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Chaouki T. Abdallah

Peter E. Hokayem

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Quasi-Monte Carlo Methods in Robust Control Design

Peter F. Hokayem†, Chaouki T. Abdallah†,
and Peter Dorato
EECE Department, MSC01 1100
1 University of New Mexico
Albuquerque, NM 87131-0001, USA
{hokayem,chaouki,peter}@eece.unm.edu

Abstract— **Many practical control problems are so complex that traditional analysis and design methods fail to solve. Consequently, in recent years sampling methods that provide approximate solutions to such difficult problems have emerged. In this paper we address the deterministic quasi-Monte Carlo method of sampling and attempt to impose bounds on the error involved in the evaluation of the quality of performance of a specific controller over the whole plant parameter uncertainty space.**

I. Introduction

Many control problems are so complex in nature that analytic techniques fail to solve them. Furthermore, even if analytic solutions are available, they generally result in very high order compensators. It is for these reasons that we accept approximate answers to provide us with certain guarantees in such control problems. This is when sampling methods come into the picture to try and remedy the "cost of solution" problem by drawing samples from a sample space, and providing an approximate answer. For many years, random sampling has dominated the afore mentioned arena [6], [7], [16]. Recently however, deterministic or quasi-Monte Carlo (QMC) methods have proven superior to random methods in several applications such as the calculation of certain integrals [10], financial derivatives [11] and motion planning in robotics [2]. They have also been used for stability analysis of high speed networks [1].

In a recent paper by the authors [4], a fairly self-contained presentation of QMC methods was given, and the performance was compared to classical random Monte Carlo method, in a robust control design setting. In this paper, we focus more on obtaining a bound on the error involved when the decision function is sufficiently differentiable. The main reason is that using a differentiable decision function with a multivariate polynomial as argument, we can place a bound on the value of the error involved in using samples from the plant parameter space instead of evaluating the stability of a certain controller over the whole plant parameter space.

The paper starts by formulating the robust control problem in Section II. Then we provide an abridged presentation of the main ideas involved in quasi-Monte Carlo sampling in Section III. In Section III-C, we present a detailed description of the notion of *total variation* of a multi-variate function in a d-dimensional space. The variation will help us in bounding the error involved in using quasi-Monte Carlo sampling methods to address the robust design problem. Finally, in Section IV, we present a detailed example that helps illustrate the theoretical notions introduced earlier in the paper.

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II. Problem Statement

Consider the control problem shown in Fig 1.

Problem 1: Given a real rational plant model $G(s,p)$, with uncertain parameter vector $p = [p_1 \quad p_2 \quad \dots \quad p_n] \in \mathbb{I}_p^n$, does there exist a controller $C(s,q)$ that can stabilize the uncertain system, where $q = [q_1 \quad q_2 \quad \dots \quad q_m] \in \mathbb{I}_q^m$ is the admissible controller parameter vector.

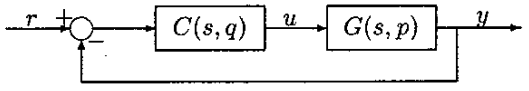

Fig. 1. Feedback Structure.

In Problem 1 above, $\mathbb{I}^i$ is the unit i-dimensional hypercube in $\mathbb{R}^i$. Without loss of generality the regions of plant uncertainty and design parameters have been scaled to the unit hypercubes $\mathbb{I}_p^n$ and $\mathbb{I}_q^m$, respectively. Let $T(s,p,q) = \frac{C(s,q)G(s,p)}{1+C(s,q)G(s,p)}$ be the closed-loop transfer function.

Problem 1 is the robust stabilization problem, and requires that the controller $C(s,q)$ stabilizes every plant inside the uncertainty interval ($\mathbb{I}_p^n$). This problem is inherently hard to solve in general, since we essentially have to check if all the plants inside the uncertainty set $\mathbb{I}_p^n$ are stabilizable, which is virtually impossible in a limited time span, due the continuity of the uncertainty interval. That is why we relax the problem into an approximate one through sampling. The method of solution is fairly simple using sampling and casting Problem 1 into an *integration* setting.

While Problem 1 requires an exact solution for the robust stabilization problem, the approximate solution requires the use of an *indicator function* (Ψ), which provides answers, regarding stability, for discrete points of the plant parameter uncertainty spectrum and admissible controller parameter space.

Definition 1: An indicator function Ψ is a decision type function that attains crisp values that belong to the discrete set $\{0,1\}$ depending on the decision criteria used to evaluate

the problem, at specific points of the sample space. Specifically for our purposes, we propose the following indicator function

$$\Psi(P_i, Q_j) = \begin{cases} 1, & T(s,p,q) \text{ is stable} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

where P_i and Q_j are sampled vectors from the plant parameter space and admissible controller parameter space, respectively.

Having defined the indicator function Ψ, we can easily cast Problem 1 into a sampling context as follows:

Problem 2: Consider Problem 1. Find vector $Q^* = [q_1^* \quad q_2^* \dots q_m^*] \in \mathbb{I}_q^m$ which stabilizes the uncertain plant with a high level of confidence, that is, Q^* maximizes

$$f_{Q^*}(P_i) = f(P_i, Q^*) = \frac{1}{N} \sum_{i=1}^{N} \Psi(P_i, Q^*) \quad (2)$$

where f is called the counting function, and N is a large number.

Problem 2 gets rid of solving the problem over a continuous plant parameter space through sampling that space, and *counting* those samples that result in $\Psi = 1$, i.e. a stable combination of P_i and Q_j. The second step is to pick $Q^* = Q_j$ that produces the largest answer for $f_Q(P)$, the counting function. The function $f_Q(P)$ can be interpreted as the average performance of the uncertain system with respect to a certain controller Q_i, in other words it is an approximation of the integration of the performance function over the plant parameter space. Hence, our problem is cast into an integration setting.

The main objective in this paper is to employ a differentiable indicator function. Hence we propose the following scaled indicator function

$$\Psi_j(P_i, Q_j) = \frac{1 + tanh\left(\frac{v(P_i,Q_j)}{\delta}\right)}{2} \quad (3)$$

where $v(.)$ is a multivariate polynomial. that meets our requirements due to the following reasons:

- $tanh(v(.))$ is a differentiable function as long as the $v(.)$ is differentiable, which is satisfied in our case since $v(.)$ is a multivariate polynomial.
- $tanh(.) \in [-1, 1]$, however according to our proposed function in (3), $\Psi(.) \in [0, 1]$ which satisfies the definition of the indicator function.
- δ determines how steep our indicator function is around the decision point 0. Figure 2 shows the indicator function Ψ for various values of δ. As δ decreases, Ψ becomes very steep and mimics the behavior of a crisp function with retention of differentiability.

III. Quasi-Monte Carlo Methods

In this section we review the basic definitions involved in qausi-Monte Carlo (QMC) methods and state the basic inequalities governing the quality of the approximation of integrals using deterministic sampling methods.

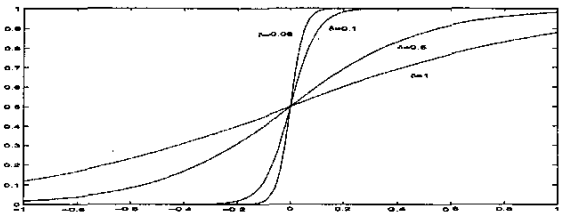

Fig. 2. Plot of $\frac{1+\tanh(\frac{x}{\delta})}{2}$ for different values of δ

The main idea in QMC methods is to evaluate an integrand at specific points and approximate the integral by the average of the results obtained at these specific points.

A. Discrepancy

The *discrepancy* is a measure of the 'regularity in distribution' of a set of points in the sample space. In order to define it mathematically, we need to define the following counting function:

$$A(B; P) = \sum_{i=1}^{N} I_B(X_i) \quad (4)$$

where $B \subset \mathbb{I}^d$ is an arbitrary set, $P = (X_1, \dots, X_N)$ is a point set, N is the number of points, and I_B is an indicator function.

Definition 2: The general formula for the evaluation of the discrepancy is given by

$$\mathcal{D}_N(\mathcal{B}, P) = \sup_{B \subset \mathcal{B}} \left| \frac{A(B,P)}{N} - \lambda_d(B) \right| \quad (5)$$

where $\lambda_d(B)$ is the d-dimensional Lebesgue measure of the arbitrary set B and $\mathcal{B}$ is the family of all lebesgue measurable subsets B of $\mathbb{I}^d$.

Definition 2 can be specialized into the following two cases:

- The *star discrepancy* $D_N^*(X_1, \dots, X_N)$ is obtained by letting $\mathcal{B}$ in (5) be defined as follows

$$\mathcal{B}^* = \{\forall B : B = \prod_{i=1}^{d} [0, u_i)\}$$

  i.e. the set of all d-dimensional subsets of $\mathbb{I}^d$ that have a vertex at the origin, and u_i's being arbitrary points in the corresponding 1-dimensional space.
- The *extreme discrepancy* $D_N(X_1, \dots, X_N)$ is obtained by letting $\mathcal{B}$ in (5) be defined as follows $\mathcal{B} = \{\forall B : B = \prod_{i=1}^{d} [v_i, u_i)\}$, where v_i's and u_i's are both arbitrary points in the corresponding 1-dimensional space.

The star discrepancy and extreme discrepancy are related through the following inequality $D_N^*(P) \leq D_N(P) \leq 2^d D_N^*(P)$.

B. Point Sets Generation

In this section we briefly describe how to generate quasi-Monte Carlo low discrepancy points in an d-dimensional sample space. Since the points result from a deterministic method of generation, they possess a certain regularity property of distribution in the sample space described by their discrepancy.

For brevity, we are not going to present the various methods used in the generation of the sample points. Instead, we refer the reader to [4] for a compact presentation and [9] for a more involved one, and present the basic methods that we are going to utilize in Section IV-A.

1) Van Der Corput: The van der Corput sequence in base b, where $b \geq 2 \in \mathbb{N}$, is a one dimensional sequence of points that possesses the property of having a low discrepancy in the unit interval $\mathbb{I} = [0,1] \subset \mathbb{R}$. *The main idea is to express every integer $n \in \mathbb{N}$ in base b and then reflect the expansion into the unit interval* $\mathbb{I}$. This is done as follows:

1) Let $R_b = \{0, 1, \ldots, b-1\}$ be the remainder set modulo b
2) Any integer $n \geq 0$ can be expanded in base b as, $n = \sum_{k=0}^{\infty} a_k(n) b^k$, where $a_k(n) \in R_b, \forall k$.
3) Finally, we get the sequence $\{X_n\}$ through $X_n = \phi_b(n) = \sum_{k=0}^{\infty} a_k(n) b^{-j-1}$.

As will be seen, the van der Corput sequence will be used to generate higher dimensional vector samples, with the variation of the expansion base b. Finally, the star discrepancy of the van der Corput sequence is given by: $D_N^*(X_1, \ldots, X_N) = \mathcal{O}(N^{-1}\log(N))$, with a constant depending on the base of expansion.

2) Halton Sequence : The Halton sequence is a generalization of the van der Corput sequence given in Section III-B.1 to span an d-dimensional sample space. The main idea is to generate d 1-dimensional sequences and form the corresponding d-dimensional vector sample points. Let $b_1, b_2, \ldots, b_d$ be the corresponding expansion bases for each dimension, preferably relatively prime[1]. Let $\phi_{b_1}, \phi_{b_2}, \ldots, \phi_{b_d}$ be the corresponding reflected expansions according to the corresponding bases. Then the d-dimensional sequences $\{X_n^d\}$ are formed as follows:

$$X_n = (\phi_{b_1}, \phi_{b_2}, \ldots, \phi_{b_d}) \in \mathbb{I}^d \tag{6}$$

Assume that the bases for the expansion are relatively prime, then the star discrepancy is given by (see [9])

$$D_N^*(X_1, \ldots, X_N) < \frac{d}{N} + \frac{1}{N} \prod_{i=1}^{d} \left(\frac{b_i - 1}{2 \log b_i} \log N + \frac{b_i + 1}{2} \right). \tag{7}$$

[1]Choosing the expansion bases relatively prime reduces the discrepancy, hence the error bound

C. Total Variation

The problem of bounding the error involved in evaluating the integral of a function using QMC methods depends on our ability to obtain the value of total variation of the function under consideration, as will be seen in the next section. Consequently, in this section we will concentrate on defining several notions of variation of a function defined on an interval $[0,1]^d$.

Definition 3: [3] A finite function $f(x)$ defined on and interval [0,1] is said to have *'bounded variation'* if there exists a number M, such that for any partition p of the interval [0,1]

$$v_p = \sum_{i=1}^{n} |f(X_i) - f(X_{i-1})| < M.$$

Moreover, the *'total variation'* of $f(x)$ on $[0,1]$ is defined as $V(f) = \sup_{p \in \mathcal{P}}(v_p)$, where $\mathcal{P}$ is the set of all partitions on [0,1].

Notice that Definition 3 pertains to functions of a single variable and does not require that the function be continuous. However, the function has to have a countable number of discontinuities on the interval under study. If it is further assumed that the function $f(x)$ is differentiable on $[0,1]$, then the total variation is defined as follows:

$$V(f) = \int_0^1 \left| \frac{df}{dx} \right| dx \tag{8}$$

Note 1: The total variation of a function can be understood as the sum of all the heights of monotone segments. That is why we integrate over the absolute value of the gradient in (8).

The total variation of a function f defined on a one-dimensional unit interval $\mathbb{I} = [0,1]$ is fairly easy to calculate. However, if f is defined on $\mathbb{I}^d$ the problem of calculating $V^{(d)}(f)$ (the d-dimensional total variation) is more involved (see [5], [9]). In what follows we only present the definitions of the total variation for continuous and differentiable functions.

Definition 4: The total variation of a function f defined on $\mathbb{I}^d$ in the sense of Vitali is defined as

$$V^{(d)} = \int_0^1 \cdots \int_0^1 \left| \frac{\partial^{(d)} f}{\partial \eta_1 \partial \eta_2 \ldots \partial \eta_d} \right| d\eta_1 d\eta_2 \ldots d\eta_d. \tag{9}$$

whenever the indicated partial derivative is continuous on $\mathbb{I}^d$. If $V^{(d)} < +\infty$, then the function f is said to have a *'bounded total variation in the sense of Vitali'*.

Note that the Definition 4 only measures the variation of f over all the variables at once. However, indicated partial derivative in (9) might be zero, but still the variation over the domain is not equal to zero as illustrated in the following example.

Example 1: Let $f(x_1, x_2) = x_1 + x_2 \Rightarrow \frac{\partial^{(2)} f}{\partial x_1 \partial x_2} = 0$ and the total variation as defined in *(9)* is equal to zero. However,

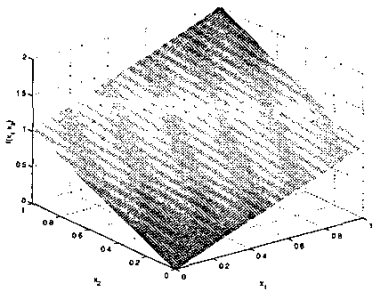

Fig. 3. Plot of $f(x_1, x_2) = x_1 + x_2$

when we plot the function $f(x_1, x_2)$, it is varying over the interval $[0, 1]^2$ as seen in Figure 3.

The problem encountered in the Definition 4 can be remedied via the following enhanced definition of the total variation.

Definition 5: [8], [9] Let f be a function defined on $\mathbb{I}^d$ with bounded variation in the sense of Vitali. Suppose that the restriction of f to each face F of $\mathbb{I}^d$ of dimension $k = 1, 2, \ldots, d-1$ is also of bounded variation on F in the sense of Vitali. Then the function f is said to be of *'bounded variation in the sense of Hardy and Krause'*.

Note 2: The restriction of the function f to the face F in definition 5 is achieved through setting the $d - k$ variables equal to 1.

Definition 5 overcomes the difficulties we encountered with Definition 4 as seen in the following example.

Example 2: Let us revisit the same function in example 1. Using definition 5 we get the following formula for the total variation of this second order function

$$V^{(2)}(f) = \int_0^1 \int_0^1 \left| \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right| dx_1 dx_2 + \int_0^1 \left| \frac{\partial f(x_1, 1)}{\partial x_1} \right| dx_1 + \int_0^1 \left| \frac{\partial f(1, x_2)}{\partial x_2} \right| dx_2 \tag{10}$$

Substituting and performing the necessary partial differentiation and integration we get $V^{(2)}(f) = 2$.

The second order total variation has been used in [14], [15], and the following intuitive bound on the variation on (10) was suggested in [14]

$$V^{(2)}(f) \leq \max_{x_1, x_2} \left| \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right| + \max_{x_1} \left| \frac{\partial f(x_1, 1)}{\partial x_1} \right| + \max_{x_2} \left| \frac{\partial f(1, x_2)}{\partial x_2} \right|. \tag{11}$$

D. Error in Quasi-Monte Carlo

The error in quasi-Monte Carlo methods integration over the unit hypercube for N samples is defined as follows,

$$e = \int_{\mathbb{I}^d} f(\eta) d\eta - \frac{1}{N} \sum_{n=1}^{N} f(X_n) \tag{12}$$

The following two theorems provide bounds on the error (12), for the cases of 1-dimensional and d-dimensional integration, respectively.

Theorem 1: Koksma's Inequality [9]
Let $f(.)$ be a function defined on $\mathbb{I} = [0, 1]$ of bounded total variation $V(f)$

$$\left| \int_{\mathbb{I}^d} f(\eta) d\eta - \frac{1}{N} \sum_{i=1}^{N} f(X_n) \right| \leq V(f) D_N^*(X_1, \ldots, X_N) \tag{13}$$

Theorem 2: Koksma-Hlawka Inequality [9]
Let $f(.)$ be a function defined on $\mathbb{I}^d$ of bounded variation in the sense of Hardy and Krause

$$\left| \int_{\mathbb{I}^d} f(\eta) d\eta - \frac{1}{N} \sum_{i=1}^{N} f(X_n) \right| \leq V^{(d)}(f) D_N^*(X_1, \ldots, X_N) \tag{14}$$

Basically, Theorems 1 and 2 state that the magnitude of the error depends on the total variation (defined in Section III-C) of the function and the star discrepancy of the point set chosen. That is why we are always after low star discrepancy point sets in quasi-Monte Carlo methods. It is also worth mentioning that the error bounds are conservative, i.e. if the variation of the function is large, we get a large bound on the error, although the actual error might be small.

IV. Example

In this section we consider an old problem first introduced by Truxal in [18], and recently revisited in [4]. The main idea is having a hypercube-like parameter space ($\mathbb{I}^n$) with a hypersphere-like region ($\mathcal{B}^n(0, \rho)$) of instability. The problem becomes challenging when the radius instability becomes close to the boundary of the sampling space. Refer to Figure 1 with the plant transfer function $G(s, p, r) = \frac{s^2+s+(3+2p_1+2p_2)}{s^3+(1+p_1+p_2)s^2+(1+p_1+p_2)s+(0.25+\rho^2+3p_1+3p_2+2p_1p_2)}$ and the simple gain controller $C(s, q) = q$, with $q \in [0, 1]$, $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$. The resulting closed-loop characteristic polynomial is

$$\begin{aligned} p(s) &= s^3 + (1 + p_1 + p_2 + q)s^2 + (1 + p_1 + p_2 + q)s \\ &\quad + (0.25 + \rho^2 + 3p_1 + 3p_2 + 3q \\ &\quad + 2p_1p_2 + 2p_1q + 2p_2q). \end{aligned} \tag{15}$$

Using Maxwell's criterion for 3^{rd}-order polynomials, we obtain the following multivariate-polynomial inequalities

(MPIs) that guarantee the stability of (15),

$$\begin{aligned} v_1(p_1,p_2,q) &= 1+p_1+p_2+q>0 \\ v_2(p_1,p_2,q) &= 0.25+\rho^2+3p_1+3p_2 \\ &\quad +3q+2p_1p_2+2p_1q+2p_2q>0 \\ v_3(p_1,p_2,q) &= p_1^2+p_2^2+q^2-p_1-p_2-q+0.75-\rho^2>0 \end{aligned} \tag{16}$$

It is easily seen that the first and second inequalities in (16) are always satisfied for the ranges of uncertainties and design regions given. However, the third inequality requires a closer look to establish the stability regions for the closed-loop system. Through completing the squares, the third inequality could be written as

$$v_3(p_1,p_2,q) = (p_1-0.5)^2+(p_2-0.5)^2+(q-0.5)^2-\rho^2>0 \tag{17}$$

It is easily seen that (17) equated to zero results in the equation of a sphere centered at $(0.5, 0.5, 0.5)$ and radius ρ. Therefore, our instability region is defined by the intersection of the unit 3-dimensional hypercube and the spherical region given in (17). Consequently, the problem is restated as follows

$$\begin{aligned} Q_{sol} &= \{q\in[0,1] : \forall p\in[0,1], r\in[0,1], p_1(p,q,r)>0 \\ &\quad \wedge p_2(p,q,r)>0 \wedge p_3(p,q,r)>0\}. \end{aligned} \tag{18}$$

Usually solution regions for problems such as the one presented in (18) are hard to obtain analytically. However, in our case the solution is fairly simple: $Q_{sol} = \{[0, 0.5-\rho) \cup (0.5+\rho, 1]\}$. For $\rho = 0.499$ we have $Q_{sol} = \{[0, 0.001) \cup (0.999, 1]\}$.

In what follows, we address the same the problem using QMC sampling. The indicator function is defined as follows

$$\Psi_j := \Psi(p_1,p_2,Q_j) = \frac{1}{2}\left(1+\tanh\left(\frac{v_3(p_1,p_2,Q_j)}{\delta}\right)\right) \tag{19}$$

where p_1, p_2 are the plant parameter variables, and $v_3(p_1, p_2, Q_j)$ is defined for a specific controller Q_j sampled from the admissible control parameter space. The main objective is to upper-bound the error involved between the actual evaluation of the indicator function f_i over the whole region of plant uncertainty space $\mathbb{I}^2$ and the empirical evaluation based to samples taken over the same space, i.e.

$$e_j = \left|\int_0^1\int_0^1 \Psi(p_1,p_2,Q_j)dp_1dp_2 - \frac{1}{N}\sum_{i=1}^{N}\Psi(p_{1i},p_{2i},Q_j)\right| \tag{20}$$

Let us first calculate the total variation of Ψ_j. The partial derivatives involved in the calculation of the total variation of Ψ_j are,

$$\begin{aligned} \frac{\partial^2\Psi_j}{\partial p_1\partial p_2} &= -4\frac{(p_1-0.5)(p_2-0.5)}{\delta^2} \\ &\quad \times\tanh\left(\frac{v_3(p_1,p_2,Q_j)}{\delta}\right) \\ &\quad \times\left[1-\tanh^2\left(\frac{v_3(p_1,p_2,Q_j)}{\delta}\right)\right] \\ \frac{\partial\Psi_j}{\partial p_1} &= \frac{(p_1-0.5)}{\delta}\left[1-\tanh^2\left(\frac{v_3(p_1,1,Q_j)}{\delta}\right)\right] \\ \frac{\partial\Psi_j}{\partial p_2} &= \frac{(p_2-0.5)}{\delta}\left[1-\tanh^2\left(\frac{v_3(1,p_2,Q_j)}{\delta}\right)\right] \end{aligned} \tag{21}$$

The corresponding error e_j in (20) is upper-bounded by $V^{(2)}(\Psi_j)D_N^\star(P_1,\ldots,P_N)$ through the use the Koksma-Hlawka inequality given in Theorem 2 and total variation on a 2-dimensional space obtained by substituting (21) in (10) and integrating.

A. Simulation

We generated using Matlab a Halton sequence of 1000 samples from the plant uncertainty space $\mathcal{P} = (P_1 = [p_{1,1}, p_{2,1}], \ldots, P_{1000} = [p_{1,1000}, p_{2,1000}])$. And from the controller parameter space we generated a van der Corput sequence of 700 samples. Applying a crisp indicator function $(\mathrm{sgn}(v_3(p_{1i}, p_{2i}, Q_j))$ as in [4], we obtain $\Psi_j^\star = 100\%$ stabilization with respect to the sampled plants for the controller $Q_j^\star = 0.00032 \in Q_{sol}$.

Now, we utilize the indicator function provided in (3) with $\delta = 0.1$. For the same controller $Q_j^\star = 0.00032$ we obtain $\Psi_j^\star = 99.95\%$. The various differentials (in absolute value) in (21) are derived for the corresponding controller.

- $\int_0^1\left|\frac{\partial\Psi_j^\star(p_1,1)}{\partial p_1}\right|dp_1 = \int_0^1\left|\frac{\partial\Psi_j^\star(1,p_2)}{\partial p_2}\right|dp_2 = 0.0131$, using the definite integration function on Matlab.
- $\int_0^1\int_0^1\left|\frac{\partial^2\Psi_j^\star(p_1,p_2)}{\partial p_1\partial p_2}\right|dp_1dp_2 \approxeq 2 \leq \max_{p_1,p_2}\left|\frac{\partial^2\Psi_j^\star(x_1,x_2)}{\partial p_1\partial p_2}\right| = 6.354$, using numerical approximation for the integral[2].

Then the total variation is bounded by $V^{(2)}(\Psi_j^\star) \leq 2 \times 0.0131 + 2 = 2.0262$. The error involved in the integration using quasi-Monte Carlo method for the optimal controller $Q_j^\star = 0.00032$ is bounded as follows,

$$e_j^\star \leq V(2)(\Psi_j^\star)D_N^\star(\mathcal{P}) < 2.0262 \times 0.0557 = 0.1128 \tag{22}$$

where $D_N^\star(\mathcal{P})$ was calculated using (7).

B. Discussion

Several notes are in order regarding the results obtained in Section IV-A.

[2]We bounded the integral of the absolute value of the second partial derivative by the maximum value in order to stress the importance of the bound introduced in (11) when the calculation of the integral is hard.

- The value of the final error in (22) says that our indicator function $\Psi_j^* = 99.95\%$ could have an approximation error of $\pm 11.28\%$.
- It should be noted that whenever the value of δ used in the indicator function decreases, the value Ψ_j^* tends more to the value obtained when using crisp indicator function, i.e. 100%.
- Also as δ decreases the peak values in the graph for the absolute value of the second partial derivative of Ψ_j, shown in Figure 4, increases tremendously due to the large variation around the origin.

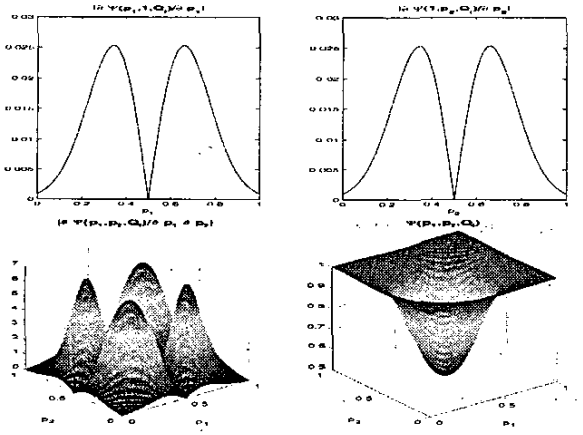

Fig. 4. Plot of absolute value of partial derivatives in (21) and indicator surface (Ψ_j^*) for $Q_i^* = 0.00032$ and $\delta = 0.1$.

V. Conclusion

In this paper we utilized deterministic sampling to address the robust control design problem in an approximate manner. We considered the notion of *total variation* of a function on a d-dimensional space and utilized it in bounding the error generated through the use of quasi-Monte Carlo sampling. It is fairly easy to calculate the error involved when the indicator function is differentiable. Hence, using a smooth indicator function allowed us to derive bounds on the error based on the value of the total variation of the indicator function.

VI. References

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