12-10-1997

Robust finite-time stability design via linear matrix inequalities

Chaouki T. Abdallah

Peter Dorato

D. Famularo

Robust Finite-time Stability Design via Linear Matrix Inequalities

P. Dorato, C.T. Abdallah and D. Famularo

EECE Dept. — DEIS Dept.

The University of New Mexico — Università della Calabria

Albuquerque, NM 87131, USA — Rende (CS), 87036, ITALY

Abstract

For linear systems with polytopic uncertainties, the problem of robust finite-time stabilization is reduced to a system of Linear Matrix Inequalities.

1 Introduction

The concept of *short-time stability*, or *finite-time stability*, [2] , deals with systems operating over finite time intervals, with given bounds on the initial and subsequent states. In particular we have the following formal definition.

Definition 1 *The time-varying linear system*

$$\dot{x} = A(t)x \tag{1}$$

is said to be finite-time stable with respect to the given triplet (c_1, c_2, T), $c_2 \geq c_1$, *if*

$$x'(0)x(0) \leq c_1 \rightarrow x'(t)x(t) \leq c_2; \quad \forall \quad 0 \leq t \leq T \tag{2}$$

Systems that are asymptotically stable may not be finite-time stable if, over the time interval of interest, a large peaking in transient behavior occurs; and systems that are classically unstable may be finite-time stable, if the state does not exceed the particular given bounds over the given time interval. In this paper we focus on systems that cannot be stabilized in the classic sense, and consider the problem of designing a fixed state-feedback controller, $u(t) = Kx(t)$, for the linear polytopic uncertain system

$$\dot{x} = A(t)x + B(t)u \tag{3}$$

where $A(t) = \sum_{i=1}^{N} \alpha_i(t) A_i$, $B(t) = \sum_{i=1}^{N} \alpha_i(t) B_i$ and $\alpha_i(t) \geq 0$, $\sum_{i=1}^{N} \alpha_i(t) = 1$ which guarantees finite-time stability with respect to the given triplet (c_1, c_2, T). Uncertainty is characterized by the time-varying parameters $\alpha_i(t)$

2 Reduction to LMI's

With feedback the closed-loop state equations are given by

$$\dot{x} = (A(t) + B(t)K)x = A_{cl}(t)x \tag{4}$$

The reduction of the robust finite-time stabilization problem to a system of LMI's is based on the following theorem.

Theorem 1 *The system (4) is finite-time stable with respect to the triplet* (c_1, c_2, T) *if the following matrix inequality*

$$-(A_{cl}(t) + A_{cl}^T(t)) + \frac{1}{T} \ln\left(\frac{c_2}{c_1}\right) I_n > 0, \;\; \forall \;\; 0 \leq t \leq T \tag{5}$$

holds.

Proof:
Let $V(x) = x^T x$, with $\dot{x} = A_{cl}(t)x$, then $\dot{V} = x^t(A_{cl}^T(t) + A_{cl}(t))x$. Now if matrix inequality

$A_{cl}^T(t) + A_{cl}(t) < \gamma I$ holds (where I denotes the identity matrix), then $\dot{V} < \gamma V$, which in turn implies $V(x(t)) < e^{\gamma t} V(x(0))$, i.e. $x^T(t)x(t) < e^{\gamma t} x^T(0)x(0)$. If we assume that the closed-loop system cannot be classically stabilized, then γ will be positive, and with $x^T(0)x(0) \leq c_1$ it then follows that $x^T(t)x(t) < e^{\gamma T} c_1$. Thus if $e^{\gamma T} c_1 = c_2$, we guarantee that $x^T(t)x(t) < c_2$, hence finite-time stability with respect to the triplet (c_1, c_2, T). If one solves for γ one obtains $\gamma = \frac{1}{T} \ln\left(\frac{c_2}{c_1}\right)$ and the inequality (5(is established as a sufficient condition for finite-time stability.

The following theorem provides the LMI's required to design a closed-loop finite-time stable system.

Theorem 2 *The closed-loop system (4) is finite-time stable, with respect to the triplet (c_1, c_2, T) if there is a feasible matrix solution K to the LMI's*

$$A_i + A_i^T + B_i K + K^T B_i^T < \frac{1}{T} \ln\left(\frac{c_2}{c_1}\right) I, \; i = 1, \ldots N \tag{6}$$

Proof:
The proof follows the standard procedure, see reference [1], for polytopic uncertainties, i.e. the inequalities in (6) are each multiplied by α_i and summed.

It should be noted that design with *static output feedback* is also possible using theorem 2, if $u(t) = Ky(t)$, $y(t) = Cx(t)$ and C is a known matrix. In this case one simply replaces K in (6) by KC. The inequalities remain LMI's in the matrix variable K. The LMI problem may be solved with existing software, e.g. MATLAB LMI Toolbox [3].

3 A Numerical Example

Let us consider the following linear time varying polytopic system

$$\dot{x}(t) = (\alpha_1 A_1 + \alpha_2 A_2)x(t) + (\alpha_1 B_1 + \alpha_2 B_2)u(t) \tag{7}$$

where, in MATLAB notation

$$A_1 = \left[\begin{array}{ccccccccc} 0 & 1 & 0; & 0 & 0 & 1; & 1 & 0 & 0 \end{array}\right],$$

$$A_2 = -\left[\begin{array}{ccccccccc} 0 & -1 & 0; & 0 & 0 & -1; & 1 & 1 & 1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{ccc} 0; & 0; & 1 \end{array}\right], B_2 = -B_1.$$

To explore the range of possible triplets (c_1, c_2, T) for this example we consider the following generalized eigenvalue problem [1].

$$\begin{gathered} \min \gamma \\ A_i + A_i^T + B_i K + K^T B_i < \gamma I, \; i = 1, 2 \end{gathered} \tag{8}$$

obtaining the following values $\gamma_{\text{opt}} = 1.0629$ and $K_{\text{opt}} = \left[\begin{array}{ccc} -1 & -0.5 & -0.5 \end{array}\right]$. The value of γ_{opt} give us a barrier of feasibility in the sense that we **cannot** have values of c_1, c_2 and T such that

$$\frac{1}{T} \ln\left(\frac{c_2}{c_1}\right) < \gamma_{\text{opt}}. \tag{9}$$

A possible triple of values which satisfies equation (9) can be, for example $c_1 = 1$, $c_2 = 2$, $T = \frac{1}{4}$, giving a value of $\frac{1}{T} \ln\left(\frac{c_2}{c_1}\right) = 2.7726$. Note that with the longer interval, $T = 1$, inequality (9) is satisfied and theorem 2 cannot be used to finite-time stabilized the closed-loop system. Note also that while each pair of systems (A_1, B_1) and (A_2, B_2) is controllable, there does not exist any fixed K which simultaneously stabilized both systems.
In reference [4] a more practical finite-time stability design problem is considered which can also be solved by theorem 2.

References

[1] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[2] P. Dorato, *Short time stability in linear time-varying systems*, Proc. IRE International Convention Record, Part 4, pp. 83-87, 1961.

[3] P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, *LMI Control Toolbox*, the MathWorks Inc, Natick, MA, 1995.

[4] F.A. San Filippo, and P. Dorato, *Short-Time Parameter Optimization With Flight Control Application*, Automatica, 1974, Vol. 10, pp. 425-430.