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# On Properties of Pair Operations

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B.S., Mathematics, High Point University, 2019
M.S., Mathematics, University of New Mexico, 2022

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# Dedication

*I dedicate this dissertation to my parents;*
*for all the dinners, card games, and little joys.*

# Acknowledgments

I would like to thank my advisor Janet Vassilev for suggesting this line of questioning and for all the time spent discussing it with me. I'd also like to thank my dissertation committee, Janet Vassilev, Alexandru Buium, Shuai Wei, and Rebecca R.G.

I couldn't have finished this without the help and support from my family and friends. They were always there for me when I needed encouragement and an excuse to take a break.

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## Abstract

For any closure operation cl and interior operation i on a class of $R$-modules, we develop the theory of cl-prereductions and i-postexpansions. A pair operation is a generalization of closure and interior operations. Using Epstein, R.G. and Vassilev's duality [ERGV23b], we show that these notions are in fact dual to each other. We discuss the relationship between the core and hull and prereductions and postexpansions. We further the thematic notion of duality and seek to understand how it arises in the context of properties pair operations can be endowed with and focus on inner product spaces and properties demonstrated by the orthogonal complement. Finally, constructions of pair operations through ring extensions and collections will be explored with relation to how the new operation can preserve certain properties.

# Contents

# List of Figures

# Glossary

| | |
|---|---|
| cl | Closure operation |
| $\mathrm{cl-core}_M(N)$ | cl-core of $N$ with respect to $N$ |
| $C'_{\mathrm{i}}(A, B)$ | Set of submodules which are not i-expansions of $A$ in $B$ |
| i | Interior operation |
| $\mathrm{i-hull}^B(A)$ | i-hull of $A$ with respect to $B$ |
| $I'_{\mathrm{cl}}(N, M)$ | Set of submodules which are not cl-reductions of $N$ in $M$ |
| $I^-$ | Integral closure of $I$ |
| $I^*$ | Tight closure of $I$ |
| $I^{\mathfrak{m}\mathrm{bf}}$ | $\mathfrak{m}$- basically full closure of $I$ |
| $\mathcal{M}$ | Class of $R$-modules closed under submodules and quotients |
| $M^\vee$ | Matlis dual of $M$ |
| $\mathcal{P}$ | Class of pairs $(N, M)$ with $N, M \in \mathcal{M}$ and $N \subseteq M$ |
| $p(N, M)$ | Pair operation of $N$ in $M$ |
| $p^\smile$ | Pair operation dual to $p$ |
| $\mathbf{P}_{\mathrm{cl}}(N, M)$ | Set of cl-prereductions of $N$ in $M$ |

# Chapter 1

# Introduction

Closure operations are used throughout algebra and connect algebra to other fields such as algebraic geometry; within algebra, closure operations connect Noetherian and non-Noetherian commutative algebra. Ask a mathematician and its likely that they've at least heard of integral closure. We can examine a closure operation through the lens of its dual interior operation and by studying interior operations, we can gain insight into its dual closure operation.

Reductions were first introduced in terms of the integral closure and have since been generalized to general closure operations ([HRR02], [Rat89], [FV10], [Eps12], [ERG21], [ERGV23a]). Then Kemp, Ratliff, and Shah introduced the concept of prereductions [KRS20]. In Chapter 3, prereductions are generalized to submodules and their properties are investigated. Epstein, R.G., and Vassilev showed that cl-reductions are dual to i-expansions [ERGV23b] and Theorem 3.4.4 shows that this duality extends to cl-prereductions and i-postexpansions.

Some closures can be difficult to compute; in that case, it can be useful to compare them to a closure that is simpler to compute; more on this and its impact to cl-prereductions is shown in Proposition 3.3.3.

In some sense, the core is a measure of the reductions of $N$. The core is known to be difficult to describe explicitly [HS06] and much work has been done to find formulas for the core of an ideal. Reductions, and thus cores, provide a way to study Rees algebras of modules. Because calculating the core is so difficult, it can be convenient to find a bound on it instead. Proposition 4.2.7 uses cl-prereductions and i-postexpansions to bound them in a few specific cases.

Some operations are studied which satisfy some or none of the definition of a closure operation. Being able to consider operations with no predetermined properties is one advantage of studying pair operations. Pair operations allow us to study operations in a broader context. For instance, in linear algebra, the orthogonal complement satisfies the requirements for a pair operation but not a closure operation. Since linear algebra is used by so many fields, it may be that the introduction of pair operations on vector spaces will open up new tools to solve problems. The orthogonal complement from a pair operations perspective is studied in Chapter 5.

Since linear algebra is used by so many fields, it may be that the introduction of pair operations on vector spaces will open up new tools to solve problems. While we don't yet know the connection between pair operations and other fields, since algebraic geometry and linear algebra already use pair operations, there are potential applications and benefits to studying them.

In his survey of closure operations [Eps12], Epstein goes over several constructions which always produce closure operations. We will examine constructions for pair operations such as ring extensions, intersections, sums, and unions in Chapter 6. Given a collection of pair operations satisfying a specific property, there is no guarantee that the resulting operation will inherit the property. In particular, to obtain an idempotent or involutive pair operation through these constructions often requires additional assumptions on the pair operations in the original collection besides the desired property. Being able to consider operations with no predetermined properties is one advantage of studying pair operations.

Our work is contributing to the development of pair operations by helping expand the framework of pair operations. It may end up being useful to people studying closure operations, particularly the work in cl-prereductions. There is also potential applications to the study of toric varities since they can be defined using inner products and my work in constructions uses inner products in a similar way.

# Chapter 2

# Background

## 2.1 Pair Operations

Pair operations were introduced by Epstein, R.G. and Vassilev as generalizations of closure and interior operations in [ERGV23b] and [ERGV23a]. Pair operations are more general than closure and interior operations, thereby providing a framework by which to consider operations with a variety of properties. The notion of pair operations allows us to use a common framework for operations defined on $R$-modules.

We start by defining pair operations; of which, closure and interior operations are specific examples.

**Definition 2.1.1.** [ERGV23b, Definition 2.1] Let $R$ be an associative ring, not necessarily commutative. Let $\mathcal{M}$ be a class of (left) $R$-modules that is closed under taking submodules and quotient modules. Let $\mathcal{P}$ be the class of pairs $(N, M)$ of $R$-modules with $N, M \in \mathcal{M}$ and $N \subseteq M$.

**Definition 2.1.2.** [ERGV23b, Definition 2.2] Let $\mathcal{M}$, be a class of $R$-modules and $\mathcal{P}$ be a collection of pairs $(L, M)$ with $L, M \in \mathcal{M}$ and $L \subseteq M$ such that whenever

$\varphi : M \to M'$ is an isomorphism and $(L, M) \in \mathcal{P}$ then $(\varphi(L), M') \in \mathcal{P}$. A *pair operation* is a function $p$ that sends each pair $(L, M) \in \mathcal{P}$ to a submodule $p(L, M)$ of $M$, in such a way that whenever $\varphi : M \to M'$ is an $R$-module isomorphism and $(L, M) \in \mathcal{P}$, then $\varphi(p(L, M)) = p(\varphi(L), M')$. When $(L, M) \in \mathcal{P}$, we say that $p$ is

- *idempotent* if whenever $(L, M), (p(L, M), M) \in \mathcal{P}$, then $p(p(L, M), M) = p(L, M)$.
- *order-preserving on submodules* if for $L \subseteq N \subseteq M$ such that $(L, M), (N, M) \in \mathcal{P}$, we have $p(L, N) \subseteq p(N, M)$.
- *extensive* if we always have $L \subseteq p(L, M)$.
- *intensive* if we always have $p(L, M) \subseteq L$.
- a *closure operation* if it is extensive, idempotent, and order-preserving on submodules.
- an *interior operation* if it is intensive, idempotent, and order-preserving on submodules.

We say $N$ is cl-*closed* in $M$ if $N = N_M^{\mathrm{cl}}$. We say $A$ is i-*open* in $B$ if $A = A_{\mathrm{i}}^B$.

**Remark 2.1.3.** If $p$ is a closure operation, we will denote $p(L, M) = L_M^{\mathrm{cl}}$ and refer to $p$ as cl. If $p$ is an interior operation, we will denote $p(L, M) = L_{\mathrm{i}}^M$ for $(L, M) \in \mathcal{P}$ and refer to $p$ as i.

**Notation 2.1.4.** We take the following notational conventions for granted,

$$p^0(L, M) = L,$$

and

$$p^n(L, M) = p(p^{n-1}(L, M), M).$$

**Remark 2.1.5.** We will usually assume that $\mathcal{M}$ is the category of all $R$-modules, finitely generated $R$-modules, Artinian $R$-modules or the class of all ideals and $\mathcal{P}$ is the class of pairs $(L, M)$ where $L \subseteq M$ and $L, M$ range over all the modules in $\mathcal{M}$. Unless otherwise noted, $\mathcal{M}$ will be the category of all $R$-modules and $\mathcal{P}$ the class of pairs ranging over $\mathcal{M}$.

The behavior of the pair operation can greatly depend on the class $\mathcal{P}$ of pairs; we note this in Example 2.3.2. It is possible that some of our proofs might break down if we restrict the set of pairs $(L, M)$ so that some pairs with $L \subseteq M$ both in $\mathcal{M}$ do not lie in $\mathcal{P}$. $\diamondsuit$

## 2.2 Matlis Duality

When discussing duality of pair operations, we will be using Matlis duality. This section covers some definitions and results for Matlis duality which will be useful later.

**Definition 2.2.1.** Let $R$ be a ring.

- Let $M$ be an $R$-module and $N \subseteq M$ a submodule; we say that $M$ is an *essential extension* of $N$ if $L \cap N \neq 0$ for every non-zero submodule $L \subseteq M$, or equivalently if

  $$0 \neq x \in M \implies \text{there exists } a \in R \text{ such that } 0 \neq ax \in N.$$

  An injective module $E$ such that $N \subseteq E$ is an essential extension is called an *injective hull* of $N$, and written $E_R(N)$. [Mat86, Appendix B]

- Let $M$ be an $R$-module and $\{M_\lambda\}_{\lambda \in \Lambda}$ a family of submodules of $M$ indexed by $\Lambda$ such that $\lambda < \mu \implies M_\lambda \supset M_\mu$. The inverse limit $\varprojlim M/M_\lambda$ is the *completion* of $M$, and is written $\hat{M}$. If $\psi : M \to \hat{M}$ is an isomorphism, we say that $M$ is *complete.* [Mat86, Chapter 8]

- A *complete local ring* $R$ with unique maximal ideal $\mathfrak{m}$ is a local ring (has a unique maximal ideal) which is complete with respect to $\mathfrak{m}$. If $R$ is a commutative ring and $\mathfrak{m}$ is a maximal ideal, then the *residue field* is the quotient ring $k = R/\mathfrak{m}$.

**Definition 2.2.2.** Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring, $E$ the injective hull of the residue field $k$. The *Matlis dual* of an $R$-module $M$ is $M^\vee := \mathrm{Hom}_R(M, E)$. A module is *Matlis-dualizable* if $M \simeq M^{\vee\vee}$.

**Theorem 2.2.3.** *[Mat86, Theorem 18.6 (iv) and (v)] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and $E = E_R(k)$ the injective hull of $k$. For each $R$-module $M$, set*

$$M^\vee = Hom_R(M, E).$$

1. *$E^\vee = Hom(E, E) = \hat{R}$. In other words, each endomorphism of the $R$-module $E$ is multiplication by a unique element of $\hat{R}$.*

2. *$E$ is Artinian as an $R$-module and also as an $\hat{R}$-module. Assume now that $R$ is complete, and write $\mathcal{N}$ (resp. $\mathcal{A}$) for the category of Noetherian (respectively Artinian) $R$-modules. Then if $M \in \mathcal{N}$ we have $M^\vee \in \mathcal{A}$ and $M \simeq M^{\vee\vee}$; if $M \in \mathcal{A}$ we have $M^\vee \in \mathcal{N}$ and $M \simeq M^{\vee\vee}$.*

This means that since the injective hull of a field $k$ is itself, the Matlis dual of a $k$-vector space is the same as the dual vector space. When vector space $V$ is finite dimensional, then the dual vector space $V^* = V^\vee$ is isomorphic to $V$.

**Lemma 2.2.4.** *[ERGV23b, Lemma 3.4] Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. Let $M$ be an $R$-module such that it and all of its quotient modules are Matlis-dualizable. Let $\{N_i\}_{i \in I}$ a collection of submodules of $M$. Then*

$$\left(\frac{M}{\sum_i N_i}\right)^\vee \cong \bigcap_i (M/N_i)^\vee$$

*and*

$$(\frac{M}{\bigcap_i N_i})^\vee \cong \sum_i (M/N_i)^\vee$$

*where all the dualized modules are considered as submodules of* $M^\vee$.

When considering the Matlis dual, we will let $R$ be a complete local ring with maximal ideal $\mathfrak{m}$, residue field $k$, and $E := E_R(k)$ the injective hull of $k$.

## 2.3 Closure and Interior Operations

Some common closure operations in commutative algebra are integral closure, tight closure and basically full closure. We will define these now for ideals of a commutative Noetherian ring to use later when we present examples. All of these closures are defined on the set of ideals of the ring. This can be extended to modules but there are multiple ways to do so (except in the case of $\mathfrak{m}$ basically full which is the same in ideals and modules) and is outside the scope of this work. All of the examples involving integral and tight closure are on ideals.

**Definition 2.3.1.** Let $R$ be a commutative ring and $I$ an ideal of $R$. The integral closure of $I$ is:

$$I^- := \{x \in R \mid x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \text{ for some } a_i \in I^i\}.$$

Note that we use $I^-$ instead of $\overline{I}$ or $I_a$ since we represent all closures as superscripts. When we refer to the closure, reduction, prereduction, etc. (i.e. without the cl- prefix), it is understood to be referring to the integral closure.

**Example 2.3.2.** The class $\mathcal{P}$ can significantly influence the properties of a pair operation. For any ideal $I \subseteq R$ with $J$ a reduction of $I$, the core of $I$ is

$$\text{core}(I) = \bigcap_{J \subseteq I \subseteq J^-} J^-.$$

Let $R$ be a ring and $\mathcal{M}$ be the class of all ideals of $R$ and $\mathcal{P}$ some class of pairs of the form $(I, R)$. Set $p(I, R) = \operatorname{core}(I)$ for $(I, R) \in \mathcal{P}$. Notice $p(I, R)$ is an intensive pair operation.

Let $R = k[x, y]$. Let $\mathcal{P}$ be the set of all pairs of the form $(I, R)$ where $I$ is an integrally closed ideal. By [HS95a, Proposition 3.15], if $I \subseteq J$ then $p(I, R) \subseteq p(J, R)$. Meaning that $p$ is order preserving on $\mathcal{P}$. Note if $R$ is not regular of dimension 2, this pair operation is not necessarily order preserving as witnessed Example 1 in [Lee08].

Suppose $I$ is an integrally closed ideal and let $\mathcal{P}'$ be the set of all pairs of the form $(J, R)$ where the integral closure of $J$ is $I$ and consider $p$ on $\mathcal{P}'$. If $K$ is basic (i.e. the only reduction of $K$ is itself) then $p(K, R) = K$. However, $K \supseteq p(J, M) = \operatorname{core}(J)$ for any $J \supseteq K$. In fact, for reductions $J_1 \subseteq J_2$ of $I$ and $K$ a reduction of $J_1$, $K$ will be also be reduction of $J_2$, implying that $\operatorname{core}(J_2) \subseteq \operatorname{core}(J_1) = \bigcap_{K \subseteq J_1 \subseteq K^-} K$. Hence, $p$ is order reversing on $\mathcal{P}'$.

In the late 80's, Hochster and Huneke [HH90] introduced tight closure, a closure operation in equicharacteristic rings. Here we will stick to the positive characteristic version.

**Definition 2.3.3.** Let $R$ be a Noetherian ring of characteristic $p > 0$ and $I \subseteq R$ an ideal. Set $I^{[p^e]} = (x^{p^e} \mid x \in I)$ and $R^o = R \setminus \bigcup \{P \mid P \text{ a minimal prime of } R\}$. The tight closure of $I$ is

$$I^* := \{x \in R \mid cx^{p^e} \in I^{[p^e]} \text{ for some } c \in R^o \text{ and all } e >> 0\}.$$

The $\mathfrak{m}$-basically full closure was introduced by Heinzer, Ratliff and Rush in [HRR02] as a closure operation on $\mathfrak{m}$-primary ideals. However, the operation is also a closure operation on the set of all ideals of a ring.

**Definition 2.3.4.** [HRR02, Definition 4.4], [ERGV23b, Definition 4.2] Let $(R, \mathfrak{m})$ be a local ring and $I$ an ideal of $R$. The $\mathfrak{m}$-basically full closure of $I$ is

$$I^{\mathrm{mbf}} := (\mathfrak{m}I : \mathfrak{m}).$$

If $N \subseteq M$, the $\mathfrak{m}$-basically full closure of $N$ in $M$ is

$$N_M^{\mathrm{mbf}} = (\mathfrak{m}N :_M \mathfrak{m}).$$

**Proposition 2.3.5.** *[Hun96, Example 1.6.1, 1.6.2] Let $R$ be a Noetherian integral domain and $R^-$ be the integral closure of $R$ in its fraction field, then*

$$(x)^* = (x)R^- \cap R = (x)^-.$$

*In particular, if $R$ is a one-dimensional Noetherian domain then $I^* = (x)^* = (x)^-$, for some $x \in I$.*

The following examples demonstrate the relationships between integral, tight, and $\mathfrak{m}$ basically full closure in numerical semigroup rings.

**Example 2.3.6.** Let $k$ be a field of characteristic $p > 0$. If $R = k[[x, y]]$, $I^* = I$ for all ideals $I \subseteq R$ since $R$ is a regular ring [HH90, Theorem 4.4]. However,

$$(x^3, y^3)^{\mathrm{mbf}} = (\mathfrak{m}(x^3, y^3) : \mathfrak{m}) = (x^3, x^2y^2, y^3).$$

So in $R$ we have $I^* = I \subseteq I^{\mathrm{mbf}}$ for all ideals $I$.

However, for $S = k[[x^2, x^5]]$,

$$(x^4)^{\mathrm{mbf}} = (\mathfrak{m}(x^4) : \mathfrak{m}) = (x^6, x^9) : (x^2, x^5) = (x^4, x^7) \subsetneq (x^4, x^5) = (x^4)^- = (x^4)^*$$

where the last equality follows from [HH90, Corollary 5.8] and the second to last equality follows from Remark A.0.1. In particular, in $S$ we have $I^{\mathrm{mbf}} \subseteq I^*$ for all ideals $I$.

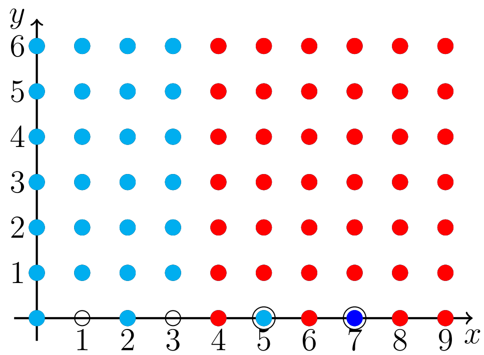


Figure 2.1: Lattice of monomials in $(x^4) \subseteq (x^4)^{\mathfrak{mbf}} \subseteq (x^4)^* \subseteq T$.

Consider the ring $T = k[[x^2, x^5, y, xy]]$. Note that $(x^4)^* = (x^4)^- = (x^4, x^5)$ by [HH90, Corollary 5.8]. However, similar to the computation in $S$, $(x^4)^{\mathfrak{mbf}} = (x^4, x^7)$ and

$$(x^4)^{\mathfrak{mbf}} \subseteq (x^4)^* = (x^4)^-.$$

We have included Figure 2.1 to illustrate the monomials in $(x^4)$ whose powers are represented by the lattice points shaded in red; the lattice points in blue represent the powers of the monomials in $T \setminus (x^4)$ . The two circled colored lattice points on the $x$-axis indicate the monomials which are in $(x^4)^* \setminus (x^4)$ where the darker blue lattice point is a monomial in $(x^4)^{\mathfrak{mbf}}$ not in $(x^4)$. The lattice points at $(1, 0)$ and $(3, 0)$ are not colored because the monomials $x$ and $x^3$ are not in $S$.

Whereas, for the ideal $J = (x^4, x^5, y^2, xy^2)$: we will see that

$$J^* = J \subseteq J^{\mathfrak{mbf}} = J + (x^2y, x^3y).$$

The first equality holds by [HH90, Lemma 4.11], since $T \subseteq k[[x, y]]$ which is a regular ring and the preimage of $(x^4, y^2)$ in $T$ is precisely $J$. Figure 2.2, helps us to understand how we obtain the $\mathfrak{m}$-basically full closure $J$. Note the red lattice points indicate monomials in $J$ and the blue lattice points indicate the monomials in $T \setminus J$. The lattice points at $(1, 0)$ and $(3, 0)$ are not colored because the monomials $x$ and $x^3$ are not in $T$. The circled red lattice points are those in $\mathfrak{m}J$. Note that every monomial in the maximal ideal multiplies the two monomials represented by the darker blue lattice points ($x^2y$ and $x^3y$) whereas for each of the remaining monomials represented by the light blue lattice points, there is at least one element of the maximal ideal that does not multiply the monomial into $\mathfrak{m}J$. Hence $J^{\mathrm{mbf}} = J + (x^2y, x^3y)$.

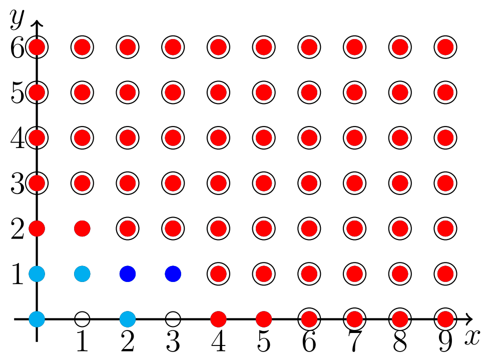


Figure 2.2: Lattice of monomials in $J \subseteq J^{\mathrm{mbf}} \subseteq T$.

Since we have obtained two ideals $I$ in $T$ one with $I^{\mathrm{mbf}} \subseteq I^*$ and the other with $I^* \subseteq I^{\mathrm{mbf}}$, we see that tight closure and $\mathfrak{m}$-basically full closure are not comparable in $T$. $\square$

We will now switch our focus to interior operations i.

**Definition 2.3.7.** [ERGV23b, Definition 2.13] Let $R$ be a Noetherian ring, $\mathcal{M}$ be a class of $R$-modules and i be an interior operation defined on $\mathcal{P}$ of pairs of modules $(A, B)$ with $A \subseteq B$ in $\mathcal{M}$. Suppose $(A, B), (C, B) \in \mathcal{P}$.

1. We say that $C$ is an i-*expansion of $A$ in $B$* if $C_{\mathrm{i}}^B \subseteq A \subseteq C$.

2. We say that a submodule $C \subseteq B$ is i-*open* if $C_{\mathrm{i}}^B = C$

3. We say that a submodule $C \subseteq B$ is i-*cobasic* if $C_{\mathrm{i}}^B = C$ and $C$ is the only i-expansion of itself.

4. If $C$ is an i-expansion of $A$ in $B$ and there is no submodule $C \subsetneq D$ such that $D_{\mathrm{i}}^B = A_{\mathrm{i}}^B$ then we say that $C$ is a *maximal* i-expansion of $A$ in $B$.

5. We define the i-*hull* by $\text{i-hull}^B(A) = \sum\{C \mid C_{\mathrm{i}}^B \subseteq A \subseteq C \text{ and } (C, B) \in \mathcal{P}\}$.

**Definition 2.3.8.** Let $(R, \mathfrak{m})$ be a local ring and i an interior operation on Artinian R-modules. We say that i is a *Nakayama interior* if for any Artinian R-modules $A \subseteq C \subseteq B$, if $(A :_C \mathfrak{m})_{\mathrm{i}}^B \subseteq A$, then $A_{\mathrm{i}}^B = C_{\mathrm{i}}^B$.

It is known that maximal i-expansions exist for a submodule $A$ of $B$ in the following cases: if $(R, \mathfrak{m})$ is a complete local ring, $\mathcal{M}$ is the class of Artinian $R$-modules and cl is a Nakayama interior [ERGV23c, Proposition 6.4] or when $R$ is an associative ring and if there exists an i-expansion $C$ of $A$ such that $B/C$ is Noetherian [ERGV23c, Proposition 6.5]. When maximal i-expansions of $A$ exist in $B$, then

$$\text{i-hull}^B(A) = \sum\{C \mid C \text{ a maximal i-expansion of } A \text{ and } (C, B) \in \mathcal{P}\}.$$

## 2.4 Reductions

Northcott and Rees were the first to define and study the reductions of an ideal in [NR54]. A *reduction* of an ideal $I$ is an ideal $J \subseteq I$ which shares the same integral closure as $I$. Rees further generalized the notion of integral closure and reductions to the setting of submodules of a module in [Ree87]. Northcott and Rees [NR54] defined an ideal $J \subseteq I$ to be a reduction of $I$ if there exists some non-negative integer $n$ with $JI^n = I^{n+1}$ and they showed that $J$ is a reduction of $I$ if and only if $J^- = I^-$. Rees generalized the notion of reduction for submodules of modules in [Ree87].

Epstein generalized reductions of ideals of a commutative Noetherian ring and submodules of finitely generated modules for closure operations cl in [Eps05] and [Eps10]. We include the definition below in the language of pair operations as well as the related notion of cl-core. The core was originally introduced for integral closure by Rees and Sally in [RS88] and then Fouli and Vassilev in [FV10] generalized it for closure operations cl.

**Definition 2.4.1.** [ERGV23b, Definition 2.10] Let $R$ be a Noetherian ring, $\mathcal{M}$ be a class of $R$-modules and cl be a closure operation defined on the class $\mathcal{P}$ of pairs of modules $(L, M)$ with $L \subseteq M$ in $\mathcal{M}$. Suppose $(L, M), (N, M) \in \mathcal{P}$.

1. We say that $L$ is a cl-*reduction of* $N$ *in* $M$ if $L \subseteq N \subseteq L_M^{\mathrm{cl}}$.

2. We say that a submodule $N \subseteq M$ is cl-*closed* if $N_M^{\mathrm{cl}} = N$.

3. We say that a submodule $N \subseteq M$ is cl-*basic* if $N_M^{\mathrm{cl}} = N$ and $N$ is a minimal cl-reduction.

4. If $L$ is a cl-reduction of $N$ in $M$ and there is no submodule $K \subsetneq L$, with $(K, M) \in \mathcal{P}$ such that $K_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}$ then we say that $L$ is a *minimal* cl-reduction of $N$ in $M$.

5. The cl-core *of* $N$ *with respect to* $M$ is the intersection of all cl-reductions of $N$ in $M$, or

$$\text{cl-core}_M(N) := \bigcap \{L | L \subseteq N \subseteq L_M^{\mathrm{cl}} \text{ and } (L, M) \in \mathcal{P}\}.$$

When we are dealing with the integral closure, we refer to the integral-reduction as the reduction and integral-core as the core.

Although we defined minimal cl-reductions of a submodule $N$ of $M$ above, we do not in general know that minimal cl-reductions exist. If they do, then

$$\text{cl-core}_M(N) = \bigcap \{L \mid L \text{ a minimal cl-reduction of } N \text{ and } (L, M) \in \mathcal{P}\}.$$

**Definition 2.4.2.** [Eps05, Definition 1.2] Let $(R, \mathfrak{m})$ be a Noetherian local ring and cl be a closure operation on the class of pairs $\mathcal{P}$ of finitely generated $R$-modules. We say that cl is a *Nakayama closure* if for $L \subseteq N \subseteq M$ finitely generated $R$-modules, if $L \subseteq N \subseteq (L + \mathfrak{m}N)_M^{\mathrm{cl}}$ then $L_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}$.

If cl is a Nakayama closure on the class of finitely generated modules, then Epstein showed that minimal cl-reductions exist first for ideals in [Eps05, Lemma 2.2] and then noted that they also exist for submodules of finitely generated modules [Eps10, Section 1].

# Chapter 3

# Prereductions and Postexpansions

## 3.1 Prereductions

In a recent work of Kemp, Ratliff and Shah [KRS20], the authors took a different direction than Epstein by looking for the ideals contained in an ideal $I$ which do not have the same integral closure as $I$. The authors define a *prereduction* to be an ideal $J \subseteq I$ if $J$ is not a (integral) reduction of $I$ but for all $K$ with $J \subsetneq K \subseteq I$, $K$ is a reduction of $I$. They used prereductions in their study of Rees rings and Ratliff-Rush equivalence. Our hope is that in the analysis of blow ups and singularities with respect to different closures cl-prereductions will be useful .

As long as $I$ is not basic, then the core of $I$ will be contained in some prereduction of $I$. As reductions have been generalized to different closure operations by Epstein in [Eps05] and [Eps10], we will generalize prereductions to other closures as well.

**Definition 3.1.1.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say that $L$ is a cl-*prereduction* of $N$ if $L \subseteq N$, $L$ not a cl-reduction of $N$ and for all $K$ with $L \subsetneq K \subseteq N$, $K$ is a cl-reduction of $N$.

Kemp, Ratliff, and Shah considered the set of ideals

$$\mathbf{I}'(I) = \{J \subseteq I \mid J \text{ is not a reduction of } I\}.$$

They note that if $\mathbf{I}'(I)$ is non-empty then the maximal elements of $\mathbf{I}'(I)$ are prereductions of $I$.

For any Nakayama closure and any submodule $N \subseteq M$ we can clearly define a set of submodules of $M$ whose closure is properly contained in the closure of $N$. As Kemp, Ratliff and Shah denoted such a set $\mathbf{I}'(I)$ for integral closure for ideals $I$ of $R$, we will modify their notation to include the closure operation cl in the subscript and the pair of modules $(N, M) \in \mathcal{P}$.

**Definition 3.1.2.** Define

$$\mathbf{I}'_{\mathrm{cl}}(N, M) := \{L \subseteq N \mid L \text{ is not a cl-reduction of } N\}.$$

Note that for any $L \subseteq N$, then $L \notin \mathbf{I}'_{\mathrm{cl}}(N, M)$ if and only if $L$ is a cl-reduction of $N$ in $M$. If $M = R$, we will omit $R$ and denote $\mathbf{I}'_{\mathrm{cl}}(I, R) = \mathbf{I}'_{\mathrm{cl}}(I)$.

It may be the case that $\mathbf{I}'_{\mathrm{cl}}(N, M)$ is empty. For instance, if $N = (0) \subseteq M$, then $(0)$ is a cl-reduction of itself and contains no proper submodules so $\mathbf{I}'_{\mathrm{cl}}((0), M) = \emptyset$. In [KRS20, Remark 3.5.1], Kemp, Ratliff and Shah note that $\mathbf{I}'(I)$ is nonempty if and only if $I$ is not a nilpotent ideal. However, unlike in the case for integral closure of ideals in a ring, $\mathbf{I}'_{\mathrm{cl}}(I)$ may not be empty when $I$ is a nilpotent ideal. For example:

**Example 3.1.3.** Let $R = k[[x, y]]/(x^2y^2)$ and $I = (xy)$. Note that $(xy)$ is the nilradical of $R$ and a nilpotent ideal. Note that $(0)^{\mathrm{mbf}} = ((0) : \mathfrak{m}) = (0)$ and

$$(xy)^{\mathrm{mbf}} = (\mathfrak{m}(xy) : \mathfrak{m}) = (x^2y, xy^2) : (x, y) = (xy).$$

We see that $(0) \in \mathbf{I}'_{\mathrm{mbf}}(xy) \neq \emptyset$ even though it is a nilpotent ideal. However, we have $(0)^- = (xy)$ since $xy$ is a zero of $t^2$ in $R[t]$ implying that $(0)$ is an integral reduction of $(xy)$.

**Example 3.1.4.** The order operation ord, defined by $\operatorname{ord}(I) = \mathfrak{m}^r$ if $I \subseteq \mathfrak{m}^r$ but $I \nsubseteq \mathfrak{m}^n$ for any $n > r$ and $\operatorname{ord}(I) = \bigcap\limits_{r \in \mathbb{N}} \mathfrak{m}^r$ if $I \subseteq \mathfrak{m}^r$ for all $r \in \mathbb{N}$ as discussed in [Vas14b] is also a Nakayama closure with $(0) \in \mathbf{I}'_{\mathrm{ord}}(xy)$ in $R$ as in Example 3.1.3 as $\operatorname{ord}(xy) = \mathfrak{m}^2$ and $(0) = \operatorname{ord}(0)$.

Although, we have given examples above of Nakayama closure operations where there are nilpotent ideals $I$ with $\mathbf{I}'_{\mathrm{cl}}(I) \neq \emptyset$, tight closure behaves more like integral closure for ideals in the sense that $\mathbf{I}'_*(I) = \emptyset$ when $I$ is nilpotent.

**Proposition 3.1.5.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$, then for any nilpotent ideal $I$, $\mathbf{I}'_*(I) = \emptyset$.*

*Proof.* Denote the nilradical by $N$. For any nilpotent ideal $I$ we have $(0) \subseteq I \subseteq N$. By [HH90, Proposition 4.1(i)], $(0)^* = N = I^*$. For any ideal $J \subseteq I$ we have $(0) \subseteq J \subseteq I$; hence it is clear that $I$ has no ideals $J$ with $J \subseteq I$ and $J^* \subsetneq I^*$. $\square$

Multiplication of elements in a module is not defined unless we extend multiplication through the tensor product; thus, we do not usually discuss nilpotent submodules of an $R$-module.

Due to the above examples and comment, we see that not all properties that Kemp, Ratliff and Shah obtained for $\mathbf{I}'(I)$ in [KRS20] generalize for $\mathbf{I}'_{\mathrm{cl}}(N, M)$ for a general Nakayama closure cl. In addition to defining the set $\mathbf{I}'_{\mathrm{cl}}(N, M)$, we can also define the notion of cl-prereductions for general Nakayama closure operations cl.

If $\mathbf{I}'_{\mathrm{cl}}(N, M) \neq \emptyset$, the maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions. We will denote the set of cl-prereductions by $\mathbf{P}_{\mathrm{cl}}(N, M)$. Note that if $\mathbf{I}'_{\mathrm{cl}}(N, M)$ is the set of all ideals properly contained in $N$, then $N$ has no cl-reductions besides $N$ itself.

Instead of the non-nilpotence assumption Kemp, Ratliff and Shah use in [KRS20, Proposition 3.7], we will require $\mathbf{I}'_{\mathrm{cl}}(N, M)$ to be nonempty for our generalization.

**Proposition 3.1.6.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and* cl *a Nakayama closure on $\mathcal{P}$, the class of finitely generated $R$-modules. Suppose $(N, M) \in \mathcal{P}$ such that $\mathbf{I}'_{\mathrm{cl}}(N, M) \neq \emptyset$. Then the following hold*

1. *Suppose $K \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ and $L \subseteq K$. Then $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.*

2. *Let $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. There exists a submodule $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ which is maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ and $\mathfrak{A} \supseteq L$.*

3. *Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are maximal submodules in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$. Then $\mathfrak{A}_1 + \mathfrak{A}_2 \notin \mathbf{I}'_{\mathrm{cl}}(N, M)$ and $\mathfrak{A}_1 + \mathfrak{A}_2$ is a* cl*-reduction of $N$ in $M$.*

4. *If $L, K \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ then either $L + K \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ or $L + K$ is a* cl*-reduction of $N$ in $M$.*

5. *If $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ then $L + \mathfrak{m}N \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.*

6. *If $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ then $L^{\mathrm{cl}}_M \cap N \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.*

7. *If $\mathfrak{A}$ is maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ then*

   *(a)* $\mathfrak{A} + \mathfrak{m}N = \mathfrak{A}$ *or* $\mathfrak{m}N \subseteq \mathfrak{A}$.

   *(b)* $\mathfrak{A}^{\mathrm{cl}}_M \cap N = \mathfrak{A}$.

   *(c)* $(\mathfrak{A}^{\mathrm{cl}}_M + (\mathfrak{m}N)^{\mathrm{cl}}_M)^{\mathrm{cl}}_M \cap N = \mathfrak{A}$.

*Proof.* (1) If $K \in \mathbf{I}'_{\mathrm{cl}}(N, M)$, then $K$ is not a cl-reduction of $N$. Since $L \subseteq K$ and

$$L^{\mathrm{cl}}_M \subseteq K^{\mathrm{cl}}_M \subsetneq N^{\mathrm{cl}}_M$$

then $L$ is also not a cl-reduction of $N$ in $M$.

(2) Since $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ and $R$ Noetherian, then there is an element $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ which is maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ and $\mathfrak{A} \supseteq L$.

(3) Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$ and

$$\mathfrak{A}_1, \mathfrak{A}_2 \subseteq \mathfrak{A}_1 + \mathfrak{A}_2 \subseteq N,$$

then $\mathfrak{A}_1 + \mathfrak{A}_2$ is a cl-reduction of $N$ in $M$.

(4) Since $L, K \subseteq N$ then $L + K \subseteq N$. If

$$L + K \in \mathbf{I}'_{\mathrm{cl}}(N, M),$$

we are done. If $L + K \notin \mathbf{I}'_{\mathrm{cl}}(N, M)$, then by definition $L + K$ is a cl-reduction of $N$ in $M$.

(5) Suppose $L + \mathfrak{m}N \notin \mathbf{I}'_{\mathrm{cl}}(N, M)$. So $L + \mathfrak{m}N$ is a cl-reduction of $N$ in $M$. Then $L \subseteq N \subseteq (L + \mathfrak{m}N)^{\mathrm{cl}}_M$. Because cl is a Nakayama closure, $L^{\mathrm{cl}}_M = N^{\mathrm{cl}}_M$ and $L$ is a cl-reduction of $N$ in $M$ which contradicts our assumption. So $L + \mathfrak{m}N \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.

(6) Suppose that $L^{\mathrm{cl}}_M \cap N \notin \mathbf{I}'_{\mathrm{cl}}(N, M)$. So $L^{\mathrm{cl}}_M \cap N$ is a cl-reduction of $N$ in $M$. Then $L^{\mathrm{cl}}_M \subseteq N^{\mathrm{cl}}_M = (L^{\mathrm{cl}}_M \cap N)^{\mathrm{cl}}_M \subseteq (L^{\mathrm{cl}}_M)^{\mathrm{cl}}_M = L^{\mathrm{cl}}_M$ and $L$ is a cl-reduction of $N$ in $M$ which contradicts $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.

(7a) Since $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. By (5), we know that $\mathfrak{A} + \mathfrak{m}N \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. Since $\mathfrak{A}$ is maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ and $\mathfrak{A} \subseteq \mathfrak{A} + \mathfrak{m}N$ then we must have $\mathfrak{A} + \mathfrak{m}N = \mathfrak{A}$ since $\mathfrak{A} + \mathfrak{m}N$ is not a cl-reduction of $N$. Also the equality, $\mathfrak{A} + \mathfrak{m}N = \mathfrak{A}$ implies that $\mathfrak{m}N \subseteq \mathfrak{A}$.

(7b) Since $\mathfrak{A}$ is maximal in $\mathbf{I}'_{\mathrm{cl}}(N, M)$ and $\mathfrak{A} \subseteq \mathfrak{A}^{\mathrm{cl}}_M \cap N$, then by (6), we have $\mathfrak{A}^{\mathrm{cl}}_M \cap N \in \mathbf{I}'_{\mathrm{cl}}(N, M)$; meaning we must have $\mathfrak{A} = \mathfrak{A}^{\mathrm{cl}}_M \cap N$.

(7c) Since $(\mathfrak{A} + \mathfrak{m}N) \subseteq (\mathfrak{A}^{\mathrm{cl}}_M + (\mathfrak{m}N)^{\mathrm{cl}}_M) \subseteq (\mathfrak{A} + \mathfrak{m}N)^{\mathrm{cl}}_M$, we have

$$(\mathfrak{A}^{\mathrm{cl}}_M + (\mathfrak{m}N)^{\mathrm{cl}}_M)^{\mathrm{cl}}_M = (\mathfrak{A} + \mathfrak{m}N)^{\mathrm{cl}}_M.$$

By (7a), we know that $\mathfrak{A} = \mathfrak{A} + \mathfrak{m}N$ and hence $\mathfrak{A}^{\mathrm{cl}}_M = (\mathfrak{A}^{\mathrm{cl}}_M + (\mathfrak{m}N)^{\mathrm{cl}}_M)^{\mathrm{cl}}_M$. By (7b), we know that $\mathfrak{A}^{\mathrm{cl}}_M \cap N = \mathfrak{A}$. Putting these equalities together we obtain $\mathfrak{A} = (\mathfrak{A}^{\mathrm{cl}}_M + (\mathfrak{m}N)^{\mathrm{cl}}_M)^{\mathrm{cl}}_M \cap N$. $\square$

The following corollary gives us properties which always hold for cl-prereductions.

**Corollary 3.1.7.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and* cl *a Nakayama closure on $\mathcal{P}$, the class of finitely generated $R$-modules. Then*

1. *Every submodule $L \subseteq N$ which is not a* cl-*reduction of $N$ in $M$ is contained in a* cl-*prereduction of $N$ in $M$.*

2. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are* cl-*prereductions of $N$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$, then $\mathfrak{A}_1 + \mathfrak{A}_2$ is a* cl-*reduction.*

3. *If $\mathfrak{A}$ is a* cl-*prereduction of $N$ in $M$. Then*

   *(a)* $\mathfrak{A} + \mathfrak{m}N = \mathfrak{A}$ *or in other words,* $\mathfrak{m}N \subseteq \mathfrak{A}$.

   *(b)* $\mathfrak{A}^{\mathrm{cl}}_M \cap N = \mathfrak{A}$.

4. *If $N$ is* cl-*closed in $M$ and $\mathfrak{A}$ is a* cl-*prereduction of $N$, then $\mathfrak{A} = \mathfrak{A}^{\mathrm{cl}}_M$.*

*Proof.* (1) By Proposition 3.1.6(2), there is some maximal element $\mathfrak{A}$ of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ which contains $L$. Such an $\mathfrak{A}$ must be a cl-prereduction since any submodule containing it must be a cl-reduction of $N$ in $M$.

(2) Since maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions, then Proposition 3.1.6(3) implies that $\mathfrak{A}_1 + \mathfrak{A}_2$ is a cl-reduction of $N$.

(3) Since maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions, then Proposition 3.1.6(7a) yields the equality in (3a) and Proposition 3.1.6(7b) yields the equality in (3b).

(4) First note that for any $L \subseteq N$, $L^{\mathrm{cl}}_M = (L \cap N)^{\mathrm{cl}}_M \subseteq L^{\mathrm{cl}}_M \cap N^{\mathrm{cl}}_M$. Hence, $\mathfrak{A}^{\mathrm{cl}}_M \subseteq \mathfrak{A}^{\mathrm{cl}}_M \cap N^{\mathrm{cl}}_M$. However since, $N$ is cl-closed in $M$, $N^{\mathrm{cl}}_M = N$. Thus $\mathfrak{A}^{\mathrm{cl}}_M \subseteq \mathfrak{A}^{\mathrm{cl}}_M \cap N$. By (3b), we see that $\mathfrak{A}^{\mathrm{cl}}_M \subseteq \mathfrak{A}$ which implies that $\mathfrak{A}^{\mathrm{cl}}_M = \mathfrak{A}$.

$\square$

**Proposition 3.1.8.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and* cl *a Nakayama closure on $\mathcal{P}$, the class of pairs of finite $R$-modules. Let $K \subseteq N \subseteq M$ be submodules of $R$ with $K$ a* cl*-reduction of $N$ in $M$. Then*

1. $\mathbf{I}'_{\mathrm{cl}}(K, M) \subseteq \mathbf{I}'_{\mathrm{cl}}(N, M)$.
2. *For each $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$, $L \cap K \in \mathbf{I}'_{\mathrm{cl}}(K, M)$.*
3. *For each maximal element of $\mathfrak{A}$ of $\mathbf{I}'_{\mathrm{cl}}(K, M)$ there exists a maximal element $\mathfrak{B}$ of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ such that $\mathfrak{B} \cap K = \mathfrak{A}$.*

*Proof.* (1) Let $L \in \mathbf{I}'_{\mathrm{cl}}(K, M)$. Since $K \subseteq N$, then $L \subseteq K \subseteq N$ and $L_M^{\mathrm{cl}} \subsetneq K_M^{\mathrm{cl}} \subseteq N_M^{\mathrm{cl}}$. Since $L$ is not a cl-reduction of $K$, it cannot be a cl-reduction of the larger module $N$. Thus $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$.

(2) If $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$, then $L \subseteq N$ and $L$ is not a cl-reduction of $N$. Note that $L \cap K \subseteq K$. To see that $L \cap K \in \mathbf{I}'_{\mathrm{cl}}(K, M)$, it is enough to see that $L \cap K$ is not a cl-reduction of $K$. Suppose that $L \cap K$ is a cl-reduction of $K$. Then $(L \cap K)_M^{\mathrm{cl}} = K_M^{\mathrm{cl}}$. Note that

$$(L \cap K)_M^{\mathrm{cl}} \subseteq L_M^{\mathrm{cl}} \cap K_M^{\mathrm{cl}}.$$

Since $K$ is a cl-reduction of $N$, then $N_M^{\mathrm{cl}} = K_M^{\mathrm{cl}} \subseteq L_M^{\mathrm{cl}} \subseteq N_M^{\mathrm{cl}}$ which gives a contradiction to $L \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. Hence, $L \cap K \in \mathbf{I}'_{\mathrm{cl}}(K, M)$.

(3) Let $\mathfrak{A}$ be a maximal element of $\mathbf{I}'_{\mathrm{cl}}(K, M)$. By (1), $\mathbf{I}'_{\mathrm{cl}}(K, M) \subseteq \mathbf{I}'_{\mathrm{cl}}(N, M)$. Thus $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ and there must exist a maximal element $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$ with $\mathfrak{A} \subseteq \mathfrak{B}$. By (2), $\mathfrak{B} \cap K \in \mathbf{I}'_{\mathrm{cl}}(K, M)$. Since $\mathfrak{A} \subseteq \mathfrak{B} \cap K$ and $\mathfrak{A}$ is maximal, we get $\mathfrak{A} = \mathfrak{B} \cap K$.

□

This leads us to the following corollary.

**Corollary 3.1.9.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* cl *a Nakayama closure on* $\mathcal{P}$*, the class of pairs of finite R-modules. Let* $L \subseteq N$ *be submodules of* $M$ *with* $L$ *a* cl-*reduction of* $N$ *in* $M$*. If* $\mathfrak{A}$ *is a* cl-*prereduction of* $L$ *then there exists a* cl-*prereduction* $\mathfrak{B}$ *of* $N$ *with* $\mathfrak{B} \cap L = \mathfrak{A}$*.*

*Proof.* This is a direct consequence of Proposition 3.1.8(3) and the fact that maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions of $N$ for any submodule $N \subseteq M$. □

We also know the relationships between cl-prereductions and other modules in the ring.

**Proposition 3.1.10.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* cl *a Nakayama closure on* $\mathcal{P}$*, the class of pairs of finite R-modules. If* $\mathfrak{A}$ *is a* cl-*prereduction of* $N$ *in* $M$ *and* $\mathfrak{A} \subseteq K \subseteq N$*, then* $\mathfrak{A}$ *is a* cl-*prereduction of* $K$ *in* $M$*.*

*Proof.* Since $\mathfrak{A}$ is a cl-prereduction of $N$ in $M$ and $\mathfrak{A} \subseteq K \subseteq N$, then $K$ is a cl-reduction of $N$ in $M$. Also $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. By Proposition 3.1.8(2), we have

$$\mathfrak{A} \cap K = \mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}}(N, M).$$

Suppose that $\mathfrak{A}$ is not a cl-prereduction of $K$ in $M$. There exists a maximal $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}}(K, M)$ with $\mathfrak{A} \subsetneq \mathfrak{B}$ and $\mathfrak{B}$ is a cl-prereduction of $K$ in $M$ by Corollary 3.1.7. Since maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions of $N$ in $M$ for any module $N$, then by Proposition 3.1.8(3) there exists a cl-prereduction $\mathfrak{C}$ of $N$ such that $\mathfrak{C} \supseteq \mathfrak{C} \cap K = \mathfrak{B} \supsetneq \mathfrak{A}$. This contradicts the maximality of $\mathfrak{A}$ in $\mathbf{I}'_{\mathrm{cl}}(N, M)$. Hence, $\mathfrak{A}$ is a cl-prereduction of $K$ in $M$. □

In [KRS20, Proposition 3.13], they show that any ideal which has no principal reductions is a union of its prereductions. Recall that a cyclic module is generated by a single element. Although a cyclic module need not be isomorphic to the ring or a principal ideal as an $R$-module, the fact that there are no cyclic cl-prereductions (i.e not generated by a single element) allows us to use a similar proof to [KRS20, Proposition 3.13] to show that submodules which do not have cyclic cl-prereductions, can be expressed as a union of their cl-prereductions.

**Proposition 3.1.11.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and* cl *a Nakayama closure on $\mathcal{P}$, the class of pairs of finite $R$-modules. Let $N \subseteq M$, be a submodule. Then*

$$N = \cup\{\mathfrak{a} \mid \mathfrak{a} \text{ a cl-prereduction of } N\}$$

*if and only if $N$ has no cyclic* cl-*reductions in $M$.*

*Proof.* Note that $N = \cup\{xR \mid x \in N\}$ as sets. Note that if $N$ has no cyclic cl-reductions then for any $x \in N$, $xR \in \mathbf{I}'_{\mathrm{cl}}(N)$ and there is a maximal element $\mathfrak{a}_x$ of $\mathbf{I}'_{\mathrm{cl}}(N)$ containing $xR$. Since

$$N = \cup\{xR \mid x \in N\} \subseteq \cup\{\mathfrak{a}_x \mid x \in N\} = \cup\{\mathfrak{a} \mid \mathfrak{a} \text{ a cl-prereduction of } N\} \subseteq N$$

we see that $N = \cup\{\mathfrak{a} \mid \mathfrak{a}$ a cl-prereduction of $N\}$ if $N$ has no cyclic cl-reductions in $M$.

Suppose now that $N$ has a cyclic cl-reduction $xR$. Then for any submodule $L$ of $M$ with $xR \subseteq L \subseteq N$, $L$ is a cl-reduction of $N$. Thus no cl-prereduction of $N$ contains $xR$, thus

$$\cup\{\mathfrak{a} \mid \mathfrak{a} \text{ a cl-prereduction of } N\} \subsetneq N.$$

□

Numerical semigroup rings give a nice source of examples where we can easily exhibit cl-prereductions for various closures. For more on the properties and computations involving $k[[x^2, x^5]]$, see Appendix A.

**Example 3.1.12.** Let $R = k[[x^2, x^5]]$, $\mathfrak{m} = (x^2, x^5)$ and $k$ a field of any characteristic. We can find the $\mathfrak{m}$bf-closures for some of the non-zero non-unital ideals of $R$ and use them to find $\mathfrak{m}$bf-prereductions and integral prereductions.

Firstly, $(x^2, x^5)_R^{\mathfrak{m}bf} = (x^2, x^5) = \mathfrak{m}$. By Proposition A.0.2, for $n \in \langle 2, 5 \rangle$,

$$(x^n)_R^{\mathfrak{m}bf} = ((x^{n+2}, x^{n+5}) :_R (x^2, x^5)) = (x^n, x^{n+3}).$$

For $n \geq 4$,

$$(x^n, x^{n+1})_R^{\mathfrak{m}bf} = ((x^{n+2}, x^{n+3}) :_R (x^2, x^5)) = (x^n, x^{n+1}).$$

For $n = 2$ and $n \geq 4$,

$$(x^n, x^{n+3})_R^{\mathfrak{m}bf} = ((x^{n+2}, x^{n+5}) :_R (x^2, x^5)) = (x^n, x^{n+3}).$$

Let $I_n = (x^n, x^{n+1})$ for $n \geq 4$. Then $I_n^- = (x^n, x^{n+1}) = I$ and $I_n^{\mathfrak{m}\mathrm{bf}} = I_n$. Note that $(x^n, x^{n+3})$ is a $\mathfrak{m}$bf-prereduction of $I_n$ and $(x^{n+1}, x^{n+2})$ is both an integral prereduction of $I_n$ and a $\mathfrak{m}$bf-prereduction of $I_n$. $\diamondsuit$

## 3.2 Expansions and Postexpansions

Following in the footsteps of Epstein and R.G. [ERG21] and Epsein, R.G., and Vassilev [ERGV23c], [ERGV23b] and [ERGV23a], we define the notion of i-postexpansions for an interior operation i on a class of modules of a Noetherian local ring $(R, \mathfrak{m})$. In Theorem 3.4.7, we see that i-postexpansions are in fact dual to cl-prereductions.

As Kemp, Ratliff and Shah defined the set of ideals $\mathbf{I}'(I)$ to be the ideals contained in $I$ which are not (integral) reductions of $I$, we can dually define for any interior operation i, the set of submodules of $B$ which contain a submodule $A$ which are not i-expansions of $A$.

$$\mathbf{C}'_{\mathrm{i}}(A,B) = \{A \subseteq C \subseteq B \mid C \text{ not an i-expansion of } A \text{ in } B\}.$$

**Definition 3.2.1.** We say $C$ is an i-*postexpansion of* $A$ *in* $B$ if $A \subseteq C \subseteq B$, $C$ not an i-expansion of $A$ in $B$, and for all submodules $D$ such that $A \subseteq D \subsetneq C \subseteq B$, $D$ is an i-expansion of $A$ in $B$.

Note that the maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N,M)$ are cl-prereductions and the minimal elements of $\mathbf{C}'_{\mathrm{i}}(A,B)$ are i-postexpansions.

The following properties hold for $\mathbf{C}'_{\mathrm{i}}(A,B)$ and the following proposition is dual to Proposition 3.1.6. However, it (and the following dual results) are proved directly rather than through properties of duality because duality only holds in the complete case and these results are more general.

**Proposition 3.2.2.** *Let $(R,\mathfrak{m})$ be an Noetherian local ring and $\mathcal{P}$ be the class of pairs of Artinian $R$-modules. Let* i *a Nakayama interior on $\mathcal{P}$. Let $A \subseteq B$ be $R$-modules such that $\mathbf{C}'_{\mathrm{i}}(A,B) \neq \emptyset$. Then the following hold:*

1. *Suppose $C \subseteq D$ are submodules of $B$ with $(C,B),(D,B) \in \mathcal{P}$, $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$. Then $D \in \mathbf{C}'_{\mathrm{i}}(A,B)$.*

2. *Let $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$. Then there exists a element $\mathfrak{A} \in \mathbf{C}'_{\mathrm{i}}(A,B)$ minimal in $\mathbf{C}'_{\mathrm{i}}(A,B)$ with $\mathfrak{A} \subseteq C$.*

3. *Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both minimal submodules in $\mathbf{C}'_{\mathrm{i}}(A,B)$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$. Then $\mathfrak{A}_1 \cap \mathfrak{A}_2 \notin \mathbf{C}'_{\mathrm{i}}(A,B)$ and $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is an* i-*expansion of $A$ in $B$.*

4. *If $C,D \in \mathbf{C}'_{\mathrm{i}}(A,B)$ then either $C \cap D \in \mathbf{C}'_{\mathrm{i}}(A,B)$ or $C \cap D$ is an* i-*expansion of $A$ in $B$.*

5. *If $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$ then $(A :_C \mathfrak{m}) \in \mathbf{C}'_{\mathrm{i}}(A,B)$.*

6. *If* $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$ *then* $C_{\mathrm{i}}^{B} + A \in \mathbf{C}'_{\mathrm{i}}(A,B)$.

7. *If* $\mathfrak{A}$ *is minimal in* $\mathbf{C}'_{\mathrm{i}}(A,B)$ *then*

   (a) $\mathfrak{A} = (A :_{\mathfrak{A}} \mathfrak{m})$ *or* $\mathfrak{m}\mathfrak{A} \subseteq A$.

   (b) $\mathfrak{A}_{\mathrm{i}}^{B} + A = \mathfrak{A}$.

   (c) $(\mathfrak{A}_{\mathrm{i}}^{B} \cap (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^{B})_{\mathrm{i}}^{B} + A = \mathfrak{A}$.

*Proof.* (1) If $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$, then $C$ is not an i-expansion of $A$. Since $C \subseteq D$ and

$$A_B^{\mathrm{i}} \subsetneq C_B^{\mathrm{i}} \subseteq D_B^{\mathrm{i}}$$

then $D$ is also not an i-expansion of $A$ in $B$.

(2) Since $C \in \mathbf{C}'_{\mathrm{i}}(A,B)$ and R Artinian, then there exists an element $\mathfrak{A} \in \mathbf{C}'_{\mathrm{i}}(A,B)$ which is minimal in $\mathbf{C}'_{\mathrm{i}}(A,B)$ and $\mathfrak{A} \subseteq C$.

(3) Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both minimal in $\mathbf{C}'_{\mathrm{i}}(A,B)$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$ and

$$A \subseteq \mathfrak{A}_1 \cap \mathfrak{A}_2 \subseteq \mathfrak{A}_1, \mathfrak{A}_2 \subseteq B,$$

then $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is an i-expansion of $A$ in $B$.

(4) Since $A \subseteq C, D \subseteq B$ then $A \subseteq C \cap D \subseteq B$. If $C \cap D \in \mathbf{C}'_{\mathrm{i}}(A,B)$, we are done. If $C \cap D \notin \mathbf{C}'_{\mathrm{i}}(A,B)$, then by definition $C \cap D$ is an i-expansion of $A$ in $B$.

(5) Suppose $(A :_C \mathfrak{m}) \notin \mathbf{C}'_{\mathrm{i}}(A,B)$. So $(A :_C \mathfrak{m})$ is an i-expansion of $A$ in $B$. Then

$$(A :_C \mathfrak{m})_{\mathrm{i}}^{B} \subseteq A \subseteq (A :_C \mathfrak{m}).$$

Because i is a Nakayama interior, $A_{\mathrm{i}}^{B} = C_{\mathrm{i}}^{B}$ and $C$ is an i-expansion of $A$ in $B$ which contradicts our assumption. So $(A :_C \mathfrak{m}) \in \mathbf{C}'_{\mathrm{i}}(A,B)$.

(6) Suppose that $C_{\mathrm{i}}^{B} + A \notin \mathbf{C}'_{\mathrm{i}}(A,B)$. Then $C_{\mathrm{i}}^{B} + A$ is an i-expansion of $A$ in $B$ and $(C_{\mathrm{i}}^{B} + A)_{\mathrm{i}}^{B} = A_{\mathrm{i}}^{B}$. Then $C_{\mathrm{i}}^{B} \subseteq (C_{\mathrm{i}}^{B} + A)_{\mathrm{i}}^{B} = A_{\mathrm{i}}^{B} \subseteq C_{\mathrm{i}}^{B}$. This is the case because

i is intensive and order preserving and $C_{\mathrm{i}}^B \subseteq C_{\mathrm{i}}^B + A$ and $A \subseteq C$. Hence, $C$ is an i-expansion of $A$ in $B$ which contradicts $C \in \mathbf{C}_{\mathrm{i}}'(A, B)$.

(7a) Since $\mathfrak{A} \in \mathbf{C}_{\mathrm{i}}'(A, B)$. By (5), we know that $(A :_{\mathfrak{A}} \mathfrak{m}) \in \mathbf{C}_{\mathrm{i}}'(A, B)$. Since $\mathfrak{A}$ is minimal in $\mathbf{C}_{\mathrm{i}}'(N, M)$ and $\mathfrak{A} \supseteq (A :_{\mathfrak{A}} \mathfrak{m})$ then we must have $(A :_{\mathfrak{A}} \mathfrak{m}) = \mathfrak{A}$ since $(A :_{\mathfrak{A}} \mathfrak{m})$ is not a i-expansion of $A$. Also the equality, $(A :_{\mathfrak{A}} \mathfrak{m}) = \mathfrak{A}$ implies that $\mathfrak{m}\mathfrak{A} \subseteq A$.

(7b) Since $\mathfrak{A}$ is minimal in $\mathbf{C}_{\mathrm{i}}'(A, B)$ and $\mathfrak{A} \supseteq \mathfrak{A}_{\mathrm{i}}^B + A$, by (6), $\mathfrak{A}_{\mathrm{i}}^B + A \in \mathbf{C}_{\mathrm{cl}}'(A, B)$; meaning we must have $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}}^B + A$.

(7c) Since $(A :_{\mathfrak{A}} \mathfrak{m}) \supseteq (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B \supseteq \mathfrak{A}_{\mathrm{i}}^B \cap (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B$, we have

$$(\mathfrak{A}_{\mathrm{i}}^B \cap (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B)_{\mathrm{i}}^B = (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B.$$

By (7a), we know that $\mathfrak{A} = (A :_{\mathfrak{A}} \mathfrak{m})$ and hence $\mathfrak{A}_{\mathrm{i}}^B = (\mathfrak{A}_{\mathrm{i}}^B \cap (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B)_{\mathrm{i}}^B$. By (7b), we know that $\mathfrak{A}_{\mathrm{i}}^B + A = \mathfrak{A}$. Putting these equalities together we obtain

$$\mathfrak{A} = (\mathfrak{A}_{\mathrm{i}}^B \cap (A :_{\mathfrak{A}} \mathfrak{m})_{\mathrm{i}}^B)_{\mathrm{i}}^B + A.$$

□

The following corollary is dual to the results found in Corollary 3.1.7.

**Corollary 3.2.3.** *Let $(R, \mathfrak{m})$ be a Noetherian ring and $\mathcal{P}$ be the class of pairs of Artinian $R$-modules. Let* i *be a Nakayama interior on $\mathcal{P}$. Then*

1. *Every submodule $C \supseteq A$ such that which is not an* i*-expansion of $A$ in $B$ contains an* i*-postexpansion of $A$ in $B$.*

2. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are* i*-postexpansions of $A$ with $\mathfrak{A}_1 \neq \mathfrak{A}_2$, then $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is a* i*-expansion.*

3. *If $\mathfrak{A}$ is an* i*-postexpansion of $A$ in $B$ then*

(a) $(A :_{\mathfrak{A}} \mathfrak{m}) = \mathfrak{A}$ *or in other words,* $\mathfrak{m}\mathfrak{A} \subseteq A$.

(b) $\mathfrak{A}_{\mathrm{i}}^{B} + A = \mathfrak{A}$.

4. *If* $A$ *is* i*-open in* $B$ *and* $\mathfrak{A}$ *is an* i*-postexpansion of* $A$*, then* $\mathfrak{A}_{\mathrm{i}}^{B} = \mathfrak{A}$.

*Proof.* (1) By Proposition 3.2.2(2), there exists some minimal element $\mathfrak{A}$ of $\mathbf{C}'_{\mathrm{i}}(A, B)$ with $\mathfrak{A} \subseteq C$. Such an $\mathfrak{A}$ must be an i-postexpansion since any submodule it contains must be an i-expansion of $A$ in $B$.

(2) Since minimal elements of $\mathbf{C}'_{\mathrm{i}}(A, B)$ are i-postexpansions, then Proposition 3.2.2(3) implies that $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is a i-expansion of $A$.

(3) Since minimal elements of $\mathbf{C}'_{\mathrm{i}}(A, B)$ are i-postexpansions, then Proposition 3.2.2(7a) yields the equality in (3a) and Proposition 3.2.2(7b) yields the equality in (3b).

(4) First note that for any $A \subseteq C \subseteq B$, $C_{\mathrm{i}}^{B} = (C + A)_{\mathrm{i}}^{B} \supseteq C_{\mathrm{i}}^{B} + A_{\mathrm{i}}^{B}$. Hence,

$$\mathfrak{A}_{\mathrm{i}}^{B} \supseteq \mathfrak{A}_{\mathrm{i}}^{B} + A_{\mathrm{i}}^{B}.$$

However since, $A$ is i-open in $B$, $A_{\mathrm{i}}^{B} = A$. Thus $\mathfrak{A}_{\mathrm{i}}^{B} \supseteq \mathfrak{A}_{\mathrm{i}}^{B} + A$. By (3b), we see that $\mathfrak{A}_{\mathrm{i}}^{B} \supseteq \mathfrak{A}$ which implies that $\mathfrak{A}_{\mathrm{i}}^{B} = \mathfrak{A}$.

□

The following proposition is dual to Proposition 3.1.8.

**Proposition 3.2.4.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* i *a Nakayama interior on* $\mathcal{P}$*, the class of Artinian* $R$*-modules. Let* $A \subseteq C \subseteq B$ *be submodules of* $R$ *with* $C$ *an* i*-expansion of* $A$ *in* $B$*. Then*

1. $\mathbf{C}'_{\mathrm{i}}(C, B) \subseteq \mathbf{C}'_{\mathrm{i}}(A, B)$.

2. *For each* $D \in \mathbf{C}'_{\mathrm{i}}(A, B)$, $D + C \in \mathbf{C}'_{\mathrm{i}}(C, B)$.

3. *For each minimal element* $\mathfrak{A}$ *of* $\mathbf{C}'_{\mathrm{i}}(C, B)$ *there exists a minimal element* $\mathfrak{B}$ *of* $\mathbf{C}'_{\mathrm{i}}(A, B)$ *such that* $\mathfrak{A} + C = \mathfrak{B}$.

*Proof.* (1) Let $D \in \mathbf{C}'_{\mathrm{i}}(C, B)$. Since $A \subseteq C$, then $A \subseteq C \subseteq D$ and $A^B_{\mathrm{i}} \subseteq C^B_{\mathrm{i}} \subsetneq D^B_{\mathrm{i}}$. Since $D$ is not an i-expansion of $C$, it cannot be an i-expansion of the smaller module $A$. Thus $D \in \mathbf{C}'_{\mathrm{i}}(A, B)$.

(2) If $D \in \mathbf{C}'_{\mathrm{i}}(A, B)$, then $A \subseteq D$ and $D$ is not an i-expansion of $A$. Note that $D \subseteq D + C$. To see that $D + C \in \mathbf{C}'_{\mathrm{i}}(C, B)$, it is enough to see that $D + C$ is not an i-expansion of $C$. Suppose that $D + C$ is an i-expansion of $C$. Then $(D + C)^B_{\mathrm{i}} = C^B_{\mathrm{i}} \subseteq D^B_{\mathrm{i}} + C^B_{\mathrm{i}}$. Since $C$ is an i-expansion of $A$ and $D \in \mathbf{C}'_{\mathrm{i}}(A, B)$, $A^B_{\mathrm{i}} = \mathbf{C}^B_{\mathrm{i}} \subseteq D^B_{\mathrm{i}} \subseteq A^B_{\mathrm{i}}$. So $D^B_i = A^B_i$ which contradicts $D \in \mathbf{C}'_{\mathrm{i}}(A, B)$. Hence $D + C \in \mathbf{C}'_{\mathrm{i}}(C, B)$.

(3) Let $\mathfrak{A}$ be a minimal element of $\mathbf{C}'_{\mathrm{i}}(C, B)$. By (1), $\mathbf{C}'_{\mathrm{i}}(C, B) \subseteq \mathbf{C}'_{\mathrm{i}}(A, B)$. Thus $\mathfrak{A} \in \mathbf{C}'_{\mathrm{i}}(A, B)$ and there must be a minimal element $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}}(A, B)$ such that $\mathfrak{B} \subseteq \mathfrak{A} \subseteq \mathfrak{A} + C$. By (2), $\mathfrak{A} + C \in \mathbf{C}'_{\mathrm{i}}(C, B)$. Since $\mathfrak{B} \subseteq \mathfrak{A} + C$ and $\mathfrak{A}$ is minimal, we get $\mathfrak{A} + C = \mathfrak{B}$.

□

The following proposition is dual to Proposition 3.1.10.

**Proposition 3.2.5.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* i *a Nakayama interior on* $\mathcal{P}$*, the class of Artinian* $R$*-modules. If* $\mathfrak{A}$ *is an* i*-postexpansion of* $A$ *in* $B$ *and* $A \subseteq C \subseteq \mathfrak{A}$*, then* $\mathfrak{A}$ *is an* i*-postexpansion of* $C$ *in* $B$.

*Proof.* Since $\mathfrak{A}$ is an i-postexpansion of $A$ in $B$ and $A \subseteq C \subseteq \mathfrak{A}$, then $C$ is an i-expansion of $A$ in $B$. Also, $\mathfrak{A} \in \mathbf{C}'_{\mathrm{i}}(A, B)$. By Proposition 3.2.4(2), we have $\mathfrak{A} + C = \mathfrak{A} \in \mathbf{C}'_{\mathrm{i}}(A, B)$.

Suppose that $\mathfrak{A}$ is not an i-postexpansion of $C$ in $B$. Then there is a minimal $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}}(C,B)$ with $\mathfrak{B} \subsetneq \mathfrak{A}$ and $\mathfrak{B}$ an i-postexpansion of $C$ in $B$. Then by Proposition 3.2.4(3) and since minimal elements of $\mathbf{C}'_{\mathrm{i}}(A,B)$ are i-postexpansions of $A$ in $B$, there exists an i-postexpansion $\mathfrak{C}$ of $A$ in $B$ such that $\mathfrak{C} \subseteq \mathfrak{C}+C = \mathfrak{B} \subsetneq \mathfrak{A}$. This contradicts the minimality of $\mathfrak{A}$ in $\mathbf{C}'_{\mathrm{i}}(A,B)$. Hence, $\mathfrak{A}$ is an i-postexpansion of $C$ in $B$. □

**Definition 3.2.6.** [ERG21, Definition 4.7] Let $(R,\mathfrak{m})$ be a local ring and $L \subseteq M$ be $R$-modules. The $\mathfrak{m}$-basically empty interior of $L$ with respect to $M$ is

$$L^M_{\mathfrak{m}\mathrm{be}} := \mathfrak{m}(L :_M \mathfrak{m}).$$

**Example 3.2.7.** Let $R = k[[x^2,x^5]]$, $\mathfrak{m} = (x^2,x^5)$ and $k$ a field of any characteristic. By using Proposition A.0.2, we can find the $\mathfrak{m}$be-interiors, and thus $\mathfrak{m}$be-expansions and $\mathfrak{m}$be-postexpansions, for some of the non-zero non-unital ideals of $R$.

$$(x^2,x^5)^R_{\mathfrak{m}be} = (x^2,x^5) = \mathfrak{m}$$

$$(x^n)^R_{\mathfrak{m}\mathrm{be}} = (x^{n+2},x^{n+5})$$

$$(x^n,x^{n+1})^R_{\mathfrak{m}be} = \begin{cases} (x^4,x^7) & \text{if } n=4 \\ (x^6,x^7) & \text{if } n=5 \\ (x^n,x^{n+1}) & \text{if } n \geq 6 \end{cases}$$

$$(x^n,x^{n+3})^R_{\mathfrak{m}be} = \begin{cases} (x^4,x^7) & \text{if } n=4 \\ (x^7,x^8) & \text{if } n=5 \\ (x^n,x^{n+3}) & \text{if } n \geq 6 \end{cases}$$

Let $I = (x^4,x^7)$. Then $(x^4,x^5)$ is a $\mathfrak{m}$be-expansion of $I$ since

$$(x^4,x^5)^R_{\mathfrak{m}\mathrm{be}} = I^R_{\mathfrak{m}\mathrm{be}} = I$$

and $I \subseteq (x^4,x^5)$. Also, $(x^4)^R_{\mathfrak{m}\mathrm{be}} = (x^6,x^9) \subseteq I^R_{\mathfrak{m}\mathrm{be}} = I$ and $I^R_{\mathfrak{m}\mathrm{be}} \not\subseteq (x^4)$, $I$ is a $\mathfrak{m}$be-postexpansion of $(x^4)$. ♢

## 3.3 Comparable Operations

Some closures, such as integral and tight closure, can be difficult to compute; in that case, it can be useful to compare it to a closure which is not so difficult. To that end, we explore how comparable operations relate to cl-reductions, cl-prereductions, i-expansions, and i-postexpansions.

**Definition 3.3.1.** Let $p_1$ and $p_2$ be pair operations defined on $\mathcal{P}$ a collection of pairs of modules in $R$. We say $p_1 \leq p_2$ if $p_1(N, M) \subseteq p_2(N, M)$ for all $(N, M) \in \mathcal{P}$. We call $p_1$ and $p_2$ *comparable* if $p_1 \leq p_2$ or $p_2 \leq p_1$.

**Remark 3.3.2.** Let $\mathrm{cl}_1$ and $\mathrm{cl}_2$ be closure operations defined on the submodules of $R$. We say $\mathrm{cl}_1 \leq \mathrm{cl}_2$ if $N_M^{\mathrm{cl}_1} \subseteq N_M^{\mathrm{cl}_2}$ for all $(N, M) \in \mathcal{P}$. So $\mathrm{cl}_1$ and $\mathrm{cl}_2$ are *comparable* if $\mathrm{cl}_1 \leq \mathrm{cl}_2$ or $\mathrm{cl}_2 \leq \mathrm{cl}_1$.

Let $\mathrm{i}_1$ and $\mathrm{i}_2$ be interior operations defined on the submodules of $R$. We say $\mathrm{i}_1 \leq \mathrm{i}_2$ if $A_{\mathrm{i}_1}^B \subseteq A_{\mathrm{i}_2}^B$ for all $(A, B) \in \mathcal{P}$. So $\mathrm{i}_1$ and $\mathrm{i}_2$ are *comparable* if $\mathrm{i}_1 \leq \mathrm{i}_2$ or $\mathrm{i}_2 \leq \mathrm{i}_1$. $\diamondsuit$

Note that integral closure and tight closure are comparable ($I^* \subseteq I^-$ by [HH90, Theorem 5.2]) and integral closure and basically full closure are comparable ($I^{\mathrm{mbf}} \subseteq I^-$ by [Rat89, Theorem 3.2.1] as noted in [HRR02, End of first full paragraph on p. 376]); however, tight closure and basically full closure are not comparable for general Noetherian rings.

We first explore relationships between comparable $\mathrm{cl}_i$-prereductions for $i = 1, 2$, with $\mathrm{cl}_1 \leq \mathrm{cl}_2$ and between comparable $\mathrm{i}_i$-postexpansions for $i = 1, 2$, with $\mathrm{i}_1 \leq \mathrm{i}_2$.

**Proposition 3.3.3.** *Let $R$ be a Noetherian ring.*

1. *Let $\mathcal{M}$ be a category of $R$-modules and $\mathcal{P}$ the class of pairs of modules in $\mathcal{M}$. Suppose $\mathrm{cl}_1$ and $\mathrm{cl}_2$ are comparable closure operations with $\mathrm{cl}_1 \leq \mathrm{cl}_2$. Then we have*

(a) $(N_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2} = (N_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_1}$, *and*

(b) $N_M^{\mathrm{cl}_1}$ *is a* $\mathrm{cl}_2$*-reduction of* $N_M^{\mathrm{cl}_2}$.

2. *Let* $\mathcal{M}$ *be the category of* $R$*-modules and* $\mathcal{P}$ *the class of pairs modules in* $\mathcal{M}$. *Suppose* $\mathrm{i}_1$ *and* $\mathrm{i}_2$ *are comparable interior operations with* $\mathrm{i}_1 \leq \mathrm{i}_2$. *Then we have*

(a) $(A_{\mathrm{i}_1}^B)_{\mathrm{i}_2}^B = A_{\mathrm{i}_1}^B = (A_{\mathrm{i}_2}^B)_{\mathrm{i}_1}^B$, *and*

(b) $A_{\mathrm{i}_2}^B$ *is an* $\mathrm{i}_1$*-expansion of* $A_{\mathrm{i}_1}^B$.

*Proof.* (1a) Since $N \subseteq N_M^{\mathrm{cl}_1} \subseteq N_M^{\mathrm{cl}_2}$, we have

$$N_M^{\mathrm{cl}_2} \subseteq (N_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} \subseteq (N_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2}$$

which implies $(N_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2}$. Also, note that

$$N_M^{\mathrm{cl}_2} \subseteq (N_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_1} \subseteq (N_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2}$$

yields $N_M^{\mathrm{cl}_2} = (N_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_1}$.

(1b) By (1a), we have $(N_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2}$ and by definition we get that $N_M^{\mathrm{cl}_1}$ is a $\mathrm{cl}_2$-reduction of $N_M^{\mathrm{cl}_2}$.

(2a) Since $A_{\mathrm{i}_1}^B \subseteq A_{\mathrm{i}_2}^B \subseteq A$, we have

$$A_{\mathrm{i}_1}^B = (A_{\mathrm{i}_1}^B)_{\mathrm{i}_1}^B \subseteq (A_{\mathrm{i}_2}^B)_{\mathrm{i}_1}^B \subseteq A_{\mathrm{i}_1}^B$$

which implies $A_{\mathrm{i}_1}^B = (A_{\mathrm{i}_2}^B)_{\mathrm{i}_1}^B$. Also, note that

$$A_{\mathrm{i}_1}^B = (A_{\mathrm{i}_1}^B)_{\mathrm{i}_1}^B \subseteq (A_{\mathrm{i}_1}^B)_{\mathrm{i}_2}^B \subseteq A_{\mathrm{i}_1}^B$$

yields $(A_{\mathrm{i}_1}^B)_{\mathrm{i}_2}^B = A_{\mathrm{i}_1}^B$.

(2b) By (2a), we have $A_{\mathrm{i}_1}^B = (A_{\mathrm{i}_2}^B)_{\mathrm{i}_1}^B$ and by definition we get that $A_{\mathrm{i}_2}^B$ is an $\mathrm{i}_1$-expansion of $A_{\mathrm{i}_1}^B$ in $B$. $\square$

**Proposition 3.3.4.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathrm{cl}_1$ and $\mathrm{cl}_2$ are comparable Nakayama closures on the class of finitely generated pairs $\mathcal{P}$ with $\mathrm{cl}_1 \leq \mathrm{cl}_2$.*

1. $\mathbf{I}'_{\mathrm{cl}_2}(N, M) \subseteq \mathbf{I}'_{\mathrm{cl}_1}(N, M)$ *for all* $(N, M) \in \mathcal{P}$.
2. *If* $\mathbf{I}'_{\mathrm{cl}_2}(N, M) \neq \emptyset$, *then* $\mathbf{I}'_{\mathrm{cl}_1}(N, M) \neq \emptyset$.
3. *If* $L \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$, *then* $L_M^{\mathrm{cl}_1} \cap N \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$ *if and only if* $L \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$.

*Proof.* (1) Suppose $L \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$. Then $L$ is not a $\mathrm{cl}_2$-reduction of $N$ in $M$. So $L \subseteq N$ and $L_M^{\mathrm{cl}_2} \subsetneq N_M^{\mathrm{cl}_2}$. If $L$ were a $\mathrm{cl}_1$-reduction of $N$ in $M$, then $N_M^{\mathrm{cl}_1} = L_M^{\mathrm{cl}_1} \subseteq L_M^{\mathrm{cl}_2}$ which implies by Proposition 3.3.3(1a) $N_M^{\mathrm{cl}_2} = (N_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} \subseteq L_M^{\mathrm{cl}_2} \subsetneq N_M^{\mathrm{cl}_2}$ which is a contradiction. Thus $L$ is not a $\mathrm{cl}_1$-reduction of $N$ in $M$ and so $L \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$.

(2) Since $\mathbf{I}'_{\mathrm{cl}_2}(N, M) \neq \emptyset$ and $\mathbf{I}'_{\mathrm{cl}_2}(N, M) \subseteq \mathbf{I}'_{\mathrm{cl}_1}(N, M)$, then $\mathbf{I}'_{\mathrm{cl}_1}(N, M) \neq \emptyset$.

(3) Suppose $L \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$. Then since $L_M^{\mathrm{cl}_1} \subseteq L_M^{\mathrm{cl}_2}$ and by Proposition 3.1.6(6)

$$L_M^{\mathrm{cl}_2} \cap N \in \mathbf{I}'_{\mathrm{cl}_2}(N, M),$$

we see that by Proposition 3.3.4(1), $L_M^{\mathrm{cl}_1} \cap N \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$.

Suppose $L \in \mathbf{I}'_{\mathrm{cl}_1}(N, M) \backslash \mathbf{I}'_{\mathrm{cl}_2}(N, M)$. Then $L_M^{\mathrm{cl}_1} \subsetneq N_M^{\mathrm{cl}_1} \subseteq N_M^{\mathrm{cl}_2}$. Since

$$L_M^{\mathrm{cl}_2} = (L_M^{\mathrm{cl}_1})_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2},$$

we see $L_M^{\mathrm{cl}_1}$ is a $\mathrm{cl}_2$-reduction of $N$ in $M$ and $L \subseteq L_M^{\mathrm{cl}_1} \cap N \subseteq L_M^{\mathrm{cl}_2} = N_M^{\mathrm{cl}_2}$. Applying $\mathrm{cl}_2$ to this chain, we see that $L_M^{\mathrm{cl}_2} \subseteq (L_M^{\mathrm{cl}_1} \cap N)_M^{\mathrm{cl}_2} \subseteq (L_M^{\mathrm{cl}_2})_M^{\mathrm{cl}_2} = L_M^{\mathrm{cl}_2}$. Thus $L_M^{\mathrm{cl}_1} \cap N$ is a $\mathrm{cl}_2$-reduction of $N$ in $M$ and $L_M^{\mathrm{cl}_1} \cap N \notin \mathbf{I}'_{\mathrm{cl}_2}(N, M)$.

□

**Proposition 3.3.5.** *Le $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathrm{cl}_1$ and $\mathrm{cl}_2$ comparable Nakayama closures defined on the class of pairs $\mathcal{P}$ with $\mathrm{cl}_1 \leq \mathrm{cl}_2$. Then*

1. *For every* $\mathrm{cl}_2$*-prereduction* $\mathfrak{A}$ *of* $N$ *in* $M$*, there exists a* $\mathrm{cl}_1$*-prereduction* $\mathfrak{B}$ *with* $\mathfrak{A} \subseteq \mathfrak{B}$.

2. *If* $N = N_M^{\mathrm{cl}_2}$ *and* $\mathfrak{A}$ *is a* $\mathrm{cl}_2$*-prereduction of* $N$ *in* $M$*, then* $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_1} = \mathfrak{A}_M^{\mathrm{cl}_2}$.

3. *If* $N = N_M^{\mathrm{cl}_1}$ *and* $\mathfrak{A}$ *is both a* $\mathrm{cl}_1$*- and* $\mathrm{cl}_2$*- prereduction of* $N$ *in* $M$*, then*
$$\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_1} = \mathfrak{A}_M^{cl_2} \cap N.$$

4. *Suppose* $\mathfrak{A}$ *is a* $\mathrm{cl}_1$*-prereduction of* $N$ *in* $M$ *and* $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_2}$*. Then* $\mathfrak{A}$ *is a* $\mathrm{cl}_2$*-prereduction of* $N$ *in* $M$*.*

*Proof.* (1) Since $\mathfrak{A}$ is a $\mathrm{cl}_2$-prereduction of $N$ in $M$, then it is a maximal element of $\mathbf{I}'_{\mathrm{cl}_2}(N, M)$. By Proposition 3.3.4(1), $\mathfrak{A} \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$. By Proposition 3.1.6(2) there then exists a maximal element $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$ with $\mathfrak{A} \subseteq \mathfrak{B}$.

(2) By Corollary 3.1.7(4) we know that $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_2}$. Since $\mathfrak{A} \subseteq \mathfrak{A}_M^{\mathrm{cl}_2} \subseteq \mathfrak{A}_M^{\mathrm{cl}_2}$, we can conclude that $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_1} = \mathfrak{A}_M^{\mathrm{cl}_2}$.

(3) By Corollary 3.1.7(4) we know that $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_1}$. By Proposition 3.1.7(3b), $\mathfrak{A} = \mathfrak{A}_M^{\mathrm{cl}_2} \cap N$. Combining the equalities gives the result.

(4) Suppose $\mathfrak{A}$ is not a $\mathrm{cl}_2$-prereduction of $N$ in $M$. Then there exists a $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}_2}(N, M)$ with $\mathfrak{A} \subsetneq \mathfrak{B}$. Since $\mathbf{I}'_{\mathrm{cl}_2}(N, M) \subseteq \mathbf{I}'_{\mathrm{cl}_1}(N, M)$, then $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$. Since $\mathfrak{A}$ is a $\mathrm{cl}_1$-prereduction of $N$ in $M$ and $\mathfrak{A} \subsetneq \mathfrak{B}$, $\mathfrak{B}$ must be a $\mathrm{cl}_1$-reduction of $N$ in $M$ which contradicts $\mathfrak{B} \in \mathbf{I}'_{\mathrm{cl}_1}(N, M)$. Thus $\mathfrak{A}$ must be a $\mathrm{cl}_2$-prereduction of $N$ in $M$. $\square$

Now we move on to the dual results for comparable $\mathrm{i}_j$-postexpansions. While they can be proved using properties of duality, they are proved directly instead.

This first proposition is dual to Proposition 3.3.4.

**Proposition 3.3.6.** $(R, \mathfrak{m})$ *a Noetherian ring and* $\mathcal{P}$ *be the class of pairs of Artinian* $R$*-modules and* $\mathrm{i}_1$ *and* $\mathrm{i}_2$ *be comparable Nakayama interiors on* $\mathcal{P}$ *with* $\mathrm{i}_1 \leq \mathrm{i}_2$*. Then*

1. $\mathbf{C}'_{\mathrm{i}_1}(A, B) \subseteq \mathbf{C}'_{\mathrm{i}_2}(A, B)$ *for all* $(A, B) \in \mathcal{P}$.
2. *If* $\mathbf{C}'_{\mathrm{i}_1}(A, B) \neq \emptyset$, *then* $\mathbf{C}'_{\mathrm{i}_2}(A, B) \neq \emptyset$.
3. *If* $C \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$, *then* $C^B_{\mathrm{i}_2} + A \in \mathbf{C}'_{\mathrm{i}_1}(A, B)$ *if and only if* $C \in \mathbf{C}'_{\mathrm{i}_1}(A, B)$.

*Proof.* (1) Suppose $C \in \mathbf{C}'_{\mathrm{i}_1}(A, B)$. Then $C$ is not an $\mathrm{i}_1$-expansion of $A$ in $B$. So $A \subseteq C \subseteq B$ and $A^B_{\mathrm{i}_1} \subsetneq C^B_{\mathrm{i}_1}$. If $C$ was an $\mathrm{i}_2$-expansion of $A$ in $B$, then

$$A^B_{\mathrm{i}_2} = C^B_{\mathrm{i}_2} \supseteq C^B_{\mathrm{i}_1}$$

which implies by 3.3.3(2a)

$$A^B_{\mathrm{i}_1} = (A^B_{\mathrm{i}_2})^B_{\mathrm{i}_1} \supseteq (C^B_{\mathrm{i}_2})^B_{\mathrm{i}_1} \supsetneq (C^B_{\mathrm{i}_1})^B_{\mathrm{i}_1} = C^B_{\mathrm{i}_1}.$$

This is a contradiction. Thus $C$ is not an $\mathrm{i}_2$-expansion of $A$ in $B$ and $C \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$.

(2) Since $\mathbf{C}'_{\mathrm{i}_1}(A, B) \neq \emptyset$ and $\mathbf{C}'_{\mathrm{i}_1}(A, B) \subseteq \mathbf{C}'_{\mathrm{i}_2}(A, B)$, then $\mathbf{C}'_{\mathrm{i}_2}(A, B) \neq \emptyset$.

(3) If $C \in \mathbf{C}'_{\mathrm{i}_1}(A, B)$, then since $C^B_{\mathrm{i}_1} \subseteq C^B_{\mathrm{i}_2}$ and by Proposition 3.2.2(6),

$$C^B_{\mathrm{i}_1} + A \in \mathbf{C}'_{\mathrm{i}_1}(A, B).$$

We see that by Proposition 3.2.2(1),

$$C^B_{\mathrm{i}_2} + A \in \mathbf{C}'_{\mathrm{i}_1}(A, B).$$

Suppose $C \in \mathbf{C}'_{\mathrm{i}_2}(A, B) \backslash \mathbf{C}'_{\mathrm{i}_1}(A, B)$. Then $A^B_{\mathrm{i}_1} \subseteq A^B_{\mathrm{i}_2} \subsetneq C^B_{\mathrm{i}_2}$. Since by 3.3.3(2a),

$$A^B_{\mathrm{i}_1} = (C^B_{\mathrm{i}_2})^B_{\mathrm{i}_1} = C^B_{\mathrm{i}_1},$$

we see $C^B_{\mathrm{i}_2}$ is an $\mathrm{i}_1$-expansions of $A$ in $B$ and

$$A^B_{\mathrm{i}_1} = C^B_{\mathrm{i}_1} \subseteq C^B_{\mathrm{i}_1} + A \subseteq C^B_{\mathrm{i}_2} + A \subseteq C.$$

Applying $\mathrm{i}_1$ to this chain, we see that $A_{\mathrm{i}_1}^B \subseteq C_{\mathrm{i}_1}^B \subseteq (C_{\mathrm{i}_2}^B + A)_{\mathrm{i}_1}^B \subseteq C_{\mathrm{i}_1}^B$. Thus $C_{\mathrm{i}_2}^B + A$ is an $\mathrm{i}_1$-expansion of $A$ in $B$ and $C_{\mathrm{i}_2}^B + A \notin \mathbf{C}'_{\mathrm{i}_1}(A, B)$.

Suppose that $C \notin \mathbf{C}'_{\mathrm{i}_1}(A, B)$. Then $(A :_C \mathfrak{m})$ is either an $\mathrm{i}_1$-expansion of $A$ in $B$ or $A_{\mathrm{i}_1}^B = (A :_C \mathfrak{m})_{\mathrm{i}_1}^B$. Since $\mathrm{i}_1$ is a Nakayama interior, then $A_{\mathrm{i}_1}^B = C_{\mathrm{i}_1}^B$. Thus $C \notin \mathbf{C}'_{\mathrm{i}_1}(A, B)$. $\square$

The following proposition is dual to Proposition 3.3.5.

**Proposition 3.3.7.** *Let $(R, \mathfrak{m})$ be a Noetherian ring and $\mathcal{P}$ be the class of pairs of Artinian $R$-modules and $\mathrm{i}_1$ and $\mathrm{i}_2$ be comparable Nakayama interiors on the submodules of $R$ with $\mathrm{i}_1 \leq \mathrm{i}_2$. Then*

1. *For every $\mathrm{i}_1$-postexpansion $\mathfrak{A}$ of $A$ in $B$, there exists an $\mathrm{i}_2$-postexpansion $\mathfrak{B}$ with $\mathfrak{B} \subseteq \mathfrak{A}$.*
2. *If $A = A_{\mathrm{i}_1}^B$ and $\mathfrak{A}$ is an $\mathrm{i}_1$-postexpansion of $A$ in $B$, then $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_1}^B = \mathfrak{A}_{\mathrm{i}_2}^B$.*
3. *If $A = A_{\mathrm{i}_2}^B$ and $\mathfrak{A}$ is both an $\mathrm{i}_1$-postexpansion and $\mathrm{i}_2$-postexpansion of $A$ in $B$, then $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_2}^B = \mathfrak{A}_{\mathrm{i}_1}^B + A$.*
4. *Suppose $\mathfrak{A}$ is an $\mathrm{i}_2$-postexpansion of $A$ in $B$ and $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_1}^B$. Then $\mathfrak{A}$ is an $\mathrm{i}_1$-postexpansion of $A$ in $B$.*

*Proof.* (1) Since $\mathfrak{A}$ is an $\mathrm{i}_1$-postexpansion of $A$ in $B$, then it is a minimal element of $\mathbf{C}'_{\mathrm{i}_1}(A, B)$. By Proposition 3.3.6(1) $\mathfrak{A} \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$. By Proposition 3.2.2(2) there then exists a minimal element $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$ with $\mathfrak{B} \subseteq \mathfrak{A}$.

(2) By Corollary 3.2.3(4), we know that $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_1}^B$. Since $\mathfrak{A}_{\mathrm{i}_1}^B \subseteq \mathfrak{A}_{\mathrm{i}_2}^B \subseteq \mathfrak{A}$, we can conclude that $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_1}^B = \mathfrak{A}_{\mathrm{i}_2}^B$.

(3) By Corollary 3.2.3(4), we know that $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_2}^B$ and by Corollary 3.2.3(3b), $\mathfrak{A} = \mathfrak{A}_{\mathrm{i}_2}^B + A$. Combining the equalities gives the result.

(4) Suppose $\mathfrak{A}$ is not an $\mathrm{i}_1$-postexpansion of $A$ in $B$. Then there exists some $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}_1}(A, B)$ with $\mathfrak{B} \subsetneq \mathfrak{A}$. Since $\mathbf{C}'_{\mathrm{i}_1}(A, B) \subseteq \mathbf{C}'_{\mathrm{i}_2}(A, B)$, then $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$. Since $\mathfrak{A}$ is an $\mathrm{i}_2$-postexpansion of $A$ in $B$ and $\mathfrak{B} \subsetneq \mathfrak{A}$, $\mathfrak{B}$ must be an $\mathrm{i}_2$-expansion of $A$ in $B$ which contradicts $\mathfrak{B} \in \mathbf{C}'_{\mathrm{i}_2}(A, B)$. Thus $\mathfrak{A}$ must be an $\mathrm{i}_1$-postexpansion of $A$ in $B$. $\square$

We provide examples illustrating the above Propositions, in particular, they demonstrates how the choice of closure determines the form of the reductions. For Example 3.3.8 we use that the $\mathfrak{m}$-basically full closure of an ideal is always contained in its integral closure [HRR02]. In Example 3.3.9, we use that the tight closure of an ideal is always contained in its integral closure [HH90].

**Example 3.3.8.** We will compare $\mathfrak{m}$bf-prereductions to integral prereductions to demonstrate how these Propositions can be applied.

Let $R = k[[x^2, x^5]]$, $\mathfrak{m} = (x^2, x^5)$ and $k$ is a field of any characteristic. For the $\mathfrak{m}$ basically full calculations, refer to Proposition A.0.2. Consider the ideal $I = (x^6, x^7)$.

Note that $I^- = (x^6, x^7)$ and $I^{\mathfrak{m}\mathrm{bf}} = (\mathfrak{m}I :_R \mathfrak{m}) = (x^6, x^7)$. Note that $(x^6, x^9)$ is an integral reduction of $I$ but not an $\mathfrak{m}$ bf-reduction of $I$ since $(x^6, x^9)^{\mathfrak{m}\mathrm{bf}} = (x^6, x^9)$. Since $I/(x^6, x^9) \cong k$, then $I$ is a non-$\mathfrak{m}$ basically full cover of $(x^6, x^9)$. (For more on covers, see Section 4.1.) Now by Remark 4.1.7, we see that $(x^6, x^9)$ is a $\mathfrak{m}$ basically full prereduction of $I$ which is not an integral prereduction of $I$.

However, the ideal $(x^7, x^8)$ is neither an integral reduction nor an $\mathfrak{m}$ basically full reduction of $I$ and $I/(x^7, x^8) \cong k$ implies that $I$ is a non-integral cover and a non-$\mathfrak{m}$ basically full cover of $(x^7, x^8)$. Again we use Remark 4.1.7 to see that $(x^7, x^8)$ is both an integral prereduction of $I$ and a $\mathfrak{m}$ basically full prereduction of $I$.

In fact, since the integrally closed ideals in $R$ have the form $(x^n)k[[x]] \cap R$, $(x^7, x^8)$ is the unique integral prereduction of $I$. Whereas, $I$ has multiple $\mathfrak{m}$-basically full prereductions.

This example also nicely illustrates Proposition 3.1.11. $I$ has principal integral reductions $(f)$ where $f = \sum_{n=6}^{\infty} a_n x^n$ and $a_6 \neq 0$, and $I \neq (x^7, x^8)$ which is the union of its integral prereductions. Whereas $(f)$ is not an $\mathfrak{m}$ basically full reduction of $I$ since

$$(f)^{\mathfrak{m}\mathrm{bf}} = (a_6 x^6 + a_7 x^7, x^9) \neq I.$$

So $I$ has no principal $\mathfrak{m}$ basically full reductions and $I$ is seen to be the union of its $\mathfrak{m}$ basically full prereductions.

**Example 3.3.9.** We will illustrate in this example that the integral prereductions and $*$-prereductions of $\mathfrak{m}^2$ can be quite different. Let $R = k[x, y, z]/(x^2 - y^3 - z^6)$ and $\mathfrak{m} = (x, y, z)$ and $k$ is a field of characteristic $p > 3$. Setting $\deg(x) = 3, \deg(y) = 2$ and $\deg(z) = 1$, $x^2 - y^3 - z^6$ is a quasihomogeneous polynomial, making $R$ into a graded ring.

The test ideal of $R$ was shown to be $\mathfrak{m}$ in [Vas97, Corollary 3.23], since

$$\deg(x) = 3 \geq 2 + 1 = \deg(y) + \deg(z).$$

Now since the test ideal is $\mathfrak{m}$ and $R$ is Gorenstein, by [Vas97, Theorem 3.1]

$$(y^2, z^2)^* = (y^2, z^2) :_R \tau = (y^2, z^2) :_R \mathfrak{m} = (xyz, y^2, z^2).$$

Similarly, by [Vas97, Theorem 3.1],

$$(y, z^2)^* = (y, z^2) :_R \mathfrak{m} = (xz, y, z^2)$$

and

$$(y^2, z)^* = (y^2, z) :_R \mathfrak{m} = (xy, y^2, z).$$

Thus, by [Vas14a, Proposition 2.4],

$$(y^2, yz, z^2)^* = ((y, z^2) \cap (y^2, z))^* = (y, z^2)^* \cap (y^2, z)^* = (xz, y, z^2) \cap (xy, y^2, z) = \mathfrak{m}^2$$

implying that $(y^2, yz, z^2)$ is a $*$-reduction of $\mathfrak{m}^2$.

As with $(y^2, z^2)^*$, any ideal generated by a system of parameters $f_1, f_2$, will have

$$(f_1, f_2)^* = (f_1, f_2) : \mathfrak{m}.$$

Since $R$ is Gorenstein such an ideal will have a single socle generator, and hence will be minimally generated by three elements. Since $\mathfrak{m}^2$ is minimally generated by 5 elements ($x^2 = y \cdot y^2 + z^4 \cdot z^2$), such an ideal cannot be a $*$-reduction of $\mathfrak{m}^2$. So $(y^2, yz, z^2)$ must be a minimal $*$-reduction of $\mathfrak{m}^2$.

By [HS06, Corollary 8.3.9], the analytic spread of a Noetherian local ring is bounded above by the dimension and bounded below by the height. Because we have $\mathrm{ht}(\mathfrak{m}^2) = 2 = \dim(R)$, any minimal reduction of $\mathfrak{m}^2$ is generated by 2 elements. It is easy to show that all elements of $\mathfrak{m}^2$ are zeros of polynomials of the form $f(t) = t^2 - g(y^2, z^2)$ for some $g(y^2, z^2) \in (y^2, z^2)^2$ and that $ax + by + cz$ will never be a zero of such a polynomial. Thus, $(y^2, z^2)$ is a minimal integral reduction of $\mathfrak{m}^2$. So although $(y^2, z^2)$ is an integral reduction of $\mathfrak{m}^2$, $(y^2, z^2) \in \mathbf{I}'_*(\mathfrak{m}^2)$ and thus not a $*$-reduction of $\mathfrak{m}^2$. Note that $(y^2, z^2)$ is not a $*$-prereduction of $\mathfrak{m}^2$ because

$$(y^2, z^2) \subsetneq (y^2, z^2, xy, xz)^* = (y^2, z^2, xy, xz)$$

by [Vra06, Theorem 2.2]. In fact, $(y^2, z^2, xy, xz)$ is a $*$-prereduction of $\mathfrak{m}^2$ since

$$\mathfrak{m}^2/(y^2, z^2, xy, xz) = ((yz) + (y^2, z^2, xy, xz))/(y^2, z^2, xy, xz) \cong R/\mathfrak{m}$$

implies that $\mathfrak{m}^2$ is a cover of $(y^2, z^2, xy, xz)$. (For more on covers, see Chapter 4.1.) By Proposition 4.1.6(2), $(y^2, z^2, xy, xz)$ is a $*$-prereduction of $\mathfrak{m}^2$, since

$$(yz + f) + (y^2, z^2, xy, xz) = \mathfrak{m}^2$$

for any $f \in (y^2, z^2, xy, xz)$. Since $(y^2, z^2) \subseteq (y^2, z^2, xy, xz)\mathfrak{m}^2$ and $(y^2, z^2)^- = \mathfrak{m}^2$, then $(y^2, z^2, xy, xz)$ is an integral reduction of $\mathfrak{m}^2$. An example of an integral prereduction of $\mathfrak{m}^2$ is $J = (xy, xz, y^3, yz, z^2)$; this is the case since $\mathfrak{m}^2/J \cong k$ and for all $f \in \mathfrak{m}^2 \setminus J$, $J + (f) = \mathfrak{m}^2$ implying that $J$ is an integral prereduction by Proposition 4.1.6(2). $\square$

## 3.4 Duality

We will see that the duality between closure and interior operations extends to a duality between cl-prereductions and i-postexpansions. To do that, we will need to formalize the notion of duality of pair operations using Matlis duality.

**Definition 3.4.1.** [ERGV23b, Definition 3.1] Let $R$ be a complete local ring. Let $p$ be a pair operation on a class of pairs of Matlis-dualizable $R$-modules $\mathcal{P}$. For any pair of $R$-modules $(A, B) \in \mathcal{P}^\vee$, set

$$\mathcal{P}^\vee := \{(A, B) \mid ((B/A)^\vee, B^\vee) \in \mathcal{P}\},$$

and define the dual of $p$ by

$$p^\smile(A, B) := \left(\frac{B^\vee}{p((B/A)^\vee, B^\vee)}\right)^\vee.$$

The following lemmas will be helpful. The first shows that closure and interior operations are dual while the second shows that this is indeed a duality that works.

**Lemma 3.4.2.** *[ERGV23b, Lemma 3.3] Let $R$ be a complete local ring and $p$ a pair operation on a class of pairs of Matlis dualizable $R$-modules $\mathcal{P}$. For any $(A, B) \in \mathcal{P}$,*

$$\left(\frac{B}{p(A, B)}\right)^\vee = p^\smile((B/A)^\vee, B^\vee).$$

*In particular, if* $\mathrm{cl}$ *is a closure operation then* $((B/A)^\vee)^{B^\vee}_{\mathrm{cl}^\smile} = (B/A^{\mathrm{cl}}_B)^\vee$*, and if* $\mathrm{i}$ *is an interior operation, then* $((B/A)^\vee)^{\mathrm{i}^\smile}_{B^\vee} = (B/A^B_{\mathrm{i}})^\vee$*.*

**Lemma 3.4.3.** *[ERGV23b, Proposition 3.6(1)] Let $(R, \mathfrak{m})$ be a complete local ring, and let $p$ be a pair operation on a class $\mathcal{P}$ of pairs of Matlis-dualizable $R$-modules. Then $p^{\smile\smile} = p$.*

**Theorem 3.4.4.** *[ERGV23b, Theorem 6.2] Let $R$ be a Noetherian complete local ring. Let* $\mathrm{i}$ *be a relative interior operation on pairs $A \subseteq B$ of $R$-modules that are*

*Noetherian or Artinian, and let* $\mathrm{cl} := \mathrm{i}^{\smile}$ *be its dual closure operation. There exists an order reversing one-to-one correspondence between the poset of* i*-expansions of* $A$ *in* $B$ *and the poset of* cl*-reductions of* $(B/A)^\vee$ *in* $B^\vee$*. Under this correspondence, an* i*-expansion* $C$ *of* $A$ *in* $B$ *maps to* $(B/C)^\vee$*, a* cl*-reduction of* $(B/A)^\vee$ *in* $B^\vee$*.*

This duality between cl-reductions and i-expansions is extended in the following theorems to cl-prereductions and i-postexpansions as well as the elements of $\mathbf{I}'_{\mathrm{cl}}((B/A)^\vee, B^\vee)$ and the elements of $\mathbf{C}'_{\mathrm{i}}(A, B)$.

**Theorem 3.4.5.** *Let* $R$ *be a Noetherian complete local ring. Let* i *be a relative interior operation on pairs* $A \subseteq B$ *of* $R$*-modules that are Noetherian or Artinian, and let* $\mathrm{cl} := \mathrm{i}^{\smile}$ *be its dual closure operation. There exists a one-to-one correspondence between the set of* i*-postexpansions of* $A$ *in* $B$ *and the set of* cl*-prereductions of* $(B/A)^\vee$ *in* $B^\vee$*. Under this correspondence, an* i*-postexpansion* $C$ *of* $A$ *in* $B$ *maps to* $(B/C)^\vee$*, a* cl*-prereduction of* $(B/A)^\vee$ *in* $B^\vee$*.*

*Proof.* $C$ is an i-postexpansion of $A$ in $B$ if and only if $A \subseteq C \subseteq B$ and $A_{\mathrm{i}}^B \subsetneq C_{\mathrm{i}}^B$ and for all submodules $D$ with $A \subseteq D \subseteq C$ we have $A_{\mathrm{i}}^B = D_{\mathrm{i}}^B$. First, $A \subseteq C \subseteq B$ if and only if $(B/C)^\vee \subseteq (B/A)^\vee \subseteq B^\vee$ by properties of Matlis duality. Next, $A_{\mathrm{i}}^B \subsetneq C_{\mathrm{i}}^B$ occurs if and only if

$$\left(\frac{B^\vee}{((B/A)^\vee)_{B^\vee}^{\mathrm{cl}}}\right)^\vee \subsetneq \left(\frac{B^\vee}{((B/C)^\vee)_{B^\vee}^{\mathrm{cl}}}\right)^\vee.$$

Since the modules in question are Matlis-dualizable and $(B/C)^\vee \subseteq (B/A)^\vee$, this happens if and only if

$$((B/C)^\vee)_{B^\vee}^{\mathrm{cl}} \subsetneq ((B/A)^\vee)_{B^\vee}^{\mathrm{cl}}.$$

If for all submodules $D$ with $A \subseteq D \subseteq C$ we have $A_{\mathrm{i}}^B = D_{\mathrm{i}}^B$ then by Theorem 3.4.4, $D$ is an i-expansion of $A$ in $B$ and thus maps to $(B/D)^\vee$ a cl-reduction of $(B/A)^\vee$ in $B^\vee$ and

$$(B/C)^\vee \subseteq (B/D)^\vee \subseteq (B/A)^\vee.$$

Similarly if for all $(B/D)^\vee$ with $(B/C)^\vee \subseteq (B/D)^\vee \subseteq (B/A)^\vee$ we have

$$((B/C)^\vee)^{\mathrm{cl}}_{B^\vee} = ((B/A)^\vee)^{\mathrm{cl}}_{B^\vee}$$

then again by Theorem 3.4.4, we have $A^B_{\mathrm{i}} = D^B_{\mathrm{i}}$.

Thus $C$ is an i-postexpansion of $A$ in $B$ if and only if $(B/C)^\vee$ is a cl-prereduction of $(B/A)^\vee$ in $B^\vee$. $\square$

**Example 3.4.6.** To see how this duality works, we return to $R = k[[x^2, x^5]]$. We use the characterization of $E = E_R(k)$ given in Proposition A.0.2(4).

Let

$$\begin{aligned} M &= kx + kx^3 + \bigoplus_{i=1}^{n} kx^{-i}, \\ N &= kx + kx^3 + \bigoplus_{i=1}^{n} kx^{-i} + kx^{-(n+2)}, \text{ and} \\ K &= kx + kx^3 + \bigoplus_{i=1}^{n} kx^{-i} + kx^{-(n+2)} + kx^{-(n+4)}. \end{aligned}$$

Let $r \in \langle 2, 5\rangle$ and $j \in \mathbb{N} \setminus \langle 2, 5\rangle$. By the definition of the action of $R$ on $E$ given in Proposition A.0.2(4), we see that $x^r \cdot x^j = 0$ when $r + j \in \langle 2, 5\rangle$.

Thus $\mathrm{Ann}_R(M) = (x^{n+4}, x^{n+5})$ since

$$x^{n+2} \cdot x^{-n+1} = x^3 \neq 0, \text{ but } x^{n+j} \cdot x^{-n+1} = x^{j+1} = 0$$

for $j \geq 4$ and all other product are clearly 0. Similarly, $\mathrm{Ann}_R(N) = (x^{n+4}, x^{n+7})$, because

$$x^{n+5} \cdot x^{-n-2} = x^3 \neq 0, \text{ but } x^{n+j} \cdot x^{-n-2} = x^{j-2} = 0$$

for all $x^{n+j} \in (x^{n+4}, x^{n+7})$ and $\mathrm{Ann}_R(K) = (x^{n+4})$ because

$$x^{n+7} \cdot x^{-n-2} = x^3 \neq 0, \text{ but } x^{n+j} \cdot x^{-n-4} = x^{j-4} = 0$$

for all $x^{n+j} \in (x^{n+4})$ . From Example 3.1.12, we can see that $\mathrm{Ann}_R(N)$ is a $\mathfrak{m}$bf-prereduction of $\mathrm{Ann}_R(M)$ and $\mathrm{Ann}_R(K)$ a $\mathfrak{m}$bf-reduction of $\mathrm{Ann}_R(N)$. Then both by duality and by Example 3.2.7, $N$ is an $\mathfrak{m}$be-postexpansion of $M$ in $E$ and $K$ is a $\mathfrak{m}$be-expansion of $N$ in $E$. $\diamondsuit$

**Theorem 3.4.7.** *Let $R$ be a Noetherian complete local ring. Let $\mathrm{i}$ be a relative interior operation on pairs $A \subseteq B$ of $R$-modules that are Noetherian or Artinian, and let $\mathrm{cl} := \mathrm{i}^{\smile}$ be its dual closure operation. There exists an order reversing one-to-one correspondence between between the elements of $\mathbf{I}'_{\mathrm{cl}}((B/A)^\vee, B^\vee)$ and the elements of $\mathbf{C}'_{\mathrm{i}}(A, B)$.*

*Proof.* Let $C \in \mathbf{C}'_{\mathrm{i}}(A, B)$. If $C$ is an i-postexpansion of $A$ in $B$ then by Theorem 3.4.5, $(B/C)^\vee$ is a cl-prereduction of $(B/A)^\vee$ in $B^\vee$, we have the one-to-one correspondence, and $(B/C)^\vee \in \mathbf{I}'_{\mathrm{cl}}((B/A)^\vee, B^\vee)$. If $C$ is not an i-postexpansion of $A$ in $B$ then $A \subseteq C \subseteq B$ and by Proposition 3.2.3(1), $C$ contains an i-postexpansion of $A$ in $B$. Let that i-postexpansion be $D$. Then by the previous theorem, $D$ maps one-to-one to $(B/D)^\vee$ a cl-prereduction of $(B/A)^\vee$ in $B^\vee$. Since $(B/C)^\vee \subseteq (B/D)^\vee$ and $(B/C)^\vee$ is not a cl-reduction of $(B/A)^\vee$ in $B^\vee$ (otherwise it would map to $C$ and $C$ would be an i-expansion of $A$ in $B$), we get that $(B/C)^\vee \in \mathbf{I}'_{\mathrm{cl}}((B/A)^\vee, B^\vee)$. The correspondence is order reversing since

$$C \subseteq D \text{ if and only if } (B/D)^\vee \subseteq (B/C)^\vee.$$

$\square$

# Chapter 4

# Other cl and i structures

## 4.1 Covers and Generating Sets

In certain cases, we can determine exactly the form of all the cl-prereductions of a submodule and the form of all the i-postexpansions of a submodule. In particular, if $N$ is a finitely generated cl-basic submodule of $M$, then every cl-prereduction can be determined in terms of the minimal generating sets of $N$. Similarly if $N$ is a finite length submodule in an Artinian module $M$ and $N$ is its only i-postexpansion in $M$, then the i-postexpansions can be determined in terms of the minimal cogenerating sets of $M/N$. In this section we discuss the relationship between covers of submodules with respect to closure operations cl and interior operations i.

The *analytic spread* of an ideal $I$ is the maximal number of algebraically independent elements in $I$. The following is a generalization of algebraic independence inspired by Vraciu's work on special tight closure and $*$-independence in [Vra02] given by Epstein [Eps05] and [Eps10].

**Definition 4.1.1.** Let $R$ be a Noetherian ring and cl be a closure operation defined on $R$-modules. We say that $f_1, \ldots, f_r \in M$ are cl-*independent* if

$$f_i \notin (f_1R + \cdots + \hat{f}_iR + \cdots + f_rR)^{\mathrm{cl}}.$$

We say a submodule $N \subseteq M$ is *strongly* cl-*independent* if every minimal set of generators of $N$ is cl-independent.

Much of Kemp, Ratliff and Shah's work is over a Noetherian local ring and uses integral closure, which is a Nakayama closure. Epstein proved that when cl is a Nakayama closure and an ideal $L \subseteq N$ is a reduction of $N$ in $M$, then $L$ is minimal cl-reduction of $N$ if and only if $L$ is a strongly cl-independent. He further generalized the notion of analytic spread to Nakayama closures cl.

**Definition 4.1.2.** [Eps05] Let $(R, \mathfrak{m})$ be a Noetherian local ring and cl a Nakayama closure defined on $R$-modules. We say $N$ has cl-*spread* if the cardinality of any minimal generating set for any minimal reduction of $N$ is the same.

In [KRS20, Definition 4.1], Kemp, Ratliff and Shah defined the notion of integral and non-integral covers. We generalize these notions to other closure and interior operations.

**Definition 4.1.3.** Let $(R, \mathfrak{m})$ be a local Noetherian ring, $\mathcal{M}$ be a class of $R$-modules, and $\mathcal{P}$ a class of pairs of $R$-modules in $\mathcal{M}$.

Suppose cl is a closure operation on pairs of modules $(L, M), (N, M)$ in $\mathcal{P}$. If $L$ is covered by $N$:

1. We say that $N$ is a cl-*cover* of $L$ if $N_M^{\mathrm{cl}} = L_M^{\mathrm{cl}}$.

2. We say that $N$ is a *non*-cl-*cover* of $L$ if $L_M^{\mathrm{cl}} \subsetneq N_M^{\mathrm{cl}}$.

Let i be an interior operation on pairs of modules $(A, B), (C, B)$ in $\mathcal{P}$. If $A$ is covered by $C$:

1. We say that $C$ is a i-*cover* of $A$ if $A_{\mathrm{i}}^{B} = C_{\mathrm{i}}^{B}$.

2. We say that $C$ is a *non*-i-*cover* of $A$ if $A_{\mathrm{i}}^{B} \subsetneq C_{\mathrm{i}}^{B}$.

**Remark 4.1.4.** Let $K \subseteq N \subseteq M$ and $L \subsetneq M$ be $R$-modules.

1. [ZS58, Theorem 28] [RR77, Remarks 2.2 and 4.2] The following are equivalent:

   (a) $N$ covers $K$.

   (b) $N = K + xR$ for any nonzero $x \in N \setminus K$ and $\mathfrak{m}x \subseteq K$.

   (c) $N = K + xR$ and $\mathfrak{m} = (K :_R xR)$ for every $x \in N \setminus K$.

2. [RR77, Remarks 2.13 and 4.2] If $N$ covers $K$, then either

   (a) $N \cap L$ covers $K \cap L$ and $N + L = K + L$ or

   (b) $N \cap L = K \cap L$ and $N + L$ covers $K + L$.

**Proposition 4.1.5.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathcal{M}$ be the category of finitely generated $R$-modules, $\mathcal{P}$ be the set of pairs $(L, M)$ with $L \subseteq M$ and $L, M \in \mathcal{M}$ and* cl *is a closure operation defined on $\mathcal{P}$.*

*Suppose $L \subseteq N \subseteq M$ with $(L, M), (N, M) \in \mathcal{P}$. The following are equivalent:*

*1. $L + xR$ is a non-*cl*-cover of $L$ in $M$ for all $x \in N \setminus L$.*

*2. $\mathfrak{m}N \subseteq L$ and $L_M^{\mathrm{cl}} \cap N = L$.*

*Proof.* $(1 \Rightarrow 2)$ If $L+xR$ is a non-cl cover of $L$ for all $x \in N \setminus L$, then $L_M^{\mathrm{cl}} \subsetneq (L+xR)_M^{\mathrm{cl}}$ for all $x \in N \setminus L$. By Remark 4.1.4 (1) $\mathfrak{m}x \subseteq L$ for every $x \in N \setminus L$. Additionally, if $x \in L$, $\mathfrak{m}x \subseteq L$. Thus $\mathfrak{m}x \subseteq L$ for all $x \in N$. Clearly $L \subseteq L_M^{\mathrm{cl}} \cap N$. Suppose $x \in L_M^{\mathrm{cl}} \cap N$. If $x \in N \setminus L$ then as $L_M^{\mathrm{cl}} \subsetneq (L + xR)_M^{\mathrm{cl}}$ then $x \notin L_M^{\mathrm{cl}}$ implying that $x \in L$.

($2 \Rightarrow 1$) If $\mathfrak{m}N \subseteq L$, then for all $x \in N \setminus L$, $\mathfrak{m}x \in L$. By Remark 4.1.4 (1) $L + xR$ is a cover of $L$. Since $L_M^{\mathrm{cl}} \cap N = L$, there is no $x \in N \setminus L$, such that $L_M^{\mathrm{cl}} = (L + xR)_M^{\mathrm{cl}}$. This means that for all $x \in N \setminus L$, is a non-cl-cover. $\square$

The following proposition is a generalization of [KRS20, Theorem 4.5] for cl a Nakayama closure.

**Proposition 4.1.6.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathcal{M}$ be the class of finitely generated $R$-modules, $\mathcal{P}$ be the class of pairs $(L, M)$ with $L \subseteq M$ and $L, M \in \mathcal{M}$ and* cl *be a Nakayama closure on $\mathcal{P}$. Let $\mathfrak{A}$ be a* cl*-prereduction of $L$ in $M$. For every $x \in L \setminus \mathfrak{A}$:*

1. *$\mathfrak{A} + xR$ is a non-*cl*-cover of $\mathfrak{A}$.*
2. *$\mathfrak{A}$ is a* cl*-prereduction of $\mathfrak{A} + xR$.*
3. *$\mathfrak{A} + xR$ is a* cl*-reduction of $L$.*

*Proof.* (1) It follows from Proposition 3.1.7(3a) that $\mathfrak{m}L \subseteq \mathfrak{A}$ and from Remark 4.1.4 that $\mathfrak{A} + xR$ is a cover of $\mathfrak{A}$. Also $\mathfrak{A}_M^{\mathrm{cl}} \cap L = \mathfrak{A}$ follows from Proposition 3.1.7(3b), so $x \notin \mathfrak{A}_M^{\mathrm{cl}}$. Thus $\mathfrak{A} + xR$ is a non-cl-cover of $\mathfrak{A}$.

(2) $\mathfrak{A}$ is a cl-prereduction of $\mathfrak{A} + xR$ by Proposition 3.1.10 since $\mathfrak{A} \subseteq \mathfrak{A} + xR \subseteq L$.

(3) $\mathfrak{A} + xR$ is a cl-reduction of $L$ in $M$ by the definition of cl-prereduction of $L$. $\square$

This allows us to relate non-cl-covers to cl-prereductions.

**Proposition 4.1.7.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

1. *If $L$ is* cl*-basic, then $L$ is a non-*cl*-cover of each* cl*-prereduction of itself.*
2. *If $K$ and $L$ are submodules of $M$ and $L$ is a non-*cl*-cover of $K$, then $K$ is a* cl*-prereduction of $L$.*

*Proof.* (1) Suppose $\mathfrak{A}$ is a cl-prereduction of $L$. By Proposition 4.1.6(3) $\forall x \in L \setminus \mathfrak{A}$, $\mathfrak{A} + xR$ is a reduction of $L$. Since $L$ is cl-basic, this implies that $\mathfrak{A} + xR = L$ for all $x \in L \setminus \mathfrak{A}$.

(2) If $L$ is a non-cl-cover of $K$, then $K$ is not a cl-reduction of $L$. Also by Remark 4.1.4(1), $K + xR = L$ for all $x \in L \setminus K$. This implies that every module $N$ with $K \subsetneq N \subseteq L$ is $L$ and since $K_M^{\mathrm{cl}} \subsetneq L_M^{\mathrm{cl}}$ then $K$ is a cl-prereduction of $L$. □

**Proposition 4.1.8.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $N$ be a strongly* cl-*independent submodule in $M$ with* cl-*spread equal to $k \geq 1$ elements. Then every* cl-*prereduction of $N$ in $M$ has the form*

$$y_1R + y_2R + \cdots + y_{k-1}R + y_k\mathfrak{m}$$

*where $y_1, ..., y_k$ are a minimal generating set for $N$.*

*Proof.* Let $y_1, ..., y_k$ be a minimal generating set of $N$ and $\mathfrak{A} = y_1R + \cdots + y_{k-1}R + y_k\mathfrak{m}$. Then $\mathfrak{A} + y_kR = y_1 + \cdots + y_kR = N$ and $y_k\mathfrak{m} \subseteq \mathfrak{A}$, so $N$ is a cover of $\mathfrak{A}$ by Remark 4.1.4. Also, the $y_i$ are strongly cl-independent, so $N$ is a minimal cl-reduction of itself by [Eps05, Proposition 2.3] and [Eps10, Page 2210] and by minimality $N$ is the only cl-reduction of itself, implying that $N$ is cl-basic. Thus $N$ is a non-cl-cover of $\mathfrak{A}$. hence, Remark 4.1.7(2) implies that $\mathfrak{A}$ is a cl-prereduction of $N$ in $M$.

Suppose $\mathfrak{A}$ is an arbitrary cl-prereduction of $N = x_1R + \cdots + x_kR$. Then there exists an $1 \leq i \leq k$ such that $x_i \notin \mathfrak{A}$. Let us assume $i = k$, then $x_k \notin \mathfrak{A}$. Note that since $\mathfrak{A}$ is a cl-prereduction of $N$ in $M$ then for any $x \in N \setminus \mathfrak{A}$, $\mathfrak{A} + xR$ is a cl-reduction of $N$ in $M$. However, since $N$ is cl-basic, this implies that $N = \mathfrak{A} + xR$ for any $x \in N \setminus \mathfrak{A}$. In particular, $N = \mathfrak{A} + x_kR$. Thus for $1 \leq i \leq k-1$, there exists $a_i \in \mathfrak{A}$ and $b_i \in R$ such that $x_i = a_i + b_ix_k$ and

$$\{a_1 + b_1x_k, ..., a_{k-1} + b_{k-1}x_k, x_k\}$$

is a minimal generating set of $N$. Thus

$$\{a_1, ..., a_{k-1}, x_k\}$$

is also a minimal generating set of $N$. Since $\mathfrak{m}N \subseteq \mathfrak{A}$ by Proposition 3.1.7(3a), we have

$$a_1R + \cdots + a_{k-1}R + x_k\mathfrak{m} \subseteq a_1R + \cdots + a_{k-1}R + \mathfrak{m}N \subseteq \mathfrak{A} \subseteq N.$$

By Proposition 4.1.7(1), $N$ is a non-cl-cover of every prereduction; in particular $N$ is a non-cl-cover of $\mathfrak{A}$. However, $N/(a_1R+\cdots+a_{k-1}R+x_k\mathfrak{m}) \cong x_kR/x_k\mathfrak{m} \cong R/\mathfrak{m}$, which implies that $N$ is a cover of $a_1R+\cdots+a_{k-1}R+x_k\mathfrak{m}$. As $a_1R+\cdots+a_{k-1}R+x_k\mathfrak{m} \subseteq \mathfrak{A}$ and $\mathfrak{A} \in \mathbf{I}'_{\text{cl}}(N, M)$, we see by Proposition 3.1.6(1) that it must be the case that $N$ is a non-cl-cover of $a_1R + \cdots + a_{k-1}R + x_k\mathfrak{m}$. By Proposition 4.1.7(2), it must be the case that

$$a_1R + \cdots + a_{k-1}R + x_k\mathfrak{m}$$

is a cl-prereduction of $N$ in $M$ implying that $\mathfrak{A} = a_1R + \cdots + a_{k-1}R + x_k\mathfrak{m}$. □

The next example illustrates how Proposition 4.1.8 can be used to express $\mathfrak{m}$bf-prereductions as a sum of minimal generators.

**Example 4.1.9.** Let $R = k[[x^2, x^5]]$ and $I = (x^6, x^7)$. By the definition of $\mathfrak{m}$bf-independent and by Proposition A.0.2, $x^6$ and $x^7$ are $\mathfrak{m}$bf-independent. This is because $x^6 \notin (x^7)^{\mathfrak{m}\text{bf}} = (x^7, x^{10})$ and $x^7 \notin (x^6)^{\mathfrak{m}\text{bf}} = (x^6, x^9)$. Note that

$$(x^6) + \mathfrak{m}(x^7) = (x^6) + (x^9, x^{12}) = (x^6, x^9)$$

and

$$(x^7) + \mathfrak{m}(x^6) = (x^7) + (x^8, x^9) = (x^7, x^8)$$

are both $\mathfrak{m}$bf-prereductions of $I$ by Proposition 4.1.8.

**Definition 4.1.10.** [ERGV23c, Definition 6.6] Let $R$ be a Noetherian local ring, $L$ an $R$-module, and $g_1, ..., g_t \in L^\vee$. We say that the *quotient of $L$ cogenerated by $g_1, ...g_t$*

is $L/(\bigcap_i \ker(g_i))$. We say that $L$ is *cogenerated by* $g_1, ...g_t$ if $\bigcap_i \ker(g_i) = 0$. We say that a cogenerating set for $L$ is *minimal* if it is irredundant, i.e., for all $1 \le j \le t$, $\bigcap_{i \neq j} \ker(g_i) \neq 0$.

We can dualize the notion of a strongly cl-independent generating set to that of a strongly i-independent cogenerating set as follows:

**Definition 4.1.11.** Let $R$ be a Noetherian local ring, $L \subseteq M$ $R$-modules, $\pi : M \to M/L$ the canonical projection, and i an interior operation defined on $R$-modules. We say that $g_1, \ldots, g_k \in (M/L)^\vee$ are an i-*independent cogenerating set* of $M/L$ if

$$\pi^{-1}(\ker(g_i)) \not\supseteq (\pi^{-1}(\bigcap_{r \neq i} \ker(g_r)))_{\mathrm{i}}^M$$

for any $1 \le i \le k$. We say that $L$ is *strongly* i-*independent* if any minimal set of cogenerators of $M/L$ is i-independent.

**Example 4.1.12.** Let $R = k[[x^2, x^5]]$. Note that $(x^{11}, x^{12}) = (x^6) \cap (x^7)$ and $(x^{11}, x^{14}) = (x^6) \cap (x^9)$. We use the formulation of $E = E_R(k)$ given in Proposition A.0.2(4). Define $g_i : R \to E$ to be the homomorphism defined by $g_i(1) = x^{-i}$ for every $i$ in the semigroup $\langle 2, 5 \rangle$. Note that $g_i$ has

$$\ker(g_i) = (x^i)$$

and

$$\mathrm{im}(g_i) = kx^3 + kx + \sum_{j=1}^{i-4} kx^{-j} + kx^{-i+2} + kx^{-i}$$

for $i > 4$. Since

$$(R/(x^{11}, x^{12}))^\vee \cong kx^3 + kx + \sum_{j=1}^{7} x^{-j}$$

and

$$(R/(x^{11}, x^{14}))^\vee \cong kx^3 + kx + \sum_{j=1}^{7} x^{-j} + kx^{-9},$$

it is easy to see $R/(x^{11}, x^{12})$ is cogenerated by the functions $g_6$ and $g_7$ whereas $R/(x^{11}, x^{14})$ is cogenerated by $g_6$ and $g_9$.

Note that $\ker(g_6) \cap \ker(g_9) = (x^{11}, x^{14})$ and $\ker(g_9) = (x^9)$ is a $\mathfrak{m}$be expansion of $(x^{11}, x^{14})$. Thus $\ker(g_6) \supseteq \ker(g_9)_{\mathfrak{m}\text{be}}$ implying that $g_6$ and $g_9$ are not strongly $\mathfrak{m}$be-independent.

However, $g_6$ and $g_7$ will be strongly $\mathfrak{m}$be-independent because

$$\ker(g_6) \not\supseteq (x^9, x^{12}) = (\ker(g_7))_{\mathfrak{m}\text{be}}.$$

$\diamondsuit$

**Proposition 4.1.13.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, $\mathcal{M}$ be the class of Artinian $R$-modules, $\mathcal{P}$ be the class of pairs $(A, B)$ with $A \subseteq B$ and $A, B \in \mathcal{M}$, $\pi : B \to B/A$ the canonical surjection and* i *be a Nakayama interior on $\mathcal{P}$. Let $\mathfrak{A}$ be an* i*-postexpansion of $A$ in $B$. For every $g \in (B/A)^\vee$ such that $\pi^{-1}(\ker(g)) \not\supseteq \mathfrak{A}$:*

1. *$\mathfrak{A}$ is an* i*-postexpansion of $\mathfrak{A} \cap \pi^{-1}(\ker(g))$.*

2. *$\mathfrak{A} \cap \pi^{-1}(\ker(g))$ is an* i*-expansion of $A$.*

*Proof.* (1) $\mathfrak{A}$ is an i-postexpansion of $\mathfrak{A} \cap \pi^{-1}(\ker(g))$ by Proposition 3.2.5 since

$$A \subseteq \mathfrak{A} \cap \pi^{-1}(\ker(g)) \subseteq \mathfrak{A}.$$

(2) $\mathfrak{A} \cap \pi^{-1}(\ker(g))$ is an i-expansion of $A$ in $B$ by the definition of i-postexpansion of $A$. $\square$

**Proposition 4.1.14.** *[ERGV23c, Proposition 6.14] Let $(R, \mathfrak{m})$ be a Noetherian local ring and* i *a Nakayama interior on Artinian $R$-modules. Let $A \subseteq B$ Artinian $R$-modules. Suppose that $C \subseteq D$ are* i*-expansions of $A$ in $B$, with $D$ a maximal* i*-expansion. Then any minimal cogenerating set of $B/D$ extends to a minimal cogenerating set for $B/C$.*

**Definition 4.1.15.** [ERGV23c, Definition 7.18] Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let i be an interior operation defined on a class of Artinian $R$-modules $\mathcal{M}$. Let $A \subseteq B$ be Artinian $R$-modules. We define the i-*cospread* $\ell_{\mathrm{i}}^{B}(A)$ of $A$ to be the minimal number of cogenerators of $B/C$ of any maximal i-expansion $C$ of $A$, if this number exists.

**Proposition 4.1.16.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring.*

1. *Suppose $C \subseteq A \subseteq B$ and $\pi : B \to B/C$ is the canonical surjection. $(B/C)^\vee$ covers $(B/A)^\vee$ if and only if $C = A \cap \pi^{-1}(\ker(g))$ for some $g \in (B/C)^\vee$ and $(\pi^{-1}(\ker(g)) :_B \mathfrak{m}) \supseteq A$.*

2. *If $C$ is* i-*cobasic then every* i-*postexpansion $\mathfrak{A}$ of $C$ is a non-*i-*cover of $C$.*

*Proof.* (1) Suppose $(B/C)^\vee$ covers $(B/A)^\vee$. Then by Remark 4.1.4, we have that $(B/C)^\vee = (B/A)^\vee + (g)$ for some $g \in (B/C)^\vee$ and $\mathfrak{m}g \subseteq (B/A)^\vee$. By [ERGV23b, Lemma 5.4],

$$(g) \cong ((B/C)/(\ker(g)))^\vee.$$

By the Third Isomorphism Theorem, $(B/C)/(\ker(g)) \cong B/(\pi^{-1}(\ker(g))$. Then by Lemma 2.2.4 we see that

$$(B/A)^\vee + (B/\pi^{-1}(\ker(g)))^\vee \cong (B/(A \cap \pi^{-1}(\ker(g))))^\vee$$

i.e., $(B/C)^\vee \cong (B/(A \cap \pi^{-1}(\ker(g))))^\vee$. Thus $C = A \cap \pi^{-1}(\ker(g))$, some $g \in (B/C)^\vee$. Furthermore, since $\mathfrak{m}g \subseteq (B/A)^\vee$ then

$$A = \left(\frac{(B/C)^\vee}{(B/A)^\vee}\right)^\vee \subseteq \left(\frac{(B/C)^\vee}{\mathfrak{m}g}\right)^\vee = \left(\frac{B^\vee}{\mathfrak{m}(g \circ \pi^\vee)}\right)^\vee = (\pi^{-1}(\ker(g)) :_B \mathfrak{m}).$$

Suppose $C = A \cap \pi^{-1}(\ker(g))$ for some $g \in (B/C)^\vee$ and $\pi^{-1}(\ker g :_{B/C} \mathfrak{m}) \supseteq A$. Then by Lemma 2.2.4

$$(B/C)^\vee = \left(\frac{B}{A \cap \pi^{-1}(\ker(g)}\right)^\vee \cong (B/A)^\vee + (B/\pi^{-1}(\ker(g)))^\vee$$

and since

$$(g) \cong ((B/C)/(\ker(g)))^\vee \cong (B/(\pi^{-1}(\ker(g)))^\vee,$$

we see that $(B/C)^\vee = (B/A)^\vee + (g)$.

Since $(\pi^{-1}(\ker(g)) :_B \mathfrak{m}) \supseteq A$, we have

$$\begin{aligned}(B/A)^\vee &\supseteq \left(\frac{B}{(\pi^{-1}(\ker(g)) :_B \mathfrak{m})}\right) \\ &= \left(\frac{B/C}{(\ker(g)) :_{B/C} \mathfrak{m})}\right)^\vee \\ &= \mathfrak{m}g\end{aligned}$$

or $\mathfrak{m}g \subseteq (B/A)^\vee$ and by Remark 4.1.4 $(B/C)^\vee$ covers $(B/A)^\vee$.

(2) Let $\mathfrak{A}$ be an i-postexpansion of $C$. Since $C$ is i-cobasic, then for all $g \in (B/C)^\vee$ such that $\pi^{-1}(\ker(g)) \not\supseteq \mathfrak{A}$, $\mathfrak{A} \cap \pi^{-1}(\ker(g))$ is an i-expansion of $C$. But this implies that $\mathfrak{A} \cap \pi^{-1}(\ker(g)) = C$ for all $g \in (B/C)^\vee$ such that $\pi^{-1}(\ker(g)) \not\supseteq \mathfrak{A}$. By Proposition 3.2.3(3a), $(A :_B \mathfrak{m}) \supseteq \mathfrak{A}$ and (1) gives that $(B/\mathfrak{A} \cap \pi^{-1}(\ker(g)))^\vee$ is a cover of $(B/\mathfrak{A})^\vee$. Also by Proposition 3.2.3(3b), $\mathfrak{A}_{\mathrm{i}}^B + A = \mathfrak{A}$. So $\ker(g) \notin \mathfrak{A}_{\mathrm{i}}^B$. Thus $(B/\mathfrak{A} \cap \ker(g))^\vee$ is a non-i-cover of $(B/\mathfrak{A})^\vee$. □

**Proposition 4.1.17.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and $A$ be a strongly* i*-independent submodule in $B$ with* i*-cospread equal to $k \geq 1$ elements and $\pi : B \to B/A$ is the canonical surjection. Then any* i*-postexpansion of $A$ in $B$ has the form*

$$\bigcap_{r \neq i} \pi^{-1}(\ker(g_r)) \cap (\pi^{-1}(\ker(g_i)) :_B \mathfrak{m}).$$

*Proof.* Let $g_1, ..., g_k$ be a minimal cogenerating set for $B/A$ and let

$$\mathfrak{A} = \bigcap_{r \neq i} \pi^{-1}(\ker(g_r)) \cap (\pi^{-1}(\ker(g_i)) :_{(B/A)} \mathfrak{m}).$$

Then $\mathfrak{A} \cap \pi^{-1}(\ker(g_k)) = \bigcap_{i=1}^{k} \pi^{-1}(\ker(g_i)) = A$ and $(\pi^{-1}(\ker(g_k)) :_{B/A} \mathfrak{m}) \supseteq A$, so $\mathfrak{A}$ is a cover of $A$ by Proposition 4.1.16(1). Also, the $g_i$ are strongly i-independent, so

$A$ is i-cobasic and $A$ is the only i-expansion of itself. Thus $\mathfrak{A}$ is a non-i-cover of $A$. Hence, Remark 4.1.16(2) implies that $\mathfrak{A}$ is an i-postexpansion of $A$ in $B$.

Suppose $\mathfrak{A}$ is an arbitrary i-postexpansion of $A$ where

$$B/A = B/(\bigcap_{i=1}^{k} \pi^{-1}(\ker(g_i))).$$

Then there exists an $1 \leq i \leq k$ such that $\pi^{-1}(\ker(g_i)) \not\supseteq \mathfrak{A}$. Let us assume $i = k$, then $\pi^{-1}(\ker(g_k)) \not\supseteq \mathfrak{A}$. Note that since $\mathfrak{A}$ is an i-postexpansion of $A$ in $B$, then for any $g \in (B/A)^\vee \setminus (B/\mathfrak{A})^\vee$, $\mathfrak{A} \cap \pi^{-1}(\ker(g))$ is an i-expansion of $A$ in $B$. However, since $A$ is i-cobasic, this implies that $A = \mathfrak{A} \cap \pi^{-1}(\ker(g))$ for any $g \in (B/A)^\vee \setminus (B/\mathfrak{A})^\vee$. In particular, $A = \mathfrak{A} \cap \pi^{-1}(\ker(g_k))$.

Thus for $1 \leq i \leq k-1$, there exists $h_i \in (B/A)^\vee$ and $b_i \in R$ such that $g_i = h_i + b_i g_k$ and

$$\{h_1 + b_1 g_k, ..., h_{k-1} + b_{k-1} g_k, g_k\}$$

is a minimal generating set of $(B/A)^\vee$. Thus $\{h_1, ..., h_{k-1}, g_k\}$ is also a minimal generating set of $(B/A)^\vee$ and hence a minimal cogenerating set for $B/A$. Since $(A :_B \mathfrak{m}) \supseteq \mathfrak{A}$ by Proposition 3.2.3(3a), we have

$$\begin{aligned}\bigcap_{i=1}^{k-1} \pi^{-1}(\ker(h_i)) \cap (\pi^{-1}(\ker(g_k)) :_B \mathfrak{m}) &\supseteq \bigcap_{i=1}^{k-1} \pi^{-1}(\ker(h_i)) \cap (A :_B \mathfrak{m}) \\ &\supseteq \mathfrak{A}.\end{aligned}$$

Since $\mathfrak{A}$ is an i-postexpansion of $A$ in $B$ and $\bigcap_{i=1}^{k-1} \ker(h_i) \cap (\pi^{-1}(\ker(g_k)) :_B \mathfrak{m})$ is an i-postexpansion of $A$ in $B$, we see that $\mathfrak{A} = \bigcap_{i=1}^{k-1} \pi^{-1}(\ker(h_i)) \cap (\pi^{-1}(\ker(g_k)) :_B \mathfrak{m})$. $\square$

## 4.2 Precore and Posthull

Generally, a submodule can have a multitude of cl-reductions, even minimal cl-reductions (minimal among the set of all cl-reductions). The cl-core of a submodule

$N$ of $M$ is the intersection of all cl-reductions of $N$ in $M$ and is in some sense a measure of the cl-reductions of $N$. If $N$ is basic, then $\mathrm{cl}-\mathrm{core}_M(N) = N$. However, when $N$ has more cl-reductions, the cl-core is never a cl-reduction of $N$.

As computing the cl-core of a submodule can be difficult and formulas for the cl-core are only known in certain settings (see [HS95b], [Moh97], [CPU01], [CPU02], [CPU03], [PU05], [HT05], [FPU08], [FV10], [FVV11], [ERGV23b] and [CFH23]), we hope that in some instances these constructions may be a helpful tool to give insight into computing the cl-cores of submodules. Alongside the cl-precore and i-posthull, we also can define the cl-prehull and i-postcore. These are sometimes comparable to the cl-core and i-hull, respectively, and can be used to provide a bound.

We can generalize the notion of cl-core discussed in papers by Fouli and Vassilev and Vraciu [FV10] and [FVV11] and Epstein, R.G. and Vassilev [ERGV23c], [ERGV23b] to that of the cl-precore of a submodule, the intersection of all cl-prereductions. One might expect that cl-precore of a submodule to be contained in its cl-core, but this is not always the case. We will examine some conditions on the submodule which will ensure the containment to be true as well as exhibit some counterexamples. In this section, we will generalize the notions of cl-core and i-hull, additionally, we will intersect and sum the cl-prereductions and the i-postexpansions to create new constructs with the goal of providing bounds on the cl-core and i-hull.

**Proposition 4.2.1.** *If $N$ has at least two* cl*-reductions in $M$, then the* cl*-core of $N$ in $M$ will be contained in some* cl*-prereduction of $N$ in $M$.*

*Proof.* Let $N$ be a submodule of $M$ with cl-reductions $L$ and $K$ with $L \neq K$.

So $\text{cl-core}_M(N) \subseteq L$ and $\text{cl-core}_M(N) \subseteq K$. Then $\text{cl-core}_M(N) \in \mathbf{I}'_{\mathrm{cl}}(N, M)$. Thus by Proposition 3.1.6(2), there exists a cl-prereduction of $N$ which contains $\text{cl-core}_M(N)$. $\square$

**Proposition 4.2.2.** *If $A$ has at least two* i*-expansions in $B$, then the* i*-hull of $A$ in $B$ will contain some* i*-postexpansion of $A$ in $B$.*

*Proof.* Let $A$ be a submodule of $B$ with i-expansions $C$ and $D$ with $C \neq D$.

So $C \subseteq \text{i-hull}^B(A)$ and $D \subseteq \text{i-hull}^B(A)$. Then $\text{i-hull}^B(A) \in \mathbf{C}'_{\mathrm{i}}(A, B)$. So by Proposition 3.2.2(2), there exists an i-postexpansion of $A$ contained in $\text{i-hull}^B(A)$. $\square$

**Theorem 4.2.3.** *[ERGV23b, Theorem 6.6] Let $R$ be a complete Noetherian local ring. Let $A \subseteq B$ be Artinian $R$-modules and let* i *be a relative Nakayama interior defined on Artinian $R$-modules. Then the* i*-hull of $A$ in $B$ is dual to the* cl*-core of $(B/A)^\vee$ in $B^\vee$, where* cl *is the closure operation dual to* i.

The following definitions are motivated by the definitions of cl-core and i-hull and utilize the duality between sums and intersections (Lemma 2.2.4) with the goal of providing simpler computations for bounds on the cl-core and i-hull.

**Definition 4.2.4.** The cl-*prehull* of $N$ with respect to $M$ is the sum of the cl-prereductions of $N$ in $M$, or

$$\text{cl-prehull}_M(N) := \sum \{L \mid L \text{ a cl-}\textit{prereduction} \text{ of } N \text{ in } M\}.$$

The i-*postcore* of $A$ with respect to $B$ is the intersection of the i-postexpansions of $A$ in $B$, or

$$\text{i-postcore}^B(A) := \bigcap \{C \mid C \text{ a i-}\textit{postexpansion} \text{ of } A \text{ in } B\}.$$

**Remark 4.2.5.** Because the maximal elements of $\mathbf{I}'_{\mathrm{cl}}(N, M)$ are cl-prereductions and the minimal elements of $\mathbf{C}'_{\mathrm{i}}(A, B)$ are i-postexpansions, we have

$$\text{cl-prehull}_M(N) = \sum \{L \mid L \text{ maximal in } \mathbf{I}'_{\mathrm{cl}}(N, M)\}$$

$$\text{i-postcore}^B(A) = \bigcap \{C \mid C \text{ minimal in } \mathbf{C}'_{\mathrm{i}}(A, B)\}$$

**Definition 4.2.6.** The cl-*precore* of $N$ with respect to $M$ is the intersection of the cl-prereductions of $N$ in $M$,

$$\text{cl-precore}_M(N) = \bigcap \{L \mid L \text{ a cl-prereduction of } N \text{ in } M\}.$$

The i-*posthull* of $A$ with respect to $B$ is the sum of the i-postexpansions of $A$ in $B$,

$$\text{i-posthull}^B(A) := \sum \{C \mid C \text{ an i-postexpansion of} A \text{ in } B\}.$$

The following proposition gives some upper and lower bounds on the cl-core in terms of the cl-precore and the cl-prehull.

**Proposition 4.2.7.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring,* cl *a Nakayama closure, and $N$ a submodule of $M$.*

1. *If $N$ is* cl*-basic or every* cl*-prereduction is contained in some minimal reduction, then*
$$\text{cl-precore}_M(N) \subseteq \text{cl-core}_M(N).$$
2. *If $N = \text{cl-prehull}_M(N)$ or $N$ has at least two* cl*-reductions in $M$, then*
$$\text{cl-core}_M(N) \subseteq \text{cl-prehull}_M(N).$$

*Proof.* (1) First, if $N$ is cl-basic, then the only cl-reduction of $N$ in $M$ is $N$ itself. So $\text{cl-core}_M(N) = N$. Since by definition, every cl-prereduction is contained in $N$, we have $\text{cl-precore}_N(M) \subseteq N$. Thus $\text{cl-precore}_M(N) \subseteq \text{cl-core}_M(N)$.

Next, suppose every cl-prereduction is contained in some minimal cl-reduction. Then minimal cl-reductions exist and we know that

$$\text{cl-core}_M(N) = \bigcap \{L \mid L \text{ a minimal cl-reduction of } N \text{ and } (L, M) \in \mathcal{P}\}.$$

Since the intersection of cl-prereductions is contained in every cl-prereduction and every cl-prereduction is contained in a minimal cl-reduction, we see that

$$\text{cl-precore}_M(N) \subseteq \text{cl-core}_M(N).$$

(2) If $N = \text{cl-prehull}_M(N) = \sum\{L \mid L \text{ a cl-prereduction of } N \text{ in } M\}$. Since by definition, every cl-reduction is contained in $N$, we have $\text{cl-core}_N(M) \subseteq N$. Thus

$$\text{cl-core}_M(N) \subseteq \text{cl-prehull}_M(N).$$

If $N$ has at least two cl-reductions in $M$, by Proposition 4.2.1 we know that

$$\text{cl-core}_M(N) \subseteq L \subseteq \text{cl-prehull}_M(N)$$

where $L$ is a cl-prereduction of $N$ in $M$. □

The following proposition gives some upper and lower bounds on the i-hull in terms of the i-postcore and the i-posthull. It is dual to Proposition 4.2.7.

**Proposition 4.2.8.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring,* i *a Nakayama interior, and $A$ a submodule of $B$.*

1. *If $A = \text{i-postcore}^B(A)$ or $A$ has at least two* i*-expansions in $B$, then*

$$\text{i-postcore}^B(A) \subseteq \text{i-hull}^B(A).$$

2. *If $A$ is* i*-cobasic or every* i*-postexpansions contains some maximal expansion, then*

$$\text{i-hull}^B(A) \subseteq \text{i-posthull}^B(A).$$

*Proof.* (1) If $A = \text{i-postcore}^B(A) = \bigcap\{C \mid C \text{ a i-postexpansion of } A \text{ in } B\}$. Since by definition, $A$ is contained in every i-expansion and the hull is the sum of all expansions, we have $A \subseteq \text{i-hull}^B(A)$. Thus $\text{i-postcore}^B(A) \subseteq \text{i-hull}^B(A)$.

If $A$ has at least two i-expansions in $B$, then by Proposition 4.2.2 we know that

$$\text{i-postcore}^B(A) \subseteq C \subseteq \text{i-hull}^B(A)$$

where $C$ is an i-postexpansion of $A$ in $B$.

(2) First, if $A$ is i-cobasic, then the only i-expansion of $A$ in $B$ is $A$ itself. So

$$A = \text{i-hull}^A(B).$$

Since $A$ is contained in every i-postexpansion of $A$ in $B$, we have

$$A \subseteq \text{i-posthull}^B(A).$$

Thus $\text{i-hull}^B(A) \subseteq \text{i-posthull}^B(A)$.

Next, suppose every i-postexpansion contains some maximal i-expansion. Then maximal i-expansions exist and we know that

$$\text{i-hull}^B(A) = \sum\{C \mid C \text{ a maximal i-expansion of } A \text{ and } (C,B) \in \mathcal{P}\}.$$

Since every i-postexpansion contains a maximal i-expansion, the sum of all maximal i-expansions is contained in the sum of all i-postexpansions. Thus

$$\text{i-hull}^B(A) \subseteq \text{i-posthull}^B(A).$$

□

We include an example motivating the bounds given in Proposition 4.2.7.

**Example 4.2.9.** Let $R = k[[x^2, x^5]]$. It may be helpful to refer to Example 3.1.12 and Proposition A.0.2.

Consider the ideal $(x^4)$. The only $\mathfrak{m}$bf-prereduction of $(x^4)$ is $(x^6, x^9)$ by Proposition 4.1.8. Thus

$$\begin{aligned}(x^6, x^9) &= \mathfrak{m}\text{bf-precore}_R(x^4) \\ &\subsetneq \mathfrak{m}\text{bf-core}_R(x^4) \\ &= (x^4) \\ &\not\subseteq \mathfrak{m}\text{bf-prehull}_R(x^4) \\ &= (x^6, x^9)\end{aligned}$$

which gives an example that the cl-precore of a submodule could be properly contained in the cl-core and the cl-core is not contained in the cl-prehull.

Consider the ideal $(x^4, x^7)$. For every $a \in k$, the ideal $(x^4 + ax^7)$ is a minimal $\mathfrak{m}$bf-reductions of $(x^4, x^7)$. Hence, $\mathfrak{m}\text{bf-core}(x^4, x^7) = \bigcap_{a \in k}(x^4 + ax^7) = (x^6, x^9)$. Since the ideals $I$ of $R$ contained in $(x^4, x^7)$ are of the form $(x^n + ax^{n+3})$ or $(x^n + ax^{n+3}, x^{n+5})$ for $n = 4$ or $n \geq 6$ and $a \in k$ or $(x^n + ax^{n+1} + bx^{n+3})$, $(x^n + ax^{n+1}, x^{n+3})$ or $(x^n, x^{n+1})$ for $n \geq 6$ and $a, b \in k$. Note the only ideals $I$ which have $(x^4, x^7)$ as a cover are $(x^4 + ax^7)$ or $(x^6, x^7)$. Since $(x^4 + ax^7)$ are $\mathfrak{m}$bf-reductions of $(x^4, x^7)$, then $(x^6, x^7)$ is the only $\mathfrak{m}$bf-prereduction of $(x^4, x^7)$ by Proposition 4.1.6. Thus

$$\begin{aligned}(x^6, x^7) &= \mathfrak{m}\text{bf-precore}_R(x^4, x^7)\\ &\not\subseteq \mathfrak{m}\text{bf-core}_R(x^4, x^7)\\ &= (x^6, x^9)\\ &\subsetneq \mathfrak{m}\text{bf-prehull}_R(x^4, x^7)\\ &= (x^6, x^7)\end{aligned}$$

gives an example where the cl-core is properly contained in the cl-prehull of a submodule and the cl-precore is incomparable with the cl-core of a submodule.

Consider the ideal $(x^4, x^5)$. Note that $(x^4, x^5)^{\mathfrak{m}\text{bf}} = (x^4, x^5)$ and in fact is the only ideal $I$ with $I^{\mathfrak{m}\text{bf}} = (x^4, x^5)$. Thus, $\mathfrak{m}\text{bf-core}(x^4, x^5) = (x^4, x^5)$. As in Example 3.1.12, the ideals $(x^4 + ax^5, x^7)$ and $(x^5, x^6)$ are $\mathfrak{m}$bf-prereductions of $(x^4, x^5)$. Since

$$\bigcap_{a \in k}(x^4 + ax^5, x^7) \cap (x^5, x^6) = (x^6, x^7)$$

and $\sum_{a \in k}(x^4 + ax^5, x^7) + (x^5, x^6) = (x^4, x^5)$, this gives an example where

$$\mathfrak{m}\text{bf-precore}_R(I) \subsetneq \mathfrak{m}\text{bf-core}_R(I) = \mathfrak{m}\text{bf-prehull}_R(I) = I.$$

**Example 4.2.10.** Let $k$ be a field of characteristic $p > 0$, $R = k[x, y, z]/(xyz)$ and $\mathfrak{m} = (x, y, z)$. In [FVV11, Example 4.1], Fouli, Vassilev and Vraciu consider the $*$-core of ideal $I = (x, yz)$. They note that $(x + ayz)$ is a minimal $*$-reduction of $I$

for any $a \in k$ and that $*\text{-core}_R(I) = \bigcap_{a \in k}(x + ayz) = (x^2, y^2z^2)$. Note that the ideals $(x, y^2z, yz^2) \subseteq (x, yz)$ are tightly closed and $(x, yz)/(x, y^2z, yz^2) \cong R/\mathfrak{m}$. Thus $(x, yz)$ is a non $*$-cover of $(x, y^2z, yz^2)$, thus $(x, y^2z, yz^2)$ is a $*$-prereduction of $(x, yz)$. Also $(x^2, xy, xz, yz) \subseteq (x, yz)$ are tightly closed ideals with

$$(x, yz)/(x^2, xy, xz, yz) \cong R/\mathfrak{m}.$$

Thus $(x^2, xy, xz, yz)$ is a $*$-prereduction of $(x, yz)$. This implies that

$$I = (x, yz) = (x, y^2, yz^2) + (x^2, xy, xz, yz) \subseteq *\text{-prehull}_R(I)$$

giving us an example where the $*\text{-core}_R(I) \subsetneq *\text{-prehull}_R(I)$.

As with the duality of the cl-core and i-hull when $\mathrm{i} = \mathrm{cl}^{\smile}$, we have duality between the cl-precore and the i-posthull and the cl-prehull and the i-postcore.

**Theorem 4.2.11.** *Let $R$ be a complete Noetherian local ring. Let $A \subseteq B$ be Artinian $R$-modules, and let* i *be a relative Nakayama interior defined on Artininian $R$-modules. Then the* i*-postcore of $A$ in $B$ is dual to the* cl*-prehull of $(B/A)^\vee$ in $B^\vee$ and the* i*-posthull of $A$ in $B$ is dual to the* cl*-precore of $(B/A)^\vee$ in $B^\vee$, where* cl *is the closure operation dual to* i.

*Proof.* Let $M = B^\vee$ and $N = (B/A)^\vee$. We need to show that

$$(M/\text{cl-prehull}_M(N))^\vee = \text{i-postcore}^B(A)$$

and

$$(M/\text{cl-precore}_M(N))^\vee = \text{i-posthull}^B(A).$$

These follows from the Definitions 4.2.4 and 4.2.6 as well as Theorem 3.4.5. $\square$

**Proposition 4.2.12.** *[ERGV23c, Proposition 7.3] Let $R$ be a local ring and $\mathrm{cl}_1 \leq \mathrm{cl}_2$ be closure operations defined on the class of finitely generated $R$-modules $\mathcal{M}$ with $\mathrm{cl}_2$ Nakayama. If $N \subseteq M$ are $R$-modules in $\mathcal{M}$, then $\mathrm{cl}_2\text{-core}_M(N) \subseteq \mathrm{cl}_1\text{-core}_M(N)$.*

**Proposition 4.2.13.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring and $\mathrm{cl}_1 \leq \mathrm{cl}_2$ Nakayama closures defined on $\mathcal{P}$. If $(N,M) \in \mathcal{P}$, then $\mathrm{cl}_2\text{-prehull}_M(N) \subseteq \mathrm{cl}_1\text{-prehull}_M(N)$.*

*Proof.* Let $L$ be a $\mathrm{cl}_2$-prereduction of $N$ in $M$. Then

$$L \in \mathrm{cl}_2\text{-prehull}_M(N) = \sum\{L \mid L \text{ maximal in } \mathbf{I}'_{\mathrm{cl}_2}(N,M)\}.$$

By Proposition 3.3.4(1), $\mathbf{I}'_{\mathrm{cl}_2}(N,M) \subseteq \mathbf{I}'_{\mathrm{cl}_1}(N,M)$ so

$$\begin{aligned}\sum\{L \mid L \text{ maximal in } \mathbf{I}'_{\mathrm{cl}_2}(N,M)\} &\subseteq \sum\{L \mid L \text{ maximal in } \mathbf{I}'_{\mathrm{cl}_1}(N,M)\} \\ &= \mathrm{cl}_1\text{-prehull}_M(N).\end{aligned}$$

□

**Proposition 4.2.14.** *[ERGV23c, Proposition 7.12] Let $R$ be an associative (i.e. not necessarily commutative) ring and $\mathrm{i}_1 \leq \mathrm{i}_2$ interior operations on a class $\mathcal{M}$ of (left) $R$-modules. Let $A \subseteq B$ be $R$-modules such that $\mathrm{i}_1$ and $\mathrm{i}_2$ are defined on all $R$-modules between $A$ and $B$. Then $\mathrm{i}_2\text{-hull}^B(A) \subseteq \mathrm{i}_1\text{-hull}^B(A)$.*

**Proposition 4.2.15.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring and $\mathcal{P}$ be Artinian $R$-modules. Let $\mathrm{i}_1 \leq \mathrm{i}_2$ be Nakayama interiors on $\mathcal{P}$. Then*

$$\mathrm{i}_1\text{-postcore}^B(A) \subseteq \mathrm{i}_2\text{-postcore}^B(A).$$

*Proof.* Let $C \in \mathrm{i}_1\text{-postcore}^B(A) = \bigcap\{C \mid C \text{ minimal in } \mathbf{C}'_{\mathrm{i}_1}(A,B)\}$.

So $C$ is an $\mathrm{i}_1$-postexpansion. By Proposition 3.3.6(1), we know that

$$\mathbf{C}'_{\mathrm{i}_1}(A,B) \subseteq \mathbf{C}'_{\mathrm{i}_2}(A,B).$$

Thus

$$\begin{aligned}\bigcap\{C \mid C \text{ minimal in } \mathbf{C}'_{\mathrm{i}_1}(A,B)\} &\subseteq \bigcap\{C \mid C \text{ minimal in } \mathbf{C}'_{\mathrm{i}_2}(A,B)\} \\ &= \mathrm{i}_2\text{-postcore}^B(A).\end{aligned}$$

□

## 4.3 Special Part of a Closure

The tight closure of an ideal can be decomposed into a minimal $*$-reduction and its *special part* [Vra02]; this was then generalized to the special part of a Nakayama closure [Eps10]. Vraciu later showed with Huneke [HV03] that in excellent normal rings with perfect residue field that the tight closure of an ideal $I$ has a special part decomposition in terms of a minimal $*$-reduction of $I$ and its special part. (See Remark 4.3.3(6) below.) We hope to make use of this decomposition to determine all the $*$-prereductions of a tightly closed ideal.

**Definition 4.3.1.** [Vra02, Definition 4.1] Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ and $I$ be an ideal. Let $R^o$ be the set of elements that are not in any minimal prime of $R$. We say $x \in R$ is in the *special part* of the tight closure of $I$ ($x \in I^{*\mathrm{sp}}$) if there exists a $c \in R^o$ such that $cx^q \in \mathfrak{m}^{q/q_0} I^{[q]}$ for all $q \geq q_0$.

**Definition 4.3.2.** [Eps05] We say that an element $c$ is a *($q_0$-)weak test element* of $R$ if it is not in any minimal prime of $R$ and if for all $x$ and all $I$, $x \in I^*$ if and only if for all powers $q \geq q_0$ of $p$, $cx^q \in I^{[}q]$.

We collect a few facts about the special part of the tight closure below:

**Remark 4.3.3.** Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ with weak test element $c$.

1. An alternate description of the elements $x$ in the special part of the tight closure is: $x \in I^{*\mathrm{sp}}$ if and only if there exists a $q_0$ such that $x^{q_0} \in (\mathfrak{m} I^{[q_0]})^*$ [Vra02, Definition 4.1].

2. For any ideal $I$, $(I^*)^{*\mathrm{sp}} = I^{*\mathrm{sp}} = (I^{*\mathrm{sp}})^*$ and if $J \subseteq I$ and $J^* = I^*$ then $J^{*\mathrm{sp}} = I^{*\mathrm{sp}}$ [Eps05, Lemma 3.4].

3. If $I \subseteq I^{*\mathrm{sp}}$ then $I$ is nilpotent [Eps05, Lemma 3.1].

4. $\mathfrak{m}I \subseteq I^{*\text{sp}} \cap I$ and equality holds if $I$ is a $*$-basic ideal [Eps05, Lemma 3.2].

5. If $x_1, \ldots, x_k$ are $*$-independent elements in $I$, then they are also $*$-independent modulo $I^{*\text{sp}}$. In particular, if $J = (x_1, \ldots, x_k)$ is a minimal $*$-reduction of $I$ and $x \in I^{*\text{sp}}$, then
$$J' = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k)$$
is not a minimal $*$-reduction of $I$ for any choice of $i$ [Vra06, Proposition 1.12].

6. If $R$ is further excellent and normal with perfect residue field, then for every ideal $I$ of an excellent normal ring of positive characteristic, $I^* = I + I^{*\text{sp}}$ [HV03, Theorem 2.1].

The following is a generalization of [KRS20, Corollary 7.2] for tight closure.

**Proposition 4.3.4.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ containing a weak test element. Suppose that $I$ is a proper principal ideal which is not nilpotent. Then the unique $*$-prereduction of $I$ is $\mathfrak{m}I$.*

*Proof.* Let $I = (x)$ where $x$ is not nilpotent. Thus, $x \notin (0)^*$ implying that $x$ is $*$-independent and the set of $*$-prereductions of $I$ is $\{\mathfrak{m}I\}$ by Proposition 4.1.8. □

As with $*$-independence, Vraciu has noted in [Vra06, Observation 1.5] that if one set of generators of an ideal $K = (J, x_1, \ldots, x_k)$ are $*$-independent modulo $J$ then any set of generators of $K$ modulo $J$ are $*$-independent. Vraciu also defined the following set in [Vra06].

**Definition 4.3.5.** Let $J \subseteq I$ be tightly closed ideals. Let $\mathcal{F}(J, I)$ be the set of all tightly closed ideals $K$ such that $J \subseteq K \subseteq I$ and $\lambda(I/K) = 1$.

**Theorem 4.3.6.** *[Vra06, Theorem 2.2] Let $(R, \mathfrak{m})$ be a local excellent normal ring of characteristic $p > 0$ and $J \subseteq I$ be tightly closed ideals. Then $\mathcal{F}(J, I)$ is equal to the* *set*

$$\{(J, x_1, \ldots, x_{k-1}) + I^{*\mathrm{sp}} \mid (J, x_1, \ldots, x_{k-1}, x_k) \textit{ min } * \textit{-reduction of } I \textit{mod} J\textit{, some } x_k\}.$$

Tight closure is not the only closure which has a special part, in fact both the Frobenius closure and integral closure have a special part. See [Eps10].

The special part of a closure operation is defined below.

**Definition 4.3.7.** [Eps10, Definition 2.1] Let $(R, \mathfrak{m})$ be a Noetherian local ring and cl be a closure operation on the ideals of $R$. Then clsp is a *special part* of cl if the following four axioms hold for ideals $I \subseteq R$.

1. $I^{\mathrm{clsp}}$ is an ideal of $R$.
2. $\mathfrak{m}I \subseteq I^{\mathrm{clsp}} \subseteq I^{\mathrm{cl}}$.
3. $(I^{\mathrm{cl}})^{\mathrm{clsp}} = I^{\mathrm{clsp}} = (I^{\mathrm{clsp}})^{\mathrm{cl}}$.
4. If $J \subseteq I \subseteq (J + I^{\mathrm{clsp}})^{\mathrm{cl}}$, then $I \subseteq J^{\mathrm{cl}}$.

**Remark 4.3.8.** Epstein showed [Eps10, Lemma 2.2] and [Eps05, Proposition 2.3] that a closure cl with a special part then

1. cl is Nakayama and hence $J$ is a minimal reduction of $I$ if and only if $J$ is strongly cl-independent.
2. If $J \subseteq I$ then $J^{\mathrm{clsp}} \subseteq I^{\mathrm{clsp}}$.
3. If $I$ is cl-independent then $\mathfrak{m}I = I \cap I^{\mathrm{clsp}}$.

Note that for all the known closures cl with a special part, an element $z \in I^{\mathrm{clsp}}$ if there exists some increasing function $f : \mathbb{N} \to \mathbb{N}$ and some $n \in \mathbb{N}$ such that $z^{f(n)} \in (\mathfrak{m}I^{f(n)})^{\mathrm{cl}}$ or $z^{f(n)} \in (\mathfrak{m}I^{[f(n)]})^{\mathrm{cl}}$. For the special part of integral closure

$f(n) = n$ [Eps10, Definition 5.1] as $z \in I^{-\mathrm{sp}}$ if $z^n \in (\mathfrak{m}I^n)^-$. For the special part of tight closure and the special part of Frobenius closure, $f(n) = p^n$ [Eps10, Definitions 3.1 and 4.1] where $p$ is the characteristic of the ring and $z \in I^{*\mathrm{sp}}$ if $z^{p^n} \in (\mathfrak{m}I^{[p^n]})^*$ and $z \in I^{F\mathrm{sp}}$ if $z^{p^n} \in (\mathfrak{m}I^{[p^n]})^F$. Note that in all three cases, the function $f$ defines a descending chain of ideals $I^{\{f(n)\}} \supseteq I^{\{f(m)\}}$ for $m \geq n$, where $I^{\{f(n)\}}$ is the appropriate $f(n)$-th "power" associated to the closure cl.

**Definition 4.3.9.** Let $R$ be a commutative ring and cl a closure defined on the ideals of of $R$. We will say that the special part of $I$ with respect to the closure cl is defined by the function $f : \mathbb{N} \to \mathbb{N}$, with values $n \geq n_0 \geq 0$ and the ideals $\{I^{\{f(n)\}}\}_{n \geq n_0}$ if $z \in I^{\mathrm{clsp}}$ when $z^{f(n)} \in (\mathfrak{m}I^{\{f(n)\}})^{\mathrm{cl}}$ for $n \geq n_0$.

We don't use this description for the proofs below as the closures are too general, but it is an interesting point that may be useful in the future as one should be able to use this function to show that $I$ satisfies the following property.

**Definition 4.3.10.** Let $(R, \mathfrak{m})$ be a local commutative ring and cl a closure defined on the ideals of of $R$. We say that an ideal has a cl-*special part decomposition on* $R$ if $I^{\mathrm{cl}} = I + I^{\mathrm{clsp}}$.

The key ingredient to generalize Theorem 4.3.6 is that strongly cl-independent ideals have a cl-special part decomposition.

**Remark 4.3.11.** As a consequence of Remark 4.3.8(3), all ideals $I$ which are generated by cl-independent elements will satisfy $I^{\mathrm{clsp}} \cap I = \mathfrak{m}I$. Combining this with the assumption $I^{\mathrm{cl}} = I + I^{\mathrm{clsp}}$, we will have the direct sum decomposition

$$\frac{I^{\mathrm{cl}}}{\mathfrak{m}I} = \frac{I}{\mathfrak{m}I} \oplus \frac{I^{\mathrm{clsp}}}{\mathfrak{m}I}$$

for ideals generated by strongly cl-independent elements. Note that if $K$ is a minimal cl-reduction of an ideal $I$, $K$ will be generated by strongly cl-independent elements by Remark 4.3.8(1). So all minimal reductions of an ideal will have this direct sum decomposition if they have a special part decomposition.

**Proposition 4.3.12.** *Let $(R, \mathfrak{m})$ be a commutative local Noetherian ring, and let* cl *be a Nakayama closure defined on the ideals of $R$. If all ideals generated by strongly* cl*-independent elements satisfy a special part decomposition, then all ideals satisfy a special part decomposition.*

*Proof.* Suppose that $K$ is an ideal generated by strongly cl-independent elements and $K \subseteq J \subseteq K^{\mathrm{cl}}$, then by Definition 4.3.7(3)

$$K^{\mathrm{clsp}} = (K^{\mathrm{cl}})^{\mathrm{clsp}} = (J^{\mathrm{cl}})^{\mathrm{clsp}} = J^{\mathrm{clsp}}.$$

Thus, if $K + K^{\mathrm{clsp}} = K^{\mathrm{cl}}$, then since $J, J^{\mathrm{clsp}} \subseteq J^{\mathrm{cl}}$ and

$$J^{\mathrm{cl}} = K^{\mathrm{cl}} = K + K^{\mathrm{clsp}} \subseteq J + J^{\mathrm{clsp}} \subseteq J^{\mathrm{cl}},$$

we have $J + J^{\mathrm{clsp}} = J^{\mathrm{cl}}$. □

We generalize the following result of [Vra06].

**Proposition 4.3.13.** *(See [Vra06, Proposition 1.12]) Let $(R, \mathfrak{m})$ be a Noetherian local ring and* cl *a closure defined on the ideals of $R$. Suppose that all strongly* cl*-independent ideals $I \subseteq R$ have a* cl*-special part decomposition. If $x_1, \ldots, x_k$ are* cl*-independent, then they are also* cl*-independent modulo $I^{\mathrm{clsp}}$.*

*In particular, if $J = (x_1, \ldots, x_k)$ is a minimal* cl*-reduction of $I$ and $x \in I^{\mathrm{clsp}}$, then*

$$J' = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k)$$

*is not a minimal* cl*-reduction of $I$ for any choice of $i$.*

*Proof.* For each $i \in 1, ..., k$, we need to show that

$$f_i \notin (I^{\mathrm{clsp}}, f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, fl)^{\mathrm{cl}}.$$

Otherwise, one could extract a minimal cl-reduction $J'$ generated by some of the $f_j$, $j \neq i$ and some of the elements in $I^{\mathrm{clsp}}$. Since $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k$ alone cannot generate a cl-reduction, it follows that we must have elements in $I^{\mathrm{clsp}}$ among the minimal generators of $J'$. But since $J'$ is a minimal cl-reduction, then $J'$ must be cl-independent. By part (3) of Definition 4.3.7, then $J^{\mathrm{clsp}} = (I^{\mathrm{cl}})^{\mathrm{clsp}} = I^{\mathrm{clsp}}$. Putting these two facts together, $I^{\mathrm{clsp}} \bigcap J' = \mathfrak{m}I$, but this is a contradiction. $\square$

The following set is modeled after a set defined by Vraciu in [Vra06] for tight closure.

**Definition 4.3.14.** Let $J \subseteq I$ be cl-closed ideals. Let $\mathcal{F}_{\mathrm{cl}}(J, I)$ be the set of all cl-closed ideals $K$ such that $J \subseteq K \subseteq I$ and $\lambda(I/K) = 1$.

Note that the ideals $K$ in $\mathcal{F}_{\mathrm{cl}}(J, I)$ will have $I$ as a non-cl-cover as both $I$ and $K$ are both cl-closed. However, this set is not the set of all ideals $K$ for which $I$ is a non-cl-cover of $K$, but only those containing an ideal $J$.

We believe that $\mathcal{F}_{\mathrm{cl}}(J, I)$ will be related to cl-prereductions and the cl-special part decompositions, in particular for the $*$-closure and Frobenius closure.

# Chapter 5

# Other Properties

Following the work of Epstein, R.G. and Vassilev [ERG21] [ERGV23c], [ERGV23b] and [ERGV23a], we further the thematic notion of duality and seek to understand how it arises in the context of other properties of pair operations. In particular, we will see in Proposition 5.1.1 that the notions of order reversing and involutive are self dual, and that independence is dual to spanning. Because involutive and idempotent are strong requirements, we also extend our analysis to the weaker notions of pre- and post- involutive and pre- and post-idempotent.

Much of the previous work on pair operations started by Epstein, R.G. and Vassilev [ERGV23b, ERGV23a] was motivated by closure and interior operations. The properties we define here are motivated by operations on inner product spaces such as the orthogonal complement and other non-closure or interior operations on modules.

**Definition 5.0.1.** Let $R$ be a ring, $\mathcal{M}$ be a category of $R$-modules, and $\mathcal{P}$ be a class of pairs $(L, M)$ with $L \subseteq M$ in $\mathcal{M}$. Let $p$ be a pair operation on pairs of modules $(L, M) \in \mathcal{P}$ as defined in Definition 2.1.2. We say $p$ is

- *order-reversing on submodules* if whenever $L \subseteq N \subseteq M$ with

$(L, M), (N, M) \in \mathcal{P}$, we have $p(N, M) \subseteq p(L, M)$.

- *$n$-periodic* if whenever $(p^i(L, M), M) \in \mathcal{P}$ for $0 \leq i \leq n-1$, $p^n(L, M) = L$.
- *pre-involutive* if when $(L, M), (p(L, M), M) \in \mathcal{P}$, then $L \subseteq p^2(L, M)$.
- *post-involutive* if when $(L, M), (p(L, M), M) \in \mathcal{P}$, then $p^2(L, M) \subseteq L$.
- *involutive* (or 2-periodic) if when $(L, M), (p(L, M), M) \in \mathcal{P}$, then $p^2(L, M) = L$.
- *independent* if whenever $(L, M) \in \mathcal{P}$, $L \cap p(L, M) = 0$.
- *spanning* if whenever $(L, M) \in \mathcal{P}$, $L + p(L, M) = M$.
- *complementary* if $p$ is both independent and spanning.
- *pre-idempotent* if when $(L, M), (p(L, M), M) \in \mathcal{P}$, then $p(L, M) \subseteq p^2(L, M)$.
- *post-idempotent* if when $(L, M), (p(L, M), M) \in \mathcal{P}$, then $p^2(L, M) \subseteq p(L, M)$.
- a *preclosure operation* if it is extensive and order-preserving on submodules.
- a *postinterior operation* if it is intensive and order-preserving on submodules.

**Remark 5.0.2.** Notice that

1. If $p$ is pre-idempotent and $(p^n(L, M), M) \in \mathcal{P}$ for all $n \geq 1$, then
$$p^n(L, M) \subseteq p^{n+1}(L, M)$$
for all $n \geq 1$.

2. If $p$ is post-idempotent and $(p^n(L, M), M) \in \mathcal{P}$ for all $n \geq 1$, then
$$p^{n+1}(L, M) \subseteq p^n(L, M)$$
for all $n \geq 1$.

3. If $p$ is pre-involutive and $(p^n(L,M),M) \in \mathcal{P}$ for all $n \geq 1$, then for all $n \geq 0$ $p^n(L,M) \subseteq p^{n+2}(L,M)$.

4. If $p$ is post-involutive and $(p^n(L,M),M) \in \mathcal{P}$ for all $n \geq 1$, then
$$p^{n+2}(L,M) \subseteq p^n(L,M)$$
for all $n \geq 0$.

Before we note some relations between these properties, we exhibit some pair operations which exhibit a selection of these properties.

The orthogonal complement is a simple and extensively studied type of pair operation. Upon comparing closure operations with the orthogonal complement, one will notice that, as pair operations, they behave in essentially opposite ways.

**Example 5.0.3.** Suppose $k$ is a field, $\mathcal{M}$ be the category of finitely generated $k$-inner product spaces $V$ with an inner product $\langle\ ,\ \rangle$ and $\mathcal{P}$ be the class of pairs of inner product spaces $(W,V)$ with $W \subseteq V$. Set
$$p(W,V) = W_V^{\perp} = \{x \in V : \langle x,w\rangle = 0\ \ \forall w \in W\},$$
the orthogonal complement of $W$ in $V$. If $U \subseteq W \subseteq V$, then $W_V^{\perp} \subseteq U_V^{\perp}$ since if $v \in W_V^{\perp}$ then $\langle v,w\rangle = 0$ for all $w \in W$. But since $U \subseteq W$, then $\langle v,u\rangle = 0$ for all $u \in U$, implying that $v \in U_V^{\perp}$. Hence $p(W,V) = W_V^{\perp}$ is a pair operation that is order reversing on submodules. Note that $p$ is also involutive and complementary since $(W_V^{\perp})_V^{\perp} = W$ and $V \cong W \oplus W_V^{\perp}$.

Notice that for $k$-vector spaces Noetherian and Artinian are the same and the Matlis dual of $V$ is the same as the dual vector space, $\operatorname{Hom}_k(V,k) =: V^*$. We need the finitely generated assumption because when $V$ is infinite, $V^*$ has larger cardinality and so duality becomes complicated. We can still define these properties on modules over complete local commutative rings.

The next few examples demonstrate that pair operations can be defined to exhibit the properties in Definition 5.0.1.

**Example 5.0.4.** Suppose $R$ is any ring and $\mathcal{M}$ is the category of $R$-modules and $\mathcal{P}$ is the class of all pairs $(L, M)$ with $L \subseteq M$ in $\mathcal{M}$. Define $p(L, M) = M$ for all $L \subseteq M$. Note that $p$ is spanning since $L + p(L, M) = L + M = M$ but not independent since $L \cap p(L, M) = L \cap M = L \neq 0$. Note that $p$ is both order preserving and order reversing on submodules and $p$ is not involutive.

**Example 5.0.5.** Let $R$ be a Noetherian ring and $I$ a regular ideal in $R$. The Ratliff-Rush ideal associated with $I$ is defined as

$$\tilde{I} := \bigcup_{n=1}^{\infty} (I^{n+1} : I^n)$$

Then $\tilde{I}$ is extensive and idempotent. However, it is not order preserving. For example, [HJLS93] showed that if $R = [[t^3, t^4]]$, $I = t^8 R$ and $J = (t^{11}, t^{12})R$ then $I \subseteq J$ but $\tilde{I} \nsubseteq \tilde{J}$. Through the use of duality, we know that the dual of $\tilde{I}$ will be intensive and idempotent, but not order preserving.

**Example 5.0.6.** Let $R$ be a commutative ring and $\mathcal{M}$ be a class of modules contained in a fixed $R$-module $M$ and $\mathcal{P}$ the set of pairs $(L, M)$ for $L \in \mathcal{M}$. Let $\sigma_M$ be an automorphism defined on the submodules of $M$ such that if $\phi : M \to M'$ is an isomorphism, then $\sigma_{\phi(M)}(\phi(L)) := \phi(\sigma_M(L))$ for any $L \subseteq M$. Then we can define $p$ to be the pair operation defined by $p(L, M) = \sigma_M(L)$. If $\sigma_M$ has finite order and $n$ in the group of automorphisms, then $p^n(L, M) = \sigma_M^n(L) = L$. And $p$ will be at most $n$-periodic. Note that it may be the case that $p$ will be the identity pair operation even if $\sigma_M$ is not the identity. For example let $\mathcal{M}$ be the class of all subfields of $\mathbb{C}$. These subfields all contain $\mathbb{Q}$ and are $\mathbb{Q}$-vector spaces. Let $\tau$ be the automorphism defined by complex conjugation. Note that although $\tau$ is not the identity map, $\tau(K) = K$ for all $K \subseteq \mathbb{C}$. However, if $\sigma$ is the extension of the embedding $\phi : \mathbb{Q}(\sqrt[n]{2}) \to \mathbb{C}$ defined by $\sigma(\sqrt[n]{2}) = \zeta_n \sqrt[n]{2}$ to all of $\mathbb{C}$ where $\zeta_n$ is a primitive $n$-th root of unity, then if

$p(K, \mathbb{C}) = \sigma(K)$, then $p$ will be $n$-periodic since $p^i(\mathbb{Q}(\sqrt[n]{2}), \mathbb{C}) = \mathbb{Q}(\zeta_n^i \sqrt[n]{2}) \neq \mathbb{Q}(\sqrt[n]{2})$ for all $1 \leq i < n$.

**Example 5.0.7.** Let $R$ be a commutative ring and suppose that $\mathcal{M}$ is the class of $R$-modules and $\mathcal{P}$ is the set of pairs of the form $(N, M)$ with $N \subseteq M$. Define $p(N, M) = \text{cl-core}_M(N)$. If $L \subseteq M$ and $L_i$ are a collection of reductions of $L$ in $M$, then $\text{cl}-\text{core}_M(L) = \bigcap L_i \subseteq L$. So $\text{cl-core}_M(L) \subseteq L_i \subseteq L$ for all $i$ and $p$ is intensive. Note that $p^2(N, M) \subseteq p(N, M) \subseteq N$ for all $N \subseteq M$ and $p$ is post-idempotent and post-involutive. However, $\text{cl-core}_M(L)$ is not order preserving nor order reversing [Lee08]. Similarly if $q(N, M) = \text{i-hull}^M(N)$,

$$N \subseteq q(N, M) \subseteq q^2(N, M)$$

implies that $q$ is extensive, pre-idempotent and pre-involutive.

**Example 5.0.8.** Let $R = k[x]/(x^n)$. Note that all the ideals of $R$ are of the form $(x^i)$ for $n \geq i \geq 0$ and they form a chain $(x^i) \subseteq (x^j)$ if $i \geq j$. Let $\mathcal{M}$ be the class of ideals of $R$ and $\mathcal{P}$ be the set of pairs $((x^i), (x^j))$ for $i \geq j$. For some fixed $1 \leq r \leq n$, define

$$p((x^i), (x^j)) = \begin{cases} (x^{j+r+n-i}) \text{ if } j + r - i \leq 0 \\ (x^n) \text{ if } j + r - i > 0. \end{cases}$$

Note that

$$p^2((x^i), (x^j)) = \begin{cases} (x^{j+r}) \text{ if } j \leq i < j + r \\ (x^i) \text{ if } j + r \leq i \leq n. \end{cases}$$

$p$ is post-involutive, order reversing but not involutive since

$$p^2((x^i), (x^j)) = (x^{j+r}) \subsetneq (x^{j+k})$$

for $j \leq i < j + r$ and $p^2((x^i), (x^j)) = (x^i)$ for $j + r \leq i \leq n$. However, $p$ is neither pre-idempotent nor post-idempotent. For example if $n = 5$ and $r = 1$, we see that

$$p^2((x^3), (x)) = (x^3) \supsetneq (x^4) = p((x^3), (x)),$$

but $p^2((x^4),(x)) = (x^4) \subseteq (x^3) = p((x^4),(x))$.

Similarly, we can construct a pre-involutive, order reversing pair operation $q$ on $\mathcal{P}$ which is neither pre-idempotent nor post-idempotent by defining

$$q((x^i),(x^j)) = \begin{cases} (x^j) \text{ if } n-r < i \\ (x^{j-r+n-i}) \text{ if } n-r \geq i. \end{cases}$$

Since

$$q^2((x^i),(x^j)) = \begin{cases} (x^{n-r}) \text{ if } n-r < i \\ (x^i) \text{ if } n-r \geq i, \end{cases}$$

we see that $(x^i) \subsetneq q^2((x^i),(x^j)) = (x^{n-r})$ for $n-r < i$ which implies that $q$ is pre-involutive. It is an easy check to see that $q$ is not pre-idempotent or post-idempotent.

**Example 5.0.9.** Let $R = k[[x]]$. All ideals have the form $(x^i)$ for $i \geq 0$ and they form a chain. Let $\mathcal{M}$ be the ideals of $R$ and $\mathcal{P}$ be the pairs $((x^i),(x^j))$ for $i \geq j$. Define

$$p((x^i),(x^j)) = \begin{cases} (x^{i+1}) \text{ if } i-j \text{ is even} \\ (x^{i-1}) \text{ if } i-j \text{ is odd.} \end{cases}$$

$p$ will be involutive but not order preserving, order reversing, pre-idempotent nor post-idempotent.

If we define a pair operation $q$ on $\mathcal{P}$ by

$$q((x^i),(x^j)) = \begin{cases} (x^{i+1}) \text{ if } i-j \text{ is even} \\ (x^i) \text{ if } i-j \text{ is odd.} \end{cases}$$

Then $q$ is order preserving, intensive, idempotent (an interior) and pre-involutive.

If we define a pair operation $r$ on $\mathcal{P}$ by

$$r((x^i),(x^j)) = \begin{cases} (x^i) \text{ if } i-j \text{ is even} \\ (x^{i-1}) \text{ if } i-j \text{ is odd.} \end{cases}$$

Then $r$ is order preserving, extensive, idempotent (a closure) and post-involutive.

If we define a pair operation $s$ on $\mathcal{P}$ by

$$s((x^i),(x^j)) = \begin{cases} (x^{i+3}) \text{ if } i-j \text{ is even} \\ (x^{i-1}) \text{ if } i-j \text{ is odd.} \end{cases}$$

$s$ is not order preserving, order reversing, pre-idempotent nor post-idempotent but is post involutive.

If we define a pair operation $t$ on $\mathcal{P}$ by

$$t((x^i),(x^j)) = \begin{cases} (x^{i+1}) \text{ if } i-j = 3n, 3n+1 \text{ for some } n \in \mathbb{N} \\ (x^{i-2}) \text{ if } i-j = 3n+2 \text{ for some } n \in \mathbb{N} \end{cases}$$

$t$ is not order preserving, order reversing, pre-idempotent, post-idempotent, nor involutive but is 3-periodic.

**Example 5.0.10.** Let $R$ be any ring, $\mathcal{M}$ be the category of all $R$-modules, $\mathcal{P}$ be the class of all pairs $(L, M)$ with $L \subseteq M$ and $p$ be a pair operation defined on $\mathcal{P}$ given by $p(0, M) = M$ and $p(L, M) = 0$ for all $L \subseteq M$, $L \neq 0$. Then $p^2(0, M) = 0$ and $p^2(L, M) = M$ for all $L \neq 0$. Note that for all $L \subseteq M$, $p^2(L, M) \supseteq L$ implying that $p$ is pre-involutive, but not involutive. Similarly, let $q$ be a pair operation defined on $\mathcal{P}$ given by $q(M, M) = 0$ and $q(L, M) = M$ for all $L \subseteq M$ with $L \neq M$. Note that $q^2(L, M) \subseteq L$ for all $L \subseteq M$; hence, $q$ is an example of a post-involutive pair operation which is not involutive. Note using Notation 2.1.4, it is easy to verity that the pair operation $p^2$ is a closure operation and $q^2$ is an interior operation.

**Example 5.0.11.** Let $R$ be any ring which is not a field, $x \in R$ be a fixed nonzero element which is not a unit. Suppose $\mathcal{M}$ is the category of all $R$-modules and $\mathcal{P}$ is the set of all pairs of submodules $(L, M)$ with $L \subseteq M$ in $\mathcal{M}$ and $p$ be a pair operation defined on $\mathcal{P}$ given by $p(0, M) = xM$ and $p(L, M) = M$ for all $L \subseteq M$, $L \neq 0$. Then

$$p^2(0, M) = M \supsetneq xM = p(0, M)$$

when $x \notin \operatorname{ann}_R(M)$ and $p^2(L,M) = M \subseteq p(L,M)$ for all $L \neq 0$ implying that for all $L \subseteq M$, $p^2(L,M) \supseteq p(L,M)$ or $p$ is pre-idempotent but not idempotent. Similarly, let $q$ be a pair operations defined on $\mathcal{P}$ given by $q(M,M) = xM$ and $q(L,M) = 0$ for all $L \subseteq M$ with $L \neq M$. Note that $q^2(L,M) \subseteq q(L,M)$ for all $L \subseteq M$ and $q^2(L,M) \subsetneq q(L,M)$ as long as $x \notin \operatorname{ann}_R(M)$; hence, $q$ is an example of a post-idempotent pair operation which is not idempotent. Note using Notation 2.1.4, it is easy to verity that, $p^2$ is a closure operation and $q^2$ is an interior operation.

**Example 5.0.12.** We can see that the isomorphism condition in the definition of pair operation can be quite strong in conjunction with some of the properties defined for pair operations by observing what happens when our ring is a field and our class of modules is subspaces of $V = \mathbb{F}_2^2$ and $\mathcal{P} = \{(W,V) \mid W \subseteq V\}$. Note that the only possible $W$ are $0$, $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,1)\rangle$, and $V$. Suppose $p$ is order preserving on submodules and because $p$ must preserve isomorphisms, there are a limited number of ways to define $p$. We can't, for instance, have $p(0,V) = \langle(1,1)\rangle$ unless $p(\langle(a,b)\rangle, V) = V$ for all $(a,b) \neq (0,0)$. Because if $p(0,V) = \langle(1,1)\rangle$, then since $\langle(1,1)\rangle \subseteq p(\langle(a,b)\rangle, V)$ for $(a,b) \neq (1,1)$, this implies that $p(\langle(a,b)\rangle, V) = v$ for $(a,b) \neq (1,1)$. However, since there is an isomorphism $\phi : V \to V$ satisfying $\phi(1,1) = (a,b)$, $(a,b) \neq (0,0)$, then this would imply that

$$\phi(p(\langle(1,1)\rangle, V)) = p(\langle\phi(1,1)\rangle, V) = p(\langle(a,b)\rangle, V) = V.$$

We can also see that not every closure operation is a pair operation. For instance, define

$$W_V^{\mathrm{cl}} = \begin{cases} \langle 1,1\rangle \text{ if } W = 0 \\ \langle 1,1\rangle \text{ if } W = \langle 1,1\rangle \\ V \text{ if } W \neq 0, \langle 1,1\rangle \end{cases}$$

Then cl is idempotent, order preserving, and extensive (and thus a closure operation by the traditional definition) but is not a pair operation using the same $\phi$ as before.

We have several possibilities for pair operations on $V$:

$$p_0(W,V) = W$$

$$p_1(W,V) = \begin{cases} 0 \text{ if } W \neq V \\ V \text{ if } W = V \end{cases}$$

$$p_2(W,V) = \begin{cases} V \text{ if } W \neq V \\ 0 \text{ if } W = V \end{cases}$$

$$p_3(W,V) = U \text{ for some } U$$

$p_0$ is the identity pair operation and $p_3$ is the constant pair operation. Notice, $p_1^{2n} = p_2^{2n-1}$ for $n \geq 1$. $p_1$ is order reversing, post-involutive, and spanning but not intensive, extensive, involutive, or independent. $p_2$ is order preserving, intensive, and idempotent but not extensive, independent, or spanning.

**Proposition 5.0.13.** *Let $R$ be a ring, $\mathcal{M}$ be a class of $R$-modules, $\mathcal{P}$ be a class of pairs of $R$-modules $(L, M)$ with $L \subseteq M$ in $\mathcal{M}$ and $p$ be a pair operation defined on $\mathcal{P}$.*

1. *$p$ is involutive if and only if $p$ is pre-involutive and post-involutive.*

2. *$p$ is idempotent if and only if $p$ is pre-idempotent and post-idempotent.*

3. *If $p$ is a preclosure operation, then $p$ is pre-idempotent and pre-involutive.*

4. *If $p$ is a postinterior operation, then $p$ is post-idempotent and post-involutive.*

5. *$p$ is a closure operation if and only if $p$ is a preclosure operation and post-idempotent.*

6. *$p$ is an interior operation if and only if $p$ is a postinterior operation and pre-idempotent.*

*Proof.* (1) Suppose $p$ is involutive and $(L, M) \in \mathcal{P}$. Then $p^2(L, M) = L$. So clearly $L \subseteq p^2(L, M) \subseteq L$ and $p$ is both pre- and post-involutive. Suppose $p$ is both pre- and post-involutive. Then $L \subseteq p^2(L, M) \subseteq L$ so $L = p^2(L, M)$ and $p$ is involutive.

(2) Suppose $p$ is idempotent and $(L, M) \in \mathcal{P}$. Then $p^2(L, M) = p(L, M)$. Clearly $p(L, M) \subseteq p^2(L, M) \subseteq p(L, M)$ and $p$ is both pre- and post-idempotent. Suppose $p$ is both pre- and post-idempotent. Then $p(L, M) \subseteq p^2(L, M) \subseteq p(L, M)$. Thus $p(L, M) = p^2(L, M)$ and $p$ is idempotent.

(3) Suppose $p$ is a preclosure operation. Then $p$ is extensive and order-preserving. So $L \subseteq p(L, M)$ implies $p(L, M) \subseteq p^2(L, M)$ and $p$ is pre-idempotent. Also,

$$L \subseteq p(L, M) \subseteq p^2(L, M)$$

implying that $p$ is pre-involutive.

(4) Suppose $p$ is a postinterior operation. Then $p$ will be intensive and order-preserving. So $p(L, M) \subseteq L$ implies $p^2(L, M) \subseteq p(L, M)$ and $p$ is post-idempotent. Also, $p^2(L, M) \subseteq L$ and $p$ is post-involutive.

(5) Suppose $p$ is a closure operation. Then $p$ being a preclosure operation follows by definition and $p$ being post-idempotent follows from (2). Suppose $p$ is a preclosure operation and post-idempotent. Then by (3), $p$ is also pre-idempotent and by (2), $p$ is idempotent. Thus $p$ is a closure operation.

(6) Suppose $p$ is an interior operation. Then $p$ being a postinterior operation follows by definition and $p$ being pre-idempotent follows from (2). Suppose $p$ is a postinterior operation and pre-idempotent. Then by (4), $p$ is also post-idempotent and by (2), $p$ is idempotent. Thus $p$ is an interior operation. $\square$

**Remark 5.0.14.** Notice that Proposition 5.0.13(3) and (4) are not "if and only if." To see this, consider Example 2.3.2 where $p(I, R) = \text{core}(I)$. $\diamondsuit$

## 5.1 Duality of Properties

For all the new properties of pair operations that we have defined, we would like to determine their duals $p^{\smile}$ when $\mathcal{M}$ is either the category of finitely generated $R$-modules or the category of Artinian $R$-modules over a complete local Noetherian ring $(R, \mathfrak{m})$.

**Proposition 5.1.1.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring, $\mathcal{M}$ be either the class of Noetherian $R$-modules or the class of Artinian $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2 and $p$ be a pair operation on $\mathcal{P}$. Also, let $p^{\smile}$ be the dual of $p$ defined on $\mathcal{P}^{\vee}$ as in Definition 3.4.1. Then*

1. *$p$ is order reversing on submodules if and only if $p^{\smile}$ is order reversing on submodules.*
2. *If $p$ is pre-involutive then $p^{\smile}$ is post-involutive.*
3. *If $p$ is post-involutive then $p^{\smile}$ is pre-involutive.*
4. *If $p$ is pre-idempotent then $p^{\smile}$ is post-idempotent.*
5. *If $p$ is post-idempotent then $p^{\smile}$ is pre-idempotent.*
6. *$p$ is involutive if and only if $p^{\smile}$ is involutive.*
7. *$p$ is $n$-periodic if and only if $p^{\smile}$ is $n$-periodic.*
8. *If $p$ is independent then $p^{\smile}$ is spanning.*
9. *If $p$ is spanning then $p^{\smile}$ is independent.*
10. *$p$ is complementary if and only if $p^{\smile}$ is complementary.*

*Proof.* (1) By Lemma 3.4.3, we only need to show one direction of the equivalence. Suppose $p$ is order reversing and let $L \subseteq N \subseteq M$ be such that $(L, M), (N, M) \in \mathcal{P}^\vee$. Then

$$(M/N)^\vee \subseteq (M/L)^\vee \subseteq M^\vee$$

and $((M/N)^\vee, M^\vee), ((M/L)^\vee, M^\vee) \in \mathcal{P}$. By assumption,

$$p((M/L)^\vee, M^\vee) \subseteq p((M/N)^\vee, M^\vee)$$

and we have a natural surjection

$$\frac{M^\vee}{p((M/L)^\vee, M^\vee)} \twoheadrightarrow \frac{M^\vee}{p((M/N)^\vee, M^\vee)}$$

Applying Matlis duality and the definition of $^\smile$,

$$\begin{aligned} p^\smile(L, M) &= \left(\frac{M^\vee}{p((M/L)^\vee, M^\vee)}\right)^\vee \\ &\supseteq \left(\frac{M^\vee}{p((M/N)^\vee, M^\vee)}\right)^\vee \\ &= p^\smile(N, M). \end{aligned}$$

Thus $p^\smile$ is order reversing.

(2) Suppose $p$ is pre-involutive and $(L, M), (p^\smile(L, M), M) \in \mathcal{P}^\vee$. Then

$$\begin{aligned} (p^\smile(p^\smile(L, M), M))^\vee &= \frac{M^\vee}{p((M/p^\smile(L, M))^\vee, M^\vee)} && \text{by definition} \\ &= \frac{M^\vee}{p(p^{\smile\smile}((M/L)^\vee, M^\vee), M^\vee)} && \text{[ERGV23b][Lemma 3.3]} \\ &= \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)} && \text{because } p^{\smile\smile} = p. \end{aligned}$$

Since $(M/L)^\vee \subseteq p^2((M/L)^\vee, M^\vee)$, then

$$\frac{M^\vee}{(M/L)^\vee} \twoheadrightarrow \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}.$$

Hence,
$$\left(\frac{M^\vee}{(M/L)^\vee}\right)^\vee \supseteq \left(\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}\right)^\vee$$
implying that $L \supseteq p^\smile(p^\smile(L,M),M)$ or $p^\smile$ is post-involutive.

(3) Suppose $p$ is post-involutive and $(L,M),(p^\smile(L,M),M) \in \mathcal{P}^\vee$. Then as in (2),
$$(p^\smile(p^\smile(L,M),M))^\vee = \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}.$$
Since $(M/L)^\vee \supseteq p^2((M/L)^\vee, M^\vee)$, then
$$\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)} \twoheadrightarrow \frac{M^\vee}{(M/L)^\vee}.$$
Hence,
$$\left(\frac{M^\vee}{(M/L)^\vee}\right)^\vee \subseteq \left(\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}\right)^\vee$$
implying that $L \subseteq p^\smile(p^\smile(L,M),M)$ or $p^\smile$ is pre-involutive.

(4) Suppose $p$ is pre-idempotent and $(L,M),(p^\smile(L,M),M) \in \mathcal{P}^\vee$. Then as in (2),
$$(p^\smile(p^\smile(L,M),M))^\vee = \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}.$$
Since $p((M/L)^\vee, M^\vee) \subseteq p^2((M/L)^\vee, M^\vee)$, then
$$\frac{M^\vee}{p((M/L)^\vee, M^\vee)} \twoheadrightarrow \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}.$$
Hence,
$$\left(\frac{M^\vee}{p((M/L)^\vee, M^\vee)}\right)^\vee \supseteq \left(\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}\right)^\vee$$
implying that $p(L,M) \supseteq (p^\smile)^2(L,M)$ or $p^\smile$ is post-idempotent.

(5) Suppose $p$ is post-idempotent and $(L,M),(p^\smile(L,M),M) \in \mathcal{P}^\vee$. Then as in (2),
$$((p^\smile)^2(L,M))^\vee = \frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}.$$
Since $p((M/L)^\vee, M^\vee) \supseteq p^2((M/L)^\vee, M^\vee)$, then
$$\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)} \twoheadrightarrow \frac{M^\vee}{p((M/L)^\vee}, M^\vee).$$

Hence,

$$\left(\frac{M^\vee}{p((M/L)^\vee, M^\vee)}\right)^\vee \subseteq \left(\frac{M^\vee}{p^2((M/L)^\vee, M^\vee)}\right)^\vee$$

implying that $p(L, M) \subseteq (p^\smile)^2(L, M)$ or $p^\smile$ is pre-idempotent.

(6) $p$ is involutive if and only if $p$ is both pre-involutive and post-involutive. Combining (2) and (3) we obtain the result.

(7) By Lemma 3.4.3, we only need to show one direction of the equivalence. Suppose $p$ is $n$-periodic and let $L \subseteq M$ be such that $(L, M), \in \mathcal{P}^\vee$. By assumption $(p^i(L, M), M) \in \mathcal{P}^\vee$ for $0 \leq i \leq n-1$ and $p^n(L, M) = L$. Then

$$\begin{aligned}
((p^\smile)^n(L, M))^\vee &= \frac{M^\vee}{p((M/(p^\smile)^{n-1}(L, M))^\vee, M^\vee)} && \text{by definition} \\
&= \frac{M^\vee}{p(p(M/(p^\smile)^{n-2}(L, M)^\vee, M^\vee), M^\vee)} && \text{[ERGV23b][Lemma 3.3]} \\
&= \frac{M^\vee}{p^n((M/L)^\vee, M^\vee)} && \text{[ERGV23b][Lemma 3.3]} \\
&= \frac{M^\vee}{(M/L)^\vee} && \text{by } n\text{-periodic} \\
&= L^\vee,
\end{aligned}$$

where the equality $p(p(M/(p^\smile)^{n-2}(L, M)^\vee, M^\vee), M^\vee) = p^n((M/L)^\vee, M^\vee)$ follows from using [ERGV23b, Lemma 3.3] $n-2$ times.

Another application of Matlis duality finishes the proof.

(8) Suppose $p$ is independent and $(L, M) \in \mathcal{P}^\vee$. Then

$$\begin{aligned}
M &= \left(\frac{M^\vee}{0}\right)^\vee \\
&= \left(\frac{M^\vee}{(M/L)^\vee \cap p((M/L)^\vee, M^\vee)}\right)^\vee && \text{By assumption} \\
&\cong \left(\frac{M^\vee}{(M/L)^\vee}\right)^\vee + \left(\frac{M^\vee}{p((M/L)^\vee, M^\vee)}\right)^\vee && \text{By 2.2.4} \\
&= L + p^\smile(L, M)
\end{aligned}$$

Thus $p^{\smile}$ is spanning.

(9) Suppose $p$ is spanning and $(L, M) \in \mathcal{P}^{\vee}$. Then

$$\begin{aligned}
0 &= \left(\frac{M^{\vee}}{M^{\vee}}\right)^{\vee} \\
&= \left(\frac{M^{\vee}}{(M/L)^{\vee} + p((M/L)^{\vee}, M^{\vee})}\right)^{\vee} && \text{By assumption} \\
&\cong \left(\frac{M^{\vee}}{(M/L)^{\vee}}\right)^{\vee} \cap \left(\frac{M^{\vee}}{p((M/L)^{\vee}, M^{\vee})}\right)^{\vee} && \text{By 2.2.4} \\
&= L \cap p^{\smile}(L, M)
\end{aligned}$$

Thus $p^{\smile}$ is independent.

(10) This follows from (8) and (9).

□

## 5.2 Interactions Between Properties

In this section, we examine pair operations that have a combination of some of the properties we defined based on the orthogonal complement (Definition 5.0.1) and those which were previously defined for closure and interior operations (Definition 2.1.2). It is quite striking that these combinations often produce two fairly rigid types of behavior: either the pair operation is constant or the pair operation is the identity.

The following propositions focus on the properties, which when combined, lead to the pair operation behaving in specific ways. In the cases where nothing of note happen, those combinations of properties are omitted. Examples of the existence of pair operations like these can be found in previous sections. For a summary of these results, see Appendix B.

**Remark 5.2.1.** In the following Propositions we have the additional assumption that the submodule which we are taking the pair operation with respect to, $M$, is fixed. To see this is necessary consider $p(L, N) = K$ for all $L \subseteq N$, but $p(L, M) = K'$ for all $L \subseteq M$ even when $L \subseteq N \subseteq M$. Then we cannot necessarily draw conclusions about $K$ and $K'$. $\diamondsuit$

**Proposition 5.2.2.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is an order reversing pair operation on $\mathcal{P}$ and $M$ is fixed.*

1. *If $p$ is idempotent, then for any fixed $M$, $p(L, M) = p(M, M)$ for all $L \subseteq M$. In other words, $p$ is constant on all submodules of $M$. In particular, we have $p^2(L, M) = p^k(L, M) = p^2(M, M)$ for $k \geq 1$.*
2. *If $p$ is pre-idempotent, then for any fixed $M$, $p^2(L, M) = p^n(L, M)$ for all $L \subseteq M$. Thus $p^k$ is idempotent for all $k \geq 2$.*
3. *If $p$ is post-idempotent, then then $p^2(L, M) = p^n(L, M)$ for all $L \subseteq M$. Thus $p^k$ is idempotent for all $k \geq 2$.*
4. *If $p$ is extensive, then $p$ is idempotent and $p(L, M) = M$ for all $L \subseteq M$.*
5. *If $p$ is intensive, then $p$ is idempotent and $p(L, M) = 0$ for all $L \subseteq M$.*

*Proof.* Since $p$ is order reversing on submodules, for $(L, M), (N, M) \in \mathcal{P}$ such that $L \subseteq N \subseteq M$ then $p(N, M) \subseteq p(L, M)$.

(1) Suppose that, in addition to $p$ being order reversing on submodules, that $p$ is idempotent. Since $p$ is idempotent, $p^2(L, M) = p(L, M)$ and $p^2(M, M) = p(M, M)$. Note that $p(L, M) \subseteq M$ by the definition of pair operation. Thus

$$p(M, M) \subseteq p^2(L, M) = p(L, M) \subseteq M.$$

Again using the order reversing property and idempotence, we see that

$$p(M, M) \subseteq p^2(L, M) = p(L, M) \subseteq p^2(M, M) = p(M, M)$$

implying that $p(L, M) = p(M, M)$ for all $L \subseteq M$. Repeatedly using idempotence and order reversing gives that $p^2(L, M) = p^k(L, M) = p^2(M, M)$ for all $k \geq 1$.

(2) Suppose that, in addition to $p$ being order reversing on submodules, that $p$ is pre-idempotent. Then

$$p(L, M) \subseteq p^2(L, M) \subseteq ... \subseteq p^n(L, M)$$

by pre-idempotence. But since $p$ is order reversing

$$p^2(L, M) \supseteq p^3(L, M) \supseteq ... \supseteq p^{n+1}(L, M).$$

Thus $p^2(L, M) = p^n(L, M)$ and $p^k$ is idempotent for $k \geq 2$.

(3) Suppose that, in addition to $p$ being order reversing on submodules, that $p$ is post-idempotent. Then

$$p(L, M) \supseteq p^2(L, M) \supseteq ... \supseteq p^n(L, M)$$

by post-idempotence. But since $p$ is order reversing

$$p^2(L, M) \subseteq p^3(L, M) \subseteq ... \subseteq p^{n+1}(L, M).$$

Thus $p^2(L, M) = p^n(L, M)$ and $p^k$ is idempotent for $k \geq 2$.

(4) Suppose that, in addition to $p$ being order reversing on submodules, that $p$ is extensive. Then $L \subseteq p(L, M) \subseteq M$. So if $(L, M)$ and $(p(L, M), M) \in \mathcal{P}$, then

$$p(L, M) \subseteq p^2(L, M) \subseteq p(L, M)$$

implying that $p$ is idempotent. Now since, $p(L, M) = p(M, M)$ for all $L \subseteq M$ and $p$ is extensive then $M \subseteq p(M, M)$ which implies that $p(L, M) = M$ for all $L \subseteq M$.

(5) Suppose that, in addition to $p$ being order reversing on submodules, that $p$ is intensive. Then $p(L,M) \subseteq L \subseteq M$. So if $(L,M)$ and $(p(L,M),M) \in \mathcal{P}$, then

$$p(L,M) \subseteq p^2(L,M) \subseteq p(L,M)$$

implying that $p$ is idempotent. Now since, $p(L,M) = p(M,M)$ for all $L \subseteq M$ and $p$ is intensive then $p(M,M) = p(0,M) \subseteq 0$ which implies that $p(L,M) = 0$ for all $L \subseteq M$. □

**Proposition 5.2.3.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L,M) \in \mathcal{P}$. Suppose $p$ is a pre-involutive pair operation on $\mathcal{P}$.*

1. *If $p$ is intensive, then $L = p^n(L,M)$ for all $(L,M) \in \mathcal{P}$, $n \geq 1$.*

2. *If $p$ is post-idempotent, then $p$ will be extensive.*

*Proof.* (1) Let $p$ be pre-involutive and intensive. Then

$$L \subseteq p^2(L,M) \subseteq p(L,M) \subseteq L$$

where the first inclusion comes from pre-involutive and the second and third from intensivity.

(2) Let $p$ be pre-involutive and post-idempotent. Then $L \subseteq p^2(L,M) \subseteq p(L,M)$ and $p$ is extensive. □

**Proposition 5.2.4.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L,M) \in \mathcal{P}$. Suppose $p$ is a post-involutive pair operation on $\mathcal{P}$.*

1. *If $p$ extensive, then $L = p^n(L,M)$ for all $(L,M) \in \mathcal{P}$, $n \geq 1$.*

2. *If $p$ is pre-idempotent, then $p$ will be intensive.*

*Proof.* (1) Let $p$ be post-involutive and extensive. Then

$$\begin{aligned} L &\subseteq p(L,M) && \text{by extensivity} \\ &\subseteq p^2(L,M) && \text{by post-involutive} \\ &\subseteq L && \text{by extensivity.} \end{aligned}$$

(2) Let $p$ be post-involutive and pre-idempotent. Then $L \supseteq p^2(L,M) \subseteq p(L,M)$ and $p$ is intensive. $\square$

**Corollary 5.2.5.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L,M) \in \mathcal{P}$. Suppose $p$ is an involutive pair operation on $\mathcal{P}$.*

1. *If $p$ is idempotent, then $p(L,M) = L$ for all $L \subseteq M$. In other words, $p$ is the identity pair operation.*

2. *If $p$ is extensive, then $p$ is idempotent and $p(L,M) = L$ for all $L \subseteq M$.*

3. *If $p$ is intensive, then $p$ is idempotent and $p(L,M) = L$ for all $L \subseteq M$.*

4. *If $p$ is pre-idempotent, then $p$ is intensive, idempotent, and $p(L,M) = L$ for all $L \subseteq M$.*

5. *If $p$ is post-idempotent, then $p$ is extensive, idempotent, and $p(L,M) = L$ for all $L \subseteq M$.*

*Proof.* (1) Suppose that, in addition to $p$ being involutive on submodules, that $p$ is idempotent. Then by Proposition 5.0.13(1), $p$ is both pre- and post-involutive. So by Proposition 5.2.3(2), $p$ is extensive and by Proposition 5.2.4(2), $p$ is intensive. Thus $p$ must be the identity pair operation.

(2) Suppose that, in addition to $p$ being involutive on submodules, that $p$ is extensive. Then by Proposition 5.0.13(1), $p$ is both pre- and post-involutive. So by Proposition 5.2.4(1), $L = p(L, M) = p^2(L, M)$ and $p$ is idempotent.

(3) Suppose that, in addition to $p$ being involutive on submodules, that $p$ is intensive. Then by Proposition 5.0.13(1), $p$ is both pre- and post-involutive. So by Proposition 5.2.3(1), $L = p(L, M) = p^2(L, M)$ and $p$ is idempotent.

(4) Suppose that $p$ is involutive and pre-idempotent. Then

$$p(L, M) \subseteq p^2(L, M) = L.$$

Thus $p$ is intensive. So by (3), $p$ is idempotent and $p(L, M) = L$ for all $L \subseteq M$.

(5) Suppose that $p$ is involutive and post-idempotent. Then

$$L = p^2(L, M) \subseteq p(L, M).$$

Thus $p$ is extensive. So by (2), $p$ is idempotent and $p(L, M) = L$ for all $L \subseteq M$. □

**Proposition 5.2.6.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is a spanning pair operation on $\mathcal{P}$.*

1. *If $p$ is extensive, then $p(N, M) = M$ for all $N \subseteq M$.*
2. *If $p$ in intensive, then $p$ is only defined for $M = 0$.*
3. *If $p$ is order preserving on submodules, then $p(L, M) = M$ for all $L \subseteq M$.*
4. *If $p$ is order reversing on submodules, then $p(L, M) + N = M$ for $L \subseteq N \subseteq M$.*

*Proof.* (1) Suppose $p$ is spanning and extensive. Then $N \subseteq p(N, M)$. So

$$M = N + p(N, M) = p(N, M).$$

(2) Suppose $p$ is spanning and intensive. Then $p(N, M) \subseteq N$. So

$$M = N + p(N, M) = N$$

for all $N \subseteq M$ which implies that there is no proper submodule $N \subseteq M$ with $(N, M) \in \mathcal{P}$.

(3) Since $p$ is order preserving on submodules, for $L \subseteq N \subseteq M$, we have $p(L, M) \subseteq p(N, M)$. Since $p$ is spanning $p(L, M) + L = M = p(N, M) + N$. Now since $L \subseteq N$, we have $M = p(L, M) + L \subseteq p(N, M) + L \subseteq p(N, M) + N = M$ or $p(N, M) + L = M$ for all $L \subseteq N \subseteq M$. In particular, if $L = 0$, then $p(N, M) + 0 = M$ implying that $p(N, M) = M$ for all $N \subseteq M$.

(4) Since $p$ is order reversing, for $L \subseteq N \subseteq M$ we have $p(N, M) \subseteq p(L, M)$. Since $p$ is spanning we have $L + p(L, M) = M = N + p(N, M)$. So

$$M \subseteq p(N, M) + N \subseteq p(L, M) + N \subseteq M$$

where the last containment follows from the definition of pair operations. $\square$

**Proposition 5.2.7.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is an independent pair operation on $\mathcal{P}$.*

1. *If $p$ is intensive, then $p(N, M) = 0$ for all $N \subseteq M$.*
2. *If $p$ in extensive, then $p$ is only defined for $M = 0$.*
3. *If $p$ is order-preserving, then for all $L \subseteq N \subseteq M$ we have $N \cap p(L, M) = 0$ and $p(L, M) = 0$ for all $L$.*
4. *If $p$ is order-reversing, then for all $L \subseteq N \subseteq M$ we have $L \cap p(N, M) = 0$.*
5. *If $p$ is involutive, then the $p^{2n}(N, M)$ will be disjoint from the $p^{2k+1}(N, M)$ for all $n, k$.*

*Proof.* (1) Suppose $p$ is independent and intensive. Then $N \supseteq p(N, M)$. So

$$0 = N \cap p(N, M) = p(N, M).$$

(2) Suppose $p$ is independent and extensive. Then $p(N, M) \supseteq N$. So

$$0 = N \cap p(N, M) = N$$

for all $N \subseteq M$ implying that $M = 0$.

(3) Suppose $p$ is independent and order-preserving and that $L \subseteq N \subseteq M$. Then $p(L, M) \subseteq p(N, M)$ and since intersections preserve order,

$$N \cap p(L, M) \subseteq N \cap p(N, M) = 0.$$

Thus $N \cap p(L, M) = 0$. In particular, $M \cap p(L, M) = p(L, M) = 0$.

(4) Suppose $p$ is independent and order-reversing and that $L \subseteq N \subseteq M$. Then $p(N, M) \subseteq p(L, M)$ and since intersections preserve order,

$$L \cap p(N, M) \subseteq L \cap p(L, M) = 0.$$

Thus $L \cap p(N, M) = 0$.

(5) Suppose $p$ is independent and involutive. Then

$$0 = L \cap p(N, M) = p^2(N, M) \cap p(N, M) = ... = p^{2n}(N, M) \cap p^{2k+1}(N, M).$$

$\square$

**Proposition 5.2.8.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is a pre-idempotent pair operation on $\mathcal{P}$.*

1. *If $p$ is independent, then $p(L, M) = 0$ for all $(L, M) \in \mathcal{P}$.*

2. *If $p$ is spanning, $p^n$ will also be spanning.*

3. *If $p$ is complementary, then $\mathcal{P} = \{(M, M)\}$.*

*Proof.* (1) Suppose $p$ is pre-idempotent and independent. Since $p$ is pre-idempotent, $p(L, M) \subseteq p^2(L, M)$ and since $p$ is independent, $L \cap p(L, M) = 0$. It follows that,

$$p(L, M) = p(L, M) \cap p(L, M) \subseteq p(L, M) \cap p^2(L, M) = 0.$$

(2) Let $p$ be pre-idempotent and spanning. Then

$$M = L + p(L, M) \subseteq L + p(L, M) \subseteq M$$

where the last inclusion follows from $L, p^2(L, M) \subseteq M$. So $L + p^2(L, M) = M$ and it is simple to see $L + p^n(L, M) = M$ for all $n$.

(3) Let $p$ be pre-idempotent and complementary. Then by (1), $p(L, M) = 0$ and by spanning $L + p(L, M) = M$. So $L + p(L, M) = L + 0 = M$ and $L = M$ for all $(L, M) \in \mathcal{P}$. □

**Remark 5.2.9.** Note that if $p$ is extensive, then $p$ is pre-idempotent and that if $p$ is intensive, then $p$ is post-idempotent. This follows from applying the definitions of extensive and intensive to $p(L, M)$. ◇

**Proposition 5.2.10.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is a post-idempotent pair operation on $\mathcal{P}$.*

1. *If $p$ independent, then $p^n$ is also independent.*

2. *If $p$ is spanning, then $p(L, M) = M$ for all $(L, M) \in \mathcal{P}$.*

3. *If $p$ is complementary, then $\mathcal{P} = \{(0, M)\}$ or $M = 0$.*

*Proof.* (1) Suppose that $p$ is post-idempotent and independent. Since $p$ is pre-idempotent, $p(L, M) \supseteq p^2(L, M)$ and since $p$ is independent, $L \cap p(L, M) = 0$. It follows that,

$$0 = L \cap p(L, M) \supseteq L \cap p^2(L, M).$$

So $L \cap p^2(L, M) = 0$ and it is simple to see $L \cap p^n(L, M) = 0$ for all $n$.

(2) Let $p$ be post-idempotent and spanning. Then

$$p(L, M) = p(L, M) + p(L, M) \supseteq p(L, M) + p^2(L, M) = M.$$

Since $p$ always chooses a submodule of $M$, $p(L, M) \subseteq M$. Thus $p(L, M) = M$.

(3) Let $p$ be post-idempotent and complementary. Then by (2) $p(L, M) = M$ and by independent $L \cap p(L, M) = 0$. So $L \cap p(L, M) = L \cap M = 0$ and either $L = 0$ or $M = 0$. $\square$

**Corollary 5.2.11.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is an idempotent pair operation on $\mathcal{P}$.*

1. *If $p$ is independent, then $p(L, M) = 0$ for all $L \subseteq M$.*

2. *If $p$ is spanning, then $p(L, M) = M$ for all $L \subseteq M$.*

3. *If $M$ has an idempotent complementary pair operation, then $M = 0$.*

*Proof.* (1) Suppose that $p$ is idempotent and independent. By Proposition 5.0.13(2), $p$ is both pre- and post-idempotent. So by Proposition 5.2.8(1) we are done.

(2) Suppose that $p$ is idempotent and spanning. By Proposition 5.0.13(2), $p$ is both pre- and post-idempotent. So by Proposition 5.2.8(2) and Proposition 5.2.10(2)

$$M \subseteq p(L, M) \subseteq M.$$

Thus $M = p(L, M)$ for all $L \subseteq M$.

(3) By (1) and (2) $M = p(L, M) = 0$ for all $L$. $\square$

**Remark 5.2.12.** If $p$ is not idempotent then a simple counter example to the above corollary may be derived by considering any simple module. $\diamondsuit$

**Proposition 5.2.13.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is a complementary pair operation on $\mathcal{P}$. Then $p$ is involutive. Also, $p(0, M) = M$ and $p(M, M) = 0$.*

*Proof.* Since $p$ is complementary, it is both independent and spanning. Therefore, for all pairs $(L, M) \in \mathcal{P}$, both $L \cap p(L, M) = 0$ and $L + p(L, M) = M$. Note that

$$p(0, M) = p(0, M) + 0 = M$$

and $p(M, M) = p(M, M) \cap M = 0$. Now, let $p^2(L, M) = p^2(L, M)$. Since $p$ is spanning, $p(L, M) + p^2(L, M) = M$. Thus,

$$p^2(L, M) = M \cap p^2(L, M) = (L + p(L, M)) \cap p^2(L, M) = L \cap p^2(L, M),$$

which implies that $p^2(L, M) \subset L$. On the other hand,

$$L = M \cap L = (p(L, M) + p^2(L, M)) \cap L = L \cap p^2(L, M),$$

thus $L \subset p^2(L, M)$. Hence $p^2(L, M) = L$ and $p$ is involutive. $\square$

**Proposition 5.2.14.** *Let $R$ be a ring, $\mathcal{M}$ the category of $R$-modules, $\mathcal{P}$ be the class of pairs as in Definition 2.1.2, and $(L, M) \in \mathcal{P}$. Suppose $p$ is an $n$-periodic pair operation on $\mathcal{P}$.*

1. *If $p$ is extensive, then $p$ is the identity.*

2. *If $p$ is intensive, then $p$ is the identity.*

3. *If $p$ is order reversing and $n$ is odd, then $p$ is only defined on the zero-module or $\mathcal{P} = \{(M, M)\}$.*

4. *If $p$ is pre-idempotent, then $p$ is intensive. Thus $p$ is the identity.*

5. *If $p$ is post-idempotent, then $p$ is extensive. Thus $p$ is the identity.*

6. *If $p$ is idempotent, then $p$ is the identity.*

7. *If $p$ is pre-involutive and $n$ is even, then $p$ is involutive.*

8. *If $p$ is post-involutive and $n$ is even, then $p$ is involutive.*

*Proof.* (1) Suppose $p$ is $n$-periodic and extensive. Then

$$p^n(L, M) = L \subseteq p(L, M) \subseteq p^2(L, M) \subseteq ... \subseteq p^n(L, M).$$

Thus $L = p(L, M)$ and $p$ is the identity pair operation.

(2) Suppose $p$ is $n$-periodic and intensive. Then

$$L = p^n(L, M) \subseteq p^{n-1}(L, M) \subseteq ... \subseteq p(L, M) \subseteq L.$$

Thus $L = p(L, M)$ and $p$ is the identity pair operation.

(3) Suppose $p$ is $n$-periodic, order reversing, and $n$ is odd. Because $L \subseteq N \subseteq M$ implies that $N = p^n(N, M) \subseteq p^n(L, M) = L$ or $L = N$ for all submodules of $M$. Notice that $n$ being odd is used to insure that the order is reversed when the operation is done $n$ times. (If $n$ is even then $L = p^n(L, M) \subseteq p^n(N, M) = N$.) This means that $M = 0$ or that the only pairs allowed are $(M, M)$.

(4) Suppose $p$ is $n$-periodic and pre-idempotent. Then

$$p(L, M) \subseteq p^2(L, M) \subseteq ... \subseteq p^n(L, M) = L.$$

Thus $p(L, M) \subseteq L$ and $p$ is intensive. So by (2) $p$ must be the identity.

(5) Suppose $p$ is $n-$periodic and post-idempotent. Then

$$L = p^n(L,M) \subseteq p^{n-1}(L,M) \subseteq ... \subseteq p(L,M).$$

Thus $L \subseteq p(L,M)$ and $p$ is extensive. So by (1) $p$ must be the identity.

(6) This follows from (5) and (6).

(7) Suppose $p$ is $n$-periodic, pre-involutive, and $n$ is even. Then

$$L \subseteq p^2(L,M) \subseteq ... \subseteq p^{n-2}(L,M) \subseteq p^n(L,M) = L.$$

Thus $L = p^2(L,M)$ and $p$ is involutive.

(8) Suppose $p$ is $n$-periodic, post-involutive, and $n$ is even. Then

$$L = p^n(L,M) \subseteq p^{n-2}(L,M) \subseteq ... \subseteq p^2(L,M) \subseteq L.$$

Thus $L = p^2(L,M)$ and $p$ is involutive. □

**Remark 5.2.15.** Notice that if $p$ is order preserving and $n$-periodic then all that can be said is if $L \subseteq N \subseteq M$ then $L = p^n(L,M) \subseteq p^n(N,M) = N$. ♢

# Chapter 6

# Constructions

We now want to consider constructions similar to Epstein's in [Eps12] but for pair operations rather than closure operations. In particular, we will focus on how they behave with the properties described in Definitions 2.1.2 and 5.0.1.

## 6.1 Through ring extension or contraction

Recall that when $f : X \to Y$ is a function and $A \subseteq X, B \subseteq Y$ we have:

$$A \subseteq f^{-1}(f(A)) \tag{6.1}$$

$$A = f^{-1}(f(A)) \text{ if } f \text{ injective} \tag{6.2}$$

$$B \supseteq f(f^{-1}(B)) \tag{6.3}$$

$$B = f(f^{-1}(B)) \text{ if } f \text{ surjective} \tag{6.4}$$

**Construction 6.1.1.** *Let $\Phi : R \to S$ be ring homomorphism and $f : M \to S \otimes_R M$ where $M$ is an $R$-module and $S$ is flat. Let $\mathcal{P}$ be a collection of pairs of $R$-modules as in Definition 2.1.2, $\mathcal{P}'$ a collection of pairs of $S$-modules as in Definition 2.1.2 such* *that*

$$\mathcal{P}' \subseteq f(\mathcal{P}) = \{(S \otimes_R L, S \otimes_R M) \mid (L, M) \in \mathcal{P}\}$$

*and $q$ a pair operation on $\mathcal{P}'$. For submodules $(N, M) \subseteq \mathcal{P}$, define*

$$p(N, M) := f^{-1}(q(f(N), S \otimes_R M)) = f^{-1}(q(S \otimes_R N, S \otimes_R M)).$$

*Then $p$ is a pair operation on $R$.*

**Proposition 6.1.2.** *Let $p$ be a pair operation as in Construction 6.1.1. Then the following hold:*

1. *If $q$ is order preserving, then $p$ is order preserving.*
2. *If $q$ is order reversing, then $p$ is order reversing.*
3. *If $q$ is intensive and $f$ is injective, then $p$ is intensive.*
4. *If $q$ is extensive, then $p$ is extensive.*
5. *If $q$ is independent and $f$ is injective, then $p$ is independent.*
6. *If $q$ is spanning and $f$ is injective, then $p$ is spanning.*
7. *If $q$ is complementary and $f$ is injective, then $p$ is complementary.*

*Proof.* (1) Let $q$ be order preserving and $L \subseteq N \subseteq M$ be $R$-modules. Then we have $f(L) \subseteq f(N)$ and since $q$ is order preserving $q(f(L), S \otimes_R M) \subseteq q(f(N), S \otimes_R M)$. Hence

$$f^{-1}(q(f(L), S \otimes_R M)) \subseteq f^{-1}(q(f(N), S \otimes_R M)).$$

Thus $p(L, M) \subseteq p(N, M)$ and $p$ is order preserving.

(2) Let $q$ be order reversing and $L \subseteq N \subseteq M$ be $R$-modules. Then $f(L) \subseteq f(N)$ and since $q$ is order reversing $q(f(L), S \otimes_R M) \supseteq q(f(N), S \otimes_R M)$. Hence

$$f^{-1}(q(f(L), S \otimes_R M)) \supseteq f^{-1}(q(f(N), S \otimes_R M)).$$

Thus $p(L, M) \supseteq p(N, M)$ and $p$ is order reversing.

(3) Let $q$ be intensive. So $q(f(N), S \otimes_R M) \subseteq f(N) = S \otimes_R N$. We know by injectivity $f^{-1}(f(N)) = N$ so $p(N, M) = f^{-1}(q(f(N), S \otimes_R M)) \subseteq f^{-1}(f(N)) = N$. Thus $p$ is intensive.

(4) Let $q$ be extensive. So $f(N) = S \otimes_R N \subseteqq q(f(N), S \otimes_R M)$. We know that

$$N \subseteq f^{-1}(f(N)) = f^{-1}(S \otimes_R N)$$

so $N \subseteq f^{-1}(S \otimes_R N) \subseteq f^{-1}(q(f(N), S \otimes_R M))$. Thus $N \subseteq p(N, M)$ and $p$ is extensive.

(5) Let $q$ be independent and $f$ injective. Then

$$\begin{aligned}
L \cap p(L, M) &= L \cap f^{-1}(q(f(L), S \otimes_R M)) \\
&= f^{-1}(f(L)) \cap f^{-1}(q(f(L), S \otimes_R M)) \\
&= f^{-1}(f(L) \cap q(f(L), S \otimes_R M)) \\
&= f^{-1}(0) \\
&= \ker(f) \\
&= 0.
\end{aligned}$$

Thus $p$ is independent.

(6) Let $q$ be spanning. Then

$$\begin{aligned}
L + p(L, M) &= L + f^{-1}(q(f(L), S \otimes_R M)) \\
&= f^{-1}(f(L) + f^{-1}(q(f(L), S \otimes_R M)) \\
&= f^{-1}(f(L) + q(f(L), S \otimes_R M)) \\
&= f^{-1}(S \otimes_R M) \\
&= M.
\end{aligned}$$

Thus $p$ is spanning.

(7) This follows from (5) and (6). $\square$

[Eps12] showed that if $q$ is a closure operation then $p$ will be a closure operation. An interesting thing to note is that to show that $p$ is idempotent, $q$ needed to be idempotent, order preserving, and extensive: removing one condition meant $p$ would no longer be idempotent.

**Proposition 6.1.3.** *Let $p$ be a pair operation as in Construction 6.1.1.*

1. *If $q$ is post-idempotent and order preserving, then $p$ will be post-idempotent.*
2. *If $q$ is pre-idempotent and order reversing, then $p$ will be pre-idempotent and $p^2(L, M) = p^n(L, M)$ for all $n \geq 2$.*
3. *If $q$ is pre-involutive and order reversing, then $p$ is pre-involutive.*
4. *If $q$ is post-involutive and order preserving and $f$ injective, then $p$ is post-involutive.*

*Proof.* (1) Let $q$ be idempotent and order preserving. Then applying the pair operation to both sides of the following:

$$q(f(N), S \otimes_R M) \supseteq f(f^{-1}(q(f(N), S \otimes_R M))),$$

we obtain

$$q^2(f(N), S \otimes_R M) \supseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

However, as $q$ is post-idempotent this implies

$$q(f(N), S \otimes_R M) \supseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

Now using the fact that preimages preserve containments, we have

$$f^{-1}(q(f(N), S \otimes_R M) \supseteq f^{-1}(q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M))$$

or

$$p(N, M) \supseteq p^2(N, M).$$

(2) Let $q$ be pre-idempotent and order reversing. Then applying the pair operation to the containment

$$q(f(N), S \otimes_R M) \supseteq f(f^{-1}(q(f(N), S \otimes_R M))),$$

we obtain

$$q^2(f(N), S \otimes_R M) \subseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

However, as $q$ is pre-idempotent this implies

$$q(f(N), S \otimes_R M) \subseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

Now using the fact that preimages preserve containments, we have

$$f^{-1}(q(f(N), S \otimes_R M) \subseteq f^{-1}(q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M))$$

or

$$p(N, M) \subseteq p^2(N, M).$$

Thus $p$ is pre-idempotent. Then by Proposition 5.2.2(1) $p^2(L, M) = p^n(L, M)$ for all $n \geq 2$.

(3) Let $q$ be pre-involutive and order reversing. Then applying the pair operation to both sides of the following:

$$q(f(N), S \otimes_R M) \supseteq f(f^{-1}(q(f(N), S \otimes_R M))),$$

we obtain

$$q^2(f(N), S \otimes_R M) \subseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

However, since $q$ is pre-involutive this implies

$$f(N) \subseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

Now, using the fact that preimages preserve containments, we have

$$f^{-1}(f(N)) \subseteq f^{-1}(q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M))$$

or

$$N \subseteq p^2(N, M).$$

Thus $p$ is pre-involutive.

(4) Let $q$ be involutive and order preserving and $f$ injective. Then applying the pair operation to the containment

$$q(f(N), S \otimes_R M) \supseteq f(f^{-1}(q(f(N), S \otimes_R M))),$$

we obtain

$$q^2(f(N), S \otimes_R M) \supseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

However, since $q$ is post-involutive, this implies

$$f(N) \supseteq q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M).$$

Now using the fact that preimages preserve containments, we have

$$f^{-1}(f(N)) \supseteq f^{-1}(q(f(f^{-1}(q(f(N), S \otimes_R M))), S \otimes_R M))$$

or

$$N \supseteq p^2(N, M).$$

Thus $p$ is post-involutive. $\square$

**Construction 6.1.4.** *Let $\Phi : R \to S$ be ring homomorphism and $f : M \to S \otimes_R M$ where $M$ is an $R$-module and $S$ is flat. Let $\mathcal{P}$ be a collection of pairs of $R$-modules as in Definition 2.1.2, $\mathcal{P}'$ a collection of $S$-modules as in Definition 2.1.2 such that*

$$\mathcal{P}' \subseteq f(\mathcal{P}) = \{(S \otimes_R L, S \otimes_R M) \mid (L, M) \in \mathcal{P}\}$$

*and $q$ a pair operation on $R$. For submodules $(N, M) \subseteq \mathcal{P}$, define*

$$p(S \otimes_R N, S \otimes_R M) := f(q(f^{-1}(S \otimes_R N), M)).$$

*Then $p$ is a pair operation on $S$.*

**Proposition 6.1.5.** *Let $p$ be a pair operation constructed as in Construction 6.1.4. Then the following hold:*

1. *If $q$ is order preserving, then $p$ is order preserving.*
2. *If $q$ is order reversing, then $p$ is order reversing.*
3. *If $q$ is intensive, then $p$ is intensive.*
4. *If $q$ is extensive and $f$ is surjective, then $p$ is extensive.*
5. *If $q$ is independent and $f$ is surjective, then $p$ is independent.*
6. *If $q$ is spanning and $f$ is surjective, then $p$ is spanning.*
7. *If $q$ is complementary and $f$ is surjective, then $p$ is complementary.*

*Proof.* (1) Suppose $q$ is order preserving and $S \otimes_R L \subseteq S \otimes_R N$. Then we have $f^{-1}(S \otimes_R L) \subseteq f^{-1}(S \otimes_R N)$ and since $q$ is order preserving

$$q(f^{-1}(S \otimes_R L), M) \subseteq q(f^{-1}(S \otimes_R N), M).$$

Thus

$$f(q(f^{-1}(S \otimes_R L), M)) \subseteq f(q(f^{-1}(S \otimes_R N), M))$$

and $p$ is order preserving.

(2) Suppose $q$ is order reversing and $S \otimes_R L \subseteq S \otimes_R N$. Then

$$f^{-1}(S \otimes_R L) \subseteq f^{-1}(S \otimes_R N)$$

and since $q$ is order reversing

$$q(f^{-1}(S \otimes_R L), M) \supseteq q(f^{-1}(S \otimes_R N), M).$$

Thus

$$f(q(f^{-1}(S \otimes_R L), M)) \supseteq f(q(f^{-1}(S \otimes_R N), M))$$

and $p$ is order reversing.

(3) Suppose $q$ is intensive. So $q(f^{-1}(S \otimes_R N), M) \subseteq f^{-1}(S \otimes_R N)$ and hence

$$f(q(f^{-1}(S \otimes_R N), M)) \subseteq f(f^{-1}(S \otimes_R N)) \subseteq S \otimes_R N.$$

Thus $p$ is intensive.

(4) Suppose $q$ is extensive and $f$ is surjective. Then

$$f^{-1}(S \otimes_R N) \subseteq q(f^{-1}(S \otimes_R N), M)$$

and by surjectivity

$$N = f(f^{-1}(S \otimes_R N)) \subseteq f(q(f^{-1}(S \otimes_R N), M)) = p(S \otimes_R N, S \otimes_R M).$$

Thus $p$ is extensive.

(5) Let $q$ be independent and $f$ be surjective. Then

$$\begin{aligned} S \otimes_R L \cap p(S \otimes_R L, S \otimes_R M) &= S \otimes_R L \cap f(q(f^{-1}(S \otimes_R L), M)) \\ &= f(f^{-1}(S \otimes_R L)) \cap f(q(f^{-1}(S \otimes_R L), M)) \\ &= f(f^{-1}(S \otimes_R L) \cap q(f^{-1}(S \otimes_R L), M)) \\ &= f(0) \\ &= 0. \end{aligned}$$

Thus $p$ is independent.

(6) Let $q$ be spanning and $f$ be surjective. Then

$$\begin{aligned}
S \otimes_R L + p(S \otimes_R L, S \otimes_R M) &= S \otimes_R L + f(q(f^{-1}(S \otimes_R L), M)) \\
&= f(f^{-1}(S \otimes_R L) + f(q(f^{-1}(S \otimes_R L), M)) \\
&= f(f^{-1}(S \otimes_R L) + q(f^{-1}(S \otimes_R L) \\
&= f(M) \\
&= S \otimes_R M.
\end{aligned}$$

Thus $p$ is spanning.

(7) This follows directly from (5) and (6).

□

**Proposition 6.1.6.** *Let $p$ be a pair operation constructed as in Construction 6.1.4.*

1. *If $q$ is pre-involutive and order preserving and $f$ surjective, then $p$ is pre-involutive.*
2. *If $q$ is post-involutive and order reversing, then $p$ is post-involutive.*

*Proof.* (1) Let $q$ be pre-involutive and order preserving and $f$ surjective. Then applying the pair operations to both sides of the the following

$$q(f^{-1}(S \otimes_R N), M) \subseteq f^{-1}(f(q(f^{-1}(S \otimes_R N), M))),$$

we obtain

$$q^2(f^{-1}(S \otimes_R N), M) \subseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

However, since $q$ is pre-involutive, this implies

$$f^{-1}(S \otimes_R N) \subseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

Now using the fact that images preserve containments, we have

$$f(f^{-1}(S \otimes_R N)) \subseteq f(q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M))$$

or

$$S \otimes_R N \subseteq p^2(S \otimes_R N, S \otimes_R M).$$

Thus $p$ is pre-involutive.

(2) Let $q$ be post-involutive and order reversing. Then applying the pair operation to the containment

$$q(f^{-1}(S \otimes_R N), M) \subseteq f^{-1}(f(q(f^{-1}(S \otimes_R N), M))),$$

we obtain

$$q^2(f^{-1}(S \otimes_R N), M) \supseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

However, since $q$ is post-involutive, this implies

$$f^{-1}(S \otimes_R N) \supseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

Now, using the fact that images preserve containments, we have

$$f(f^{-1}(S \otimes_R N)) \supseteq f(q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M))$$

or

$$S \otimes_R N \supseteq p^2(S \otimes_R N, S \otimes_R M).$$

Thus $p$ is post-involutive. □

**Proposition 6.1.7.** *Let $p$ be a pair operation constructed as in Construction 6.1.4.*

1. *If $q$ is pre-idempotent and order preserving, then $p$ will be pre-idempotent.*

2. *If $q$ is post-idempotent and order reversing, then $p$ will be post-idempotent.*

*Proof.* (1) Let $q$ be pre-idempotent and order preserving. Then applying the pair operation to both sides of the following

$$q(f^{-1}(S \otimes_R N), M) \subseteq f^{-1}(f(q(f^{-1}(S \otimes_R N), M)))$$

we obtain

$$q^2(f^{-1}(S \otimes_R N), M) \subseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

However, since $q$ is pre-idempotent, this implies

$$q(f^{-1}(S \otimes_R N), M) \subseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

Now using the fact that images preserve containments, we have

$$f(q(f^{-1}(S \otimes_R N), M)) \subseteq f(q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M))$$

or

$$p(S \otimes_R N, S \otimes_R M) \subseteq p^2(S \otimes_R N, S \otimes_R M).$$

Thus $p$ is pre-idempotent.

(2) Let $q$ be post-idempotent and order reversing. Then applying the pair operation to the containment

$$q(f^{-1}(S \otimes_R N), M) \subseteq f^{-1}(f(q(f^{-1}(S \otimes_R N), M))),$$

we obtain

$$q^2(f^{-1}(S \otimes_R N), M) \supseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

However, since $q$ is post-idempotent, this implies

$$q(f^{-1}(S \otimes_R N), M) \supseteq q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M).$$

Now, using the fact that images preserve containments, we have

$$f(q(f^{-1}(S \otimes_R N), M)) \supseteq f(q(f^{-1}(f(q(f^{-1}(S \otimes_R N), M))), M))$$

or

$$p(S \otimes_R N, S \otimes_R M) \supseteq p^2(S \otimes_R N, S \otimes_R M).$$

Thus $p$ is post-idempotent. $\square$

## 6.2 Intersection, Sum, and Union

**Construction 6.2.1.** *Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of pair operations on submodules of $R$ defined on $\mathcal{P}$. Then*

$$p(N, M) := \bigcap_{\lambda\in\Lambda} p_\lambda(N, M)$$

*is a pair operation defined on $\mathcal{P}$.*

[Eps12] showed that if $p_\lambda$ are closure operations then $p$ will be a closure operation.

**Proposition 6.2.2.** *Let $p$ be a pair operation constructed as in Construction 6.2.1. Then the following hold:*

1. *If $p_\lambda$ are order preserving, then $p$ is order preserving.*
2. *If $p_\lambda$ are order reversing, then $p$ is order reversing.*
3. *If $p_\lambda$ are extensive, then $p$ is extensive.*
4. *If $p_\lambda$ are intensive, then $p$ is intensive.*
5. *If $p_\lambda$ are post-idempotent and order preserving, then $p$ is post-idempotent.*
6. *If $p_\lambda$ are pre-idempotent and order reversing, then $p$ is pre-idempotent.*
7. *If $p_\lambda$ are post-involutive and order preserving, then $p$ is post-involutive.*
8. *If $p_\lambda$ are pre-involutive and order reversing, then $p$ is pre-involutive.*
9. *If $p_\lambda$ are independent, then $p$ is independent.*

*Proof.* (1) Suppose $p_\lambda$ are order preserving and $L \subseteq N$. So $p_\lambda(L, M) \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Then

$$\bigcap_{\lambda\in\Lambda} p_\lambda(L, M) \subseteq \bigcap_{\lambda\in\Lambda} p_\lambda(N, M)$$

and $p$ is order preserving.

(2) Suppose $p_\lambda$ are order reversing and $L \subseteq N$. So $p_\lambda(L, M) \supseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Then

$$\bigcap_{\lambda\in\Lambda} p_\lambda(L, M) \supseteq \bigcap_{\lambda\in\Lambda} p_\lambda(N, M)$$

and $p$ is order reversing.

(3) Suppose $p_\lambda$ are extensive. Then $N \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \subseteq \bigcap_{\lambda\in\Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is extensive.

(4) Suppose $p_\lambda$ are intensive. Then $N \supseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \supseteq \bigcap_{\lambda\in\Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is intensive.

(5) Suppose $p_\lambda$ are post-idempotent and order preserving. Suppose $f \in p^2(N, M)$. Then for every $\lambda \in \Lambda$, we have $f \in p_\lambda(p(N, M), M)$. But since $p(N, M) \subseteq p_\lambda(N, M)$ and $p_\lambda$ is order preserving, we have

$$f \in p_\lambda(p(N, M), M) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq p_\lambda(N, M)$$

where the last containment follows from the post-idempotence of $p_\lambda$. Since $\lambda \in \Lambda$ was arbitrary, $f \in p(N, M)$ as required. Thus $p$ is post-idempotent.

(6) Suppose $p_\lambda$ are pre-idempotent and order reversing. Suppose $f \in p(N, M)$. Then for every $\lambda \in \Lambda$, we have $f \in p_\lambda(N, M)$. But since $p(N, M) \subseteq p_\lambda(N, M)$ and $p_\lambda$ is order reversing, we have

$$f \in p_\lambda(, M) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq p_\lambda(p(N, M), M)$$

where the first containment follows from the pre-idempotence of $p_\lambda$. Since $\lambda \in \Lambda$ was arbitrary, $f \in p^2(N, M)$ as required. Thus $p$ is pre-idempotent.

(7) Suppose $p_\lambda$ are post-involutive and order preserving. Suppose $f \in p^2(N, M)$. Then for every $\lambda \in \Lambda$, we have $f \in p_\lambda(p(N, M), M)$. But since $p(N, M) \subseteq p_\lambda(N, M)$ and $p_\lambda$ is order preserving, we have

$$f \in p_\lambda(p(N, M), M) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq N$$

where the last containment follows from $p_\lambda$ being post-involutive. Since $\lambda \in \Lambda$ was arbitrary, $f \in N$ as required. Thus $p$ is post-involutive.

(8) Suppose $p_\lambda$ are pre-involutive and order reversing. Suppose $f \in N$. Then for every $\lambda \in \Lambda$ and since $p_\lambda$ are pre-involutive, we have $f \in N \subseteq p_\lambda(p_\lambda(N, M), M)$. But since $p(N, M) \subseteq p_\lambda(N, M)$ and $p_\lambda$ is order reversing, we have

$$f \in p_\lambda(p_\lambda(N, M), M) \subseteq p_\lambda(p(N, M), M).$$

Since $\lambda \in \Lambda$ was arbitrary, $f \in p^2(N, M)$ as required. Thus $p$ is pre-involutive.

(9) Suppose $p_\lambda$ are independent. Then

$$\begin{aligned} N \cap p(N, M) &= N \cap \bigcap_{\lambda \in \Lambda} p_\lambda(N, M) \\ &= \bigcap_{\lambda \in \Lambda} (N \cap p_\lambda(N, M)) \\ &= \bigcap_{\lambda \in \Lambda} 0 \\ &= 0. \end{aligned}$$

Thus $p$ is independent. $\square$

**Remark 6.2.3.** The proof of Proposition 6.2.2 is based on [Eps12, Construction 3.1.3]. There Epstein proved that if $p_\lambda$ are closure operations then $p$ will be a closure operation. His proof of $p^2(N, M) \subseteq p(N, M)$ uses idempotence and order preservation but does not rely on extensivity. The extensivity is used to prove the other direction, $p(N, M) \subseteq p^2(N, M)$. $\diamondsuit$

**Example 6.2.4.** This example illustrates that even if $p_\lambda$ are spanning for all $\lambda$ it is not necessarily the case that $p$ is spanning (since sums do not necessarily distribute over intersections).

Let $V = F^2$ and define $p_1(\langle e_1\rangle, V) = \langle e_1 + e_2\rangle$ and $p_2(\langle e_i\rangle, V) = \begin{cases} \langle e_1\rangle & \text{if } i = 2 \\ \langle e_2\rangle & \text{if } i = 1 \end{cases}$. Then $\langle e_1\rangle + p_2(\langle e_1\rangle, V) = \langle e_1\rangle + \langle e_2\rangle = F^2$ and $\langle e_1\rangle + p_1(\langle e_1\rangle, V) = \langle e_1\rangle + \langle e_1 + e_2\rangle = F^2$, but $\langle e_1\rangle + (p_1 \cap p_2)(\langle e_1\rangle, V) = \langle e_1\rangle + (\langle e_2\rangle \cap \langle e_1 + e_2\rangle) = \langle e_1\rangle$. This means Construction 6.2.1 does not generally work well with spanning or complementary $p_\lambda$.

**Remark 6.2.5.** Note that we do not have a way to force $p$ to be involutive or idempotent using order reversing the way we can with order preserving. $\diamondsuit$

**Example 6.2.6.** In this example, we see a pair operation that satisfies Construction 6.2.1. Let $R(A)$ be the range subspace of the matrix $A$ in $R$. Let $A$ and $B$ be Hermitian semi-definite matrices.

Define the parallel sum of $A$ and $B$ as $A : B = A(A + B)^\dagger B$ where $A^\dagger$ is the Moore-Penrose generalized inverse. We can define a pair operation as

$$p(R(A), \mathbb{R}^n) = R(A : B).$$

Then $p(R(A), \mathbb{R}^n) = R(A) \cap R(B)$ [AD69, Lemma 3]. If $P$ and $Q$ are projections, then the projection onto $R(P) \cap R(Q)$ is $2P : Q$ [AD69, Theorem 8] (this is written as $P_{L\cap M} = 2P_L(P_L + P_M)^\dagger P_M = 2(P_L : P_M)$ where $P_L$ is the projector of $L$ in [BI15]. Parallel sum appears as an interior operation (intensive, order preserving, and idempotent). But not involutive, independent, or spanning.

**Construction 6.2.7.** *Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of pair operations on a* *class of pairs $\mathcal{P}$ of $R$. Then*

$$p(N, M) := \sum_{\lambda \in \Lambda} p_\lambda(N, M)$$

*is a pair operation defined on $\mathcal{P}$.*

**Proposition 6.2.8.** *Let $p$ be a pair operation constructed as in Construction 6.2.7. Then the following hold:*

1. *If $p_\lambda$ are order preserving, then $p$ is order preserving.*
2. *If $p_\lambda$ are order reversing, then $p$ is order reversing.*
3. *If $p_\lambda$ are extensive, then $p$ is extensive.*
4. *If $p_\lambda$ are intensive, then $p$ is intensive.*
5. *If $p_\lambda$ are post-idempotent and order reversing, then $p$ is post-idempotent.*
6. *If $p_\lambda$ are pre-idempotent and order preserving, then $p$ is pre-idempotent.*
7. *If $p_\lambda$ are post-involutive and order reversing, then $p$ is post-involutive.*
8. *If $p_\lambda$ are pre-involutive and order preserving, then $p$ is pre-involutive.*
9. *If $p_\lambda$ are spanning, then $p$ is spanning.*

*Proof.* Recall that the sum of submodules is the smallest submodule which contains all summands so $p_\lambda(N, M) \subseteq \sum_{\lambda \in \Lambda} p_\lambda(N, M)$ for every $\lambda \in \Lambda$.

(1) Suppose $p_\lambda$ are order preserving and $L \subseteq N$. So $p_\lambda(L, M) \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Then $\sum_{\lambda \in \Lambda} p_\lambda(L, M) \subseteq \sum_{\lambda \in \Lambda} p_\lambda(N, M)$ and $p$ is order preserving.

(2) Suppose $p_\lambda$ are order reversing and $L \subseteq N$. So $p_\lambda(N, M) \subseteq p_\lambda(L, M)$ for all $\lambda \in \Lambda$. Then $\sum_{\lambda \in \Lambda} p_\lambda(N, M) \subseteq \sum_{\lambda \in \Lambda} p_\lambda(L, M)$ and $p$ is order reversing.

(3) Suppose $p_\lambda$ are extensive. Then $N \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \subseteq \sum_{\lambda \in \Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is extensive.

(4) Suppose $p_\lambda$ are intensive. Then $N \supseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \supseteq \sum_{\lambda \in \Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is intensive.

(5) Suppose $p_\lambda$ are post-idempotent and order reversing and $f \in p^2(N, M)$. Then $f = \sum_{\lambda \in \Lambda'} f_\lambda$ where $f_\lambda \in p_\lambda(p(N, M), M)$ and $\Lambda' \subseteq \Lambda$. Since $p_\lambda(N, M) \subseteq p(N, M)$ and $p_\lambda$ are order reversing, we have

$$f_\lambda \in p_\lambda(p(N, M)) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq p_\lambda(N, M)$$

where the last property follows from $p_\lambda$ being post-idempotent. Then

$$f = \sum_\lambda f_\lambda \in \sum_\lambda p_\lambda(N, M) = p(N, M).$$

Since $f \in p_\lambda(N, M) \subseteq p(N, M)$, $p^2(N, M) \subseteq p(N, M)$ and $p$ is post-idempotent.

(6) Suppose $p_\lambda$ are pre-idempotent and order preserving. Suppose $f \in p(N, M)$. Then $f = \sum_{\lambda \in \Lambda'} f_\lambda$ where $f_\lambda \in p_\lambda(N, M)$ and $\Lambda' \subseteq \Lambda$. Since $p_\lambda(N, M) \subseteq p(N, M)$ and $p_\lambda$ are order preserving, we have

$$f_\lambda \in p_\lambda(N, M) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq p_\lambda(p(N, M), M).$$

Then $f \in \sum_{\lambda \in \Lambda'} p_\lambda(p(N, M), M) = p^2(N, M)$ and $p$ is pre-idempotent.

(7) Suppose $p_\lambda$ are post-involutive and order reversing. Suppose $f \in p^2(N, M)$. Then $f = \sum_{\lambda \in \Lambda'} f_\lambda$ where $f_\lambda \in p_\lambda(p(N, M), M)$ and $\Lambda' \subseteq \Lambda$.But because $p_\lambda$ is order reversing and $p_\lambda(N, M) \subseteq p(N, M)$, we have

$$f \in p_\lambda(p(N, M)) \subseteq p_\lambda(p_\lambda(N, M), M) \subseteq N$$

where the last property follows from $p_\lambda$ being post-involutive. Thus

$$f \in p^2(N,M) \subseteq N$$

as required and $p$ is post-involutive.

(8) Suppose $p_\lambda$ are pre-involutive and order preserving. Suppose $f = \sum_{\lambda\in\Lambda'} f_\lambda$ where $f_\lambda \in N$ and $\Lambda' \subseteq \Lambda$. Since $p_\lambda(N,M) \subseteq p(N,M)$ and $p_\lambda$ are order preserving, we have

$$f_\lambda \in N \subseteq p_\lambda^2(N,M) \subseteq p_\lambda(p(N,M),M)$$

where the first property follows from $p_\lambda$ being pre-involutive. Then

$$f = \sum_{\lambda\in\Lambda'} f_\lambda \in \sum_{\lambda\in\Lambda'} p_\lambda(p(N,M),M) = p^2(N,M)$$

and $p$ is pre-involutive.

(9) Suppose $p_\lambda$ are spanning. Then

$$\begin{aligned} N + p(N,M) &= N + \sum_{\lambda\in\Lambda} p_\lambda(N,M) \\ &= \sum_{\lambda\in\Lambda}(N + p_\lambda(N,M)) \\ &= \sum_{\lambda\in\Lambda} M \\ &= M. \end{aligned}$$

Thus $p$ is spanning.

□

**Remark 6.2.9.** It is not always the case that if $p_\lambda$ being independent then $p$ will be independent. First note that if $Q \subseteq N$ then $N \cap (P+Q) = (N\cap P) + (N \cap Q)$. So in this case to get

$$0 = L \cap p(L,M) = L \cap (\sum_{\lambda\in\Lambda} p_\lambda(L,M)) = (L \cap p_1(L,M)) + ... + (L \cap p_i(L,M)) + ...$$

we'd need $p_i(L, M) \subseteq L \forall i \in \Lambda$. i.e. $p_i$ is intensive. By Proposition 5.2.7(1) this implies $p_i(L, M) = 0$ for all $L \subseteq M, i \in \Lambda$. Thus $p(L, M) = 0$. $\diamondsuit$

**Definition 6.2.10.** Let $p_1$ and $p_2$ be pair operations defined on $\mathcal{P}$ a collection of pairs of modules in $R$. We say $p_1 \leq p_2$ if $p_1(N, M) \subseteq p_2(N, M)$ for all $(N, M) \in \mathcal{P}$. We say $p_1$ and $p_2$ are *comparable* if $p_1 \leq p_2$ or $p_2 \leq p_1$.

The next construction uses a partial order on pair operations.

**Construction 6.2.11.** *Let $R$ be Noetherian and let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a directed set of pair operations on a class of pairs $\mathcal{P}$ of $R$-modules. That is, for any $\lambda_1, \lambda_2 \in \Lambda$, there exists some $\mu \in \Lambda$ such that $p_{\lambda_i} \leq p_\mu$ for $i = 1, 2$. Then*

$$p(N, M) := \bigcup_{\lambda \in \Lambda} p_\lambda(N, M)$$

*is a pair operation defined on $\mathcal{P}$.*

**Proposition 6.2.12.** *Let $p$ be a pair operation constructed as in Construction 6.2.11. Then the following hold:*

1. *If $p_\lambda$ are order preserving, then $p$ is order preserving.*
2. *If $p_\lambda$ are order reversing, then $p$ is order reversing.*
3. *If $p_\lambda$ are extensive, then $p$ is extensive.*
4. *If $p_\lambda$ are intensive, then $p$ is intensive.*
5. *If $p_\lambda$ are post-idempotent and order preserving, then $p$ is post-idempotent.*
6. *If $p_\lambda$ are pre-idempotent and order reversing, then $p$ is pre-idempotent.*
7. *If $p_\lambda$ are post-involutive and order preserving, then $p$ is post-involutive.*
8. *If $p_\lambda$ are pre-involutive and order reversing, then $p$ is pre-involutive.*

9. *If $p_\lambda$ are independent, then $p$ is independent.*

10. *If $p_\lambda$ are spanning, then $p$ is spanning.*

11. *If $p_\lambda$ are complementary, then $p$ is complementary.*

*Proof.* (1) Suppose $p_\lambda$ are order preserving and $L \subseteq N$. So $p_\lambda(L, M) \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Then

$$\bigcup_{\lambda\in\Lambda} p_\lambda(L, M) \subseteq \bigcup_{\lambda\in\Lambda} p_\lambda(N, M)$$

and $p$ is order preserving.

(2) Suppose $p_\lambda$ are order reversing and $L \subseteq N$. So $p_\lambda(L, M) \supseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Then

$$\bigcup_{\lambda\in\Lambda} p_\lambda(L, M) \supseteq \bigcup_{\lambda\in\Lambda} p_\lambda(N, M)$$

and $p$ is order reversing.

(3) Suppose $p_\lambda$ are extensive. Then $N \subseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \subseteq \bigcup_{\lambda\in\Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is extensive.

(4) Suppose $p_\lambda$ are intensive. Then $N \supseteq p_\lambda(N, M)$ for all $\lambda \in \Lambda$. Thus

$$N \supseteq \bigcup_{\lambda\in\Lambda} p_\lambda(N, M) = p(N, M)$$

and $p$ is intensive.

(5) Suppose $p_\lambda$ are post-idempotent and order preserving. By the directedness of the set $\{p_\lambda | \lambda \in \Lambda\}$, for any $(L, M)$ there exists $\mu \in \Lambda$ such that $p(L, M) = p_\mu(L, M)$. So there exists $\lambda_1, \lambda_2 \in \Lambda$ such that $p^2(L, M) = p_{\lambda_1}(L, M)$ and $p(L, M) = p_{\lambda_2}(L, M)$.

Choose $\mu \in \Lambda$ such that $p_{\lambda_1} \leq p_\mu$ and $p_{\lambda_2} \leq p_\mu$. Then

$$\begin{aligned}
p(p(L,M),M) &= p_{\lambda_1}(p_{\lambda_2}(L,M),M) \\
&\subseteq p_\mu(p_\mu(L,M),M) && \text{By choice of } \mu \text{ and order preserving} \\
&\subseteq p_\mu(L,M) && \text{By post-idempotence} \\
&\subseteq p(L,M).
\end{aligned}$$

Thus $p$ is post-idempotence.

(6) Suppose $p_\lambda$ are pre-idempotent and order reversing. Let $\lambda_1, \lambda_2, \mu \in \Lambda$ be as in (5). Then

$$\begin{aligned}
p(L,M) &= p_{\lambda_2}(L,M) \\
&\subseteq p_\mu(L,M) \\
&\subseteq p_\mu(p_\mu(L,M),M) && \text{By pre-idempotence} \\
&\subseteq p_\mu(p_{\lambda_2}(L,M),M) && \text{By order reversing} \\
&= p_\mu(p(L,M),M) \\
&\subseteq p(p(L,M),M)
\end{aligned}$$

Thus $p$ is pre-idempotent.

(7) Suppose $p_\lambda$ are post-involutive and order preserving. Then

$$\begin{aligned}
p(p(L,M),M) &= p_{\lambda_1}(p_{\lambda_2}(L,M),M) \\
&\subseteq p_\mu(p_\mu(L,M),M) && \text{By order preserving} \\
&\subseteq L && \text{By post-involutive}
\end{aligned}$$

So $p$ is post-involutive.

(8) Suppose $p_\lambda$ are pre-involutive and order reversing. Then

$$\begin{aligned}
L &\subseteq p_\mu(p_\mu(L,M),M) \\
&\subseteq p_\mu(p_{\lambda_2}(L,M),M) && \text{By order reversing} \\
&= p_\mu(p(L,M),M) \\
&\subseteq p(p(L,M),M)
\end{aligned}$$

So $p$ is pre-involutive.

(9) Suppose $p_\lambda$ are independent. Then

$$\begin{aligned}
N \cap p(N,M) &= N \cap \bigcup_{\lambda\in\Lambda} p_\lambda(N,M) \\
&= \bigcup_{\lambda\in\Lambda} (N \cap p_\lambda(N,M)) \\
&= \bigcup_{\lambda\in\Lambda} 0 \\
&= 0.
\end{aligned}$$

Thus $p$ is independent.

(10) Suppose $p_\lambda$ are spanning. Then

$$\begin{aligned}
N + p(N,M) &= N + \bigcup_{\lambda\in\Lambda} p_\lambda(N,M) \\
&= \bigcup_{\lambda\in\Lambda} (N + p_\lambda(N,M)) \\
&= \bigcup_{\lambda\in\Lambda} M \\
&= M.
\end{aligned}$$

The equality of the second line can be shown as follows.

Let $x \in N + \bigcup_{\lambda\in\Lambda} p_\lambda(N,M)$. Then $x = n + p$ where $n \in N$ and $p \in p_\lambda(N,M)$ for some $\lambda \in \Lambda$. So for at least one $\lambda$, $x \in N + p_\lambda(N, M)$ and thus

$$x \in \bigcup_{\lambda \in \Lambda} (N + p_\lambda(N, M)).$$

If $x \in \bigcup_{\lambda\in\Lambda}(N + p_\lambda(N, M))$, then $x = n + p$ for $n \in N$ and $p \in p_\lambda(N, M)$ some $\lambda \in \Lambda$. So $p \in \bigcup_{\lambda\in\Lambda} p_\lambda(N, M)$ and $x \in N + \bigcup_{\lambda\in\Lambda} p_\lambda(N, M)$.

Thus $p$ is spanning.

(11) This follows directly from (9) and (10). $\square$

**Remark 6.2.13.** The proof of Proposition 6.2.12 is based on [Eps12, Construction 3.1.4]. There Epstein proved that if $p_\lambda$ are closure operations then $p$ will be a closure operation. His proof of $p^2(N, M) \subseteq p(N, M)$ uses idempotence and order preservation and does not rely on extensivity. The extensivity is used to prove the other direction, $p(N, M) \subseteq p^2(N, M)$. $\diamondsuit$

# Chapter 7

# Future Work

Both tight closure and $\mathfrak{m}$-basically full closure have well understood dual interior formulations ([ERG21], [ERGV23b]). However, there is not yet a nice equational description for the integral interior. Analyzing a description of the integral interior could enable us to compute integral interiors not only of Artinian modules, but also of other modules.

Because the core is so widely studied, it would be worthwhile to refine the prehull, posthull, precore, and postcore bounds in Propositions 4.2.7 and 4.2.8 to be applicable in more general cases. This could be done by eliminating or including elements that are known to be in the core or hull.

There are other potential properties to explore in the general pair operation setting- for instance, a closure operation is *star* if for every ideal $J$ and nonzero divisor $x$ in $R$, $(xJ)^{\mathrm{cl}} = x(J^{\mathrm{cl}})$ and a closure operation is *semi-prime* if for all ideal $I, J$ of $R$, we have $I \cdot J^{\mathrm{cl}} \subseteq (IJ)^{\mathrm{cl}}$. These have been defined mostly for ideals of rings, however, a semiprime closure on modules has been defined ($xL_M^{\mathrm{cl}} \subseteq (xL)_M^{\mathrm{cl}}$ for all $x \in R$) and is currently being studied. When constructing closure operations, the methods in Chapter 6 will often yield semi-prime closure operations [Eps12]. This can be extended to constructions of pair operations.

Epstein, R.G., and Vassilev have been studying absolute operations (when $L \subseteq N \subseteq M$ then $p(L, M) = p(L, N)$), restrictable operations (when $L \subseteq M$ and $N \subseteq M$ then $p(L \cap N, N) \subseteq p(L, M)$), and residual operations (when $L \subseteq N \subseteq M$ then $p(N, M) = \pi^{-1}(p(N/L, M/L))$ where $\pi : M \twoheadrightarrow M/L$ is the natural surjection) pair operations so these could also be considered in conjunction with the other properties. There is a duality relation between residual closure operations and absolute interior operations ([ERG21], [ERG21]) so it would be interesting to see how these concepts relate outside of closure and interior operations.

Alongside these other properties, pair operations that are independent, spanning, and/ or involutive can be looked at with greater depth. In particular, we can look for existing operations in vector spaces and linear algebra which can be studied through the lens of pair operations.

One of the next things we will be looking at is the infinite sum and the infinite intersection of pair operations when equipped with some variety of properties. Unions can be used to turn preclosures (an operation that is extensive and order preserving but not idempotent) into closures by considering $\cup_{i=1}^{\infty} p^i(N, M)$. A similar process could potentially be used to turn operations that are almost involutive or idempotent into operations that are. Intersections could be used to turn a postinterior operation (one that is intensive and order preserving but not idempotent) into an interior operation.

Using arguments similar to [Vra02], we will use the special part decomposition and prereductions to form nice chains of ideals. When it comes to the special part of a closure operation, the $f$ described in Definition 4.3.9 is known for integral ($f(n) = n$) and tight ($f(n) = p^n$) closure. There is a closure operation between integral and tight closure: the $s$-closure [Tay21]. It will be interesting to discover whether the $s$-closure has a special part that is describable with a function $f(n)$ such that $n \leq f(n) \leq p^n$ and whether it would be dependent on $s$. Another direction regarding the special part would be to extend the results of Epstein in the special part of the integral closure of monomial ideals [Eps10] into a affine semigroup setting. Some minor results, such as when $M$ is a monomial in $R$ with degree $(n_1, 0)$ and $J$ a homogeneous ideal with $(n_1, n_2)$ an exponent of a monomial of $J$ such that $n_1$ is of smallest degree, have been found but stronger results could be interesting.

# Appendix A

# Numerical Semigroup Results

As many of our examples are in numerical semigroup rings, we include some results here.

**Remark A.0.1.** Let $R = k[[t^S]]$ where $S$ is a subsemigroup of $\mathbb{N}$ satisfying $\gcd(S) = 1$.

1. $(k[[t^S]])^- = k[[t]]$. This is the case because for all $n \in \mathbb{N} \setminus S$, there is an $m = kn \in S$ with $k \in \mathbb{N}$ so that $x^k - t^m \in R[x]$ and $t^n$ is a zero.

2. [ADGS20, Page 6] The gaps of $S$, denoted $G(S) := \mathbb{N} \setminus S$. The Frobenius of $S$ is $f = \max(H(S))$ and the conductor is $c(S) = F(S) + 1$.

3. $(t^n)^* = (t^n)^- = (t^n)(k[[t]])^- \cap R = (t^r \mid r \geq n \text{ and } r \in S)$ by Proposition 2.3.5.

$\diamondsuit$

**Proposition A.0.2.** *Let $R = k[[x^2, x^5]]$ and $\mathfrak{m} = (x^2, x^5)$. Then*

*1.* $(x^s) :_R \mathfrak{m} = \begin{cases} R \text{ for } s = 0, \\ (x^s, x^{s+3}) \text{ for } s \in \langle 2, 5\rangle \setminus 0. \end{cases}$

*and in particular,* $\mathfrak{m}((x^s) :_R \mathfrak{m}) = (x^{s+2}, x^{s+5})$ *for all* $s \in \langle 2, 5\rangle$.

2. $$(x^s, x^{s+3}) :_R \mathfrak{m} = \begin{cases} R \text{ for } s = 2, \\ (x^5, x^6) \text{ for } s = 5, \\ (x^{s-2}, x^{s+1}) \text{ for } s = 4, s \geq 6 \end{cases}$$

   *and in particular,* $\mathfrak{m}(x^s) :_R \mathfrak{m} = \mathfrak{m}(x^s, x^{s+3}) :_R \mathfrak{m} = (x^s, x^{s+3})$ *for all* $s \in \langle 2, 5\rangle \backslash 0$ *and*

$$\mathfrak{m}((x^s, x^{s+3}) :_R \mathfrak{m}) = \begin{cases} (x^7, x^8) \text{ for } s = 5, \\ (x^s, x^{s+3}) \text{ for } s = 2, 4, s \geq 6. \end{cases}$$

3. $$(x^s, x^{s+1}) :_R \mathfrak{m} = \begin{cases} (x^2, x^5) \text{ for } s = 4, \\ (x^4, x^5) \text{ for } s = 5, \\ (x^{s-2}, x^{s-1}) \text{ for } s \geq 6. \end{cases}$$

   *and in particular,* $\mathfrak{m}(x^s, x^{s+1}) :_R \mathfrak{m} = (x^s, x^{s+1})$ *for all* $s \geq 4$ *and*

$$\mathfrak{m}((x^s, x^{s+1}) :_R \mathfrak{m}) = \begin{cases} (x^4, x^7) \text{ for } s = 4, \\ (x^6, x^7) \text{ for } s = 5 \\ (x^s, x^{s+1}) \text{ for } s \geq 6. \end{cases}$$

4. $E_R(k) = kx \oplus kx^3 \oplus \bigoplus_{i=1}^{\infty} kx^{-i}$ *where the action determined by the monomials of* $R$ *is given by*

$$x^j x^{-i} = \begin{cases} 0 & \text{if } j - i = 0, 2 \text{ or } j - i \geq 4 \\ x^{j-i} & \text{if } j - i = 1, 3 \text{ or } j - i \leq -1 \end{cases}.$$

*Proof.* (1) Since $\mathfrak{m} \subseteq (x^0) = R$, then $(x^0) :_R \mathfrak{m} = R$. If $s \neq 0$, then since $x^{s+i} \in (x^s)$ for $i \in \langle 2, 5\rangle$ and $x^{s+3} \in R$, then we see that $x^j x^{s+3} = x^{s+j+3} \in (x^s)$ when $x^j \in \mathfrak{m}$. Even if $x^{s+1} \in R$, $x^2 x^{s+1} = x^{s+3} \notin (x^s)$ which implies that $(x^2) :_R \mathfrak{m} = (x^s, x^{s+3})$.

(2) Note that $x^{s+i} \in (x^s, x^{s+3})$ for all $i \in \mathbb{N} \setminus \{1\}$ and if $s \geq 4$, $x^{s+i} \in R$ for all $i \in \mathbb{N}$. If $x^{s-4} \in R$, then

$$x^2 \cdot x^{s-4} = x^{s-2}, x^5 \cdot x^{s-4} = x^{s+1} \notin (x^s, x^{s+3});$$

thus $x^{s-4} \notin (x^s, x^{s+3}) :_R \mathfrak{m}$ for any $s \in \langle 2, 5\rangle$. If $x^{s-2} \in R$,

$$x^2 \cdot x^{s-2} = x^s, x^5 \cdot x^{s-2} = x^{s+3} \in (x^s, x^{s+3});$$

thus, $x^{s-2} \in (x^s, x^{s+3}) :_R \mathfrak{m}$ if $x^{s-2} \in R$. This is the case as long as $s \in \langle 2, 5\rangle \setminus \{5\}$. When $s = 2$, since $s - 2 = 0$, $(x^s, x^{s+3}) : \mathfrak{m} = R$. When $s = 4$ or $s \geq 6$, then $(x^s, x^{s+3}) :_R \mathfrak{m} = (x^{s-2}, x^{s+1})$. For $s = 5$, $x^{s-2} \notin R$; in this case,

$$x^2 \cdot x^5 = x^7, x^2 \cdot x^6 = x^8, x^5 \cdot x^5 = x^{10}, x^5 \cdot x^6 = x^{11} \in (x^5, x^8)$$

which will imply that $(x^5, x^8) :_R \mathfrak{m} = (x^5, x^6)$. The statements $\mathfrak{m}(x^s) :_R \mathfrak{m} = \mathfrak{m}(x^s, x^{s+3}) :_R \mathfrak{m} = (x^s, x^{s+3})$ for all $s \in \langle 2, 5\rangle \setminus 0$ and

$$\mathfrak{m}((x^s, x^{s+3}) :_R \mathfrak{m}) = \begin{cases} (x^7, x^8) \text{ for } s = 5, \\ (x^s, x^{s+3}) \text{ for } s = 2, 4, s \geq 6 \end{cases}$$

follow as in (1) from straightforward multiplication of ideals.

(3) Note that $x^{s+i} \in (x^s, x^{s+1})$ for all $i \in \mathbb{N}$ and if $s \geq 4$, $x^{s+i} \in R$ for all $i \in \mathbb{N}$. If $x^{s-i} \in R$ for $i \geq 3$, then

$$x^2 \cdot x^{s-i} = x^{s-i+2} \notin (x^s, x^{s+1});$$

thus $x^{s-i} \notin (x^s, x^{s+3}) :_R \mathfrak{m}$ for any $i \geq 3$ and any $s \in \langle 2, 5\rangle$. If $x^{s-1}, x^{s-2} \in R$,

$$x^2 \cdot x^{s-1} = x^{s+1}, x^5 \cdot x^{s-1} = x^{s+4}, x^2 \cdot x^{s-2} = x^s, x^5 \cdot x^{s-2} = x^{s+3} \in (x^s, x^{s+1});$$

thus, $x^{s-1}, x^{s-2} \in (x^s, x^{s+1}) :_R \mathfrak{m}$ if $x^{s-1}, x^{s-2} \in R$. This is the case as long as $s \geq 6$. When $s \geq 6$, then $(x^s, x^{s+1}) :_R \mathfrak{m} = (x^{s-2}, x^{s-1})$. For $s = 5$, $x^{s-2} \notin R$; in this case,

$$x^2 \cdot x^4 = x^6, x^2 \cdot x^5 = x^7, x^5 \cdot x^4 = x^9, x^5 \cdot x^5 = x^{10} \in (x^5, x^6)$$

which will imply that $(x^5, x^6) :_R \mathfrak{m} = (x^4, x^5)$. For $s = 4$, $x^{s-1} \notin R$; in this case,

$$x^2 \cdot x^2 = x^4, x^2 \cdot x^5 = x^7, x^5 \cdot x^2 = x^7, x^5 \cdot x^5 = x^{10} \in (x^5, x^6)$$

which will imply that $(x^4, x^5) :_R \mathfrak{m} = (x^2, x^5)$. The statements $\mathfrak{m}(x^s) :_R \mathfrak{m} = \mathfrak{m}(x^s, x^{s+3}) :_R \mathfrak{m} = (x^s, x^{s+3})$ for all $s \in \langle 2, 5\rangle \setminus 0$ and

$$\mathfrak{m}((x^s, x^{s+3}) :_R \mathfrak{m}) = \begin{cases} (x^7, x^8) \text{ for } s = 5, \\ (x^s, x^{s+3}) \text{ for } s = 2, 4, s \geq 6 \end{cases}$$

follow as in (1) from straightforward multiplication of ideals.

(4) By [BS13, Lemma 11.2.3], $E_R(k) = H^1_{\mathfrak{m}}(R) = H^1_{(x^2)}(R)$ since $R$ is Gorenstein and $x^2$ is a minimal reduction of $\mathfrak{m}$. By [BS13, 5.1.19] $H^1_{\mathfrak{m}}(R) \cong H^1(C^\bullet(x^2))$, where $C^\bullet(x^2) : 0 \to R \to R_{x^2} \to 0$ is the Čech complex. It is straightforward to see that $C^\bullet(x^2) : 0 \to R \to k[x, x^{-1}] \to 0$ and thus

$$E_R(k) = H^1(C^\bullet(x^2)) = k[x, x^{-1}]/R = kx \oplus kx^3 \oplus \bigoplus_{i=1}^{\infty} kx^{-i}$$

where the $R$-action on $kx \oplus kx^3 \oplus \bigoplus_{i=1}^{\infty} kx^{-i}$ determined by the monomials of $R$ is

$$x^j x^{-i} = \begin{cases} 0 & \text{if } j - i = 0, 2 \text{ or } j - i \geq 4 \\ x^{j-i} & \text{if } j - i = 1, 3 \text{ or } j - i \leq -1 \end{cases}$$

$\square$

# Appendix B

# Reference Table

The following table can be used as a quick reference on how pair operations endowed with two properties interact.

For the tables we will assume $L \subseteq N \subseteq M$.

| Meaning | Color | Code |
|---|---|---|
| Same/Same | | "black" |
| Somewhere else on table | | "gray" |
| Nontrivial examples exist | | "green" |
| Constant/Redundant/Identity | | "orange" |

| | Extensive | Intensive | Order Preserving | Pre-Idempotent | Post-Idempotent | Idempotent |
|---|---|---|---|---|---|---|
| Definition | $L \subseteq p(L,M)$ | $p(L,M) \subseteq L$ | $p(L,M) \subseteq p(N,M)$ | $p(L,M) \subseteq p^2(L,M)$ | $p^2(L,M) \subseteq p(L,M)$ | $p(L,M) = p^2(L,M)$ |
| Extensive | | | | | | |
| Intensive | | | | | | |
| Order Preserving | Pre-Closure by Definition, Ex is $\alpha$-tight closure | Post-Interior by Definition | | | | |
| Pre-Idempotent | Ex 5.0.7, Remark 5.2.9 | | | | | |
| Post-Idempotent | | Ex 5.0.7, Remark 5.2.9 | | Idempotent by Definition | | |
| Idempotent | Ex 5.0.5 | Ex 5.0.5 | | Redundant by 5.0.13(2) | Redundant by 5.0.13(2) | |
| Order Reversing | Idempotent and $p(L,M) = M$ by 5.2.2(4) | Idempotent and $p(L,M) = 0$ by 5.2.2(5). Ex 2.3.2 | Constant | $p^2(L,M) = p^n(L,M)$ by 5.2.2(2) | $p^2(L,M) = p^n(L,M)$ by 5.2.2(3) | $p(L,M) = p(M,M)$ by 5.2.2(1) |
| Pre-Involutive | Ex 5.0.7 | $L = p^n(L,M)$ by 5.2.3(1). Ex 5.0.9 | Ex 5.0.9 | Ex 5.0.7 | Extensive by 5.2.3(2) | Ex 5.0.9 |
| Post-Involutive | $L = p^n(L,M), n \geq 1$ by 5.2.4(1) | Ex 5.0.7,5.0.9 | Ex 5.0.9 | Intensive by 5.2.4(2) | Ex 5.0.7 | Ex 5.0.9 |
| Involutive | Idempotent and $L = p(L,M)$ by 5.2.5(2) | Idempotent and $L = p(L,M)$ by 5.2.5(3) | | Intensive by 5.2.5(4) | Extensive by 5.2.5(5) | $p(L,M) = L$ by 5.2.5(1) |
| Independent | Only defined for $M = 0$ by 5.2.7(2) | $p(L,M) = 0$ by 5.2.7(1) | $N \cap p(L,M) = 0$ by 5.2.7(3) | $p(L,M) = 0$ by 5.2.8(1) | $p^n$ will be independent by 5.2.10(1) | $p(L,M) = 0$ by 5.2.11(1) |
| Spanning | $p(L,M) = M$ by 5.2.6(1) | Only defined for $M = 0$ by 5.2.6(2) | $p(L,M) = M$ by 5.2.6(3) | $p^n$ will be spanning by 5.2.8(2) | $p(L,M) = M$ by 5.2.10(2) | $p(L,M) = M$ by 5.2.11(2) |
| Complementary | | | | $\mathcal{P} = \{(M,M)\}$ by | $M = 0$ or $\mathcal{P} = \{(0,M)\}$ by 5.2.10(3) | $M = 0$ by 5.2.11(3) |
| $n$-Periodic | $p(L,M) = L$ by 5.2.14(1) | $p(L,M) = L$ by 5.2.14(2) | | $p(L,M) = L$ by 5.2.14(4) | $p(L,M) = L$ by 5.2.14(5) | $p(L,M) = L$ by 5.2.14(6) |

| | Order Reversing | Pre-Involutive | Post-Involutive | Involutive | Independent | Spanning | Complementary |
|---|---|---|---|---|---|---|---|
| Definition | $p(N,M) \subseteq p(L,M)$ | $L \subseteq p^2(L,M)$ | $p^2(L,M) \subseteq L$ | $L = p^2(L,M)$ | $L \cap p(L,M) = 0$ | $L + p(L,M) = M$ | Independent + Spanning |
| Extensive | | | | | | | |
| Intensive | | | | | | | |
| Order Preserving | | | | | | | |
| Pre-Idempotent | | | | | | | |
| Post-Idempotent | | | | | | | |
| Idempotent | | | | | | | |
| Order Reversing | | | | | | | |
| Pre-Involutive | Ex 5.0.8 | | | | | | |
| Post-Involutive | Ex 5.0.8 | Involutive by Definition | | | | | |
| Involutive | Ex 5.0.3 | Redundant 5.0.13 by (1) | Redundant by 5.0.13(1) | | | | |
| Independent | $L \cap p(N,M) = 0$ by 5.2.7(4) | | | $p^{2n} \cap p^{2k+1} = 0$ by 5.2.7(5) | | | |
| Spanning | $p(L,M) + N = M$ by 5.2.6(4) | | 5.0.12 | | | | |
| Complementary | Ex 5.0.3 | | | | | | |
| $n$-Periodic | $n$ odd: $p$ only defined on 0-module by 5.2.14(3) | $n$ even: Involutive by 5.2.14(7) | $n$ even: Involutive by 5.2.14(8) | | | | |

# References

[AD69] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594. MR 242573

[ADGS20] Abdallah Assi, Marco D'Anna, and Pedro A. García-Sánchez, *Numerical semigroups and applications*, RSME Springer Series, vol. 3, Springer, Cham, [2020] ©2020, Second edition [of 3558713].

[BI15] Adi Ben-Israel, *Projectors on intersections of subspaces*, Infinite products of operators and their applications, Contemp. Math., vol. 636, Amer. Math. Soc., Providence, RI, 2015, pp. 41–50. MR 3155359

[BS13] M. P. Brodmann and R. Y. Sharp, *Local cohomology*, second ed., Cambridge Studies in Advanced Mathematics, vol. 136, Cambridge University Press, Cambridge, 2013, An algebraic introduction with geometric applications. MR 3014449

[CFH23] Alessandra Costantini, Louiza Fouli, and Jooyoun Hong, *Residual intersections and core of modules*, J. Algebra **629** (2023), 227–246.

[CPU01] Alberto Corso, Claudia Polini, and Bernd Ulrich, *The structure of the core of ideals*, Math. Ann. **321** (2001), no. 1, 89–105.

[CPU02] ______, *Core and residual intersections of ideals*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2579–2594.

[CPU03] ______, *Core of projective dimension one modules*, Manuscripta Math. **111** (2003), no. 4, 427–433.

[Eps05] Neil M. Epstein, *A tight closure analogue of analytic spread*, Math. Proc. Cambridge Philos. Soc. **139** (2005), no. 2, 371–383.

[Eps10] Neil Epstein, *Reductions and special parts of closures*, J. Algebra **323** (2010), no. 8, 2209–2225.

*References*


[Eps12] Neil Epstein, *A guide to closure operations in commutative algebra*, Progress in commutative algebra 2, Walter de Gruyter, Berlin, 2012, pp. 1–37.

[ERG21] Neil Epstein and Rebecca R. G., *Closure-interior duality over complete local rings*, Rocky Mountain J. Math. **51** (2021), no. 3, 823–853.

[ERGV23a] Neil Epstein, Rebecca R. G., and Janet Vassilev, *How to extend closure and interior operations to more modules*, Nagoya Mathematical Journal (2023), 1–48.

[ERGV23b] ———, *Integral closure, basically full closure, and duals of nonresidual closure operations*, J. Pure Appl. Algebra **227** (2023), no. 4, Paper No. 107256.

[ERGV23c] ———, *Nakayama closures, interior operations, and core-hull duality - with applications to tight closure theory*, J. Algebra **613** (2023), 46–86.

[FPU08] Louiza Fouli, Claudia Polini, and Bernd Ulrich, *The core of ideals in arbitrary characteristic*, Michigan Math. J. **57** (2008), 305–319, Special volume in honor of Melvin Hochster.

[FV10] Louiza Fouli and Janet C. Vassilev, *The cl-core of an ideal*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 2, 247–262.

[FVV11] Louiza Fouli, Janet C. Vassilev, and Adela N. Vraciu, *A formula for the $*$-core of an ideal*, Proc. Amer. Math. Soc. **139** (2011), no. 12, 4235–4245.

[HH90] Melvin Hochster and Craig Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.

[HJLS93] William Heinzer, Bernard Johnston, David Lantz, and Kishor Shah, *Coefficient ideals in and blowups of a commutative Noetherian domain*, J. Algebra **162** (1993), no. 2, 355–391. MR 1254782

[HRR02] William J. Heinzer, Louis J. Ratliff, Jr., and David E. Rush, *Basically full ideals in local rings*, J. Algebra **250** (2002), no. 1, 371–396.

[HS95a] Craig Huneke and Irena Swanson, *Cores of ideals in 2-dimensional regular local rings*, Michigan Math. J. **42** (1995), no. 1, 193–208.

[HS95b] ———, *Cores of ideals in 2-dimensional regular local rings*, Michigan Math. J. **42** (1995), no. 1, 193–208.

[HS06] ______, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.

[HT05] Craig Huneke and Ngô Viêt Trung, *On the core of ideals*, Compos. Math. **141** (2005), no. 1, 1–18.

[Hun96] Craig Huneke, *Tight closure and its applications*, CBMS Regional Conference Series in Mathematics, vol. 88, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, With an appendix by Melvin Hochster.

[HV03] Craig Huneke and Adela Vraciu, *Special tight closure*, Nagoya Math. J. **170** (2003), 175–183.

[KRS20] Paula Kemp, Louis J. Ratliff, Jr., and Kishor Shah, *Prereductions of ideals in local rings*, Comm. Algebra **48** (2020), no. 7, 2798–2817.

[Lee08] Kyungyong Lee, *A short note on containment of cores*, Comm. Algebra **36** (2008), no. 10, 3890–3892.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid. MR 879273

[Moh97] Radha Mohan, *The core of a module over a two-dimensional regular local ring*, J. Algebra **189** (1997), no. 1, 1–22.

[NR54] Douglas Northcott and David Rees, *Reductions of ideals in local rings*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 145 — 158.

[PU05] Claudia Polini and Bernd Ulrich, *A formula for the core of an ideal*, Math. Ann. **331** (2005), no. 3, 487–503.

[Rat89] Louis J. Ratliff, Jr., $\Delta$*-closures of ideals and rings*, Trans. Amer. Math. Soc. **313** (1989), no. 1, 221–247.

[Ree87] D. Rees, *Reduction of modules*, Math. Proc. Cambridge Philos. Soc. **101** (1987), no. 3, 431–449.

[RR77] L. J. Ratliff, Jr. and David E. Rush, *Notes on ideal covers and associated primes*, Pacific J. Math. **73** (1977), no. 1, 169–191.

[RS88] D. Rees and Judith D. Sally, *General elements and joint reductions*, Michigan Math. J. **35** (1988), no. 2, 241–254.

[Tay21] William D. Taylor, *Fundamental results on s-closures*, J. Pure Appl. Algebra **225** (2021), no. 4, Paper No. 106565, 14. MR 4150820

[Vas97] Janet Cowden Vassilev, *Test ideals in Gorenstein isolated singularities and F-finite reduced rings*, ProQuest LLC, Ann Arbor, MI, 1997, Thesis (Ph.D.)–University of California, Los Angeles. MR 2696487

[Vas14a] Janet Vassilev, *$m$-full and basically full ideals in rings of characteristic $p$*, J. Rocky Mountain Math. Soc. **44** (2014), no. 2, 691 —- 704.

[Vas14b] Janet C. Vassilev, *When is a Nakayama closure semiprime?*, J. Commut. Algebra **6** (2014), no. 3, 439–454.

[Vra02] Adela Vraciu, *$*$-independence and special tight closure*, J. Algebra **249** (2002), no. 2, 544 — 565.

[Vra06] Adela Vraciu, *Chains and families of tightly closed ideals*, Bull. London Math. Soc. **38** (2006), no. 2, 201–208.

[ZS58] Oscar Zariski and Pierre Samuel, *Commutative algebra, Volume I*, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, NJ, 1958, With the cooperation of I. S. Cohen. MR 90581