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Determining the idealizers of principal monomial ideals over a rational normal curve

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ABSTRACT

Given an ideal J generated by an element of the form $s^{m_1}t^{m_2}$, where $m_1 \geq 2$ and $m_2 \geq 0$, we illustrate how to compute the idealizer $\mathbb{I}(J)$ over the ring of the rational normal curve of degree n and we give a formula for it using the graded pieces of the sets of differential operators.

Contents

1. **Introduction**

In this project, we compute $\mathbb{I}(J)_{\boldsymbol{d}} := \{\delta \in D(R) | \delta * J \subseteq J\}$, which is called the idealizer of J, where $J = \langle s^{m_1} t^{m_2} \rangle$ is an ideal, $D(R)$ denotes the ring of differential operator and $*$ denotes an action of a differential operator. The idealizer of an ideal J in a regular ring R has been important for determining the ring of differential operators, $D(R)$, in a quotient of the regular ring modulo J. In particular, when R is not regular ring, Berkesch, et al. in [2] gave some examples where the ring of differential operators of a quotient of R is not a quotient of the idealizer. However, the idealizer does help us determine the differential operators in R which are differential operators in the quotient. Techniques to determine the idealizer in the toric setting are becoming important in developing work of Miller, Taylor and Vassilev [6] to get a characteristic 0 notion of differentially fixed ideals.

The computations we provide are over the ring of the rational normal curve of degree 2 and 3, $R_{A_2} = \mathbb{C}[s, st, st^2]$ and $R_{A_3} = \mathbb{C}[s, st, st^2, st^3]$, respectively, and we generalize it to the ring of the rational normal curve of degree n, $R_{A_n} = \mathbb{C}[s, st, st^2, \cdots, st^n]$. We include illustrative examples and visualizations in order to aid our computations and we provide an explicit formula for these computations. In particular, the pictures that we include explain how to compute differential operators of the form $D(I, I)_{\boldsymbol{d}} = \{\delta \in D(R) | \delta * J \subseteq J\}$.

1.1. **Background.** In order to contextualize the following sections, we need to recall and introduce some important mathematical definitions and structures. One of the most fundamental and important structures in mathematics is a ring.

Definition 1. *Let A a non-empty set. A binary operation $*$ on the set A is a function $* : A \times A \longrightarrow A$ such that for any two elements a and b in A, $(a, b) \in A \times A$, determines a third element $a * b$ in A.*

For instance, addition $+$, subtraction $-$, and multiplication $\times$ are some basic examples of binary operations on the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. We will usually write the multiplication of a and b by ab instead of $a \times b$. In particular, there are some important structures called rings that have two binary operations.

Definition 2. *A ring R is a non-empty set with two binary operations, addition and multiplication, such that for all a, b and c in R, we have:*

- *R is closed under addition: $a + b \in R$*
- *Addition is associative: $(a + b) + c = a + (b + c)$*
- *Addition is commutative: $a + b = b + a$*
- *R has an additive identity element, the zero and usually denoted by 0 such that $a + 0 = 0 + a = a$*
- *Every element in R has an additive inverse: for each a, there exists $b \in R$ such that $a + b = 0 = b + a$ and usually it is denoted as $b = -a$*
- *R is closed under multiplication: $ab \in R$*

- *Multiplication is associative:* $(ab)c = a(bc)$
- *Multiplication distributes over addition:* $a(b+c) = ab + ac$ *and* $(a+b)c = ac + bc$.

There are special types of rings that include some extra properties. Some well known examples of rings are: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{F}_2$.

If multiplication in R commutes, then the ring R is called *commutative ring.* Not all the rings are commutative; for example, a non-commutative ring is the set of all $n \times n$ matrices with entries in R, $M_{n\times n}(R)$, called matrix ring.

Definition 3. *A ring R with unity is a ring that has a multiplicative identity element, generally denoted by 1 or 1_R such that $1_R \times r = r \times 1_R$ for all $r \in R$.*

Throughout, we will be assuming that all rings have unity.

Definition 4. *Let R be a ring with identity $1 \neq 0$. An element u of R is called a unit if there is some w in R such that $uw = 1 = wu$.*

Fields are important examples of commutative rings.

Definition 5. *A field F is a commutative ring with unity in which $1 \neq 0$ and every nonzero element has a multiplicative inverse.*

An example of field is the set of rational numbers $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.

Definition 6. *Let R be a ring. A nonzero element r of R is called zero divisors if there is a nonzero element s in R such that either $rs = 0$ or $sr = 0$.*

A commutative ring with unity $1 \neq 0$ is called an *integral domain* if it has no zero divisors. Another specific case of rings are the polynomial rings.

Definition 7. *Given a ring R with unity and a indeterminate x, the ring $R[x]$ is the ring of all polynomials in x with coefficients in R. That is, it is the ring of all sums of the form $\sum_{k=0}^{m} a_i x^i$ where m is a non-negative integer.*

It can be generalized for indeterminate $x_1, x_2, \cdots, x_r$. For instance, $\mathbb{C}[x_1, \cdots, x_n]$ is the polynomial ring in indeterminates $x_1, \cdots, x_n$ with coefficients in $\mathbb{C}$.

A subring $S \subseteq R$ is an additive subgroup of R which is also closed under multiplication.

Definition 8. *Let R be a ring and let I be an additive subring, which is an additive subgroup, then I is an ideal if $rI \subseteq I$ and $Ir \subseteq I$ for all $r \in R$.*

Multiplying any element $r \in R$ by an element of the ideal I, s, produces another element of the ideal I. However, rs may not equal sr, multiplication does not have to be commutative.

An ideal can be generated by a single element or by many elements.

Definition 9. *An ideal generated by a single element is called a principal ideal and an ideal generated by a finite set is called a finitely generated ideal.*

For instance, the trivial ideal zero, 0, and the ideal R are both principal since the ideal 0 is generated by itself and R is generated by 1. There are some other properties that allow us to define and characterize another ideals.

Definition 10. *An ideal M in a ring R is called **maximal ideal** if $M \neq R$ and if J is an ideal of R such that $M \subset J \subset R$, then $J = M$ or $J = R$.*

Another important type of ideals are called prime ideals

Definition 11. *An ideal P in a ring R is said to be a prime ideal if whenever $xy \in P$, then either $x \in P$ or $y \in P$.*

In particular, $P \neq R$. A well known example is when $R = \mathbb{Z}$ and $P = p\mathbb{Z}$, where p is a prime. If $xy \in P$ then $xy = pq$ for some $q \in \mathbb{Z}$, then p divides xy and since p is prime then p divides x or p divides y, which implies that $x \in P$ or $y \in P$.

Proposition 1. [5, Proposition 12 Sec.7.4] *If R is a commutative ring with 1. An ideal M is maximal if and only if R/M is a field.*

Proposition 2. [5, Proposition 13 Sec.7.4] *P is a prime ideal if and only if R/P is an integral domain.*

From the previous statements, it easy to conclude that if R is a commutative ring with 1, every maximal ideal in R is a prime ideal.

Definition 12. *Let R be a commutative ring and suppose that $r \in R$ is nonzero and is not a unit. Then, r is called an irreducible element if whenever $r = ab$, $a, b \in R$, then a or b is a unit of R.*

Definition 13. *Let R be a commutative ring. A nonzero element $p \in R$ is called a prime element if (p) is a prime ideal.*

Definition 14. *Let R be an integral domain. We say that R is a unique factorization domain or UFD when the following two conditions are satisfied:*

- *$r \in R$, which is not zero and not a unit, can be written as a finite product of irreducibles: $r = p_1 \cdots p_n$.*
- *The decomposition is unique up to reordering and up to associates: if $r = p_1 \cdots p_n = q_1 \cdots q_m$ and all p_i and q_j are irreducibles. Then $n = m$ and there exist a permutation such that it sends every p_i to some q_j.*

For instance, $\mathbb{Z}[x]$ is a UFD.

The previous definitions and structures are basic algebraic concepts; however, these concepts are useful in all mathematics disciplines. The following definitions are some examples of how algebra and geometry are closely related.

Definition 15. *An ideal I in the ring $\mathbb{C}[x_1, \cdots, x_n]$ defines an affine variety, which is the set of common zeros of the complex polynomials on the complex n-space $\mathbb{C}^n$:*

$$V(I) := \{y \in \mathbb{C}^n | f(y) = 0 \textit{ for all } f \in I\},$$

and on the other hand, an affine variety $V \subseteq \mathbb{C}^n$ gives the ideal

$$I(V) := \{f \in \mathbb{C}[x_1, \cdots, x_n] | f(y) = 0 \textit{ for all } y \in V\}.$$

Some examples are: $V(x^2+y^2-1) \subseteq \mathbb{C}^2$ which is the unit circle centered at the origin, $V(y-x^2) \subset \mathbb{C}^2$, a parabola, and a the cone $V(z^2-x^2-y^2) \subseteq \mathbb{C}^3$ in $\mathbb{R}^3$.

Definition 16. *The coordinate ring of an affine variety* $V \subseteq \mathbb{C}[x_1, \cdots, x_n]$ *is the ring* $\mathbb{C}[V] \cong \mathbb{C}[x_1, \cdots, x_n]/I(V)$.

Definition 17. *Let* R *be an integral domain with field of fractions* K. *We say that* K *is integral over* R *if every element in* K *is a root of a monic polynomial in* $R[x]$.

Definition 18. *Let* R *be an integral domain with field of fractions* K. *Then* R *is normal if* K *is integral over* R.

Definition 19. *Let* R *be a ring and* S *be a commutative subring such that* $1_r \in S$. S *is called integrally closed in* R *if* S *is equal to its integral closure in* R.

Definition 20. *An affine variety* V *is normal if* $\mathbb{C}[V]$ *is integrally closed.*

For example, any UFD is normal. Sometimes geometric objects are defined algebraically, the next definition is an example of that.

Definition 21. *An irreducible affine variety* V *is normal if its coordinate ring* $\mathbb{C}[V]$ *is normal.*

For example, $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1, \cdots, x_n]$ is a UFD and hence normal.

1.2. **Projective Space and Rational Normal Curves.** In this section, we introduce the concept of projective space and rational normal curves since one of our goals in this project is to compute the differential operators for an affine semigroup ring R_{A_n} and radical monomial ideal J of R_{A_n}. A rational normal curve of degree n can be viewed as the image of of a map from the projective spaces or as the closure in $\mathbb{P}^n$ or the image of complex spaces.

Definition 22. *The n-dimensional projective space $\mathbb{P}^n$ is defined as the quotient space*

$$\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

where $\sim$ is a relation: two elements x and y in $\mathbb{C}^{n+1} \setminus \{0\}$ are equivalent if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. If x and y are equivalent, we write $x \sim y$.

Definition 23. *A projective variety $V \subseteq \mathbb{P}^n$ is defined by the vanishing of finitely many homogeneous polynomials in the polynomial ring $\mathbb{C}[x_0, \cdots, x_n]$. The homogeneous coordinate ring of V is the quotient ring*

$$\mathbb{C}[V] = \mathbb{C}[x_0, \cdots, x_n]/I(V)$$

where $I(V)$ is generated by all homogeneous polynomials that vanish on V.

[4] The ideal $I(V) \subseteq \mathbb{C}[x_0, \cdots, x_n]$ also defines an affine variety $\hat{V} \subseteq \mathbb{C}^{n+1}$ called the affine cone of V and the variety $\hat{V}$ satisfies:

$$V = \hat{V} \setminus \{0\}/\mathbb{C}^*$$

and its coordinate ring is the homogeneous coordinate ring of V:

$$\mathbb{C}[\hat{V}] = \mathbb{C}[V].$$

Therefore, the polynomial ring is the homogeneous coordinate ring of the projective space itself, and the variables are the homogeneous coordinates, for some basis.

For example, consider the ideal $I = \langle x_i x_{j+1} - x_{i+1} xj | 0 \leq i < j \leq n-1 \rangle \subseteq \mathbb{C}[x_0, \cdots, x_n]$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \end{pmatrix}$$

I defines a projective variety, C_n, since it is homogeneous and it is the image of the map $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$ which is defined below.

Definition 24. *A* ***rational normal curve*** *of degree n, C_n, in a projective space $\mathbb{P}^n$ is defined as the image of the map*

$$v_n : \mathbb{P}^1 \longrightarrow \mathbb{P}^n,$$

$$v_n : [s:t] \longrightarrow [s^n : s^{n-1}t : \cdots : st^{n-1} : t^n],$$

which is defined in homogeneous coordinates.

Remark: The image of v_n is the conic curve in $\mathbb{P}^n$ and we can see this rational normal curve of degree n as the closure in $\mathbb{P}^n$ of the image of ϕ_n:

$$\phi_n : (\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^{n+1}$$

$$\phi_n : (s,t) \longrightarrow (s, st, st^2, ..., st^n),$$

since the projective space $\mathbb{P}^n$ is the quotient space $\mathbb{C}^{n+1} \setminus \{0\}/\sim$ and there is a bijective relation between the images of v_n and ϕ_n given by: $s^n \mapsto s$, $s^{n-1}t \mapsto st$, $\cdots$ $t^n \mapsto st^n$. In general,

$$s^{n-i}t^i \mapsto st^i$$

for all $i \in \{0, 1, \cdots, n\}$. Therefore, we call $\mathbb{C}[s, st, st^2, \cdots, st^n]$ the ring of the rational normal curve of degree n since it is the coordinate ring of the cone of the projective rational normal curve and it is denoted by R_{A_n}. In particular, rational normal curves are examples of toric rings.

1.3. **Differential Operators.** Differential operators play an important role in different branches of mathematics; for instance, in differential equations and topology.

Rings of differential operators were studied by Musson [7], Smith and Stafford [10] and many others, and they obtained important results and although there are many equivalent definitions of these rings, in this section we will give some of the definitions and some of its characterizations that will be useful for our purposes.

In particular, we will be assuming that k is an algebraically closed field of characteristic zero, and R is a $k-$algebra.

In order to define differential operators and rings of differential operators, we need a preamble. Let R be a commutative $k-$algebra, and let M and N be $R-$modules. The linear space of all $k-$linear maps ϕ from M to N is denoted by $\mathrm{Hom}_k(M, N)$ and this space contains $\mathrm{Hom}_R(M, N)$ as a subset.

Let R be a commutative $k-$algebra, and M and N defined as above. For $\phi \in \text{Hom}_k(M,N)$ and $r \in R$, we let $[\phi, r]$ denote the element $\phi r - r\phi \in \text{Hom}_k(M,N)$, which is called the commutator of ϕ and r. A particular case occurs when $M = N$ since we have an endomorphism, which usually is denoted by $\text{End}_k(M)$, instead of $\text{Hom}_k(M,M)$, and the commutator is denoted by $[\phi_1, \phi_2] := \phi_1\phi_2 - \phi_2\phi_1$ for $\phi_1, \phi_2 \in \text{End}_k(M)$.

Let R be a commutative $k-$algebra and let M be an $R-$module. A $k-$derivation is an homomorphism $\delta \in \text{Hom}_k(R,M)$ that satisfies Leibniz's law: $\delta * (ab) = a(\delta * b) + (\delta * a)b$ for all $a, b \in R$.

Let R be a k-algebra and let M and N be $R-$modules. The differential operators from M to N of order n, $D^n_R(M,N)$, are defined using induction as follows:

(i) $D^0_R(M,N) = \text{Hom}_R(M,N)$

(ii) If r_M and r_N denote the multiplication by r in M and N respectively, then

$$D^n_R(M,N) = \{\delta \in \text{Hom}_R(M,N) | (\delta r_M - r_N\delta) \in D^{n-1}_R(M,N)\ \forall r \in R\}.$$

The ring of differential operators from M to N is given by

$$D_R(M,N) := \bigcup_{n \in \mathbb{N}} D^n_R(M,N).$$

The subscript R will be dropped when there is no likelihood of confusion.

In particular, the ring structure on D satisfies that $D^m_R D^n_R \subseteq D^{m+n}_R$ for all $m, n \in \mathbb{N} \cup \{0\}$, and $D_R(M,M) = D_R(M)$ [3].

Easily we can notice that for any M and N modules such that $\text{Hom}_R(M,N) = 0$ then $D^n_R(M,N) = 0$ by induction for all n and consequently $D_R(M,N) = 0$.

Similarly, let R be a k-algebra. The ring of the differential operators is defined as follows:

$$D_k(R) = \bigcup_{n\in\mathbb{N}} D_k^n(R),$$

where $D_k(R) = D_k(R, R)$.

[7, Definition 1.2] Let M be an $R-$module and I and J subsets of M then we define

$$\Delta_M(I, J) := \{\delta \in D_R(M) | \delta * I \subseteq J\},$$

where $*$ denotes the action of a differential operator on I. In addition, we will denote as $\Delta(I, J)$, and it is important to point out that $\Delta(I, J)$ is different from $D(I, J)$ when both can be defined. For instance, $D(I, 0) = 0$ for all ideal I, but $\Delta(I, 0)$ is not zero. However, if I is an ideal of R, then $D(R, I) = \Delta(R, I)$. We dropped the subscript above and we will drop it as well in $D_k(R)$, which will be represented by $D(R)$.

Definition 25. *Let J be any ideal in a ring R, the idealizer of J in the ring of the differential operators of R, $D(R)$, is the ring:*

$$\mathbb{I}(J)_{\boldsymbol{d}} = \{\delta \in D(R) | \delta * J \subseteq J\}.$$

The idealizer is the largest subring of R in which J is a two-sided ideal. One important fact that we point out is that $\mathbb{I}(J)_{\boldsymbol{d}}$ coincides with the definition of $D(J, J) := D(J)$ for any ideal of $D(R)$ since

$$D(J) = \{\delta \in D(R) | \delta * J \subseteq J\}.$$

1.4. **Differential operators in semigroup rings.** Let A be a $m \times n$ matrix with entries in $\mathbb{Z}$, and let $\mathbb{N}A$ denote the semigroup in $\mathbb{Z}^n$ generated by the columns of A,

$$R_A := \mathbb{C}[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} \mathbb{C} \cdot \mathbf{t}^{\mathbf{a}}$$

where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ and $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ which is in $\mathbb{N}A$. R_A is the ring of regular functions on the affine toric variety determined by the columns of A. For example, consider the matrix

$$A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & n-1 & n \end{bmatrix}$$

$R_{A_n} = \mathbb{C}[\mathbb{N}A_n] = \mathbb{C}[s, st, st^2, \cdots, st^n]$ which is the ring of the rational normal curve of degree n.

Now, consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

the coordinate ring is $R := R_A = \mathbb{C}[st, t^2, s^2, s^3]$, which is determined by the columns of A: the values in the columns give us the powers i and j in $s^i t^j$. The semigroup $\mathbb{N}A$ is illustrated in Figure 1.

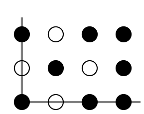


Figure 1. Semigroup $\mathbb{N}A$

In Figure 1, we have some lattice points without color in the coordinates $(0,1)$, $(1,0)$, $(1,2)$, and $(2,1)$ that correspond to the exponents of the multidegrees

t, s, st^2 and s^2t, respectively. This is because it is not possible generate those elements in R.

Consider the matrix

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

whose associated coordinate ring is $R = R_{A_2} = \mathbb{C}[s, st, st^2]$. Figure 2 illustrates the semigroup $\mathbb{N}A_2$. It is important to point out that in Figure 2 we are not representing all the elements in the semigroup $\mathbb{N}A_2$, just some of them in order to provide a visualization.

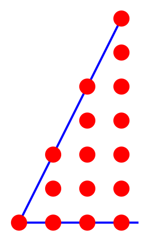


FIGURE 2. The semigroup $\mathbb{N}A_2$

In this case, $\mathbb{N}A_2$ is a cone bounded by $h_1 = \theta_2 = 0$ and $h_2 = 2\theta_1 - \theta_2 = 0$ that are called **primitive integral support functions**.

Definition 26. *Consider a facet σ of A. The primitive integral support function of the facet σ of the cone $\mathbb{R}_{\geq 0}A$, h_σ, is a uniquely determined linear form on $\mathbb{R}^n$ satisfying:*

(1) $h_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$

(2) $h_\sigma(\sigma) = 0$

(3) $h_\sigma(\mathbb{Z}^n) = \mathbb{Z}$.

Next example illustrates the facets and the support functions.

[9, Example 2.2] Consider the matrix

$$A := (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The facets of the cone $\mathbb{R}A$ are: σ_{23}, σ_{24}, σ_{13} and σ_{14} where $\sigma_{mn} = \mathbb{R}_{\geq 0}a_m + \mathbb{R}_{\geq 0}a_n$. Then, the respective primitive integral support functions are:

$$h_{\sigma_{23}}(\theta) = \theta_1, \qquad h_{\sigma_{24}}(\theta) = \theta_1 + \theta_3, \qquad h_{\sigma_{13}}(\theta) = \theta_2, \qquad h_{\sigma_{14}}(\theta) = \theta_2 + \theta_3$$

or simply $h_{23}(\theta)$, $h_{24}(\theta)$, $h_{13}(\theta)$, $h_{14}(\theta)$.

And now back to the ring of the rational normal curve of degree n; $R_{A_n} := \mathbb{C}[\mathbb{N}A_n] = \mathbb{C}[s, st, st^2, \cdots, st^n]$ which is associated to the matrix

$A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & n-1 & n \end{bmatrix}.$

The coordinate ring of the affine cone of the projective rational normal curve has two facets that are:

$$\sigma_1 = \mathbb{N}\mathbf{a}_0 = \{(x, y) \in \mathbb{N}^2 | x \geq 0,\ y = 0\},$$

$$\sigma_2 = \mathbb{N}\mathbf{a}_n = \{(x, y) \in \mathbb{N}^2 | x \geq 0,\ y = nx\},$$

and its respective primitive integral support functions are

$$h_1 = \theta_2 \qquad \textit{and} \qquad h_2 = n\theta_1 - \theta_2.$$

Now, we can describe the factorial function using the h functions,

$$\prod_{i=0}^{n}(h - i) = h(h-1)(h-2)\cdots(h-n) =: (h, n)!$$

Then for $j = 1, 2$ and any multidegree $\boldsymbol{d}$ we have

$$\prod_{i=0}^{h_j(-\boldsymbol{d})-1} (h_j - i) = (h_j, 0)(h_j, 1) \cdots (h_j, h_j(-\boldsymbol{d})-2)(h_j, h_j(-\boldsymbol{d})-1) = (h_j, h_j(-\boldsymbol{d})-1)!$$

such that if $n < 0$ then $(h, n)! = 1$.

In addition, we define

$$(h, 0)! = h.$$

One important convention that we are going to follow throughout is if $m_2 \leq m_1$ and $h_i - m_2$ is a factor of $(h_i, m_1)!$; we do not need to multiply it by $h_i - m_2$. We only need to multiply by equations of the lines if they were not already present in the descending factorial which expresses the operator of that multidegree.

Theorem 1. [2, Theorem 2.3] *and* [8, Theorem 3.2.2] *If R_A is a pointed, normal affine semigroup ring with $\mathbb{Z}A = \mathbb{Z}^n$, then*

$$D(R_A) = \bigoplus_{\boldsymbol{d} \in \mathbb{Z}^d} \boldsymbol{t}^{\boldsymbol{d}} \cdot \mathbb{I}(\Omega(\boldsymbol{d})) = \bigoplus_{\boldsymbol{d} \in \mathbb{Z}^d} \boldsymbol{t}^{\boldsymbol{d}} \cdot \langle (h_j, h_j(\text{-}\boldsymbol{d}) - 1)! \rangle.$$

where $\mathbf{d} = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$ and the outer product runs over those $j = 1, 2, \cdots, n$ with $h_j(d) < 0$.

Throughout, we will use $\langle - \rangle$ to indicate the ideals in the ring R_{A_n} or in the polynomial ring $\mathbb{C}[\theta] := \mathbb{C}[\theta_1, \cdots, \theta_n]$.

For our purposes, we will consider $\boldsymbol{d} = (d_1, d_2) \in \mathbb{Z}^2$, which is generated by an element in the form of $s^{d_1} t^{d_2} \cdot p(\theta_1, \theta_2)$ and recall that $p(\theta_1, \theta_2) = \prod_{j=1}^{2} (h_j, h_j(\text{-}\mathbf{d}) - 1)!$, which is a polynomial polynomial and h_1 and h_2 are the primitive integral support functions.

2. **Differential operators and idealizers in** R_{A_2}

In this section, we compute the idealizer $\mathbb{I}(J)_{\boldsymbol{d}}$ for some ideals J over the ring of the rational normal curve of degree 2, $R_{A_2} = \mathbb{C}[s, st, st^2]$. In order to aid our computations, we include some illustrations of the lattice representing the multidegrees in the plane, which is divided into four chambers; C1, C2, C3 and C4. The facets of A_2 are given by $\sigma_1 = \{(x, y) \in \mathbb{N}^2 | x \geq 0, y = 0\}$ and $\sigma_2 = \{(x, y) \in \mathbb{N}^2 | x, y \geq 0, y = 2x\}$ that have primitive integral support functions $h_1 = \theta_2$ and $h_2 = 2\theta_1 - \theta_2$. Figure 3 illustrates the integer lattice, divided into

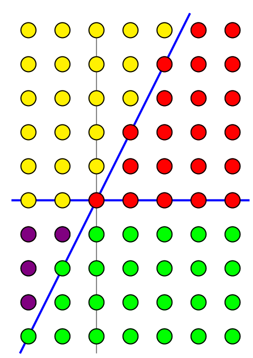


FIGURE 3. Chambers of the differential operators with the extra differentiators

four chambers that are colored as follows:

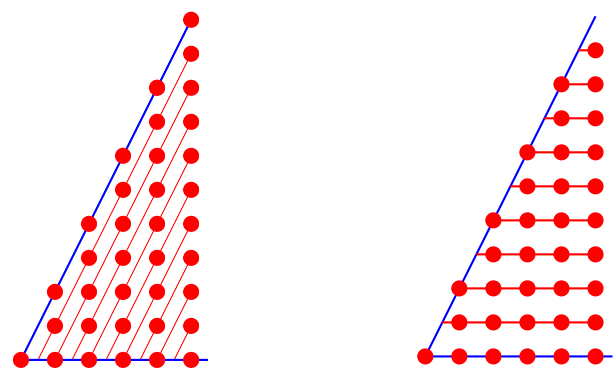
C1: The red multidegrees correspond to monomials in R_{A_2}

C2: The yellow multidegrees are the $\boldsymbol{d}$ such that $h_1(\boldsymbol{d}) \geq 0$ and $h_2(\boldsymbol{d}) < 0$

C3: The violet multidegrees are the $\boldsymbol{d}$ such that $h_1(\boldsymbol{d}) < 0$ and $h_2(\boldsymbol{d}) < 0$, and

C4: The green multidegrees are the $\boldsymbol{d}$ such that $h_1(\boldsymbol{d}) < 0$ and $h_2(\boldsymbol{d}) \geq 0$.

On the other hand, the lines parallel to the facets σ_1 and σ_2 will be a guide in order to compute $\mathbb{I}(J)_{\boldsymbol{d}}$ since we can identify the lines parallel that contain the monomials in J and the monomials in $R_{A_2} \setminus J$. See Figure 4



(A) Lines parallel to σ_2 (B) Lines parallel to σ_1

FIGURE 4. Lines parallel to the facets

Before presenting a general formula for the idealizer of a principal monomial ideal in R_{A_2}, we will exhibit how to find the idealizer for some specific examples of principal ideals J in the subsections below.

2.1. **The idealizer for $J = \langle s^2 \rangle$ over the ring R_{A_2}.** In this example we will compute the idealizer, $\mathbb{I}(J)_{\boldsymbol{d}}$, for $J = \langle s^2 \rangle$ and some different values of $\boldsymbol{d}$ in the ring of the rational normal curve of degree 2. The monomials in J are the product of multiplying s^2 by the elements of R_{A_2} and by any monomials in R_{A_2} that correspond to the lattice points shaded in red in Figure 5. It is important to point out that in Figure 5 the monomials in red are not all the monomials in J.

We have infinitely many monomials that keep going to the right forever; however, we just included some of them in order to give a visualization.

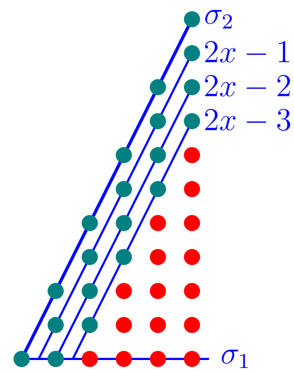


FIGURE 5. Elements in $R_{A_2} \setminus J$ lie on $y = 2x - i$ for $i = 0, 1, 2, 3$

In Figure 5, the red multidegrees correspond to monomials in J. The teal multidegrees are all the multidegrees outside the ideal but in the ring. They are in the region: $R_{A_2} \setminus J$ and they lie on the half-lines $y = 2x$, $y = 2x - 1$, $y = 2x - 2$ and $y = 2x - 3$ that can be observed in Figure 5. C2, C3 and C4 are defined as in subsection 2.1.

[2, Theorem 2.3] gives us the graded pieces of $D(R_{A_2})$ that are given by

$$D(R_{A_2})_{\boldsymbol{d}} = s^{d_1} t^{d_2} \cdot \langle (h_1, h_1(-\boldsymbol{d}) - 1)!(h_2, h_2(-\boldsymbol{d}) - 1)! \rangle \tag{1}$$

where $\boldsymbol{d} = (d_1, d_2)$. Thus, if we break down (1) by chambers, we obtain:

$$D(R_{A_2})_{\boldsymbol{d}} = \begin{cases} s^{d_1} t^{d_2} \cdot \mathbb{C}[\theta_1, \theta_2] & \text{if } \boldsymbol{d} \in C1 \\ s^{d_1} t^{d_2} \cdot \langle (h_2, -2d_1 + d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C2 \\ s^{d_1} t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_2, -2d_1 + d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C3 \\ s^{d_1} t^{d_2} \cdot \langle (h_1, -d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C4 \end{cases}$$

where the primitive integral support functions at $\boldsymbol{d}$ are: $h_1(\boldsymbol{d}) = d_2$ and $h_2(\boldsymbol{d}) = 2d_1 - d_2$. Now, in order to compute the idealizer $\mathbb{I}(J)_{\boldsymbol{d}} = \{\delta \in D(R_{A_2}) | \delta * J \subseteq J\}$, we can consider graded pieces of $D(R_{A_2})$ at the different chambers; C1, C2, C3 and C4. In these first computations, we will consider $\boldsymbol{d} = (-1, 0)$, which is in C2, $\boldsymbol{d} = (-1, -1)$ in C3 and $\boldsymbol{d} = (-1, -2)$ in C4 as in [2]. However, in order to see and describe the behavior of $\mathbb{I}(J)_{\boldsymbol{d}}$, in the following subsection we will included some tables that contain various values of $\boldsymbol{d}$ in the different chambers.

Consider the graded piece of $D(R_{A_2})$ at $(-1, 0)$ in C2. Then $(-1, 0) : s^{-1}t^0 \cdot \langle (h_2, -(-2) + 0 - 1)! \rangle = s^{-1} \cdot \langle (h_2, 1)! \rangle$ and applying $s^{-1} \cdot (h_2, 1)!$ to a monomial whose exponent lies on the half-lines $y = 2x$ and $y = 2x - 1$ will yield 0 which is in J; however, the half-lines $y = 2x$ and $y = 2x - 1$ contain exponent of monomials that are not contained in the ideal generated by s^2.

When we apply $s^{-1} \cdot (h_2, 1)!$ to a monomial whose exponent lies on the half-lines $y = 2x - 2$ and $y = 2x - 3$ we obtain an integer multiple of a monomial whose exponent lies in the facet σ_2 or in the half-line $y = 2x - 1$, respectively, and those lines are not in the ideal J which makes clear that we do not need to consider the monomials on the lines outside the ideal. Nevertheless, when $s^{-1} \cdot \langle (h_2, 1)! \rangle$ acts on a monomial whose exponent is a member of the half-lines $y = 2x - 4$ or $y = 2x - 5$ lying inside $\mathbb{N}A_2$, we obtain an integer multiple of a monomial whose exponent lies in $y = 2x - 2$ or $y = 2x - 3$, respectively, and these are not elements of J.

In order to correct it, we should multiply the terms in $D(R_{A_2})_{(-1,0)}$ by $(h_2 - 4)(h_2 - 5)$ and applying $s^{-1} \cdot \langle (h_2, 1)!(h_2 - 4)(h_2 - 5) \rangle$ to these monomials yields 0.

In Figure 6, the light blue lines indicate the half-lines representing the multidegrees of monomials in R_{A_2} that after application of an element in $D(R_{A_2})_{(-1,0)}$, fails to yield an element in J. The blue lines are representing the multidegrees of monomials in the ring, but outside J, $R_{A_2} \setminus J$, that after the application of an element in $D(R_{A_2})_{(-1,0)}$ fails to yield an element in J, but it does not affect our computations. Thus,

$$D(J)_{(-1,0)} = s^{-1} \cdot \langle (h_1, 1)!(h_2 - 4)(h_2 - 5) \rangle$$

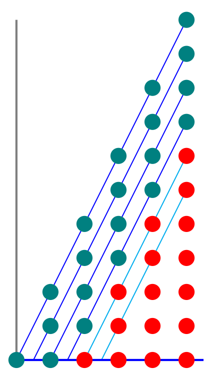


FIGURE 6. Vanishing for $\boldsymbol{d} = (-1, 0)$ in $\langle s^2 \rangle$

Now, consider the graded piece of $D(R_{A_2})$ at $(-1, -2)$ in C4. Then $(-1, -2)$: $s^{-1}t^{-2} \cdot \langle (h_1, -(-2) - 1)! \rangle = s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$. After applying $s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$ to a monomial whose exponent lies on the half-lines σ_1 and $y = 1$ will yield 0, which is inside J, then these lines do not need our attention or any correction, but we will identify those lines with the red color. Then, the red line indicates that their respective monomials are in the ideal J and they go to 0 after the application of $s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$.

When we apply $s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$ to any monomial corresponding to a multidegree $\boldsymbol{d}$ in J along $y = n$, for $n = 2, 3, \cdots$, we obtain monomials on $y = n - 2$, respectively. Thus, after this application; monomials inside the ideal J are still elements in J. Therefore, no correction is needed.

In Figure 7, the blue lines correspond to the half-lines: σ_2, $y = 2-1$, $y = 2x-2$, and $y = 2x - 3$ that contain the exponents of monomials that are not in J, but no change or correction is needed. Thus, we would not modify our formula.

$$D(J)_{(-1,-2)} = s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$$

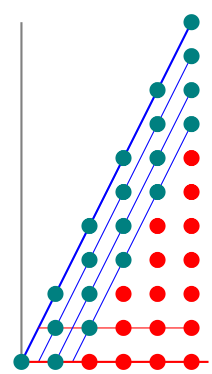


FIGURE 7. Vanishing for $\boldsymbol{d} = (-1, -2)$ in $\langle s^2 \rangle$

Similarly, consider the graded piece of $D(R_{A_2})$ at $(-1, -1)$ in C3. $(-1, -1)$: $s^{-1}t^{-1} \cdot \langle (h_1, 1-1)!(h_2, 2-1-1)! := s^{-1}t^{-1}h_1h_2$. If the operator $s^{-1}t^{-1}h_1h_2$, is applied to any of the monomials corresponding to multidegree $\boldsymbol{d} \in \mathbb{N}A_2$ along the light blue half-lines in Figure 8: $y = 2x - 4$, we obtain an integer multiple of a monomial with exponent in $y = 2x - 3$, which is not in J, and no problems are created for the remaining monomials in J.

In Figure 8, the blue lines are in R_{A_2}, but not in the ideal J and the application of $s^{-1}t^{-1}h_1h_2$ does not affect the monomials corresponding to multidegree in those lines. In addition, the application of $s^{-1}t^{-1}h_1h_2$ to the monomials with exponent on the facet σ_1, which is in the ideal, yield 0 and it is in J and in Figure 8 it is represented by a red line. However, multiplying by $(h_2 - 4)$ yields a new operator $s^{-1}t^{-1} \cdot \langle h_1h_2(h_2-4)\rangle$ that will send to 0 all monomials with multidegrees $\boldsymbol{d} \in \mathbb{N}A_2$ along the half-line $y = 2x-4$. No problems are created for the remaining monomials in R_{A_2} and we obtain:

$$D(J)_{(-1,-1)} = s^{-1}t^{-1} \cdot \langle h_1h_2(h_2 - 4)\rangle$$

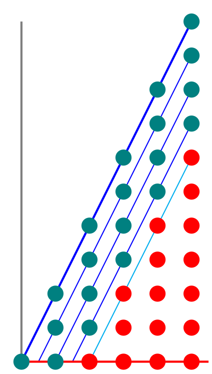


Figure 8. Vanishing for $\boldsymbol{d} = (-1, -1)$ in $\langle s^2 \rangle$

Thus, we obtained

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2,1)!(h_2-4)(h_2-5)\rangle & \text{if } \boldsymbol{d} = (-1,0) \\ s^{d_1}t^{d_2} \cdot \langle h_1h_2(h_2-4)\rangle & \text{if } \boldsymbol{d} = (-1,-1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1,1)!\rangle & \text{if } \boldsymbol{d} = (-1,-2) \end{cases}$$

This expression is incomplete. We have many more elements in each chamber, but we are presenting this table just for the multidegrees that we computed here; however, we have $s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$ for any $\boldsymbol{d} \in$ C1.

Definition 27. *For a given multidegree $\boldsymbol{d}$, we say that the differential operators of multidegree $\boldsymbol{d}$ satisfy a **factorial-continuous formula**, if for each support function h_j, there exists an integer $n_{j,\boldsymbol{d}}$ such that $\mathbb{I}(J)_{\boldsymbol{d}} = s^{d_1}t^{d_2} \cdot \langle \prod_{j=1}^{2} (h_j, n_{j,\boldsymbol{d}})!\rangle$.*

Throughout, the light blue lines indicate the half-lines representing the multidegrees of monomials in J that after application of an element in $D(R_{A_2})_{(d_1,d_2)}$, fails to yield an element in J and we have to correct this lack of membership. The blue lines represent the multidegrees of monomials in the ring, but outside J, $R_{A_2} \setminus J$, and after the application of an element in $D(R_{A_2})_{(d_1,d_2)}$ fail to yield an element in J, but no changes or corrections are needed. Finally, the red lines indicate the multidegrees of monomials in the ideal that after application of an element in $D(R_{A_2})_{(d_1,d_2)}$ yield 0 and as 0 is in any ideal then no changes are required.

2.2. **The idealizer for $J = \langle s^2t\rangle$ over the ring R_{A_2}.** In this example, we will compute the idealizer for the ideal $J = \langle s^2t\rangle$, $\mathbb{I}(J)_{\boldsymbol{d}}$, over the ring of the rational normal curve of degree 2, $R_{A_2} = \mathbb{C}[s, st, st^2]$. In Figure 9, the red multidegrees correspond to monomials in J. The teal multidegrees are all the multidegrees

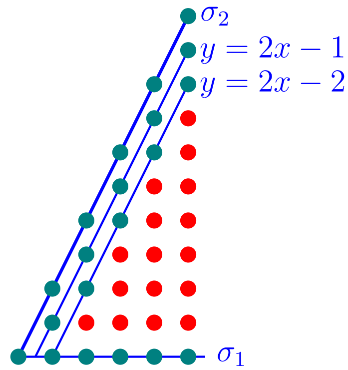


FIGURE 9. Elements in $R_{A_2} \setminus J$ lie on σ_2, $y = 2x - 1$, $y = -2x - 2$ and σ_1

outside the ideal but in the ring. They are in the region: $R_{A_2} \setminus J$ and they lie on the half-lines σ_2, $y = 2x - 1$, $y = 2x - 2$ and σ_1.

C2, C3 and C4 are defined as above.

Again, consider the graded piece of $D(R_{A_2})$ at $(-1, 0)$ in C2. In Figure 10, the monomials whose exponents are member of the light blue half-lines need a correction since after application of an element in $D(R_{A_2})_{(-1,0)}$, fails to yield an element in J they go to $y = 2x - 1$ or $y = 2x - 2$, respectively, but those lines have monomials that are not in J. In order to correct this lack of membership in J for the monomials on $y = 2x - 3$ and $y = 2x - 4$, every element of $D(R_{A_2})_{(-1,0)}$ should be multiplied by $(h_2 - 3)(h_2 - 4)$. Then, applying $s^{-1} \cdot \langle (h_2, 1)!(h_2 - 3)(h_2 - 4) \rangle$ to the monomials in the half-lines $y = 2x - 4$, and $y = 2x - 3$ yield 0. The application of $s^{-1} \cdot (h_2, 1)!(h_2 - 3)(h_2 - 4)$ to the remaining monomials in J will output a term inside J.

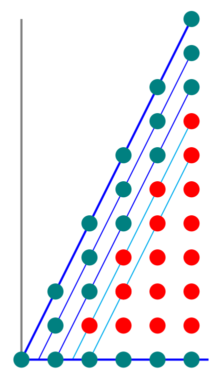


FIGURE 10. Vanishing for $\boldsymbol{d} = (-1, 0)$ in $\langle s^2 t \rangle$

Thus,

$$D(J)_{(-1,0)} = s^{-1} \cdot \langle (h_2, 1)!(h_2 - 3)(h_2 - 4) \rangle.$$

Consider the graded piece of $D(R_{A_2})$ at $(-1, -2)$ in C4. The monomials corresponding to the multidegrees that lie on the light blue line $y = 2$ in Figure 11 are the exponents of monomials that fail to land inside J after the application of $s^{-1}t^{-2}\langle (h_1, 1)! \rangle$. In order to correct this, right-multiplying by $(h_1 - 2)$ yields $s^{-1}t^{-2} \cdot \langle (h_1, 2)! \rangle$ that will send to 0 all monomials with multidegrees $\boldsymbol{d} \in \mathbb{N}A_2$ along the half-line $y = 2$, and no problems are created for the remaining monomials in J then

$$D(J)_{(-1,-2)} = s^{-1}t^{-2} \cdot \langle (h_1, 2)! \rangle$$

Similarly, we consider the graded piece of $D(R_{A_2})$ at $(-1, -1)$ in C3: $s^{-1}t^{-1}h_1h_2$. If the operator $s^{-1}t^{-1}h_1h_2$, is applied to any of the monomials corresponding to multidegree $\boldsymbol{d} \in C1$ along the light blue half-lines in Figure 12, $y = 2x - 3$ and $y = 1$, we obtain an integer multiple of a monomial with exponent in $y = 2x - 2$ or

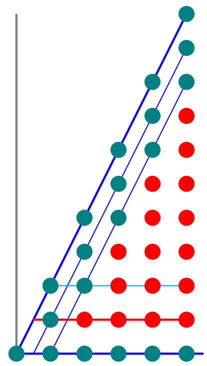


FIGURE 11. Vanishing for $\boldsymbol{d} = (-1,-2)$ in $\langle s^2t \rangle$

σ_1 respectively, which are not in J, and no problems are created for the remaining monomials in J. However, multiplying by $(h_1-1)(h_2-3)$ yields a new operator $s^{-1}t^{-1} \cdot \langle (h_1-1)!h_2(h_2-3) \rangle$ that will send to 0 all monomials with multidegrees $\boldsymbol{d} \in \mathbb{N}A_2$ along the half-lines $y=1$ and $y=2x-3$. Thus,

$$D(J)_{(-1,-1)} = s^{-1}t^{-1} \cdot \langle (h_1,1)!h_2(h_2-3) \rangle$$

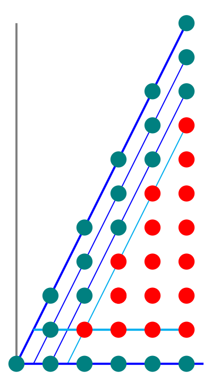


FIGURE 12. Vanishing for $\boldsymbol{d} = (-1,-1)$ in $\langle s^2t \rangle$

Therefore, we obtain:

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2}\cdot\langle (h_2,1)!(h_2-3)(h_2-4)\rangle & \text{if } \boldsymbol{d}=(-1,0)\\ s^{d_1}t^{d_2}\cdot\langle (h_1,1)!h_2(h_2-3)\rangle & \text{if } \boldsymbol{d}=(-1,-1)\\ s^{d_1}t^{d_2}\cdot\langle (h_1-2)!\rangle & \text{if } \boldsymbol{d}=(-1,-2)\end{cases}$$

This expression is incomplete since we have many more elements in each chamber; however, it summarizes the results that we determined in this subsection. In addition, we have $s^{d_1}t^{d_2}\cdot\mathbb{C}[\theta]$ for any $\boldsymbol{d}\in$ C1. For example, see Figure 27 which exhibits the multidegrees $\boldsymbol{d}$ where the idealizer does not have factorial-continuous behavior.

2.3. **The idealizer for** $J=\langle s^2t^2\rangle$ **over the ring** R_{A_2}**.** In this example we compute the idealizer for $J=\langle s^2t^2\rangle$, $\mathbb{I}(J)_{\boldsymbol{d}}$, Considering the same graded pieces

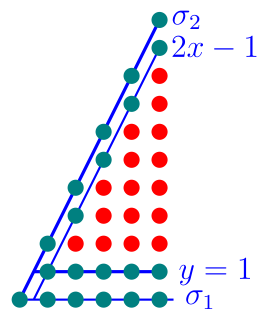
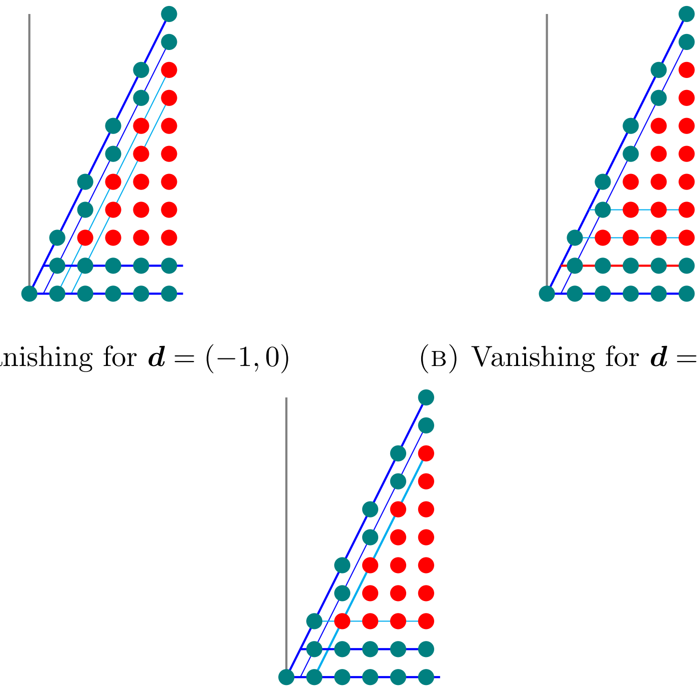


FIGURE 13. Elements in $R_{A_2}\setminus J$ lie on σ_2, $y=2x-1$, σ_1 and $y=1$

of $D(R_{A_2})$ at the different chambers; $(-1,0)$ in C2, $(-1,-1)$ in C3 and $(-1,-2)$ in C4, we obtain the following figures:

For $d=(-1,0)$ we obtain $s^{-1}\cdot\langle (h_2,1)!\rangle$ and when $s^{-1}\cdot\langle (h_2,1)!\rangle$ acts on a monomial whose exponent is a member of the half-line $y=2x-2$ or $y=2x-3$ we obtain an integer multiple of a monomial whose exponent lies in σ_2 or $y=2x-1$ respectively, which means that it fails to land inside J. Then every element of



(A) Vanishing for $\boldsymbol{d} = (-1, 0)$ (B) Vanishing for $\boldsymbol{d} = (-1, -2)$

(C) Vanishing for $\boldsymbol{d} = (-1, -1)$

FIGURE 14. $\boldsymbol{d}$ in the different chambers for $\langle s^2t^2 \rangle$

$D(R_{A_2})_{(-1,0)}$ should be multiplied by $(h_2-2)(h_2-3)$. Thus, applying $s^{-1}\cdot\langle(h_2, 3)!\rangle$ these monomials now yields 0.

For $d = (-1, -2)$ we obtain $s^{-1}t^{-2} \cdot \langle(h_1, 1)!\rangle$ and when $s^{-1}t^{-2} \cdot \langle(h_1, 1)!\rangle$ acts on a monomial whose exponent is a member of the half-lines $y = 2$ and $y = 3$ we obtain an integer multiple of a monomial whose exponent lies in σ_1 and $y = 1$ respectively. To correct this, we should multiply the elements in $D(R_{A_2})_{(-1,-2)}$ by $(h_1 - 2)(h_1 - 3)$. Then applying $s^{-1}t^{-2} \cdot \langle(h_1, 3)!\rangle$ to a monomial corresponding to $\boldsymbol{d} \in \mathbb{N}A_2$ along $y = 2$ or $y = 3$ yields 0.

For $d = (-1,-1)$, when $s^{-1}t^{-1} \cdot h_1h_2$ acts on a monomial whose exponent is a member of the half-lines $y = 2x-2$ or $y = 2$ we obtain an integer multiple of a monomial whose exponent lies in $y = 2x-1$ or $y = 1$, respectively, which are not in J. Then, we should multiply by $(h_1-2)(h_2-2)$ and applying $s^{-1}t^{-1} \cdot \langle h_1(h_1-2)h_2(h_2-2)\rangle$ to any element of $D(R_{A_2})_{(-1,-1)}$ yields 0.

Then, we obtain

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2,3)!\rangle & \text{if } \boldsymbol{d} = (-1,0) \\ s^{d_1}t^{d_2} \cdot \langle h_1(h_1-2)h_2(h_2-2)\rangle & \text{if } \boldsymbol{d} = (-1,-1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1,3)!\rangle & \text{if } \boldsymbol{d} = (-1,-2) \end{cases}$$

This expression is not complete. We have many more elements in each chamber, but we are presenting this table just for the multidegrees that we computed here; In addition, we have $s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$ for any $\boldsymbol{d} \in$ C1.

2.4. **The idealizer for $J = \langle s^2t^3\rangle$ over the ring R_{A_2}.** In this example we compute $\mathbb{I}(J)_{\boldsymbol{d}}$ for the ideal $J = \langle s^2t^3\rangle$. Thus, in an analogous way than in the

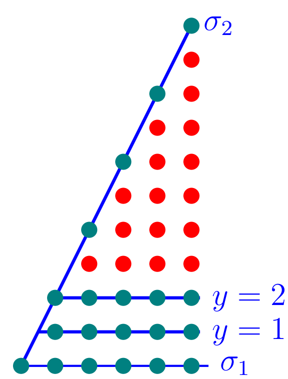
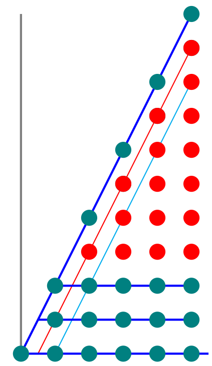
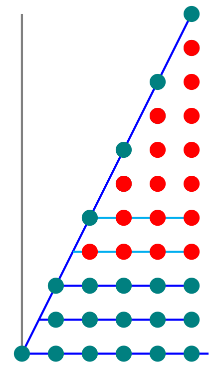


FIGURE 15. Elements in $R_{A_2} \setminus J$

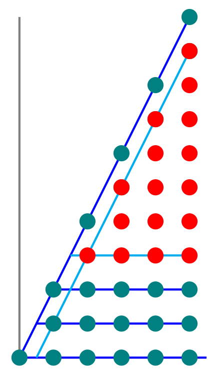
previous subsections, we obtain the following figures



(A) Vanishing for $\boldsymbol{d} = (-1,0)$ in $\langle s^2t^3\rangle$



(B) Vanishing for $\boldsymbol{d} = (-1,-2)$ in $\langle s^2t^3\rangle$



(C) Vanishing for $\boldsymbol{d} = (-1,-1)$ in $\langle s^2t^3\rangle$

FIGURE 16. $\boldsymbol{d}$ in the different chambers in $\langle s^2t^3\rangle$

Applying $s^{-1}\cdot\langle(h_2,1)!\rangle$ to the monomials in the half-line $y = 2x-1$ and $y = 2x-2$ we obtain 0 or an integer multiple of a monomial whose exponent lies in σ_2, respectively, but the monomials corresponding to the multidegrees which lie on σ_2 are not in J. Then we should multiply the terms in $D(R_{A_2})_{(-1,0)}$ by (h_2-2) and applying $s^{-1}\cdot\langle(h_2,2)!\rangle$ yields 0.

Applying $s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$ to the monomials in the half-line $y = 3$ and $y = 4$ we obtain an integer multiple of a monomial whose exponent lies in $y = 1$ or $y = 2$ respectively, which are not in J. Then we should multiply the terms in $D(R_{A_2})_{(-1,-2)}$ by $(h_1 - 3)(h_1 - 4)$ and applying $s^{-1} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4) \rangle$ yields 0.

Similarly, applying $s^{-1}t^{-1} \cdot h_1 h_2$ to the monomials in the half-line $y = 2x - 1$ and $y = 3$ we obtain an integer multiple of a monomial whose exponent lies in σ_2 or $y = 2$, respectively, which are not in J. Then we should multiply the terms in $D(R_{A_2})_{(-1,-1)}$ by $(h_1 - 3)(h_2 - 1)$ and applying $s^{-1}t^{-1} \cdot \langle h_1(h_1 - 3)(h_2, 1)! \rangle$ yields 0. Thus, we obtain

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, 2)! \rangle & \text{if } \boldsymbol{d} = (-1, 0) \\ s^{d_1}t^{d_2} \cdot \langle h_1(h_1 - 3)(h_2, 1)! \rangle & \text{if } \boldsymbol{d} = (-1, -1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4) \rangle & \text{if } \boldsymbol{d} = (-1, -2) \end{cases}$$

This expression is incomplete. We have many more elements in each chamber; however, it summarizes the results that we determined in this subsection. In addition, we have $s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$ for any $\boldsymbol{d} \in$ C1.

2.5. **The idealizer for $J = \langle s^3 \rangle$ over the ring R_{A_2}.** In this example we want to compute $\mathbb{I}(J)_{\boldsymbol{d}}$ for $J = \langle s^3 \rangle$.

Then, we obtain the following figures

Therefore, we obtain

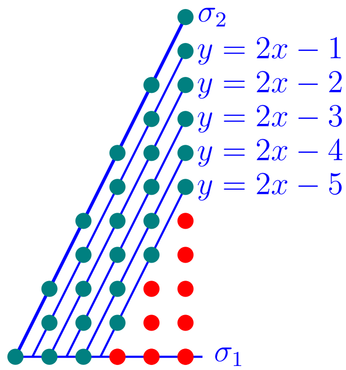
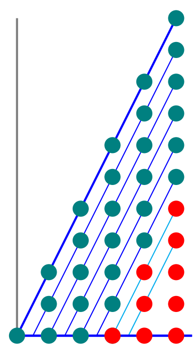
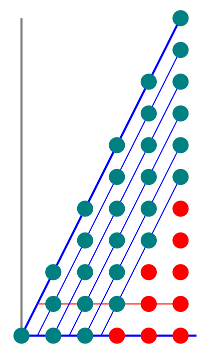


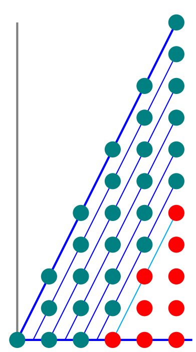
FIGURE 17. Elements in $R_{A_2} \setminus J$



(A) Vanishing for $\boldsymbol{d} = (-1, 0)$



(B) Vanishing for $\boldsymbol{d} = (-1, -2)$



(C) Vanishing for $\boldsymbol{d} = (-1, -1)$

FIGURE 18. $\boldsymbol{d}$ in the different chambers for $\langle s^3 \rangle$

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, 1)!(h_2-6)(h_2-7) \rangle & \text{if } \boldsymbol{d} = (-1, 0) \\ s^{d_1}t^{d_2} \cdot \langle h_1 h_2 (h_2-6) \rangle & \text{if } \boldsymbol{d} = (-1, -1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 1)! \rangle & \text{if } \boldsymbol{d} = (-1, -2) \end{cases}$$

This expression is incomplete. It only includes the multidegrees that we computed for this subsection. Notice that these figures are similar to the case when $J = \langle s^2 \rangle$. We have omitted computations since they are similar to to the case $J = \langle s^2 \rangle$; however, it is important to mention that for $\langle s^3 \rangle$ the multiples of the lines $h_2 - i$ are shifted to the right by 2.

2.6. **The idealizer for $J = \langle s^3 t^3 \rangle$ over the ring R_{A_2}.** In this example we show some figures that help to compute $\mathbb{I}(J)_{\boldsymbol{d}}$ for $J = \langle s^3 t^3 \rangle$.

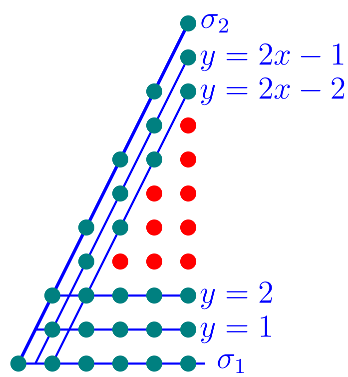
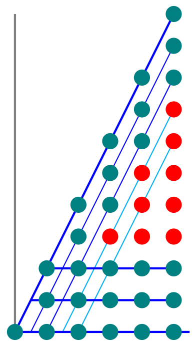


FIGURE 19. Elements in $R_{A_2} \setminus J$

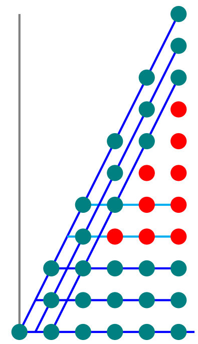
We have that

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, 1)!(h_2-3)(h_2-4) \rangle & \text{if } \boldsymbol{d} = (-1, 0) \\ s^{d_1}t^{d_2} \cdot \langle h_1(h_1-3) h_2 (h_2-3) \rangle & \text{if } \boldsymbol{d} = (-1, -1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 1)!(h_1-3)(h_1-4) \rangle & \text{if } \boldsymbol{d} = (-1, -2) \end{cases}$$

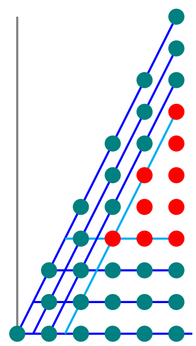
This expression is incomplete. It only includes the multidegrees that we computed for this subsection. We did not include computations for this subsection. However,



(A) Vanishing for $\boldsymbol{d} = (-1, 0)$



(B) Vanishing for $\boldsymbol{d} = (-1, -2)$



(C) Vanishing for $\boldsymbol{d} = (-1, -1)$

FIGURE 20. $\boldsymbol{d}$ in the different chambers for $\langle s^3t^3 \rangle$

we can compare it to the case when the ideal is generated by s^2t^2. For $\langle s^3t^3 \rangle$ the multiples of the lines $h_2 - i$ are shifted to the right by 1 and the multiples of the lines $h_1 - j$ are shifted up by 1. The next subsection shows some examples that help to determine a more general formula which will come in subsection 2.8.

2.7. **Toward a formula for $\mathbb{I}(J)$.** In this subsection, we include tables that show some ideals for different values $\boldsymbol{d}$. Those tables help to visualize the formulas in the different chambers. In the tables we provide in this section, we will consider ideals J as above: $\langle s^2 \rangle$, $\langle s^2t \rangle$, $\langle s^2t^2 \rangle$, $\langle s^2t^3 \rangle$, $\langle s^3 \rangle$, $\langle s^3t^3 \rangle$ that are generated by elements of the form $s^{m_1}t^{m_2}$, where $deg(s^{m_1}) = m_1$ and $deg(t^{m_2}) = m_2$ and we include various values $\boldsymbol{d}$ in the different chambers.

Table 1 will help to illustrate how we found the differential operators of degree $\boldsymbol{d} = (-3, 0)$ which is in C2. The expressions in the top row: $2x - 6$, $2x - 7$, $\cdots$, $2x - 11$ represent the half-lines $y = 2x - 6$, $y = 2x - 7$, $\cdots$, $y = 2x - 11$, respectively. In general, the mononmials on these top half-lines will map to the monomials on the half-lines given each successive row. If a half-line has been omitted in a row below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal.

The differential operator in multidegree $\boldsymbol{d} = (-3, 0)$ is $s^{-3} \cdot \langle (h_2, 5)! \rangle$. Consider the ideal $J = \langle s^2 \rangle$ and the monomials with exponents in the half-lines $y = 2x - 6$, $y = 2x - 7$, $y = 2x - 8$ and $y = 2x - 9$. If we apply $s^{-3} \cdot \langle (h_2, 5)! \rangle$ to those exponents, we obtain the exponents of the monomials that lie on σ_2, $y = 2x - 1$, $y = 2x - 2$, or $y = 2x - 3$ respectively, which are not in J. Thus, we should multiply $s^{-3} \cdot \langle (h_2, 5)! \rangle$ by $(h_2 - 6)(h_2 - 7)(h_2 - 8)(h_2 - 9)$ in order to correct this lack of membership.

Similarly, if we take the ideal $J = \langle s^2 t \rangle$ and the monomials with exponents in the half-lines $y = 2x - 6$, $y = 2x - 7$ and $y = 2x - 8$ and we applying $s^{-3} \cdot \langle (h_2, 5)!$ to those exponents then we obtain the exponents of the monomials that lie on

σ_2, $y = 2x - 1$ or $y = 2x - 2$ respectively, which are not in J. Thus, we should multiply $s^{-3} \cdot \langle (h_2, 5)! \rangle$ by $(h_2 - 6)(h_2 - 7)(h_2 - 8)$ in order to correct this.

In a similar way, we obtained all the values in Table 1

TABLE 1. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-3, 0)$ along with formula

Ideal	$2x-6$	$2x-7$	$2x-8$	$2x-9$	$2x-10$	$2x-11$	Formula for $(-3,0)$
$\langle s^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$			$s^{-3} \cdot \langle (h_2, 9)! \rangle$
$\langle s^2 t \rangle$	σ_2	$2x-1$	$2x-2$				$s^{-3} \cdot \langle (h_2, 8)! \rangle$
$\langle s^2 t^2 \rangle$	σ_2	$2x-1$					$s^{-3} \cdot \langle (h_2, 7)! \rangle$
$\langle s^2 t^3 \rangle$	σ_2						$s^{-3} \cdot \langle (h_2, 6)! \rangle$
$\langle s^2 t^4 \rangle$							$s^{-3} \cdot \langle (h_2, 5)! \rangle$
$\langle s^3 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$	$2x-4$	$2x-5$	$s^{-3} \cdot \langle (h_2, 11)! \rangle$
$\langle s^3 t \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$	$2x-4$		$s^{-3} \cdot \langle (h_2, 10)! \rangle$
$\langle s^3 t^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$			$s^{-3} \cdot \langle (h_2, 9)! \rangle$
$\langle s^3 t^3 \rangle$	σ_2	$2x-1$	$2x-2$				$s^{-3} \cdot \langle (h_2, 8)! \rangle$
$\langle s^3 t^4 \rangle$	σ_2	$2x-1$					$s^{-3} \cdot \langle (h_2, 7)! \rangle$

Notice that all the values in the last column in Table 1 satisfy the formula $s^{d_1} t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$. Then, when the exponent is $\boldsymbol{d} = (-3, 0)$ the differentials operators for all the ideals in that table will satisfy the following formula

$$s^{d_1} t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle.$$

In the next pages, we will present two more examples for differential operators in degree $\boldsymbol{d}$ in C2 to illustrate that the proposed formula for the graded piece of idealizer of the ideals of the form $\langle s^{m_1} t^{m_2} \rangle$ in degree $\boldsymbol{d}$ holds.

Now, Table 2 refers to the differential operator of degree $\boldsymbol{d} = (-1, 3)$ which is in C2. The expressions in the top row: $2x - 5$, $2x - 6$, $\cdots$, $2x - 10$ represent the half-lines $y = 2x - 5$, $y = 2x - 6$, $\cdots$, $y = 2x - 10$, respectively. In general, the mononmials on half-lines on the top row will map to monomials on the half-lines of each successive row. If a half-line has been omitted in any row below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal. For $\boldsymbol{d} = (-1, 3)$ in C2; the differential operator is $s^{-1}t^3 \cdot \langle (h_2, 4)! \rangle$. Then, in the ideal $J = \langle s^2 \rangle$ if we apply $s^{-1}s^3 \cdot \langle (h_2, 4)! \rangle$ to the monomials with exponents in the half-lines $y = 2x - 5$, $y = 2x - 6$, $y = 2x - 7$ and $y = 2x - 8$, we obtain the exponents of the monomials that lie on σ_2, $y = 2x - 1$, $y = 2x - 2$, or $y = 2x - 3$ respectively, which are not in J. Thus, we should multiply $s^{-1}t^3 \cdot \langle (h_2, 4)! \rangle$ by $(h_2 - 5)(h_2 - 6)(h_2 - 7)(h_2 - 8)$ in order to correct this lack of membership.

Using a similar argument for all the ideals, we obtain all the values in Table 2

TABLE 2. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-1, 3)$ along with formula

Ideal	$2x-5$	$2x-6$	$2x-7$	$2x-8$	$2x-9$	$2x-10$	Formula for (-1,3)
$\langle s^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$			$s^{-1}t^3 \cdot \langle (h_2, 8)! \rangle$
$\langle s^2t \rangle$	σ_2	$2x-1$	$2x-2$				$s^{-1}t^3 \cdot \langle (h_2, 7)! \rangle$
$\langle s^2t^2 \rangle$	σ_2	$2x-1$					$s^{-1}t^3 \cdot \langle (h_2, 6)! \rangle$
$\langle s^2t^3 \rangle$	σ_2						$s^{-1}t^3 \cdot \langle (h_2, 5)! \rangle$
$\langle s^2t^4 \rangle$							$s^{-1}t^3 \cdot \langle (h_2, 4)! \rangle$
$\langle s^3 \rangle$		$2x-1$	$2x-2$	$2x-3$	$2x-4$	$2x-5$	$s^{-1}t^3 \cdot \langle (h_2, 4)!(h_2-6) \cdots (h_2-10) \rangle$
$\langle s^3t \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$	$2x-4$		$s^{-1}t^3 \cdot \langle (h_2, 9)! \rangle$
$\langle s^3t^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$			$s^{-1}t^3 \cdot \langle (h_2, 8)! \rangle$
$\langle s^3t^3 \rangle$	σ_2	$2x-1$	$2x-2$				$s^{-1}t^3 \cdot \langle (h_2, 7)! \rangle$
$\langle s^3t^4 \rangle$	σ_2	$2x-1$					$s^{-1}t^3 \cdot \langle (h_2, 6)! \rangle$

In Table 2, for the ideal generated by s^3 the formula does not include $(h_2 - 5)$ since the half-line $y = 2x - 5$ is not in the ideal and after applying $s^{-1}t^3 \cdot \langle (h_2, 4)! \rangle$

to the exponents of the monomials that lie on $y = 2x - 5$, we obtain the exponents of the monomials that lie on σ_2, which is not in the ideal. Except for the ideal generated by s^3, the values in the last column in Table 2 satisfy the formula $s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$.

The following table, Table 3, is the last example of a differential operator in multidegree $\boldsymbol{d}$ in C2. The table was filled using the values obtained in the precious subsections. Again, the expressions in the top row: $2x - 2$, $2x - 3$, $\cdots$, $2x - 7$ represent the half-lines $y = 2x - 2$, $y = 2x - 3$, $\cdots$, $y = 2x - 7$, respectively. The mononmials on the top half-lines map to monomials on the half-lines on each successive row. If a half-line is omitted on any row below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal.

TABLE 3. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-1, 0)$ along with formula

Ideal	$2x-2$	$2x-3$	$2x-4$	$2x-5$	$2x-6$	$2x-7$	Formula for (-1,0)
$\langle s^2 \rangle$			$2x-2$	$2x-3$			$s^{-1} \cdot \langle (h_2, 1)!(h_2-4)(h_2-5) \rangle$
$\langle s^2t \rangle$		$2x-1$	$2x-2$				$s^{-1} \cdot \langle (h_2, 1)!(h_2-3)(h_2-4) \rangle$
$\langle s^2t^2 \rangle$	σ_2	$2x-1$					$s^{-1} \cdot \langle (h_2, 3)! \rangle$
$\langle s^2t^3 \rangle$	σ_2						$s^{-1} \cdot \langle (h_2, 2)! \rangle$
$\langle s^2t^4 \rangle$							$s^{-1} \cdot \langle (h_2, 1)! \rangle$
$\langle s^3 \rangle$					$2x-4$	$2x-5$	$s^{-1} \cdot \langle (h_2, 1)!(h_2-6)(h_2-7) \rangle$
$\langle s^3t \rangle$				$2x-3$	$2x-4$		$s^{-1} \cdot \langle (h_2, 1)!(h_2-5)(h_2-6) \rangle$
$\langle s^3t^2 \rangle$			$2x-2$	$2x-3$			$s^{-1} \cdot \langle (h_2, 1)!(h_2-4)(h_2-5) \rangle$
$\langle s^3t^3 \rangle$		$2x-1$	$2x-2$				$s^{-1} \cdot \langle (h_2, 1)!(h_2-3)(h_2-4) \rangle$
$\langle s^3t^4 \rangle$	σ_2	$2x-1$					$s^{-1} \cdot \langle (h_2, 3)! \rangle$

Notice that the ideals that have a continuous factorial formula are $\langle s^2t^2\rangle$, $\langle s^2t^3\rangle$, $\langle s^2t^4\rangle$ and $\langle s^3t^4\rangle$, and for $\boldsymbol{d} = (-1, 0)$ their formula is given by

$$s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)!\rangle.$$

The rest of the ideals, in the table, skip some terms and they do not have a factorial-continuous formula. For example, see Figure 27 and Figure 28 that exhibit the multidegrees $\boldsymbol{d}$ where the idealizer does not have a factorial-continuous behavior for $\langle s^2t\rangle$ and $\langle s^3t^2\rangle$, respectively.

In the following pages, 3 examples for differential operators in degree $\boldsymbol{d}$ in C4 are included in order to give a formula for the graded piece of idealizer.

In Table 4, the mononmials on the half-lines $y = 4, \cdots, y = 7$ will map to monomials on the half-lines in each successive row. If a half-line is omitted in a row below the half-line on the top line, it is because the monomials on the line remain in the ideal.

TABLE 4. Half-lines that shift out of the ideal for $\boldsymbol{d} = (0, -4)$ along with formula

Ideal	$y = 4$	$y = 5$	$y = 6$	$y = 7$	Formula for (0,-4)
$\langle s^2\rangle$					$t^{-4} \cdot \langle (h_1, 3)!\rangle$
$\langle s^2t\rangle$	σ_1				$t^{-4} \cdot \langle (h_1, 4)!\rangle$
$\langle s^2t^2\rangle$	σ_1	$y = 1$			$t^{-4} \cdot \langle (h_1, 5)!\rangle$
$\langle s^2t^3\rangle$	σ_1	$y = 1$	$y = 2$		$t^{-4} \cdot \langle (h_1, 6)!\rangle$
$\langle s^2t^4\rangle$	σ_1	$y = 1$	$y = 2$	$y = 3$	$t^{-4} \cdot \langle (h_1, 7)!\rangle$
$\langle s^3\rangle$					$t^{-4} \cdot \langle (h_1, 3)!\rangle$
$\langle s^3t\rangle$	σ_1				$t^{-4} \cdot \langle (h_1, 4)!\rangle$
$\langle s^3t^2\rangle$	σ_1	$y = 1$			$t^{-4} \cdot \langle (h_1, 5)!\rangle$
$\langle s^3t^3\rangle$	σ_1	$y = 1$	$y = 2$		$t^{-4} \cdot \langle (h_1, 6)!\rangle$
$\langle s^3t^4\rangle$	σ_1	$y = 1$	$y = 2$	$y = 3$	$t^{-4} \cdot \langle (h_1, 7)!\rangle$

Table 4 has continuous formulas for all ideals and those formulas satisfy

$$s^{d_1}t^{d_2}\cdot\langle(h_1,-d_2-1+m_2)!\rangle.$$

We will present a couple more tables for differential operators in C4 to illustrate that the proposed formula for the graded piece of idealizer of the ideals of the form $\langle s^{m_1}t^{m_2}\rangle$ in degree $\boldsymbol{d}$ holds. In Table 5, the mononmials on the half-lines $y=3$, $y=4$, $y=5$, $y=6$ will map to monomials on the half-lines in each successive row. If a half-line is omitted in any row below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal.

TABLE 5. Half-lines that shift out of the ideal for $\boldsymbol{d}=(-1,-3)$ along with formula

Ideal	$y=3$	$y=4$	$y=5$	$y=6$	Formula for (-1,-3)
$\langle s^2\rangle$					$s^{-1}t^{-3}\cdot\langle(h_1,2)!\rangle$
$\langle s^2t\rangle$	σ_1				$s^{-1}t^{-3}\cdot\langle(h_1,3)!\rangle$
$\langle s^2t^2\rangle$	σ_1	$y=1$			$s^{-1}t^{-3}\cdot\langle(h_1,4)!\rangle$
$\langle s^2t^3\rangle$	σ_1	$y=1$	$y=2$		$s^{-1}t^{-3}\cdot\langle(h_1,5)!\rangle$
$\langle s^2t^4\rangle$		$y=1$	$y=2$	$y=3$	$s^{-1}t^{-3}\cdot\langle(h_1,2)!(h_2-4)(h_2-5)(h_2-6)\rangle$
$\langle s^3\rangle$					$s^{-1}t^{-3}\cdot\langle(h_1,2)!$
$\langle s^3t\rangle$	σ_1				$s^{-1}t^{-3}\cdot\langle(h_1,3)!\rangle$
$\langle s^3t^2\rangle$	σ_1	$y=1$			$s^{-1}t^{-3}\cdot\langle(h_1,4)!\rangle$
$\langle s^3t^3\rangle$	σ_1	$y=1$	$y=2$		$s^{-1}t^{-3}\cdot\langle(h_1,5)!\rangle$
$\langle s^3t^4\rangle$		$y=1$	$y=2$	$y=3$	$s^{-1}t^{-3}\cdot\langle(h_1,2)!(h_1-4)(h_1-5)(h_1-6)\rangle$

In Table 5, the ideals $\langle s^2t^4\rangle$ and $\langle s^3t^4\rangle$ skip a term, the term (h_1-3), in both cases it is because after applying $s^{-1}t^{-3}\cdot\langle(h_1,2)!\rangle$ to a monomial whose exponent lie on the half-line $y=3$, we obtain an exponent of the line σ_1, but the exponents in $y=3$ correspond to some monomials that are not part of that ideal. However, the remaining formulas satisfy

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle.$$

In the next table, we include some values obtained in the previous subsections.In Table 6, the mononmials on the half-lines $y = 2$, $y = 3$, $y = 4$, and $y = 5$ will map to monomials on the half-lines in the rows below. If a half-line is omitted below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal. In Table 6, the ideals generated by s^2t^4, s^2t^4, s^3t^3 and s^3t^4 skip

TABLE 6. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-1, -2)$ along with formula

Ideal	$y=2$	$y=3$	$y=4$	$y=5$	Formula for (-1,-2)
$\langle s^2 \rangle$					$s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$
$\langle s^2t \rangle$	σ_1				$s^{-1}t^{-2} \cdot \langle (h_1, 2)! \rangle$
$\langle s^2t^2 \rangle$	σ_1	$y=1$			$s^{-1}t^{-2} \cdot \langle (h_1, 3)! \rangle$
$\langle s^2t^3 \rangle$		$y=1$	$y=2$		$s^{-1}t^{-2} \cdot \langle (h_1, 1)!(h_1-3)(h_1-4) \rangle$
$\langle s^2t^4 \rangle$			$y=2$	$y=3$	$s^{-1}t^{-2} \cdot \langle (h_1, 1)!(h_1-4)(h_1-5) \rangle$
$\langle s^3 \rangle$					$s^{-1}t^{-2} \cdot \langle (h_1, 1)! \rangle$
$\langle s^3t \rangle$	σ_1				$s^{-1}t^{-2} \cdot \langle (h_1, 2)! \rangle$
$\langle s^3t^2 \rangle$	σ_1	$y=1$			$s^{-1}t^{-2} \cdot \langle (h_1, 3)! \rangle$
$\langle s^3t^3 \rangle$		$y=1$	$y=2$		$s^{-1}t^{-2} \cdot \langle (h_1, 1)!(h_1-3)(h_1-4) \rangle$
$\langle s^3t^4 \rangle$			$y=2$	$y=3$	$s^{-1}t^{-2} \cdot \langle (h_1, 1)!(h_1-4)(h_1-5) \rangle$

some terms, but the remaining formulas satisfy

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle.$$

In the following pages we have included tables for some different values $\boldsymbol{d}$ in C3. For C3, we have formulas that involve h_1 and h_2, as we could see in the first subsections. It is because we need to shift some half-lines that are parallel to the facet σ_1 and some half-lines that are parallel to σ_2. Therefore, tables for C3 will contain more columns. For values $\boldsymbol{d}$ in C3, we will present the information in two different tables: one table that includes the lines with the monomials whose images do not lie in J and a second table with formulas for each ideal.

Table 7 and Table 8 correspond to the differential operator of degree $\boldsymbol{d} = (-2,-2)$ in C3. Table 7 indicates the half-lines that need to be sent to zero as the corresponding lines map outside of the ideal and Table 8 shows the formulas for each ideal. The expressions in the top row: $2x-2$, $2x-3$, $\cdots$, $2x-7$ represent the half-lines $y = 2x-2$, $y = 2x-3$, $\cdots$, $y = 2x-7$, respectively. The mononmials on the half-lines on the top row will map to monomials on the half-lines in each successive row. If a half-line has been omitted below the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the ideal.

TABLE 7. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-2,-2)$

Ideal	$2x-2$	$2x-3$	$2x-4$	$2x-5$	$2x-6$	$2x-7$	$y=2$	$y=3$	$y=4$	$y=5$
$\langle s^2\rangle$			$2x-2$	$2x-3$						
$\langle s^2t\rangle$		$2x-1$	$2x-2$				σ_1			
$\langle s^2t^2\rangle$	σ_2	$2x-1$					σ_1	$y=1$		
$\langle s^2t^3\rangle$	σ_2							$y=1$	$y=2$	
$\langle s^2t^4\rangle$									$y=2$	$y=3$
$\langle s^3\rangle$					$2x-4$	$2x-5$				
$\langle s^3t\rangle$				$2x-3$	$2x-4$		σ_1			
$\langle s^3t^2\rangle$			$2x-2$	$2x-3$			σ_1	$y=1$		
$\langle s^3t^3\rangle$		$2x-1$	$2x-2$					$y=1$	$y=2$	
$\langle s^3t^4\rangle$	σ_2	$2x-1$							$y=2$	$y=3$

Table 8 has an interesting behavior. The ideals generated by s^2, s^2t, s^2t^2, s^3, s^3t, and s^3t^2 have a factorial-continuous expression for the part of the formula that

TABLE 8. Formula for $\boldsymbol{d} = (-2, -2)$ in C3

Ideal	Formula for (-2,-2)
$\langle s^2 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_2, 1)!(h_2 - 4)(h_2 - 5) \rangle$
$\langle s^2 t \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 2)!(h_2, 1)!(h_2 - 3)(h_2 - 4) \rangle$
$\langle s^2 t^2 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 3)!(h_2, 3)! \rangle$
$\langle s^2 t^3 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4)(h_2, 2)! \rangle$
$\langle s^2 t^4 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_1 - 4)(h_1 - 5)(h_2, 1)! \rangle$
$\langle s^3 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_2, 1)!(h_2 - 6)(h_2 - 7) \rangle$
$\langle s^3 t \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 2)!(h_2, 1)!(h_2 - 5)(h_2 - 6) \rangle$
$\langle s^3 t^2 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 3)!(h_2, 1)!(h_2 - 4)(h_2 - 5) \rangle$
$\langle s^3 t^3 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4)(h_2, 1)!(h_2 - 3)(h_2 - 4) \rangle$
$\langle s^3 t^4 \rangle$	$s^{-2}t^{-2} \cdot \langle (h_1, 1)!(h_1 - 4)(h_1 - 5)(h_2, 3)! \rangle$

involves h_1 and they satisfy $s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$ and the ideals generated by s^2t^2, s^2t^23, s^2t^4 and s^3t^4 have a factorial-continuous expression for the part of the formula that involves h_2 and their formulas satisfy $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle$. Nevertheless, only the ideal generated by s^2t^2 has factorial-continuous expressions for both parts of the formula and that formula satisfies

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)!(h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle.$$

The following two tables correspond to the differential operator of degree $\boldsymbol{d} = (-5, -4)$ in C3. One indicates the half-lines that need to be changed and the another shows the formulas for each ideal. The expressions in the top row: $2x - 6$, $2x - 7$, $\cdots$, $2x - 11$ represent the half-lines $y = 2x - 6$, $y = 2x - 7$, $\cdots$, $y = 2x - 11$, respectively. The monomials on the half-lines on the top row will map to monomials on the half-lines in each successive row. If a half-line has been omitted the half-line on the top line, it is because the monomials on the line that would have appeared on the half-line in that row of the table already lie in the respective ideal.

TABLE 9. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-5, -4)$

Ideal	$2x-6$	$2x-7$	$2x-8$	$2x-9$	$2x-10$	$2x-11$	$y=4$	$y=5$	$y=6$	$y=7$
$\langle s^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$						
$\langle s^2 t \rangle$	σ_2	$2x-1$	$2x-2$				σ_1			
$\langle s^2 t^2 \rangle$	σ_2	$2x-1$					σ_1	$y=1$		
$\langle s^2 t^3 \rangle$	σ_2						σ_1	$y=1$	$y=2$	
$\langle s^2 t^4 \rangle$							σ_1	$y=1$	$y=2$	$y=3$
$\langle s^3 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$	$2x-4$	$2x-5$				
$\langle s^3 t \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$	$2x-4$		σ_1			
$\langle s^3 t^2 \rangle$	σ_2	$2x-1$	$2x-2$	$2x-3$			σ_1	$y=1$		
$\langle s^3 t^3 \rangle$	σ_2	$2x-1$	$2x-2$				σ_1	$y=1$	$y=2$	
$\langle s^3 t^4 \rangle$	σ_2	$2x-1$					σ_1	$y=1$	$y=2$	$y=3$

TABLE 10. Formula for $\boldsymbol{d} = (-5, -4)$ in C3

Ideal	Formula for (-5,-4)
$\langle s^2 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 3)!(h_2, 9)! \rangle$
$\langle s^2 t \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 4)!(h_2, 8)! \rangle$
$\langle s^2 t^2 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 5)!(h_2, 7)! \rangle$
$\langle s^2 t^3 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 6)!(h_2, 6)! \rangle$
$\langle s^2 t^4 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 7)!(h_2, 5)! \rangle$
$\langle s^3 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 3)!(h_2, 11)! \rangle$
$\langle s^3 t \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 4)!(h_2, 10)! \rangle$
$\langle s^3 t^2 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 5)!(h_2, 9)! \rangle$
$\langle s^3 t^3 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 6)!(h_2, 8)! \rangle$
$\langle s^3 t^4 \rangle$	$s^{-5}t^{-4} \cdot \langle (h_1, 7)!(h_2, 7)! \rangle$

Notice that all the formulas in Table 10 do not skip any term for both components of the formula and the formulas satisfy:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)!(h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle.$$

Finally, in Table 11, we have included the values that we obtained in the previous subsections.

Table 11. Formula for $\boldsymbol{d} = (-1, -1)$ in C3

Ideal	Formula for (-1,-1)
$\langle s^2 \rangle$	$s^{-1}t^{-1} \cdot \langle h_1 h_2(h_2 - 4) \rangle$
$\langle s^2 t \rangle$	$s^{-1}t^{-1} \cdot \langle (h_1, 1)! h_2(h_2 - 3) \rangle$
$\langle s^2 t^2 \rangle$	$s^{-1}t^{-1} \cdot h_1(h_1 - 2)h_2(h_2 - 2) \rangle$
$\langle s^2 t^3 \rangle$	$s^{-1}t^{-1} \cdot h_1(h_1 - 3)(h_2, 1)! \rangle$
$\langle s^3 \rangle$	$s^{-1}t^{-1} \cdot h_1 h_2(h_2 - 6) \rangle$
$\langle s^3 t \rangle$	$s^{-1}t^{-1} \cdot (h_1, 1)h_2(h_2 - 5) \rangle$
$\langle s^3 t^2 \rangle$	$s^{-1}t^{-1} \cdot h_1(h_1 - 2)h_2(h_2 - 4) \rangle$
$\langle s^3 t^3 \rangle$	$s^{-1}t^{-1} \cdot h_1(h_1 - 3)h_2(h_2 - 3) \rangle$
$\langle s^3 t^4 \rangle$	$s^{-1}t^{-1} \cdot h_1(h_1 - 4)h_2(h_2 - 2) \rangle$

2.8. **Formulas for R_{A_2}.** If an ideal J is generated by an element of the form $s^{m_1}t^{m_2}$, where $deg(s^{m_1}) = m_1$ and $deg(t^{m_2}) = m_2$ for $m_1 \geq 2$ and $m_2 \geq 0$, over the ring of the rational normal curve of degree 2, $R_{A_2} = \mathbb{C}[s, st, st^2]$, then we can give a general formula for $\mathbb{I}(J)_{\boldsymbol{d}}$. The number of lines parallel to σ_2 that are not in the ideal J, but are in R_{A_2}, is given by $2m_1 - m_2$. For instance, see Figure 5, Figure 9, Figure 13, Figure 15, Figure 17 and Figure 19. Therefore, in order to have a formula for C2 that includes all the continuous terms, without skipping any term, we need that $-\boldsymbol{d}$ does not lie on one of the lines parallel to σ_2 in $R_{A_2} \setminus J$, that is, $-\boldsymbol{d}$ should lie in a line parallel to σ_2 inside J. Then, the condition

$$-2d_2 + d_1 \geq 2m_1 - m_2$$

guarantees the formula

$$s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle \tag{2}$$

For instance, in Figure 21 the black lattice points represent $\boldsymbol{d}$ and $-\boldsymbol{d}$. Observe that $-\boldsymbol{d} = (1, 0)$ lies in the half-line $y = 2x - 2$ which is a line parallel to σ_2 and this line belongs to the ideal J. The inequality $-2d_2 + d_1 \geq 2m_1 - m_2$ is satisfied; therefore, we would have the continuous formula $s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle = s^{-1} \cdot \langle (h_2, 3)! \rangle$

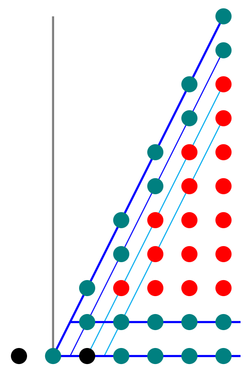


FIGURE 21. $\boldsymbol{d} = (-1, 0)$ and $-\boldsymbol{d} = (1, 0)$ in $\langle s^2t^2 \rangle$

For the next example, we will consider $\boldsymbol{d} = (1, 3)$ in C2 and its table, Table 12,

TABLE 12. Half-lines that shift out of the ideal for $\boldsymbol{d} = (1, 3)$ along with formula

Ideal	$2x-1$	$2x-2$	$2x-3$	$2x-4$	$2x-5$	$2x-6$	Formula for (1,3)
$\langle s^2 \rangle$				$2x-3$			$st^3 \cdot \langle h_2(h_2-4) \rangle$
$\langle s^2t \rangle$			$2x-2$				$st^3 \cdot \langle h_2(h_2-3) \rangle$
$\langle s^2t^2 \rangle$		$2x-1$					$st^3 \cdot \langle h_2(h_2-2) \rangle$
$\langle s^2t^3 \rangle$	σ_2						$st^3 \cdot \langle (h_2, 1)! \rangle$
$\langle s^2t^4 \rangle$	σ_2						$st^3 \cdot \langle h_2 \rangle$
$\langle s^3 \rangle$						$2x-5$	$st^3 \cdot \langle h_2(h_2-6) \rangle$
$\langle s^3t \rangle$					$2x-4$		$st^3 \cdot \langle h_2(h_2-5) \rangle$
$\langle s^3t^2 \rangle$				$2x-3$			$st^3 \cdot \langle h_2(h_2-4) \rangle$
$\langle s^3t^3 \rangle$			$2x-2$				$st^3 \cdot \langle h_2(h_2-3) \rangle$
$\langle s^3t^4 \rangle$		$2x-1$					$st^3 \cdot \langle h_2(h_2-2) \rangle$

For $\boldsymbol{d} = (1, 3)$: $\langle s^2t^3 \rangle$ and $\langle s^2t^4 \rangle$ are the only ideals in the list that satisfy $-2d_1 + d_2 \geq 2m_2 - m_1$, then our formula $s^{d_1}t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$ is applicable just for these ideals. Otherwise $-2d_1+d_2 < 2m_2-m_1$ and $-2d_1+d_2 = 1$ that requires the formula

$$s^{d_1}t^{d_2} \cdot \langle (h_2, -2d_1 + d_2 - 1)!(h_2 - (2m_1 - 2d_1 - m_2 + d_2 - 1))\rangle.$$

Denote

$$\phi_{2,1}(\boldsymbol{d}) := \phi_{2,1} = -d_2 - 1 + m_2 \tag{3}$$

and

$$\phi_{2,2}(\boldsymbol{d}) := \phi_{2,2} = 2m_1 - 2d_1 - m_2 + d_2 - 1. \tag{4}$$

and

$$h'_{2,1} := (h_1, -d_2 - 1)!$$

$$h'_{2,2} := (h_2, -2d_1 + d_2 - 1)!$$

If $-2d_2 + d_1 < 2m_1 - m_2$ and $-2d_1 + d_2 = 1$, then the formula for C2 becomes:

$$s^{d_1}t^{d_2} \cdot \langle h'_{2,2}(h_2 - \phi_{2,2})\rangle.$$

In general, if $-2d_2 + d_1 < 2m_1 - m_2$ and $-2d_1 + d_2 = n$,

$$s^{d_1}t^{d_2} \cdot \langle h'_{2,2}(h_2 - (\phi_{2,2} - (n-1))) \cdots (h_2 - (\phi_{2,2} - 1))(h_2 - \phi_{2,2})\rangle. \tag{5}$$

We can denote $(h_2 - (\phi_{2,2} - (n-1))) \cdots (h_2 - (\phi_{2,2} - 1))(h_2 - \phi_{2,2})$ as $(h_2 - \phi_{2,2} - n + 1, n - 1)!$. In the following pages, we will try to use this notation in some expressions in order to show the formulas in a simpler way. Thus, the previous expression is equivalent to

$$s^{d_1}t^{d_2} \cdot \langle h'_{2,2}(h_2 - \phi_{2,2} - n + 1, n - 1)!\rangle.$$

In the following example represented in Figure 22, notice that $-\boldsymbol{d}$ is outside the ideal and $-2d_1 + d_2 = 2$

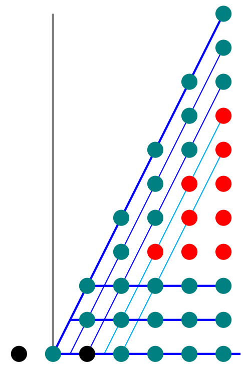


FIGURE 22. $\boldsymbol{d} = (-1, 0)$ and $-\boldsymbol{d} = (1, 0)$ in $\langle s^3t^3 \rangle$

Therefore, as $-2d_1+d_2 = 2$, then from equation (5), the formula $s^{d_1}t^{d_2} \cdot \langle h'_{2,2}(h_2 - (\phi_{2,2} - 1))(h_2 - \phi_{2,2}) \rangle$ is applicable for this $\boldsymbol{d}$, and it becomes $s^{-1} \cdot \langle (h_2, 1)!(h_2 - 3)(h_2 - 4) \rangle$. On the other hand, in Figure 22, the two half-lines that need to be shifted are $y = 2x-3$ and $y = 2x-4$ and it agrees with $s^{-1} \cdot \langle (h_2, 1)!(h_2-3)(h_2-4) \rangle$.

The elements in C4 only involve the formula in terms of h_1 and the number of lines parallel to σ_1 that are in R_{A_2}, but are not in J is given by m_2. See Figure 5, Figure 9, Figure 13, Figure 15, Figure 17 and Figure 19. Then, in order to have a continuous-factorial formula for C4, we need that $-d_2$ lies in a line parallel to σ_1 inside J. Then, the condition

$$-d_2 \geq m_2$$

gives us the factorial formula for C4

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle$$

For example, consider $\boldsymbol{d} = (-1, -2)$ which is represented in the following figure Notice that $-\boldsymbol{d} = (1, 2)$ is not in the ideal, but it lies on the half-line $y = 2$ and

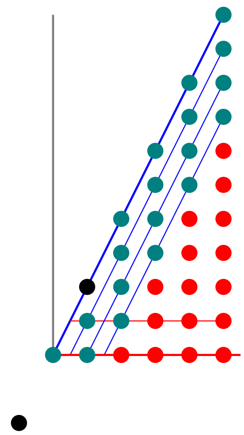


FIGURE 23. $\boldsymbol{d} = (-1, -2)$ and $-\boldsymbol{d} = (1, 2)$ in $\langle s^2 \rangle$

that line is inside the ideal. In addition, the inequality $-d_2 = 2 \geq 0$ is satisfied. Thus, the formula $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle = \langle s^{-1}t^{-2}(h_1, 1)! \rangle$ works well in this case.

For C4, if $-d_2 < m_2$ and $-d_2 = 1$, then the formula is given by:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_1 - \phi_{2,1}) \rangle$$

In general, if $-d_2 < m_2$ and $-d_2 = n$, then the formula is given by:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_1 - (\phi_{2,1} - (n - 1))) \cdots (h_1 - (\phi_{2,1} - 1))(h_1 - \phi_{2,1}) \rangle.$$

For example, $-\boldsymbol{d}$ is in a line parallel to σ_1 outside the ideal and notice that $-d_2 < m_2$ and $-d_2 = 2$. Therefore, we have the formula $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1) \rangle !(h_1 - (\phi_{2,1} - (1)))(h_1 - \phi_{2,1}) \rangle = s^{-1}t^{-2} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4) \rangle$.

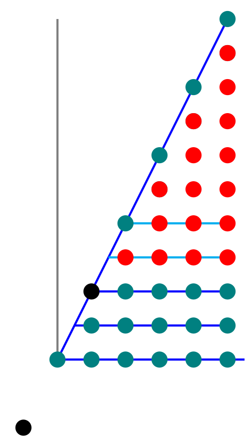


FIGURE 24. $\boldsymbol{d} = (-1,-2)$ and $-\boldsymbol{d} = (1,2)$ in $\langle s^2t^3\rangle$

Now, consider $\boldsymbol{d} = (5,-2)$ in C4 and its table which was obtained as in previous pages. The ideals $\langle s^2\rangle$, $\langle s^2t\rangle$, $\langle s^2t^2\rangle$, $\langle s^3\rangle$, $\langle s^3t\rangle$ and $\langle s^3t^2\rangle$ satisfy the condition $-d_2 \geq m_2$, and the formula $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)!\rangle$ is applicable; however, $\langle s^2t^3\rangle$, $\langle s^3t^3\rangle$ and $\langle s^3t^4\rangle$ do not satisfy it and since $-d_2 = 2$ then their formula is given by $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_1 - (\phi_{2,1} - 1))(h_1 - \phi_{2,1})\rangle$. In general, we can apply this formula to the elements in $\mathrm{C4} \cap \sigma_2$.

TABLE 13. Half-lines that shift out of the ideal for $\boldsymbol{d} = (5,-2)$ along with formula

Ideal	$y=2$	$y=3$	$y=4$	$y=5$	Formula for (5,-2)
$\langle s^2\rangle$					$s^5t^{-2} \cdot \langle (h_1, 1)!\rangle$
$\langle s^2t\rangle$	σ_1				$s^5t^{-2} \cdot \langle (h_1, 2)!\rangle$
$\langle s^2t^2\rangle$	σ_1	$y=1$			$s^5t^{-2} \cdot \langle (h_1, 3)!\rangle$
$\langle s^2t^3\rangle$		$y=1$	$y=2$		$s^5t^{-2} \cdot \langle (h_1, 1)!(h_1-3)(h_1-4)\rangle$
$\langle s^2t^4\rangle$			$y=2$	$y=3$	$s^5t^{-2} \cdot \langle (h_1, 1)!(h_1-4)(h_1-5)\rangle$
$\langle s^3\rangle$					$s^5t^{-2} \cdot \langle (h_2, 1)!\rangle$
$\langle s^3t\rangle$	σ_1				$s^5t^{-2} \cdot \langle (h_2, 2)!\rangle$
$\langle s^3t^2\rangle$	σ_1	$y=1$			$s^5t^{-2} \cdot \langle (h_2, 3)!\rangle$
$\langle s^3t^3\rangle$		$y=1$	$y=2$		$s^5t^{-2} \cdot \langle (h_1, 1)!(h_1-3)(h_1-4)\rangle$
$\langle s^3t^4\rangle$			$y=2$	$y=3$	$s^5t^{-2} \cdot \langle (h_1, 1)!(h_1-4)(h_1-5)\rangle$

Any $\boldsymbol{d}$ in C3 satisfies that $-2d_1 + 2d_2 > 0$ and $-d_2 > 0$; therefore, the formula for C3 should involve terms of h_1 and h_2 and be related to the previous formulas for C2 and C4.

For C3, if $-d_2 \geq m_2$ and $-2d_2 + d_1 \geq 2m_1 + m_2$, then the formula is given by:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)!(h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$$

For instance, consider $\boldsymbol{d} = (-5, -4)$, which is represented in the following figure, The line parallel to σ_1 in which $-\boldsymbol{d}$ lies is in the ideal J and the line parallel to

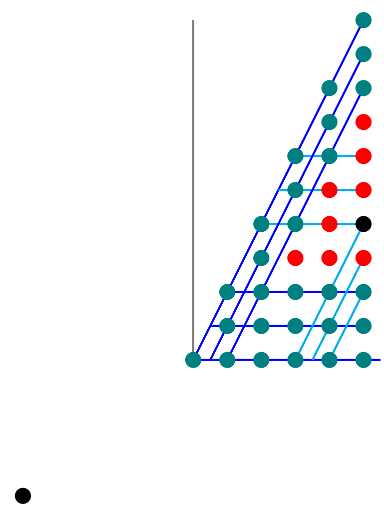


FIGURE 25. $\boldsymbol{d} = (-5, -4)$ and $-\boldsymbol{d} = (5, 4)$ in $\langle s^3t^3 \rangle$

σ_2 in which $-\boldsymbol{d}$ lies is in the ideal J as well, that implies that both inequalities are satisfied $-d_2 \geq m_2$ and $-2d_2 + d_1 \geq 2m_1 + m_2$. Therefore, we have: $s^{-5}t^{-4} \cdot \langle (h_1, 6)!(h_2, 8)! \rangle$, which is a continuous-factorial formula.

In addition, we have the combinations from the previous pages. If $-d_2 \geq m_2$ and $-2d_2 + d_1 = q < 2m_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{2,1})! h'_{2,2}(h_2 - \phi_{2,2} - q + 1, q - 1)! \rangle$$

If $-d_2 = p < m_2$ and $-2d_2 + d_1 \geq 2m_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle h'_{2,1}(h_1 - \phi_{2,1} - p + 1, p + 1)! \rangle.$$

Finally, if $-d_2 < m_2$ and $-d_2 = p$ and $-2d_2 + d_1 < 2m_1 - m_2$ and $-2d_2 + d_1 = q$ then

$$s^{d_1}t^{d_2} \cdot \langle h'_{2,1}(h_1 - \phi_{2,1} - p + 1, p + 1)! h'_{2,2}(h_2 - \phi_{2,2} - q + 1, q + 1)! \rangle$$

Consider $\boldsymbol{d} = (-1, -1)$ in $\langle s^2 t \rangle$, which is represented in the following figure, Observe that $\boldsymbol{d} = (-1, -1)$ lies on $y = 1$ which is parallel to σ_1 and belongs to the

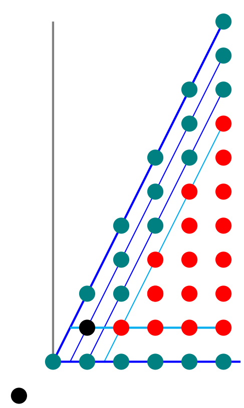


FIGURE 26. $\boldsymbol{d} = (-1, -1)$ and $-\boldsymbol{d} = (1, 1)$ in $\langle s^2 t \rangle$

ideal and on the half-line $y = 2x - 1$ which is parallel to σ_2, but does not belong to the idea. That implies $-d_2 \geq m_2$ and $-2d_2 + d_1 = 1 < 2m_1 + m_2$. Therefore, the formula in terms of h_1 is a continuous-factorial, but the formula in terms of

h_2 does not include some terms. Then, the formula for $\boldsymbol{d} = (-1,-1)$ is given by:

$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{2,1})! h'_{2,2}(h_2 - \phi_{2,2}) \rangle = \langle s^{-1}t^{-1}(h_1, 1)! h_2(h_2 - 3) \rangle$.

Now, consider $\boldsymbol{d} = (-2,-1)$: $-2d_1 + d_2 = -2(-2) - 1 = 3$ and $-d_2 = 1$. $\boldsymbol{d} = (-2,-1)$ is represented in the following two tables: Table 14 and Table 15. The first table contains the half-lines that need to be shifted as in the previous subsection and the second table contains the formulas for $\boldsymbol{d} = (-2,-1)$.

Table 14. Half-lines that shift out of the ideal for $\boldsymbol{d} = (-2,-1)$

Ideal	$2x-3$	$2x-4$	$2x-5$	$2x-6$	$2x-7$	$2x-8$	$y=1$	$y=2$	$y=3$	$y=4$
$\langle s^2 \rangle$		$2x-1$	$2x-2$	$2x-3$						
$\langle s^2t \rangle$	σ_2	$2x-1$	$2x-2$				σ_1			
$\langle s^2t^2 \rangle$	σ_2	$2x-1$						$y=1$		
$\langle s^2t^3 \rangle$	σ_2								$y=2$	
$\langle s^2t^4 \rangle$										$y=3$
$\langle s^3 \rangle$				$2x-3$	$2x-4$	$2x-5$				
$\langle s^3t \rangle$			$2x-2$	$2x-3$	$2x-4$		σ_1			
$\langle s^3t^2 \rangle$		$2x-1$	$2x-2$	$2x-3$				$y=1$		
$\langle s^3t^3 \rangle$	σ_2	$2x-1$	$2x-2$						$y=2$	
$\langle s^3t^4 \rangle$	σ_2	$2x-1$								$y=3$

Table 15. Formula for $\boldsymbol{d} = (-2,-1)$ in C3

Ideal	Formula for (-2,-1)
$\langle s^2 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_2, 2)!(h_2-4)(h_2-5)(h_2-6) \rangle$
$\langle s^2t \rangle$	$s^{-2}t^{-1} \cdot \langle (h_1, 1)!(h_2, 5)! \rangle$
$\langle s^2t^2 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-2)(h_2, 4)! \rangle$
$\langle s^2t^3 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-3)(h_2, 3)! \rangle$
$\langle s^2t^4 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-4)(h_2, 2)! \rangle$
$\langle s^3 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_2, 2)!(h_2-6)(h_2-7)(h_2-8) \rangle$
$\langle s^3t \rangle$	$s^{-2}t^{-1} \cdot \langle (h_1, 1)!(h_2, 2)!(h_2-5)(h_2-6)(h_2-7) \rangle$
$\langle s^3t^2 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-2)(h_2, 2)!(h_2-4)(h_2-5)(h_2-6) \rangle$
$\langle s^3t^3 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-3)(h_2, 5)! \rangle$
$\langle s^3t^4 \rangle$	$s^{-2}t^{-1} \cdot \langle h_1(h_1-4)(h_2, 4)! \rangle$

For $\langle s^2 \rangle$, $\langle s^2t \rangle$ and $\langle s^3 \rangle$ the formula in terms of h_1 is continuous-factorial since in those cases $-d_2 \geq m_2$ and the rest of the formulas involving h_1 are of the form $s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_1 - \phi_{2,1}) \rangle$ since $-d_2 = 1$. On the other hand, $-2d_2 + d_1 \geq$

$2m_1 - m_2$ is satisfied for the ideals $\langle s^2t\rangle$, $\langle s^2t^2\rangle$, $\langle s^2t^3\rangle$, $\langle s^2t^4\rangle$, $\langle s^3t^3\rangle$ and $\langle s^3t^4\rangle$ and the part of the formula involving h_2 do not skip any terms and since $-2d_2+d_1 = 3$, then the remaining ideals satisfy:

$$s^{d_1}t^{d_2} \cdot \langle(-2d_1 + d_2 - 1)!(h_2 - (\phi_{2,2} - 2))(h_2 - (\phi_{2,2} - 1))(h_2 - \phi_{2,2})\rangle$$

We can verify that tables in the previous subsections satisfy and behave according to these formulas. The formulas are given by: where

Table 16. Formula for $\mathbb{I}(J)_{\boldsymbol{d}}$ in R_{A_2}

C	Condition 1	Condition 2	Formula
1			$s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$
2	$-2d_1 + d_2 \geq 2m_1 - m_2$		$s^{d_1}t^{d_2} \cdot \langle(h_2, \phi_{2,2})!\rangle$
	$-2d_1 + d_2 < 2m_1 - m_2$	$-2d_1 + d_2 = q$	$s^{d_1}t^{d_2} \cdot \langle h'_{2,2}(h_2 - \phi_{2,2} - q + 1, q - 1)!\rangle$
4	$-d_2 \geq m_2$		$s^{d_1}t^{d_2} \cdot \langle(h_1, \phi_{2,1})!\rangle$
	$-d_2 < m_2$	$-d_2 = p$	$s^{d_1}t^{d_2} \cdot \langle h'_{2,1}(h_1 - \phi_{2,1} - p + 1, p - 1)!\rangle$
3	$-2d_1 + d_2 \geq 2m_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle(h_1, \phi_{2,1})!(h_2, \phi_{2,2})!\rangle$
	$q = -2d_1 + d_2 < 2m_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle(h_1, \phi_{2,1})!h_{2,2,q}\rangle$
	$-2d_1 + d_2 \geq 2m_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{2,1,p}(h_2, \phi_{2,2})!\rangle$
	$q = -2d_1 + d_2 < 2m_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{2,1,p}h_{2,2,q}\rangle$

$$h_{2,2,q} := h'_{2,2}(h_2 - \phi_{2,2} - q + 1, q - 1)!$$

and

$$h_{2,1,p} := h'_{2,1}(h_1 - \phi_{2,1} - p + 1)!.$$

Finally, we can consider an expression, which does not include all the $\boldsymbol{d}$'s:

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta] & \text{if } \boldsymbol{d} \in C1 \\ s^{d_1}t^{d_2} \cdot \langle(h_2, \phi_{2,2})!\rangle & \text{if } \boldsymbol{d} \in C2 \text{ and } -2d_1 + d_2 \geq 2m_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle(h_1, \phi_{2,1})!(h_2, \phi_{2,2})!\rangle & \text{if } \boldsymbol{d} \in C3 \text{ and } -d_2 \geq m_2 \text{ and} \\ & \qquad -2d_1 + d_2 \geq 2m_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle(h_1, \phi_{2,2})!\rangle & \text{if } \boldsymbol{d} \in C4 \text{ and } -d_2 \geq m_2 \end{cases}$$

This expression exhibits the conditions on the multidegrees $\boldsymbol{d}$ where the differential operators of degree $\boldsymbol{d}$ have factorial-continuous behavior occurs, from Definition 27. For example, the following two figures illustrate the regions that have a factorial-continuous behavior for the ideals generated by s^2t and s^3t^2, respectively. In Figure 27, the multidegrees in the lines $y = 2x + 1$ and $y = 2x + 2$ are the only

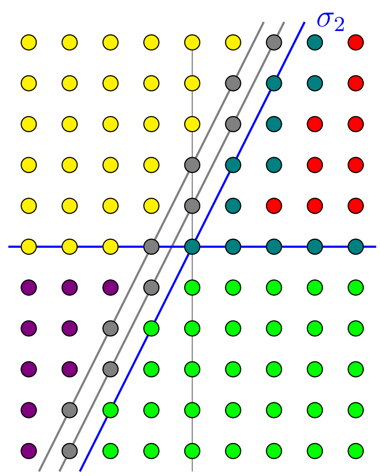


FIGURE 27. Regions with a factorial-continuous behavior for $\langle s^2t\rangle$

elements that do not have a factorial-continuous behavior for $\langle s^2t\rangle$ and degree 2.

In Figure 28, the multidegrees in the lines $y = 2x + 1$, $y = 2x + 2$, $y = 2x + 3$ and $y = -1$ are the elements that do not have a factorial-continuous behavior for $\langle s^3t^2\rangle$ and degree 2.

3. Differential operators and idealizers in R_{A_3}

In this section, we will compute the idealizer $\mathbb{I}(J)_{\boldsymbol{d}}$ for some ideals J over the ring of the rational normal curve of degree 3, $R_{A_3} = \mathbb{C}[s, st, st^2, st^3]$. In order to

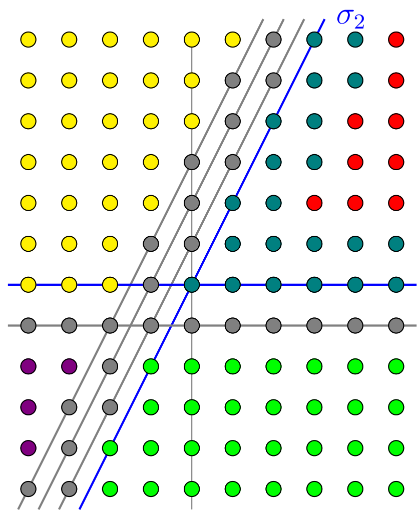


FIGURE 28. Regions with a factorial-continuous behavior for s^3t^2

aid our computations, we include some illustrations of the lattice representing the multidegrees in the plane, which is divided into four chambers; C1, C2, C3 and C4. For the rational curves of degree 3, A_3, the facets are given by $\sigma_1 = \{(x, y) \in \mathbb{N}^2 | x \geq 0, y = 0\}$ and $\sigma_2 = \{(x, y) \in \mathbb{N}^2 | x, y \geq 0, y = 3x\}$ that have primitive integral support functions $h_1 = \theta_2$ and $h_2 = 3\theta_1 - \theta_2$.

Figure 29 illustrates the integer lattice, divided into four chambers that are colored as follows:

C1: The red multidegrees correspond to monomials in R_{A_3}

C2: The yellow multidegrees are the d such that $h_1(\boldsymbol{d}) \geq 0$ and $h_2(\boldsymbol{d}) < 0$

C3: The violet multidegrees are the d such that $h_1(\boldsymbol{d}) < 0$ and $h_2(\boldsymbol{d}) < 0$, and

C4: The green multidegrees are the d such that $h_1(\boldsymbol{d}) < 0$ and $h_2(\boldsymbol{d}) \geq 0$.

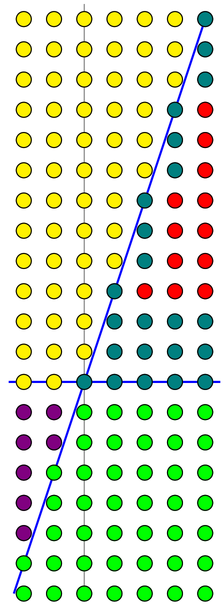


FIGURE 29. Chambers for R_{A_3}

Still following the convention $(h,n)! = 1$ if $n \geq 0$, then the graded pieces of $D(R_{A_3})$ are given by:

$$D(R_{A_2})_{\boldsymbol{d}} = s^{d_1}t^{d_2} \cdot \langle (h_1, h_1(-\boldsymbol{d}) - 1)!(h_2, h_2(-\boldsymbol{d}) - 1)! \rangle \tag{6}$$

This equation follows from Theorem 1 which comes from [2, Theorem 2.3] and [8, Theorem 3.2.2], which is the Saito-Traves description of differential operators., and broken down by chambers gives us

$$D(R_{A_3})_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, -3d_1 + d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C2 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_2, -3d_1 + d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C3 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)! \rangle & \text{if } \boldsymbol{d} \in C4 \end{cases}$$

3.1. **The idealizer for $J = \langle s^2t^3 \rangle$ over the ring R_{A_3}.** In this example we want to compute $\mathbb{I}(J)_{\boldsymbol{d}}$ for $J = \langle s^2t^3 \rangle$ and as in the previous section, we will show some figures in order to illustrate the computations.

Consider the following figure which contains the ideal generated by s^2t^3 represented by the red lattice points and the teal lattice points represent the elements that are in $R_{A_3} \setminus J$. Consider $\boldsymbol{d} = (-2, 1)$ in C2, $\boldsymbol{d} = (-3, -3)$ in C3 and

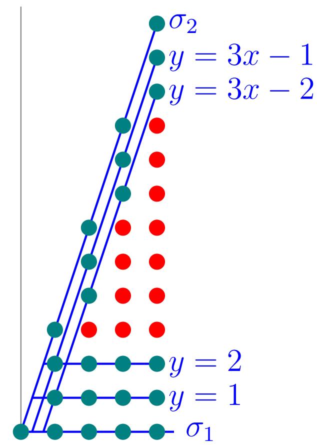
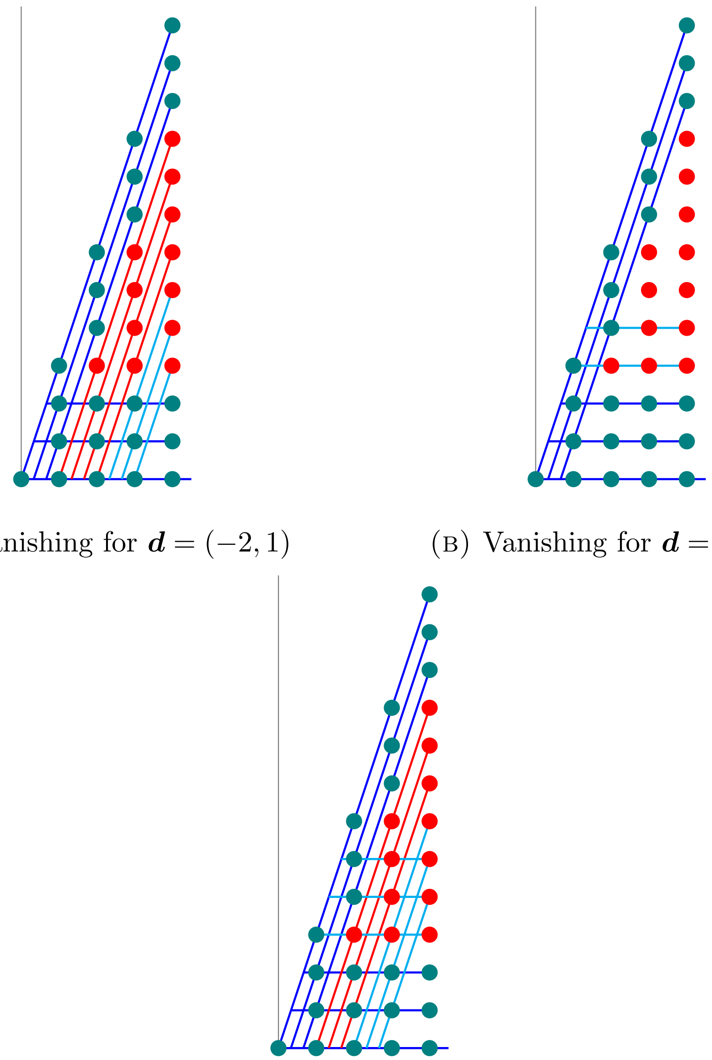


FIGURE 30. Elements in $R_{A_3} \setminus J$

$\boldsymbol{d} = (3, -2)$ in C4. Using an argument as in the previous section, we obtain:

Again, the light blue lines indicate the half-lines representing the multidegrees of monomials in J that after application of an element in $D(R_{A_3})_{(d_1,d_2)}$, fails to yield an element in J and we have to correct this lack of membership. The blue lines represent the multidegrees of monomials in the ring, but outside J, $R_{A_3} \setminus J$, but no changes or corrections are needed, and red lines indicate the multidegrees of monomials in the ideal that after application of an element in $D(R_{A_3})_{(d_1,d_2)}$ yield 0 and as 0 is in any ideal then no changes are required. Therefore, we obtain the following formulas for the multidegrees which are described in Figure 31.



(A) Vanishing for $\boldsymbol{d} = (-2, 1)$

(B) Vanishing for $\boldsymbol{d} = (3, -2)$

(C) Vanishing for $\boldsymbol{d} = (-3, -3)$

FIGURE 31. Different values $\boldsymbol{d}$ vanishing in $\langle s^2t^3 \rangle$

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, 9)! \rangle & \text{if } \boldsymbol{d} = (-2, 1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 5)!(h_2, 8)! \rangle & \text{if } \boldsymbol{d} = (-3, -3) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 1)!(h_1 - 3)(h_1 - 4) \rangle & \text{if } \boldsymbol{d} = (3, -2) \end{cases}$$

3.2. **The idealizer for $J = \langle s^3t^4 \rangle$ over the ring R_{A_3}.** In this example, we compute $\mathbb{I}(J)_{\boldsymbol{d}}$ for $J = \langle s^3t^4 \rangle$ using its figures. Figure 32 shows the ideal generated by s^3t^4 and it is represented by the red lattice points and teal lattice points represent the elements that are in $R_{A_3} \setminus J$.

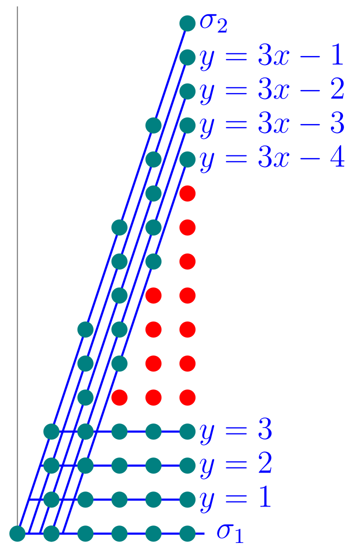
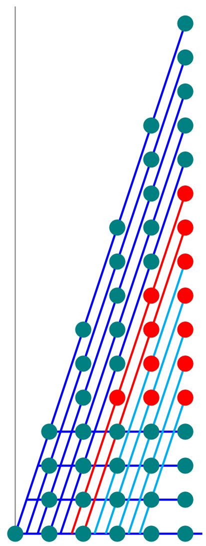


FIGURE 32. Elements in $R_{A_3} \setminus J$

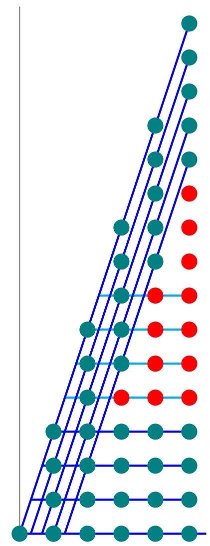
Consider $\boldsymbol{d} = (-2, -1)$ in C2, $(3, -4)$ in C3, and $(-3, -4)$ in C4, represented in Figure 33. Only the light blue lines need to be corrected since they indicate the half-lines representing the multidegrees of monomials in J that after application of an element in $D(R_{A_3})_{(d_1,d_2)}$, fails to yield an element in J. Thus, we can obtain the following formulas for the multidegrees which are described in 33.

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \langle (h_2, 11)! \rangle & \text{if } \boldsymbol{d} = (-2, 1) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 7)!(h_2, 9)! \rangle & \text{if } \boldsymbol{d} = (-3, -4) \\ s^{d_1}t^{d_2} \cdot \langle (h_1, 7)! \rangle & \text{if } \boldsymbol{d} = (3, -4) \end{cases}$$

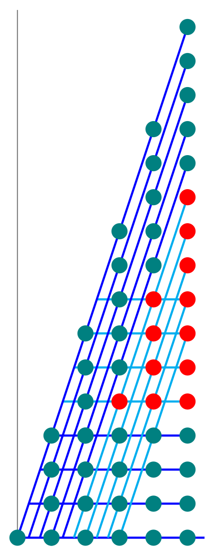
Observe that in both cases, the number of lines parallel to σ_2 is determined by $3m_1 - m_2$ and the number of lines parallel to σ_1 is determined by m_2.



(A) Vanishing for $\boldsymbol{d} = (-2, 1)$



(B) Vanishing for $\boldsymbol{d} = (3, -4)$



(C) Vanishing for $\boldsymbol{d} = (-3, -4)$

FIGURE 33. Different values $\boldsymbol{d}$ vanishing in $\langle s^3 t^4 \rangle$

3.3. **Formulas for R_{A_3}.** In this subsection, we will give a formula for the rational normal curves of degree 3. In particular, we will not include tables as we did for degree two because the same logic that was used to derive the formulas for degree two can be used to devise the formulas for degree three. However, the idea of how to determine the formulas is the same. The equation of one of the facets has changed and thus our formulas are altered.

Define

$$\phi_{3,1}(\boldsymbol{d}) := \phi_{3,1} = -d_2 - 1 + m_2 \tag{7}$$

and

$$\phi_{3,2}(\boldsymbol{d}) := \phi_{3,2} = 3m_1 - 3d_1 - m_2 + d_2 - 1 \tag{8}$$

The number of lines parallel to σ_2 that are not in the ideal J, but are in R_{A_3}, is given by $3m_1 - m_2$. Therefore, in order to have a formula for C2 that includes all the continuous terms, without skipping any term, we need that $-\boldsymbol{d}$ does not lie on one of the lines parallel to σ_2 in $R_{A_3} \setminus J$, that is, $-\boldsymbol{d}$ should lie in a line parallel to σ_2 inside J. Then, the condition

$$-3d_2 + d_1 \geq 3m_1 - m_2$$

guarantees the formula

$$s^{d_1}t^{d_2} \cdot \langle (h_2, 3m_1 - 3d_1 - m_2 + d_2 - 1)! \rangle$$

Define $h'_{3,1} = (h_1, -d_2 - 1)$ and $h'_{3,2} = (h_2, -3d_1 + d_2 - 1)!$, then, if $-3d_2 + d_1 < 3m_1 - m_2$ and $-3d_1 + d_2 = q$:

$$s^{d_1}t^{d_2} \cdot \langle h'_{3,2}(h_2 - \phi_{3,2} - q + 1, q - 1)! \rangle$$

In C4, as the case for degree 2, just consider the formula in terms of h_1 and the number of lines parallel to σ_1 that are in R_{A_3}, but are not in J is given by m_2.

Then, in order to have a continuous-factorial formula for C4, we need that $-d_2$ lies in a line parallel to σ_1 inside J. Then, the condition

$$-d_2 \geq m_2$$

gives us the factorial formula for C4

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle$$

For C4, if $-d_2 < m_2$ and $-d_2 = p$, then the formula is given by:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1)!(h_1 - \phi_{3,1} - p + 1, p - 1)! \rangle.$$

For C3, if $-d_2 \geq m_2$ and $-3d_2 + d_1 \geq 3m_1 - m_2$, then the formula is given by:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)!(h_2, 3m_1 - 3d_1 - m_2 + d_2 - 1)! \rangle$$

and we can have some combinations from the previous cases.

If $-d_2 \geq m_2$ and $-3d_2 + d_1 = q < 3m_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,1})! h'_{3,2}(h_2 - \phi_{3,2} - q + 1, q - 1)! \rangle$$

If $-d_2 = p < m_2$ and $-3d_2 + d_1 \geq 3m_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle h'_{3,1}(h_1 - \phi_{3,1} - p + 1, p - 1)!(h_2, \phi_{3,2})! \rangle.$$

Finally, if $-d_2 < m_2$ and $-d_2 = p$ and $-3d_2 + d_1 < 3m_1 - m_2$ and $-3d_2 + d_1 = q$ then

$$s^{d_1}t^{d_2} \cdot \langle h'_{3,1}(h_1 - \phi_{3,1} - p + 1, p - 1)! h'_{3,2}(h_2 - \phi_{3,2} - q + 1, q - 1)! \rangle$$

The formulas are given by: where $h_{3,2,q} := h'_{3,2}(h_2 - \phi_{3,2} - q + 1, q - 1)!$ and $h_{3,1,p} := h'_{3,1}(h_1 - \phi_{3,1} - p + 1, p - 1)!$.

Table 17. Formula for $\mathbb{I}(J)_{\boldsymbol{d}}$ in R_{A_3}

C	Condition 1	Condition 2	Formula
1			$s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$
2	$-3d_1 + d_2 \geq 3m_1 - m_2$		$s^{d_1}t^{d_2} \cdot \langle (h_2, \phi_{3,2})! \rangle$
	$-3d_1 + d_2 < 3m_1 - m_2$	$-3d_1 + d_2 = q$	$s^{d_1}t^{d_2} \cdot \langle h'_{3,2}(h_2 - \phi_{3,2} - q + 1, q - 1)! \rangle$
4	$-d_2 \geq m_2$		$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,1})! \rangle$
	$-d_2 < m_2$	$-d_2 = p$	$s^{d_1}t^{d_2} \cdot \langle h'_{3,1}(h_1 - \phi_{3,1} - p + 1, p - 1)! \rangle$
3	$-3d_1 + d_2 \geq 3m_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,1})!(h_2, \phi_{3,2})! \rangle$
	$q = -3d_1 + d_2 < 3m_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,1})! h_{3,2,q} \rangle$
	$-3d_1 + d_2 \geq 3m_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{3,1,p}(h_2, \phi_{3,2})! \rangle$
	$q = -3d_1 + d_2 < 3m_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{3,1,p} h_{3,2,q} \rangle$

In addition, we can consider the following expression that only contains factorial-continuous formulas:,

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta] & \text{if } \boldsymbol{d} \in C1 \\ s^{d_1}t^{d_2} \cdot \langle (h_2, \phi_{3,2})! \rangle & \text{if } \boldsymbol{d} \in C2 \text{ and } -3d_1 + d_2 \geq 3m_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,1})!(h_2, \phi_{3,2})! \rangle & \text{if } \boldsymbol{d} \in C3 \text{ and } -d_2 \geq m_2 \text{ and} \\ & \qquad -3d_1 + d_2 \geq 3m_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{3,2})! \rangle & \text{if } \boldsymbol{d} \in C4 \text{ and } -d_2 \geq m_2 \end{cases}$$

This expression exhibits the conditions on the multidegrees $\boldsymbol{d}$ where the differential operators of degree $\boldsymbol{d}$ have factorial-continuous behavior occurs, from Definition 27.

4. Formulas for the idealizer $\mathbb{I}(J)_{\boldsymbol{d}}$ for the rational normal curve in degree n

For any ideal $J = \langle s^{m_1}t^{m_2} \rangle$ such that $m_1 \geq 2$ and $m_2 \geq 0$ over the ring of the rational normal curve of degree n, $R_{A_n} = \mathbb{C}[s, st, st^2, \cdots, st^n]$, $n \geq 2$, we can give a formula for the idealizer.

The plane including R_{A_n} can be divided into four chambers in a similar way that we divided the plane for degrees two and three. The facets of A_n are given by $\sigma_1 = \{(x, y) \in \mathbb{N}^2 | x \geq 0, y = 0\}$ and $\sigma_2 = \{(x, y) \in \mathbb{N}^2 | x, y \geq 0, y = nx\}$ that have primitive integral support functions $h_1 = \theta_2$ and $h_2 = n\theta_1 - \theta_2$. The primitive integral support functions $h_1 = \theta_2$ and $h_2 = n\theta_1 - \theta_2$ are very important elements in order to determine the formula since they modify the facets and consequently it modifies the inequalities. Thus, the number of lines parallel to σ_2 that are not in the ideal J, but are in R_{A_n}, is given by $nm_1 - m_2$ and the the number of lines parallel to σ_1 that are in R_{A_n}, but are not in J is given by m_2. Then, for all $n \geq 2$ and ideal $J = \langle s^{m_1} t^{m_2} \rangle$ we will have plots similar to those obtained in the previous sections. The facet σ_2 is the only significant modification as n changes.

Therefore, to have a formula for C2 that includes all the continuous terms, we need that $-\boldsymbol{d}$ does not lie on one of the lines parallel to σ_2 in $R_{A_n} \setminus J$, that is, $-\boldsymbol{d}$ should lie in a line parallel to σ_2 inside J. Then, the condition

$$-nd_2 + d_1 \geq nm_1 - m_2$$

guarantees the formula

$$s^{d_1} t^{d_2} \cdot \langle (h_2, nm_1 - nd_1 - m_2 + d_2 - 1)! \rangle$$

for C2, which is similar to $s^{d_1} t^{d_2} \cdot \langle (h_2, 2m_1 - 2d_1 - m_2 + d_2 - 1)! \rangle$ and $s^{d_1} t^{d_2} \cdot \langle (h_2, 3m_1 - 3d_1 - m_2 + d_2 - 1)! \rangle$ that are the formulas that we obtained for degree 2 and 3, respectively.

Now, define the following functions:

$$\phi_{n,1}(\boldsymbol{d}) := \phi_{n,1} = -d_2 - 1 + m_2 \tag{9}$$

$$\phi_{n,2}(\boldsymbol{d}) := \phi_{n,2} = nm_1 - nd_1 - m_2 + d_2 - 1. \tag{10}$$

$$h'_{n,1} = (h_1, -d_2 - 1)$$

$$h'_{n,2} = (h_2, -nd_1 + d_2 - 1)!$$

If $-nd_2 + d_1 < nm_1 - m_2$ and $-nd_1 + d_2 = q$; thus,

$$s^{d_1} t^{d_2} \cdot \langle h'_{n,2}(h_2 - \phi_{n,2} - q + 1, q - 1)! \rangle.$$

For the formula for C4: the number of lines parallel to σ_1 that are in R_{A_n}, but are not in J is given by m_2. Then, in order to have a continuous-factorial formula for C4, we need that $-d_2$ lies in a line parallel to σ_1 inside J. Then, the condition

$$-d_2 \geq m_2$$

gives us the factorial formula for C4

$$s^{d_1} t^{d_2} \cdot \langle (h_1, -d_2 - 1 + m_2)! \rangle$$

if $-d_2 < m_2$ and $-d_2 = p$, then the formula is given by:

$$s^{d_1} t^{d_2} \cdot \langle h'_{n,1}(h_1 - \phi_{n,1} - p + 1, p - 1)! \rangle.$$

For the formula for C3: a combination of the conditions $-nd_2 + d_1 \geq nm_1 - m_2$ and $-d_2 \geq m_2$ guarantees the continuous formula for C3; if $-d_2 \geq m_2$ and $-nd_2 + d_1 \geq nm_1 + m_2$, then the formula is given by:

$$s^{d_1} t^{d_2} \cdot \langle (h_1, \phi_{n,1})!(h_2, \phi_{n,2})! \rangle$$

and we have combinations from the previous cases.

If $-d_2 \geq m_2$ and $-nd_2 + d_1 = q < nm_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,1})! h'_{n,2}(h_2 - \phi_{n,2} - q + 1, q - 1)! \rangle$$

If $-d_2 = p < m_2$ and $-nd_2 + d_1 \geq nm_1 - m_2$ then:

$$s^{d_1}t^{d_2} \cdot \langle h'_{n,1}(h_1 - \phi_{n,1} - p + 1, p - 1)!(h_2, \phi_{n,2})! \rangle.$$

Finally, if $-d_2 = p < m_2$ and $-nd_2 + d_1 = q < nm_1 - m_2$ then the formula is:

$$s^{d_1}t^{d_2} \cdot \langle h'_{n,1}(h_1 - \phi_{n,1} - p + 1, p - 1)! h'_{n,2}(h_2 - \phi_{n,2} - q + 1, q - 1)! \rangle.$$

Therefore, from the computations that we have done above, we conclude the following proposition.

Proposition 3. *Let $R_{A_n} = \mathbb{C}[s, st, st^2, \cdots, st^n]$, the ring of the rational normal curve of degree n for $n \geq 2$. For any multidegree $\boldsymbol{d}$ and ideal $J = \langle s^{m_1}t^{m_2} \rangle$ such that $m_1 \geq 1$, $m_2 \geq 0$ and $nm_1 - m_2 \geq 0$, the formula for the idealizer in multidegree is given by the entries in the last column of Table 18.*

For each chamber, as long as condition 1 and 2 are satisfied then the formula is given by the respective formula in the last column. where $h_{n,2,q} := h'_{n,2}(h_2 - (\phi_{n,2} - (q-1))) \cdots (h_2 - \phi_{n,2})$
and $h_{n,1,p} := h'_{n,1}(h_1 - (\phi_{n,1} - (p-1))) \cdots (h_1 - \phi_{n,1})$.

Corollary 1. *Let $R_{A_n} = \mathbb{C}[s, st, st^2, \cdots, st^n]$, the ring of the rational normal curve of degree n for $n \geq 2$. Given the same hypothesis as in the last proposition*

TABLE 18. Formula for $\mathbb{I}(J)_{\boldsymbol{d}}$ in R_{A_n}

C	Condition 1	Condition 2	Formula
1			$s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta]$
2	$-nd_1 + d_2 \geq nm_1 - m_2$		$s^{d_1}t^{d_2} \cdot \langle (h_2, \phi_{n,2})! \rangle$
	$-nd_1 + d_2 < nm_1 - m_2$	$-nd_1 + d_2 = q$	$s^{d_1}t^{d_2} \cdot \langle h'_{n,2}(h_2 - \phi_{n,2} - q + 1, q - 1)! \rangle$
4	$-d_2 \geq m_2$		$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,1})! \rangle$
	$-d_2 < m_2$	$-d_2 = p$	$s^{d_1}t^{d_2} \cdot \langle h'_{n,1}(h_1 - \phi_{n,1} - p + 1, p - 1)! \rangle$
3	$-nd_1 + d_2 \geq nm_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,1})!(h_2, \phi_{n,2})! \rangle$
	$-nd_1 + d_2 = q < nm_1 - m_2$	$-d_2 \geq m_2$	$s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,1})! h_{n,2,q} \rangle$
	$-nd_1 + d_2 \geq nm_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{n,1,p}(h_2, \phi_{n,2})! \rangle$
	$-nd_1 + d_2 = q < nm_1 - m_2$	$-d_2 = p < m_2$	$s^{d_1}t^{d_2} \cdot \langle h_{n,1,p} h_{n,2,q} \rangle$

and as long as $-nd_1 + d_2 \geq nm_1 - m_2$ *and/or* $-d_2 \geq m_2$ *are satisfied, the idealizer can be expressed as follows:*

$$\mathbb{I}(J)_{\boldsymbol{d}} = \begin{cases} s^{d_1}t^{d_2} \cdot \mathbb{C}[\theta] & \text{if } \boldsymbol{d} \in C1 \\ s^{d_1}t^{d_2} \cdot \langle (h_2, \phi_{n,2})! \rangle & \text{if } \boldsymbol{d} \in C2 \text{ and } -nd_1 + d_2 \geq nm_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,1})!(h_2, \phi_{n,2})! \rangle & \text{if } \boldsymbol{d} \in C3 \text{ and } -d_2 \geq m_2 \\ & \qquad -nm_1 + d_2 \geq nm_1 - m_2 \\ s^{d_1}t^{d_2} \cdot \langle (h_1, \phi_{n,2})! \rangle & \text{if } \boldsymbol{d} \in C4 \text{ and } -d_2 \geq m_2 \end{cases}$$

This expression is a special case that exhibits the conditions for the multidegrees $\boldsymbol{d}$ where the differential operators of degree $\boldsymbol{d}$ have factorial-continuous behavior occurs.

Again, the primitive integral support functions $h_1 = \theta_2$ and $h_2 = n\theta_1 - \theta_2$ are very important elements in order to determine the formula because they modify the facets and the inequalities are modified by those.

5. **Future directions and concluding remarks**

In future directions, similar techniques could be used to compute $D(R, J)$ for principal ideals J. Also similar analysis could be used to determine $\mathbb{I}(J)$ and $D(R, J)$ for other monomial ideals. For example, in [1]; differential operators, retracts, and toric face rings, it is determined that $\mathbb{I}(J)/D(R, J)$ is a subring of $D(R/J)$ when J is the special monomial ideal which includes all monomials associated to the interior of the cone and a similar statement is true for any monomial ideal J.

In this document, we computed the idealizer for monomials in ideals J that have two generators; however, it is possible to compute it for ideals with more generators and similar techniques can be useful in order to determine these objects, or compute $\mathbb{I}(J)$ for more complicated regions or rings. For instance, Saito and Traves in [8] have determined the ring of differential operators for saturated affine semigroup rings, and we can use their work to compute $\mathbb{I}(J)$, $D(R_A, J)$ and $\mathbb{I}(J)/D(R_A, J)$ for non-normal rational normal curves R_A and a graded radical R_A-ideal J.

On the other hand, determining the idealizer for monomial ideals J of R_{A_n} will be helpful to find differential operators δ for which J is δ-compatible and δ-fixed as Lance Miller, William Taylor and Janet Vassilev are developing in [6].

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