 minutes long). Please complete the following survey to the best of your ability if you think you may be interested in participating in this research. I appreciate your time and dedication to teaching. I look forward to hearing from you! Thank you.

* Required

I strategically choose and use math representations and examples to build students' understanding and remediate misconceptions, use math language carefully, highlight core ideas, and make my mathematical thinking visible. *

	0	1	2	3	4	5	
Strongly Disagree	○	○	○	○	○	○	Strongly Agree

I carefully phrase questions that elicit, probe, and advance students' thinking about math content. *

	0	1	2	3	4	5	
Strongly Disagree	○	○	○	○	○	○	Strongly Agree

I pose questions or tasks that provoke students to share their thinking about specific academic content to evaluate student understanding, guide my instructional decisions, and surface ideas that will benefit other students. *

	0	1	2	3	4	5	
Strongly Disagree	○	○	○	○	○	○	Strongly Agree

I am familiar with common patterns of student thinking and development and am fluent in anticipating or identifying them as I implement instruction and evaluate student learning. *

	0	1	2	3	4	5	
Strongly Disagree	○	○	○	○	○	○	Strongly Agree

I know specific instructional strategies that are effective in response to particular common patterns of student thinking and use them to support, extend, or begin to change student thinking. *

	0	1	2	3	4	5	
Strongly Disagree	○	○	○	○	○	○	Strongly Agree

Provide an example of when you asked a student to explain their thinking. What did you learn (if anything) from your student's explanation? *

Your answer

Have you ever noticed a pattern in how students thinking in math develops? If so, please describe what you have noticed. *

Your answer

Provide an example of a time you adjusted math instruction to meet the needs of your learners. *

Your answer

Please share the number of years you have been teaching.

Your answer

Please share the grade you teach.

Your answer

Are you interested in participating in my research which seeks to understand what teachers notice about their students' mathematical thinking? *

- ◯ Yes
- ◯ Maybe
- ◯ No

If you answered yes or maybe above, please indicate your name, phone number, and a time/ day of the week I may contact you, to share more information about my research. *

Your answer

Appendix E

THE UNIVERSITY *of* NEW MEXICO

Elementary Math Teachers' Narrated Noticing

Seeking people who teach elementary math in a public or charter school setting to participate in a research study.

The purpose of this study is to look at the complex ways teachers use knowledge in mathematical interactions with their students.

If you decide to join the study, you will be asked to:

- Complete an online survey
- Participate in three (3) 90 minute interviews
- Be observed during classroom math instruction 2-4 days

This study is being conducted by Allison Stansbury (ABD) from UNM.

Contact Information: Please call 505-385-0105 or email amoch@unm.edu to get more information.

Leave space for IRB Stamp

Appendix F

WHEN REACHING AN ANSWERING MACHINE OR VOICE MAIL
DO NOT LEAVE TELEPHONE MESSAGES REGARDING RESEARCH RECRUITMENT

IF SOMEONE OTHER THAN PARTICIANT ANSWERS THE PHONE
Hello,

Am I speaking to (*potential participant*)?

- *If NO, ask if the desired person is available. If not available, then indicate you will call back, say Thank You and hang up. Do not provide any information that might violate the potential subject's privacy.*

ONCE THE POTENTIAL PARTICIPANT IS ON THE LINE
Hello,

Am I speaking to (*potential participant*)?

If YES, then continue:

My name is Allison Stansbury. I am a researcher at the University of New Mexico. I am doing a study about teacher knowledge use in mathematics. I am contacting you because I received a message from you that you might be interested in this study and I wanted to call you back to talk to you about the study.

May I have your permission to talk to you about my study?

- *If no, say Thank you for your time and end the call.*
- *If yes, continue as below.*

The purpose of this research is to learn about the types of knowledge you use when you respond to your students' participation during math instruction.

If you agree to participate, this study will involve an initial interview where you tell me about your past, present, and future educational experiences. This will take about 30-45 minutes. The initial interview will be followed by 10 consecutive academic days of me observing your math instruction, during the first or second trimester of the school year. I will observe and audio record your instruction during the first five days to better understand your teaching context. I will pay close attention to the interactions you have with students during the second five days. We will schedule 2 interviews closely following the second five days of instruction, dependent upon your schedule, to discuss the various interactions you have with your students while you teach. These interviews may take 45-60 minutes in total. Additionally, I will ask for a copy of your math lesson plans, and blank copies of all instructional worksheets.

The benefit of participating in this research is that you will have the opportunity to analyze your practice. Your narratives of how you use your knowledge to respond to students during math instruction will enable us to better understand the complexities of practice, as well as develop a language of analysis that can describe and analyze said knowledge and practice.

There is a risk of becoming upset, stressed, or fatigued as a result of participating in my research. There is also a minimal risk that data could be disclosed outside the research setting. I have taken steps to minimize this risk; you will choose a pseudonym, I will remove all identifiers as quickly as possible, paper data will be stored in a keyed file cabinet, and electronic data documents will be password protected, as well as the computer itself.

You do not have to be in this study, your decision to be in any study is totally voluntary.

Do you have any questions? (*Answer any questions*)

"OK very good. Are you interested in being part of this study?

- *If no, say Thank you for your time and end the call.*
- *If yes then set up appointment for an initial visit with participant*

Appendix G

THE UNIVERSITY of NEW MEXICO

ELEMENTARY MATH TEACHERS' NARRATED NOTICING
Consent to Participate in Research
10/30/17

Purpose of the study: You are being asked to participate in a research study that is being done by Allison Stansbury (ABD) from the Teacher Education, Educational Leadership, and Policy Department. The purpose of this study is to understand how teachers adjust instruction in response to students' mathematical thinking. My goal is to describe how teachers organize and bring to bear multiple types of knowledge in their responses to students. You are being asked to take part in this study because you are a public/ charter school elementary teacher who teaches math.

This form will explain what to expect when joining the research, as well as the possible risks and benefits of participation. If you have any questions, please ask me.

What you will do in the study: Participation will occur during the first, second, or third trimester of the academic year. The study begins with a 90 minute interview in which you share your educational past, present and future. You can skip any question that makes you uncomfortable and can stop the interview at any time. Next, we will schedule 2-4 consecutive days of observation. I will observe your math instruction, as well as video record it. I will pay close attention to the interactions you have with students. We will schedule two more 90 minute interviews. The second will be after your observation period, the third, 3-5 days after the second interview. Again, you can skip any question that makes you uncomfortable and can stop the interview at any time. Additionally, I will ask for a copy of your math lesson plans, and blank copies of all instructional worksheets for all days I am observing your math instruction. Participation in this study will take a total of 5.5 hours or less of observation over a period of 2-4 academic days and 4.5 hours of interviewing.

Risks: There are risks of stress, emotional distress, inconvenience and possible loss of privacy and confidentiality associated with participating in a research study.

Benefits: You will have the opportunity to analyze your practice. Additionally, I hope that your narratives of how you use your knowledge to respond to students during math instruction will enable us to better understand the complexities of practice, as well as develop a language of analysis that can describe and analyze said knowledge and practice.

Confidentiality of your information: I will ask you to choose a pseudonym. I will remove all identifiers as quickly as possible (immediately following all observation periods). Paper data will be stored in keyed file cabinets in my home office. Electronic data documents will be password protected. My computer will be password protected as well. I will take measures to protect the security of all your personal information, but I cannot guarantee confidentiality of all study data. The University of New Mexico Institutional Review Board (IRB) that oversees human subject research and/ or my dissertation chair may be permitted to access your records. Your name will not be used in any published reports about this study.

You should understand that the researcher is not prevented from taking steps, including reporting to authorities, to prevent serious harm of yourself or others.

Payment: You will not be paid for participating in this study.

Right to withdraw from the study: Your participation in this study is completely voluntary. You have the right to choose not to participate or to withdraw your participation at any point in this study without penalty. Your observation data, interview data, and audio data will be destroyed should you choose to withdraw.

If you have any questions, concerns, or complaints about the research study, please contact:

Allison Stansbury: 505-385-0105. amoch@unm.edu

If you would like to speak with someone other than myself to obtain information or offer input or if you have questions regarding your rights as a research participant, please contact the IRB. The IRB is a group of people from UNM and the community who provide independent oversight of safety and ethical issues related to research involving people:

UNM Office of the IRB, (505) 277-2644, irbmaincampus@unm.edu. Website: http://irb.unm.edu/

CONSENT

You are making a decision whether to participate in this study. Your signature below indicates that you have read this form (or the form was read to you) and that all questions have been answered to your satisfaction. By signing this consent form, you are not waiving any of your legal rights as a research participant. A copy of this consent form will be provided to you.

I agree to participate in this study.

______________________	______________________	______
Name of Adult Participant	Signature of Adult Participant	Date

Researcher Signature (to be completed at time of informed consent)

I have explained the research to the participant and answered all of his/her questions. I believe that he/she understands the information described in this consent form and freely consents to participate.

______________________	______________________	______
Name of Research Team Member	Signature of Research Team Member	Date

Appendix H

THE UNIVERSITY of NEW MEXICO

/

ELEMENTARY MATH TEACHERS' NARRATED NOTICING
Informed Consent for Parents/ Guardians
10/30/17

Allison Stansbury, from the Department of Teacher Education, Educational Leadership and Policy is conducting a research study. The purpose of the research is to understand how teachers adjust instruction in response to students' mathematical thinking. The research is focused on the teacher and his/ her narratives of their interactions with students. However, your students' responses (verbal and nonverbal) will precede all these interactions. You are being asked to provide consent for your student/s verbal and nonverbal responses in math to be used in this study.

The purpose of the observation of students' responses during mathematics instruction is to use this data with the teacher during interviews. Students will only be identified by a number with their response to be used as a prompt in teacher interviews. Other student variables identified by the teacher will be described as part of a knowledge pattern, and not linked to an individual student. This data will be used to provide a context for understanding the described teacher knowledge pattern. Your child's teacher will read the analysis of data to ensure that no student is identifiable.

I will be in your students' classroom for two to four academic math classes. Your child's involvement in the study is voluntary, and you may choose to not have your child participate. If you choose to not have your child participate, your child's responses will not be recorded for use in teacher interviews. All parent/ guardian consent/ dissent forms will be stored in a keyed file cabinet. A second keyed file cabinet will be used to store all observations from your child's classroom. Your child's teacher has chosen a pseudonym, and all student responses will be stored in this file. All data and consent forms will be destroyed a year after the completion of my dissertation.

The findings from this project will provide information on how teachers use different types of knowledge to respond to students during math instruction. We will be able to better understand the complexities of teaching, as well as develop a language of analysis that can describe and analyze teaching.

If you have any questions about this research project, please feel free to call Allison Stansbury at 505-385-0105. If you have questions regarding your child's rights as a research subject, or about what you should do in case of any harm to your child, or if you want to obtain information or offer input you may call the UNM Office of the IRB (OIRB) at (505) 277-2644 or irb.unm.edu.

By signing below you will be providing consent for your child to participate in the above described research study.

______________________________	______________________________	________
Name of Child Participant	Signature of Adult Participant	Date
______________________________	______________________________	________
Name of Research Team Member	Signature of Research Team Member	Date

Appendix I

THE UNIVERSITY of NEW MEXICO

ELEMENTARY MATH TEACHERS' NARRATED NOTICING
Assent to Participate in Research
10/30/17

You are being asked to join a research study by Allison Stansbury from the Department of Teacher Education, Educational Leadership and Policy. The purpose of this study is to understand how teachers teach.

If you join the project, you will be asked to participate in math class the way you usually would. If you join, there may be some risks, bad things that happen, like feeling stressed while I am in your classroom. There may also be some benefits, or good thing that happen, like learning more about what your teacher knows that helps you learn.

If you do not want to join the project, please do not sign or return this form. I will not write down anything you say or write while I am in your classroom. If you want to be a part of the research, please sign and return this form.

I have also sent a parent/ guardian consent form home. Please talk with your parents/ guardians before you decide to join or not join this study. I will also ask your parents/ guardians to sign a consent form if they want you to be in this study.

If you have any questions at any time, please call or email Allison Stansbury at 505-385-0105. If you would like to talk to someone else, you can call the Office of the IRB at (505) 277-2644 or email at IRBMainCampus@unm.edu.

You do not have to be in this study. If you do choose to be in the study, you can change your mind at any time. I won't care if you change your mind or if you don't want to join this study.

Signing this form means you have read this form and all of your questions have been answered. You and your parents/ guardians will be given a copy of this form.

I agree to join this study.

______________________________	______________________________	______
Name of Child Participant	Signature of Child Participant	Date

Researcher Signature (to be completed at time of informed consent)

I have explained the research to the participant and answered all of his/her questions. I believe that he/she understands the information described in this consent form and freely consents to participate.

______________________________	______________________________	______
Name of Research Team Member	Signature of Research Team Member	Date

Appendix J

Differentiated Lesson Plan Template & Instructions

1. General Lesson Information

- Teacher Candidate Name:
- Date & Timeframe of Lesson:
- Title:
- Grade Level:
- Subject Area:

2. Standards & Benchmarks

Provide the number and written standard for:

- Standard(s) appropriate to the lesson:
- Standard(s) fully cited:
- Rationale for selecting standard:
  [Standards should come from one of the following: 1) Common Core State Standards (for Math or English Language Arts); 2) New Mexico State Standards (Science, Social Studies), 3) New Mexico Expanded Grade Band Expectations/Performance Standards (for students with significant disabilities), 4) NM CCC for ECE, and/or 5) approved by instructors]
- **EFFECTIVE: The teacher's lesson plans and resulting lessons: DEVELOP INSTRUCTION THAT REFLECT SOLID KNOWLEDGE OF THE CONTENT AREA AND ACADEMIC LANGUAGE DEMANDS AT A GRADE APPROPRIATE LEVEL; ARE DIRECTLY ALIGNED TO ALL NM ADOPTED STANDARDS

3. Specific Learning Goals & Objectives of the Lesson

What do you want the student to learn during this lesson?

- Goals & objectives appropriate to lesson/activity that are measurable and observable
- Goals & objectives clearly link to standards
- Goals & objectives are directly linked to and measured by the lesson's assessment(s)
- Specific IEP goals & objectives for students with disabilities
- Specific goals for second language learners

4. Preparation

Materials and resources

List specific materials and resources you will need, including adapted materials and assistive tools, etc.

The Environment

Explain any special considerations you might need when setting up the classroom environment (i.e., how will students be grouped? Special arrangements?)

- EFFECTIVE: * Classroom interactions between teacher and students and among students: EXHIBIT POLITENESS AND RESPECT; CREATE AND MAINTAIN AN ENVIRONMENT IN WHICH STUDENTS' DIVERSE BACKGROUNDS, IDENTITIES, STRENGTHS, AND CHALLENGES ARE RESPECTED; AND DISPLAY RESPECT AND VALUE FOR THE LANGUAGES AND CULTURES OF THE SCHOOL COMMUNITIES THROUGH CLASSROOM ARTIFACTS AND INTERACTIONS.

- EFFECTIVE: *The teacher organizes the classroom in such a way that: ESSENTIAL LEARNING IS ACCESSIBLE TO ALL STUDENTS; THE PHYSICAL ARRANGEMENT ENCOURAGES TEACHER[STUDENT AND STUDENT–STUDENT INTERACTION IN A VARIETY OF SETTINGS AND STUDENT GROUPINGS; EVIDENCE OF STUDENT LEARNING IS POSTED; LEARNING OUTCOMES ARE POSTED AND EASILY ACCESSIBLE DURING THE LESSON FOR THE TEACHER AND STUDENTS TO REFERENCE; VISUALS, GRAPHICS, ANCHOR CHARTS, AND TECHNOLOGY ARE READILY ACCESSIBLE TO ENHANCE LEARNING OPPORTUNITIES; AND PROVIDES TECHNIQUES TO ENHANCE LEARNING OPPORTUNITIES

Diversity

- Address diversity considerations including language, second language, culture, community assets, etc.
- EFFECTIVE: * The teacher: CONDUCTS EFFECTIVE COMMUNICATION FROM SCHOOL-TO-HOME ABOUT APPROPRIATE STUDENT PROGRESS; COMMUNICATES IN A MANNER THAT IS CULTURALLY SENSITIVE, RESPONSIVE, AND AFFIRMS THE POSITIVE WORTH OF STUDENTS AND FAMILIES; USES CLEAR, ACCURATE, AND UNDERSTANDABLE LANGUAGE.
- Describe classroom demographics and present levels of performance (academic, behavior, language)
- EFFECTIVE: * The teacher: PROVIDES APPROPRIATE INFORMATION SUCH AS STUDENT STRENGTHS, WEAKNESSES, PREFERRED MODALITIES, ENVIRONMENTAL MODIFICATIONS
- EFFECTIVE: *The teacher designs standards of conduct that: ARE SUPPORTED BY AN EFFECTIVE STUDENT BEHAVIOR MANAGEMENT PLAN WITH STUDENTS' KNOWLEDGE OF THEIR ROLES; CREATE AN ATMOSPHERE CONDUCIVE TO LEARNING WITH A FOCUS ON SELF-DISCIPLINE, RESPECT FOR THE RIGHTS OF OTHERS, AND COOPERATION; ARE COMMUNICATED CLEARLY AND MODELED TO ALL STUDENTS; AND ENSURE RESPONSES TO STUDENT MISBEHAVIOR ARE CONSISTENT, RESPECT THE STUDENTS' DIGNITY, ARE SENSITIVE TO CULTURAL DIFFERENCES, AND ARE IN ACCORDANCE TO THE STUDENT'S FBA/BIP STRATEGIES, WHEN APPLICABLE.

Differentiation

How have you specifically differentiated for CONTENT, PROCESS, and PRODUCT as a way of maximizing student engagement in, participation in, and access to what is being taught and learned throughout the lesson?

Technology

Describe and explain use of technology that is needed for teaching, learning, and assessment.

Collaboration

Provide evidence of collaboration between general ed, special ed, bilingual ed, art ed, and/or others

- EFFECTIVE: * The teacher: PARTICIPATES IN INTERACTIONS WITH COLLEAGUES THAT ARE CHARACTERIZED BY A WILLINGNESS TO LISTEN AND CONSIDER MULTIPLE POINTS OF VIEW; CONSULTS WITH APPROPRIATE PERSONNEL ABOUT INSTRUCTIONAL, ENVIRONMENTAL, AND BEHAVIORAL MODIFICATIONS ABOUT INSTRUCTIONAL STRATEGIES FOR CULTURALLY AND LINGUISTICALLY DIVERSE STUDENTS.

5. Instructional Procedures (Flexible depending on preferred content model of teaching)

Introduction (Time Allotted)

- What information will you be introducing AND what will be your essential question for this activity?

  EFFECTIVE: * The teacher's questioning techniques: PROVIDE FREQUENT OPPORTUNITIES FOR INTERACTION BETWEEN TEACHER AND STUDENT; USE SCAFFOLDS; USE PRE-PLANNED QUESTIONS/TASKS; CONSISTENTLY ENGAGE STUDENTS; ALLOW STUDENTS TO RESPOND IN A VARIETY OF WAYS.
- Provide a "hook" for the students to raise their interest and draw them into the lesson.
- How will you connect with students' prior knowledge and backgrounds?
- How will you address the language demands of the content?

  EFFECTIVE: * Activities, assignments, materials, pacing, and grouping of students are fully appropriate to the learning outcomes resulting in good student engagement: THE TEACHER CONNECTIONS THE LESSON TP PRIOR UNDERSTANDING AND STUDENT BACKGROUND EXPERIENCE; THE LESSON SUPPORTS ACTIVE ENGAGEMENT OF ALL STUDENTS; THE TEACHER DELIVERS LESSON COHERENTLY; THE TEACHER INCORPORATES EXPERIENCES TO SUPPORT LEARNING; THE TEACHER ADAPTS METHODS FOR IMPROVED LEARNING; STUDENTS ARE STRATEGICALLY GROUPED.
- How will you differentiate for CONTENT? EFFECTIVE: * The teacher: PROVIDES SPECIFIC SUGGESTIONS AS TO HOW INSTRUCTIONAL PRACTICES MIGHT BE IMPROVED, BASED ON STUDENTS' PROGRESS AND USE OF ASSESSMENT DATA.

Instruction (Time Allotted) EFFECTIVE: Little instructional time is lost because of: ESTABLISH ROUTINES AND PROCEDURES THAT ARE DEVELOPMENTALLY APPROPRIATE FOR ALL STUDENTS AND MAY INCLUDE MODIFYING SPEECH AND WAIT TIME TO ENSURE UNDERSTANDING OF THE ROUTINES; ROUTINES AND PROCEDURES THAT ARE DESIGNED TO KEEP STUDENTS' INTEREST, MAXIMIZE LEARNING, AND ASSIST IN TRANSITIONS; AND IMPLEMENTATION OF A WELL-ORGANIZED SYSTEM FOR ACCESSING MATERIALS, INCLUDING SUPPLIES AND MANIPULATIVES.

- How specifically – step by step – will you present the information and carry out this lesson?
- EFFECTIVE: * The communication and delivery of expectations for learning, directions, procedures, and explanations of content with students include: DESIRED LEARNING OUTCOMES POSTED AND REFERRED TO DURING THE LESSON CYCLE; USE OF CLEAR COMMUNICATION AND A RANGE OF VOCABULARY WITH SCAFFOLDS TO ENSURE LEARNING OUTCOMES ARE UNDERSTANDABLE, INCLUDING THE SOLICITATION OF FEEDBACK AND ALLOWING FOR CLARIFICATION FROM ALL STUDENTS BY USING MULTIPLE STRATEGIES; INSTRUCTIONS AND PROCEDURES THAT ARE CONSISTENT AND ANTICIPATE POSSIBLE STUDENT MISCONCEPTIONS; CONTENT THAT IS DELIVERED AND DIFFERENTIATED BY INCLUDING THE STUDENT OF THE STUDENTS ACADEMIC ENGLISH LANGUAGE PROFICIENCY LEVEL AND/OR IEP GOALS, AS APPLICABLE; USE OF OPPORTUNITIES TO CONNECT TO STUDENTS CULTURAL AND LINGUISTIC BACKGROUND KNOWLEDGE.
- What specifically will you do to make sure students understand, engage in and apply what they are learning?
- How will you differentiate for CONTENT and PROCESS?

Guided/Independent Practice (Time Allotted)

- What activities will allow the students to independently use/apply the information they have learned (i.e., learning centers, additional projects/activities, worksheets, homework, etc.)?
- How will you differentiate for PROCESS and PRODUCT?

	• EFFECTIVE: *Activities, assignments, materials, pacing, and grouping of student are fully appropriate to the learning outcomes, language proficiency levels, and applicable IEP goals, resulting in good student engagement in which: THE TEACHER EXPLICITLY CONNECTS THE LESSON TO PRIOR UNDERSTANDING AND STUDENT BACKGROUND EXPERIENCE; THE LESSON SUPPORTS ACTIVE ENGAGEMENT OF ALL STUDENTS AND MAINTAINS AN AWARENESS OF THE EFFECTIVE AMOUNT OF STUDENT TALK VS. TEACHER TALK; THE TEACHER DELIVERS LESSONS COHERENTLY WITH ATTENTION TO SCAFFOLDING, PACING, SEQUENCING, FLEXIBLE GROUPING, STUDENT REFLECTION, AND CLOSURE; THE TEACHER INCORPORATES COGNITIVE, DEVELOPMENTAL, LINGUISTIC, AND CULTURAL EXPERIENCES TO SUPPORT LEARNING **Closure** • How will you summarize or close the lesson? • What questions will you ask for clarity of comprehension?
6.	**Assessment**
	• How will you use formative assessment to inform your teaching and check for understanding throughout the lesson? • EFFECTIVE: * THE TEACHER ASSESSES STUDENT ENGAGEMENT AND UNDERSTANDING AND ADAPTS METHODS OF IMPROVED LEARNING WHEN NEEDED • Assessment tools have been clearly identified/created. This tool accurately measures and records student performance aligned to NM State Standards, IEP Goals, and language proficiency levels (rubric, checklist, student work sample/portfolio, self-assessment, etc.). ***Be sure to clearly describe the assessment being used and include actual assessment tools with your lesson***. • EFFECTIVE: * The teacher: ENSURES THAT GRADING, ASSESSMENTS PRACTICES, AND RECORD-KEEPING SYSTEMS ARE EFFECTIVE IN SERVING ACADEMIC LEARNING GOALS. • Assessment links directly back and is appropriate to stated goals and objectives of the lesson. • Assessment establishes a clear and measurable "criteria for success" (80% accuracy, 5/10 trials, etc.). • EFFECTIVE: * Assessments are consistently used to inform instruction and: CONTAIN DIFFERENTIATED ASSESSMENT STRATEGIES/INSTRUCTION; ALLOW THE TEACHER TO CHECK FOR UNDERSTANDING THROUGHOUT THE LESSON AND USE TECHNIQUES THAT ARE BASED ON STUDENTS ACADEMIC LANGUAGE NEEDS AND DEVELOPMENTAL LEVEL OF READINESS
7.	**Accommodations/Modifications**
	• What accommodations/modifications will be needed for students with IEPs and English learners? • Be sure to consider what accommodations/modifications might be needed *at every stage of the lesson plan.* • EFFECTIVE: * The teacher modifies the instruction within the lesson/class period by: PROMOTING SUCCESSFUL LEARNING OF ALL STUDENTS; MODIFYING INSTRUCTION ACCORDING TO APPLICABLE IEPS; ADJUSTING INSTRUCTIONAL PLANS AND MAKING ACCOMMODATIONS FOR STUDENT QUESTIONS, NEEDS, AND INTERESTS, WHILE TAKING INTO ACCOUNT THE LANGUAGE DEMANDS AND GRADE LEVEL APPROPRIATENESS OF THE CONTENT AND INSTRUCTION; ADJUSTING INSTRUCTIONAL PLANS BY EMPLOYING A VARIETY OF STRATEGIES AND TECHNIQUES THAT ARE RESPONSIVE TO STUDENTS' NEEDS, PROFICIENCY, CULTURES AND/OR EXPERIENCES; AND ADJUSTING THE LESSON BASED ON

	PERIODIC CHECKING FOR UNDERSTANDING AND/OR FORMATIVE ASSESSMENTS OF ALL STUDENTS.
8.	**Reflection**
	• If you actually taught this lesson, reflect on your experience and the students' experiences. EFFECTIVE: * The teacher: DISPLAYS A HIGH LEVEL OF PROFESSIONALISM BY MAKING DECISIONS AND RECOMMENDATIONS BASED ON THE NEEDS OF ALL STUDENTS; PROMOTES A POSITIVE WORKING/LEARNING ENVIRONMENT FOR STUDENTS, COLLEAGUES, AND COMMUNITY MEMBERS; DEMONSTRATES KNOWLEDGE OF APPLICABLE LAWS, POLICIES, REGULATIONS, AND PROCEDURES RELATED TO STUDENTS • What new knowledge will students gain from this lesson? • What, if anything, would you change about this lesson? Why? EFFECTIVE: * The teacher: ACTIVELY READS AND APPLIES CURRENT RESEARCH IN AREAS OF GREATEST IMPACT FOR ALL STUDENTS.

❖ Differentiation should be considered for CONTENT, PROCESS, and PRODUCT of student learning. Differentiation should be considered at every stage of your lesson plan's development and implementation.

Appendix K

CODES LISTED ALPHABETICALLY

A	Abstract	Summaries of the story to come or an interviewer asked question to elicit a narrative
ACS	Attending to child's strategies	Teacher interacts verbally or nonverbally with student(s) to indicate recognition of student mathematical sense-making
ASE	Analysis of student errors	Teacher narrates how they are aware of and make sense of student mistakes
AW	Awareness	Teacher noticing students' mathematical strategies/thinking
AWA	Awareness in action	Teacher recognition of the opportunity to interact differently given what they've noticed about students' mathematical sense-making
AWD	Awareness in discipline	What a teacher notices about student sense-making is used to alter their instructional interaction with students
AWC	Awareness in counsel	Teacher ability to assist fellow teachers in the process of noticing student mathematical thinking
BN	Barely noticing	May recognize what is being described, although otherwise forgotten
C	Connect a topic	A teacher demonstrates how two or more math concepts are related (ex: counting backward on a number line and solving subtraction number stories are related)
C	Coda	Narrator returns to the present time of the story
c	Comparator	Compares what did occur to what did not
CA	Completing actions	Relates the events of a story
Co	Coding	Teacher transforms student mathematical thinking they have noticed in a specific interaction into parts they can talk about
CSC	Common student computational strategies	Teachers narrate how their noticing of students' common computational strategies help or hinder their instruction

CSC&M	Common student conception or misconception	A teacher can narrate what they know or noticed about what students may or may not understand about math content
DCS	Deciding how to respond based on CU	Teacher interacts verbally or nonverbally with a student(s) based on what they think they understand about a student(s) mathematical sense-making
DD	Discerning details	Picking out bits and distinguishing
DM	Developmental sequence	Teacher narrates how their instruction is influenced by the readiness of the student
E	Evaluation	Crucial point of a narrative
EMP	Find an example to make a specific mathematical point	How a teacher leads up to or responds to student mathematical sense-making; the teacher may choose to use a curriculum prompt or come up with an in-the-moment response
EPSC	Evaluate the plausibility of students' claims	Teacher asks student to demonstrate how they got their answer (student may respond verbally, written representations, models or manipulatives)
Ex	Explicatives	Explaining why something happened
Hi	Highlighting	Teacher marks as salient specific aspects of student thinking
HW	Holding wholes	Gazing at something without particularly discerning details
I	Intensifiers	Expressive quantifier
ICS	Interpreting child's understandings	Teacher interacts verbally or nonverbally with student(s) to try to understand a student(s) mathematical sense-making
M	Marking	Think to make a remark about something noticed
max	Maxims	Knowledge of what teachers do/past experiences
Na	Narrate	Teacher can describe how what they notice assists them in forming an appropriate interaction with their student(s)
NN	Not noticing	Entirely oblivious to a referenced experience
norm	Norms	Knowledge of what teachers do/present experiences
O	Orientations	Provide a setting
para	Parables	Specific instance of practice/story

PMI	Presenting mathematical ideas	How a teacher enacts the math content in a specific student–teacher–content interaction
PP	Perceiving properties	Becoming aware of particular interactions that could hold in other situations
prec	Precedent	Specific instance of practice/story that happened before
princ	Principles	Knowledge of what teachers do/based on research
Q	Ask productive mathematical questions	A teacher prompt that may assist the teacher in recognizing, identifying, and/or interpreting patterns of student thinking/strategies
R	Result	Tells the listener how the story ends
R	Recording	So struck by an incident to make some type of record to enable a re-entrance to the experience at a future time
RBAP	Reasoning on the basis of agreed properties	Using the assembled things noticed and applying to new interactions
RR	Recognizing relationships	Becoming aware of the sameness and difference or other relationship among the noticed details in an interaction
RSQIU	Respond to student questions, ideas, understandings	How the teacher chooses to engage the student with the content (ex: questions asked, representations used, problems presented); what the teacher notices about the students' mathematical sense-making and how the teacher chooses to respond to the students' mathematical thinking
Sequence content	Highlighting the order content is taught in	A teacher narrates why they teach content in the order they do
SIC	Students' interaction with the content	How the student responds to the content (ex: verbal feedback, nonverbal feedback, written representations, manipulatives used)
strat	Strategy	Teachers use propositional and case knowledge to make decisions
TIC	Teacher's interaction with the content	How the teacher chooses to present specific aspects of the content to students (ex: lesson plans, lesson plans as actually enacted)

TIS	Teacher's interaction with the student	How the teacher chooses to engage the student with the content (ex: questions asked, representations used, problems presented)
TISIC	Teacher's interaction with students' interactions with the content	What the teacher notices about the students' feedback and how the teacher chooses to respond to the students' mathematical thinking
UDIM	Using different instructional models	Teacher narrates how their instruction is different from the "status quo"

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