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The Elastic Scattering of Deuterons by Deuterons

Richard John Runge

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THE ELASTIC SCATTERING OF DEUTERONS BY DEUTERONS

A Dissertation

Presented to

the Faculty of the Graduate School

University of New Mexico

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in Physics

by

Richard John Range

July 1951

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This dissertation, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

E. H. Castetter
DEAN

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THE ELASTIC SCATTERING OF DEUTERONS BY DEUTERONS

By

Richard John Runge

Committee

Roy Thomas
CHAIRMAN

John R. Green

J. L. Riebsamer

Victor H. Regeuer

Laurel Froman

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1.1. The problem is stated as follows. The main theorem is that

in the case of the point at which the function is zero.

1.2. The first part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$.

1.3. The second part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.4. The third part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.5. The fourth part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

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1.7. The sixth part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.8. The seventh part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.9. The eighth part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.10. The ninth part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.11. The tenth part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

1.12. The eleventh part of the problem is the following. Let $f(x)$ be a function of x which is continuous at x_0 and has a finite limit L as $x \rightarrow x_0$. Let $g(x)$ be a function of x which is continuous at x_0 and has a finite limit M as $x \rightarrow x_0$.

INTRODUCTION

The concern of this work is the calculation of elastic scattering of deuterons by deuterons. The solution of the problem will be given in the usual center of mass system of coordinates, the conversion of results from this to the laboratory system being standard.⁵ Since the deuteron itself consists of a neutron and a proton, the problem is thus one of four bodies. The nuclear three body problem of the scattering of a neutron by a deuteron has been treated by previous authors^{1,2,3} and we will have occasion to employ methods used by these authors extending them to the four body problem. There are a number of possibilities in a deuteron-deuteron collision, most of them inelastic. However, we shall restrict ourselves to the elastic case.

The elastic scattering will constitute a certain fraction of the total collision process and should presumably be the simplest one for which a calculation can be made. The assumption of elastic scattering implies that initially both deuterons are in their ground state and that, after scattering, this will also be true, none of the energy available in the center of mass system being used to produce excited states. In addition, we shall simplify the analysis by considering the ground state to be a 3S state, although it has been shown that the deuteron is actually a 96% 3S and 4% 3D mixture.^{4,8} This allows us to use a spherically symmetric ground state wave function for the deuterons. As will be indicated later, a choice of

Gaussian functions of the type $ae^{-\lambda^2 r^2}$ (a and λ being constants) for these wave functions enables us to evaluate all subsequent integrations and obtain an answer in terms of known functions.

The nuclear interaction assumed to hold between the nucleons will be of the form $F_{ij} J(r_{ij})$ where r_{ij} is the magnitude of the distance between nucleons i and j , $J(r_{ij})$ is a function of this magnitude alone and F_{ij} depends only on spin and coordinate exchange operators Q_{ij} and P_{ij} . The function J will be taken as a Gaussian function^{1,2}, or the sum of a Gaussian function and Coulomb interaction in case i and j are protons. Thus, we shall assume that J is the same for the neutron-neutron and the neutron-proton interaction as well as for the nuclear part of the proton-proton interaction.

The procedure employed in the calculation is to set up the wave equation, express it in suitable coordinates in the center of mass system and then proceed to a simplification of the problem by introducing a properly symmetrized solution consistent with Pauli's principle for like particles. After this, a summation over possible spin states is made and the deuteron coordinates are integrated out. This leaves one variable, the radius vector between the centers of mass of the two deuterons. A solution of this equation in the form of an incident plane wave together with a scattered wave can then be readily obtained by means of Born's approximation.⁶ The theory developed will thus not be applicable to the case of low

Generation function of the type $Q_2 = \lambda \cdot \lambda^2$

where λ is the wave number and λ^2 is the wave number squared.

The model function is assumed to be of the form

$$f(\lambda) = \frac{1}{\lambda^2} \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^4} \right)$$

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of the form $f(\lambda) = \frac{1}{\lambda^2} \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^4} \right)$

energy collisions. Now the solution for the scattered wave gives us the angular dependence of the scattering, that is, the differential cross section. These differential cross sections must be determined for the three spin states, the singlet, triplet and quintet corresponding to a total spin quantum number $S = 0, 1, 2$ for a four body system. These cross sections are then combined in the properly weighted ratio to give the average differential cross section over all three states.

For the type of interaction we shall consider, it will not be difficult to show that the total spin of the system is a constant of the motion and that transitions between the different spin states will not occur. In the final chapter, a comparison of the theoretical averaged differential cross section with some experimental data is made.

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CHAPTER I

THE WAVE EQUATION AND THE WAVE FUNCTIONS

1. The Wave Equation for the Problem.

In this section we shall consider the Schrodinger equation for the scattering. We take the mass of the neutron and proton equal and consider the system to have a total energy E_{LAB} (including binding energies) in the laboratory system. If E is the total energy in the center of mass system and $\mathcal{E}$ the kinetic energy of the center of mass in the laboratory system then

$$E_{LAB} = E + \mathcal{E} \quad (1.1)$$

Figure 1-Coordinates in the laboratory system.

In figure 1, let $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and $\vec{r}_4$ represent the position vectors of nucleons 1, 2, 3 and 4 respectively where;
 nucleon 1 = the neutron in the first deuteron,

nucleon 2 = the proton in the first deuteron,

nucleon 3 = the neutron in the second deuteron,

nucleon 4 = the proton in the second deuteron.

Then if $V(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)$ represents the total interaction between the four nucleons, the wave equation can be written as

$$-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2 + \nabla_3^2 + \nabla_4^2) \Psi + V \Psi = E_{LAB} \Psi \quad (1.2)$$

in the laboratory coordinates. In (1.2), m is the mass of a nucleon. An equivalent analysis may be obtained in the center of mass coordinates. Setting (see fig.1),

$$\begin{aligned} \vec{\rho}_1 &= \vec{r}_1 - \vec{r}_2, & \vec{r} &= \frac{\vec{r}_1 + \vec{r}_2}{2} - \frac{\vec{r}_3 + \vec{r}_4}{2}, \\ \vec{\rho}_2 &= \vec{r}_3 - \vec{r}_4, & \vec{R} &= \frac{1}{2} \left[\frac{\vec{r}_1 + \vec{r}_2}{2} + \frac{\vec{r}_3 + \vec{r}_4}{2} \right], \end{aligned} \quad (1.3)$$

we see that the system can be described by this new set of independent variables as in figure 2.,

Figure 2-The center of mass system.

motion 2 = the motion of the first motion

motion 3 = the motion of the second motion

motion 4 = the motion of the third motion

Then if $V(\vec{r}, \vec{p})$ is the Hamiltonian function

the four motions, the four equations can be written as

$$-\frac{\partial V}{\partial p_i} = \dot{q}_i, \quad \frac{\partial V}{\partial q_i} = \dot{p}_i$$

in the laboratory coordinate system, i.e.,

An equivalent analysis can be made in the center of mass

system. Defining the relative

$$\vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \quad \vec{p} = \frac{m_1 \vec{p}_1 + m_2 \vec{p}_2}{m_1 + m_2}$$

$$\vec{r} = \frac{1}{2} \left[\frac{m_1 + m_2}{m_1} \vec{r}_1 + \frac{m_1 + m_2}{m_2} \vec{r}_2 \right]$$

we see that the system can be described by the set of independent

variables as in Figure 1.

Figure 1 - Diagram of the system

In this new system, $\vec{R}$, describes the motion of the center of mass. It is then possible to factor out the motion of the center of mass (a plane wave) and obtain an equation in $\vec{r}$, $\vec{p}_1$ and $\vec{p}_2$.

In order to write equation (1.2) in terms of these variables, observe that the total kinetic energy can be expressed as the sum of four terms

$$T_R + T_r + T_{p_1} + T_{p_2},$$

where T_R is the kinetic energy of the center of mass in the laboratory system, T_r is the kinetic energy of the two deuteron units in the center of mass system while T_{p_1} and T_{p_2} are the kinetic energies of the nucleon pairs within the deuterons 1 and 2. The total interaction V can be written as

$$V = V_{INT} + V_{12} + V_{34},$$

where V_{INT} represents interactions between the deuterons, i.e.,

$$V_{INT} = V_{13} + V_{14} + V_{23} + V_{24},$$

and V_{12} , V_{34} represent the interaction within the deuterons.

The total Hamiltonian is then

$$T_R + T_r + V_{INT} + (T_{p_1} + V_{12}) + (T_{p_2} + V_{34}) \quad (1.4)$$

In this case, $\bar{R}$, is the mean of the vector $\mathbf{R}$. It is then possible to express the mean of the vector $\mathbf{R}$ in terms of the mean of the vector $\mathbf{R}$ and the mean of the vector $\mathbf{R}$.

in order to solve equation (1.1) for the mean of the vector $\mathbf{R}$, we observe that the total kinetic energy can be expressed as the sum of four terms

$$\bar{T}_R + \bar{T}_L + \bar{T}_A + \bar{T}_B$$

where $\bar{T}_R$ is the mean of the vector $\mathbf{R}$ in the laboratory system, $\bar{T}_L$ is the mean of the vector $\mathbf{L}$ in the laboratory system, $\bar{T}_A$ is the mean of the vector $\mathbf{A}$ in the laboratory system, and $\bar{T}_B$ is the mean of the vector $\mathbf{B}$ in the laboratory system. The total interaction V can be written as

$$V = V_{int} + V_1 + V_2 + V_3 + V_4$$

where V_{int} represents the interaction between the particles, V_1 , V_2 , V_3 , and V_4 represent the interaction between the particles and the walls.

$$V_{int} = V_1 + V_2 + V_3 + V_4$$

and V_1 , V_2 , V_3 , and V_4 represent the interaction between the particles and the walls. The total Hamiltonian is then

$$(1.1) \quad \bar{T}_R + \bar{T}_L + \bar{T}_A + \bar{T}_B + (V_{int} + V_1 + V_2 + V_3 + V_4)$$

Setting

$$\begin{aligned} T_{p_1} + V_{12} &= H_1, \\ T_{p_2} + V_{34} &= H_2, \end{aligned} \quad (1.5)$$

we have the two deuteron Hamiltonians. Likewise setting

$$H_R = T_R, \quad (1.6)$$

one obtains the Hamiltonian for the center of mass motion. Setting

$$H_r = V_{INT} + T_r, \quad (1.7)$$

the interaction Hamiltonian in the center of mass system is obtained.

The wave equation in these new variables is obtained from the Hamiltonian by the operator substitutions $\hbar/L \nabla \rightarrow \vec{P}$ in the kinetic energy terms of the Hamiltonian; thus one obtains

$$(H_R + H_r + H_1 + H_2) \bar{\Psi}(\vec{R}, \vec{r}, \vec{p}_1, \vec{p}_2) = E_{LAB} \bar{\Psi}, \quad (1.8)$$

or explicitly

$$\left(-\frac{\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2M_r} \nabla_r^2 + H_1 + H_2 + V_{INT} \right) \bar{\Psi} = E_{LAB} \bar{\Psi}, \quad (1.9)$$

$$T_1 + V_1 = H_1$$

(1.2)

$$T_2 + V_2 = H_2$$

we have the two identical Hamiltonians. Likewise setting

(1.3)

$$H_R = T_R$$

one obtains the Hamiltonian for the center of mass motion. Setting

(1.4)

$$H_T = V_T + T_T$$

the interaction Hamiltonian in the center of mass system is obtained.

The wave equation for these two particles is obtained from the

Hamiltonian by the operator substitution $\mathbf{p} \rightarrow \nabla$ in the

kinetic energy terms of the Hamiltonian; thus one obtains

$$(H_R + H_T + H_1 + H_2) \Psi(\mathbf{R}, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) = E_{\text{lab}} \Psi \quad (1.5)$$

or explicitly

$$(-\frac{\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2m} \nabla_r^2 + V_1 + V_2 + H_1 + H_2 + V_T) \Psi = E_{\text{lab}} \Psi \quad (1.6)$$

where

$$\mu = 4m$$

and (1.10)

$$\mu_r = \frac{(2m)^2}{2m + 2m} = m.$$

Now Ψ can be expressed in the form

$$\Psi = \Phi(\vec{R}) \psi(\vec{r}, \vec{p}_1, \vec{p}_2), \quad (1.11)$$

where Φ is the plane wave satisfying

$$-\frac{\hbar^2}{8m} \nabla_R^2 \Phi = E \Phi, \quad (1.12)$$

E being given in equation (1.1). Hence cancelling out the $\vec{R}$ part of Ψ , one obtains

$$\left(-\frac{\hbar^2}{2m} \nabla_r^2 + H_1 + H_2 + V_{INT}\right) \psi(\vec{r}, \vec{p}_1, \vec{p}_2) = E \psi. \quad (1.13)$$

We also point out that if $\psi_0(\vec{p}_1)$ and $\psi_0(\vec{p}_2)$ are the ground state wave functions of the two deuterons, then the following equations will hold,

$$\begin{aligned} H_1 \psi_0(\vec{p}_1) &= E_B \psi_0(\vec{p}_1), \\ H_2 \psi_0(\vec{p}_2) &= E_B \psi_0(\vec{p}_2), \end{aligned} \quad (1.14)$$

where $E_B = -2.2 \text{ MEV.}$ is the deuteron binding energy.¹⁰

2. The Wave Functions with Spin and the Symmetrized Solution.

The wave function ψ in (1.13) includes a spin wave func-

The wave function Ψ in (1.1) is a function of position and time.

2. The wave function with respect to the system is given by

$$\Psi = \Psi(\vec{r}, t) \quad \text{where } \vec{r} = (x, y, z) \text{ is the position vector.}$$

$$H_1 \Psi(\vec{r}, t) = E_1 \Psi(\vec{r}, t)$$

$$H_2 \Psi(\vec{r}, t) = E_2 \Psi(\vec{r}, t)$$

equations will hold.

These wave functions are the eigenfunctions of the Hamiltonian.

As also point out that $\Psi(\vec{r}, t)$ and $\Psi(\vec{r}, t)$ are the same

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right) \Psi(\vec{r}, t) = E \Psi(\vec{r}, t) \quad (1.1)$$

part of Ψ is a constant.

being given by equation (1.1).

$$\nabla^2 \Psi = -\frac{2m}{\hbar^2} (E - V) \Psi$$

where Ψ is the wave function.

$$\Psi = \Phi(\vec{r}) \psi(\vec{r}, t) \quad (1.2)$$

Now

$$\Psi = \frac{\psi(\vec{r}, t)}{\psi(\vec{r}, 0)} = \psi(\vec{r}, t)$$

or

$$\Psi = \psi(\vec{r}, t)$$

where

tion χ_L describing the spin state of the system. Hence in what follows we shall refer to the wave function $\psi_L = \psi \chi_L$ as the solution of (1.13). A detailed discussion of the spin wave functions χ_L will be given.

If we denote by P_{ij} an operator which exchanges the coordinates of the i th and j th nucleon and by Q_{ij} an operator which exchanges their spins, i.e.,

$$P_{ij} G(\dots \vec{r}_i \dots \vec{r}_j \dots) = G(\dots \vec{r}_j \dots \vec{r}_i \dots),$$

$$Q_{ij} \chi_m(\dots i \dots j \dots) = \chi_m(\dots j \dots i \dots),$$

then it is easy to see that the operators $P_{13} Q_{13}$, $P_{24} Q_{24}$ and $P_{13} Q_{13} P_{24} Q_{24}$ all commute with the total Hamiltonian (1.2) and hence with (1.9) provided V commutes with these operators. Now each V_{ij} in V has the form $F_{ij} J(r_{ij})$ where F_{ij} depends only on coordinate and spin exchange operators. In this case the commutation is not difficult to show. Hence $P_{13} Q_{13}$ and $P_{24} Q_{24}$ will commute with the Hamiltonian in (1.9). Since $\vec{R}$ is symmetric in 1, 2, 3 and 4 we can conclude that these operators will therefore commute with

$H_r + H_1 + H_2$. This fact enables us to write the following four equations;

$$(H_r + H_1 + H_2) \psi_L = E \psi_L,$$

$$(H_r + H_1 + H_2) P_{13} Q_{13} \psi_L = E P_{13} Q_{13} \psi_L,$$

along X_i describing the spin state of the system. Hence

what follows we shall refer to the wave function $\psi = \psi(X_i)$

as the solution of (1.13). A detailed discussion of the spin wave

functions X_i will be given.

If we denote by P_{ij} an operator which exchanges the

coordinates of the i th and j th nucleus and by Q_{ij} an operator

which exchanges their spins, i.e.,

$$P_{ij} \psi(\dots r_i \dots r_j \dots) = \psi(\dots r_j \dots r_i \dots)$$

$$Q_{ij} \psi(\dots \uparrow \dots \downarrow \dots) = \psi(\dots \downarrow \dots \uparrow \dots)$$

then it is easy to see that the operators $P_{ij} Q_{ij}$ and $P_{ij} Q_{ij}^{-1}$

and $P_{ij} Q_{ij} P_{ij} Q_{ij}^{-1}$ all commute with the total Hamiltonian

(1.2) and hence with (1.3) provided V commutes with them.

operator. Now each V_{ij} is in V has the form $F_{ij}(r_i, r_j)$

where F_{ij} depends only on coordinates and spin exchange oper-

ators. In this case the commutation is not difficult to show.

hence $P_{ij} Q_{ij}$ and $P_{ij} Q_{ij}^{-1}$ will commute with the Hamil-

tonian in (1.3). Since $\bar{R}$ is symmetric in $1, 2, 3$ and $1, 2$

can conclude that these operators will therefore commute with

$$H_v + H_1 + H_2$$

following four equations:

$$(H_v + H_1 + H_2) \psi_e = E \psi_e$$

$$(H_v + H_1 + H_2) P_{12} Q_{12} \psi_e = E P_{12} Q_{12} \psi_e$$

$$(H_r + H_1 + H_2) P_{24} Q_{24} \psi_c = E P_{24} Q_{24} \psi_c ,$$

$$(H_r + H_1 + H_2) P_{13} Q_{13} P_{24} Q_{24} \psi_c = E P_{13} Q_{13} P_{24} Q_{24} \psi_c . \quad (1.15)$$

Hence ψ_c , $P_{13} Q_{13} \psi_c$, $P_{24} Q_{24} \psi_c$ and $P_{13} Q_{13} P_{24} Q_{24} \psi_c$ are all solutions of the center of mass wave equation (1.13), and exchange of like particles is therefore a constant of motion in the center of mass system.

We now can construct a solution of (1.13) which satisfies the Pauli principle in being antisymmetric in coordinate and spin exchange of like particles, thus taking into account the allowable exchanges in the scattering. Such a function would be,

$$\psi_{\text{ANTI-SYMM}} = (1 - P_{13} Q_{13})(1 - P_{24} Q_{24}) \psi_c , \quad (1.16)$$

or abbreviating $\psi_{\text{ANTI-SYMM}} = K \psi_c$ where

$$K = (1 - P_{13} Q_{13})(1 - P_{24} Q_{24}) . \quad (1.17)$$

This is the properly symmetrized solution of (1.13), since combining equations (1.15) yields

$$(H_r + H_1 + H_2) K \psi_c = E K \psi_c . \quad (1.18)$$

$$(H_1 + H_2) P_{21} \psi = E P_{21} \psi$$

$$(H_1 + H_2) P_{21} P_{13} P_{21} \psi = E P_{21} P_{13} P_{21} \psi \quad (1.12)$$

$$\text{Hence } \psi, P_{13} \psi, P_{21} \psi, \text{ and } P_{13} P_{21} \psi$$

are all solutions of the center of mass wave equation (1.12), and

exchange of like particles is therefore a constant of motion in

the center of mass system.

We now construct a solution of (1.12) which satisfies the

Pauli principle in being antisymmetric in coordinate and spin ex-

change of like particles, thus taking into account the antisym-

metrization in the wavefunction. Such a function would be

$$\psi_{\text{anti-sym}} = (1 - P_{12})(1 - P_{21}) \psi \quad (1.13)$$

$$\text{or abbreviating } \psi_{\text{anti-sym}} = K \psi \quad \text{where}$$

$$K = (1 - P_{12})(1 - P_{21}) \quad (1.14)$$

This is the properly symmetrized solution of (1.12), since combining

equation (1.12) yields

$$(H_1 + H_2) K \psi = E K \psi \quad (1.15)$$

3. The Wave Equation in $\vec{r}$

Each of the solutions of (1.15) can be expanded in terms of an orthogonal set of deuteron ground state plus continuum wave functions and spin wave functions as follows:⁵

$$\psi_j = \sum f_i(\vec{r}) \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i \int_0^{E-E_B} f_i(\epsilon, \vec{r}) \psi_\epsilon(1) \psi_0(2) d\epsilon$$

$$+ \sum \chi_i \int_0^{E-E_B} f_i'(\epsilon, \vec{r}) \psi_0(1) \psi_\epsilon(2) d\epsilon \quad (1.19)$$

$$+ \sum \chi_i \int_0^{E-E^*} \int_0^{E^*} f_i''(\epsilon_1, \epsilon_2, \vec{r}) \psi_{\epsilon_1}(1) \psi_{\epsilon_2}(2) d\epsilon_1 d\epsilon_2,$$

(1.20)

$$P_{13} Q_{13} \psi_j = \sum g_i(\vec{r}) \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i (g\text{-INTEGRALS}),$$

(1.21)

$$P_{24} Q_{24} \psi_j = \sum h_i(\vec{r}) \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i (h\text{-INTEGRALS}),$$

$$P_{13} Q_{13} P_{24} Q_{24} \psi_j = \sum k_i(\vec{r}) \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i (k\text{-INTEGRALS}),$$

(1.22)

where the χ_i are spin wave functions, $\psi_0(1)$ and $\psi_0(2)$ are the deuteron ground state wave functions while ψ_ϵ are

Each of the sets of ψ_i is an eigenfunction of the Hamiltonian and forms an orthogonal set of functions. The functions and eigenvalues are denoted by

$$\psi_i = \sum_k f_k(\mathbf{r}) \phi_k(\mathbf{r}) + \sum_k g_k(\mathbf{r}) \phi_k(\mathbf{r})$$

$$+ \sum_k h_k(\mathbf{r}) \phi_k(\mathbf{r}) + \sum_k i_k(\mathbf{r}) \phi_k(\mathbf{r})$$

$$\psi_i = \sum_k f_k(\mathbf{r}) \phi_k(\mathbf{r}) + \sum_k g_k(\mathbf{r}) \phi_k(\mathbf{r})$$

$$\psi_i = \sum_k f_k(\mathbf{r}) \phi_k(\mathbf{r}) + \sum_k g_k(\mathbf{r}) \phi_k(\mathbf{r})$$

$$\psi_i = \sum_k f_k(\mathbf{r}) \phi_k(\mathbf{r}) + \sum_k g_k(\mathbf{r}) \phi_k(\mathbf{r})$$

where the ϕ_k are spin wave functions, $\phi_k(\mathbf{r})$ and $\phi_k(\mathbf{r})$ are the two spin components of the wave function ϕ_k .

the continuum wave functions of the deuteron. The summations are over the set of orthogonal spin wave functions χ_i . Setting

$$H_D = H_1 + H_2 \quad \text{so that}$$

$$H_D \psi_0(1) \psi_0(2) = 2 E_B \psi_0(1) \psi_0(2) ,$$

and defining E_0 as the kinetic energy of the system in the center of mass coordinates when the deuterons are infinitely far apart, we obtain

$$E = E_0 + 2 E_B \quad . \quad (1.23)$$

Taking the expansion for ψ_f and substituting in the first equation of (1.15) one obtains,

$$\begin{aligned} (H_r + H_D) \psi_f &= E \psi_f = \\ & \left(-\frac{\hbar^2}{2m} \nabla_r^2 + V_{INT} \right) \left[\sum f_i \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i \text{ INTEGRALS} \right] \\ & + \left[2 E_B \sum f_i \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i H_D \text{ INTEGRALS} \right] \\ & = E_0 \left[\sum f_i \psi_0(1) \psi_0(2) \chi_i + \sum \chi_i \text{ INTEGRALS} \right] \\ & + \left[2 E_B \sum f_i \psi_0(1) \psi_0(2) \chi_i + 2 E_B \sum \chi_i \text{ INTEGRALS} \right] . \end{aligned} \quad (1.24)$$

the optimum wave function of the system. The minimum is over the set of orthogonal wave functions.

$$H_0 = H_1 + H_2$$

$$H_0 \psi_0(\mathbf{r}) = E_0 \psi_0(\mathbf{r})$$

and defining E_0 as the lowest energy of the system the center of mass coordinate $\mathbf{R}$ is separated and for space, we obtain

$$E = E_0 + E_1$$

Taking the operation ∇^2 on both sides of (1.15) and using

$$(H_1 + H_2) \psi = E \psi$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \left[\sum_i A_i \psi_i(\mathbf{r}) \right] + \left[\sum_i A_i \psi_i(\mathbf{r}) \right] \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{R}) \right) = E_0 \left[\sum_i A_i \psi_i(\mathbf{r}) \right] + \left[\sum_i A_i \psi_i(\mathbf{r}) \right] E_1$$

$$+ \left[\sum_i A_i \psi_i(\mathbf{r}) \right] \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{R}) \right) = E_0 \left[\sum_i A_i \psi_i(\mathbf{r}) \right] + \left[\sum_i A_i \psi_i(\mathbf{r}) \right] E_1$$

$$+ \left[\sum_i A_i \psi_i(\mathbf{r}) \right] \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{R}) \right) = E_0 \left[\sum_i A_i \psi_i(\mathbf{r}) \right] + \left[\sum_i A_i \psi_i(\mathbf{r}) \right] E_1$$

$$+ \left[\sum_i A_i \psi_i(\mathbf{r}) \right] \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{R}) \right) = E_0 \left[\sum_i A_i \psi_i(\mathbf{r}) \right] + \left[\sum_i A_i \psi_i(\mathbf{r}) \right] E_1$$

Cancelling the two terms in $2E_0$ and rearranging (1.24) yields

$$-\frac{\hbar^2}{2m} \nabla_r^2 \sum f_c \psi_0(1) \psi_0(2) \chi_c - \frac{\hbar^2}{2m} \sum \chi_c \nabla_r^2 \text{INTEGRALS} \\ + V_{\text{INT}} \psi_f + \sum \chi_c H_0 \text{INTEGRALS} = \quad (1.25)$$

$$E_0 \sum f_c \psi_0(1) \psi_0(2) \chi_c + E_0 \sum \chi_c \text{INTEGRALS} \\ + 2E_0 \sum \chi_c \text{INTEGRALS} .$$

Now the effect of ∇_r^2 on the integrals is to alter their $\vec{r}$ dependence only. The effect of H_0 can be gotten from the equations,

$$H_0 \psi_{\epsilon_1} \psi_{\epsilon_2} = (\epsilon_1 + \epsilon_2) \psi_{\epsilon_1} \psi_{\epsilon_2} ,$$

$$H_0 \psi_0(1) \psi_{\epsilon}(2) = (E_0 + \epsilon) \psi_0(1) \psi_{\epsilon}(2) ,$$

so that the $\vec{p}_1$ and $\vec{p}_2$ dependence in the integrals is unchanged by ∇_r^2 and H_0 . Multiplying (1.25) on the left by $\bar{\psi}_0(1) \bar{\psi}_0(2)$ and integrating over $\vec{p}_1$ and $\vec{p}_2$ gives

$$-\frac{\hbar^2}{2m} \nabla_r^2 \sum f_c \chi_c + \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{\text{INT}} \psi_f d\vec{p}_1 d\vec{p}_2 \\ = E_0 \sum f_c \chi_c , \quad (1.26)$$

since all the integrals contain $\psi_{\epsilon} \psi_0$ or $\psi_{\epsilon_1} \psi_{\epsilon_2}$ products which are orthogonal to $\psi_0(1) \psi_0(2)$. Multiplying (1.26) on the left by $\bar{\chi}_c$ one obtains the equation,

Cancelling the two terms in $2E_0$ and rearranging (1.25)

yields

$$-\frac{\hbar^2}{2m} \nabla^2 \sum_i \psi_i^2 + \sum_i \psi_i^2 V = -\frac{\hbar^2}{2m} \nabla^2 \sum_i \psi_i^2 + \sum_i \psi_i^2 V$$

$$(1.26) \quad + \sum_i \psi_i^2 V = \sum_i \psi_i^2 V + \sum_i \psi_i^2 V$$

$$E_0 \sum_i \psi_i^2 + E_0 \sum_i \psi_i^2 + E_0 \sum_i \psi_i^2 + 2E_0 \sum_i \psi_i^2$$

Now the effect of ∇^2 on the integrals is to alter their

dependence only. The effect of H_0 can be gotten from the

equations,

$$H_0 \psi_i = (E_i + E_0) \psi_i$$

$$H_0 \psi_0 = (E_0 + E_0) \psi_0$$

so that the ψ_i and ψ_0 dependence in the integrals is un-

changed by ∇^2 and H_0 . Multiplying (1.25) on the left by

$$\psi_0^*(1) \bar{\psi}_0(1) \text{ and integrating over } \vec{r}_1 \text{ and } \vec{r}_2 \text{ gives}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \sum_i \psi_i^2 + \sum_i \psi_i^2 V = -\frac{\hbar^2}{2m} \nabla^2 \sum_i \psi_i^2 + \sum_i \psi_i^2 V$$

(1.26)

$$= E_0 \sum_i \psi_i^2$$

since all the integrals contain $\psi_0^* \psi_0$ or $\psi_0^* \psi_0$

products which are orthogonal to $\psi_0(1) \bar{\psi}_0(1)$. Multiplying

(1.26) on the left by $\bar{\psi}_0$ one obtains the equation,

$$-\frac{\hbar^2}{2m} \nabla_r^2 f_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} \psi_f d\vec{p}_1 d\vec{p}_2 = E_0 f_c . \quad (1.27)$$

A similar procedure applied to the other equations in (1.15) yields

$$-\frac{\hbar^2}{2m} \nabla_r^2 g_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} p_{13} q_{13} \psi_f d\vec{p}_1 d\vec{p}_2 = E_0 g_c , \quad (1.28)$$

$$-\frac{\hbar^2}{2m} \nabla_r^2 h_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} p_{24} q_{24} \psi_f d\vec{p}_1 d\vec{p}_2 = E_0 h_c ,$$

$$-\frac{\hbar^2}{2m} \nabla_r^2 k_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} p_{13} q_{13} p_{24} q_{24} \psi_f d\vec{p}_1 d\vec{p}_2 = E_0 k_c .$$

Combining (1.27) and (1.28) after setting

$$\phi_c(\vec{r}) = f_c - g_c - h_c + k_c , \quad (1.29)$$

one obtains

$$-\frac{\hbar^2}{2m} \nabla_r^2 \phi_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} K \psi_f d\vec{p}_1 d\vec{p}_2 = E_0 \phi_c . \quad (1.30)$$

Equation (1.30) when solved gives the $\vec{r}$ dependence of the symmetrized wave function for our problem.

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \bar{V} \psi = E_0 \psi \quad (1.27)$$

(1.27)

A similar procedure applied to the other equations in (1.25) yields

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \bar{V} \psi = E_0 \psi \quad (1.28)$$

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$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \bar{V} \psi = E_0 \psi \quad (1.30)$$

Combining (1.27) and (1.28) after setting

(1.29)

$$\psi = \psi_1 + \psi_2 + \dots$$

one obtains

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \bar{V} \psi = E_0 \psi \quad (1.30)$$

(1.30)

Equation (1.30) when solved gives the dependence of the

symmetrized wave function for our problem.

Since $V_{INT} K$ in (1.30) contains spin operators, (1.30) involves matrix elements of the type

$$\bar{\chi}_i \text{ spin exchange operators } \chi_j .$$

The off-diagonal elements, ($i \neq j$), do not in general vanish.

We will later show that for our case only the diagonal elements will contribute so that (1.30) can be written

$$-\frac{\hbar^2}{2m} \nabla^2 \phi_i + \bar{\chi}_i \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} K \psi_i d\vec{p}_1 d\vec{p}_2 = E_0 \phi_i . \quad (1.31)$$

Equation (1.31) will be valid if there are no transitions between spin states, i.e., the spin is a constant of the motion.

4. Spin States and the Spin Wave Functions.

Let S be the total spin quantum number of the system of four nucleons. Since each nucleon has a spin of $\frac{1}{2}$ (in units of $\hbar$), S can take on values $S = 2, 1, 0$ called the quintet, triplet and singlet respectively. We require an orthogonal normalized set of spin wave functions, χ_S^m (where m is the magnetic quantum number $-S \leq m \leq S$), which are symmetric in exchanges within the deuterons and which have the proper eigenvalues with respect to the operators $\vec{S}$ and S_z . These properties are the following:

Since $V_{int} K$ in (1.30) contains spin operators,

(1.30) involves matrix elements of the type

$$\langle \bar{\chi}_i | \text{spin exchange operators } \chi_j$$

The off-diagonal elements, $i \neq j$, do not in general vanish.

We will later show that for our case only the diagonal elements will

contribute so that (1.30) can be written

$$-\frac{\hbar^2}{2m} \nabla^2 \phi + \chi_i \int \bar{\psi}_0(\omega) \psi_0(\omega) V_{int} K \psi_j \psi_j d\omega = E_0 \phi, \quad (1.31)$$

Equation (1.31) will be valid if there are no transitions between

spin states, i.e., the spin is a constant of the motion.

1. Spin States and the Spin Wave Functions.

Let S be the total spin quantum number of the system of fermions.

Since each nucleon has a spin of $\frac{1}{2}$ (in units of $\hbar$),

S can take on values $S = 0, 1, 2, \dots$ called the spin, triplet and

singlet respectively. We require an orthogonal normalized set of

spin wave functions, χ_2^{SM} (where M is the magnetic quantum

number, $-2 \leq M \leq 2$), which are symmetric in exchanges within the

fermions and which have the proper eigenvalues with respect to the

operators S^2 and S_z . These properties are the

following:

$$\text{I-} \quad \bar{\chi}_s^m \chi_{s'}^{m'} = \delta_{ss'} \delta_{mm'} ,$$

$$\text{II-} \quad Q_{12} \chi_s^m = Q_{34} \chi_s^m = \chi_s^m ,$$

(1.32)

$$\text{III-} \quad \vec{S}^2 \chi_s^m = s(s+1) \chi_s^m ,$$

$$\text{IV-} \quad S_z \chi_s^m = m \chi_s^m , \quad -s \leq m \leq s .$$

In (1.32), $\vec{S}^2$ and S_z are the operators,

$$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4)^2 ,$$

(1.33)

$$S_z = S_{1z} + S_{2z} + S_{3z} + S_{4z} ,$$

where

$$\vec{S}_i = \frac{1}{2} \vec{\sigma}_i \quad \text{and} \quad S_{iz} = \frac{1}{2} \sigma_{iz}$$

Symmetry with respect to Q_{12} and Q_{34} (property II) follows from the fact that the deuteron ground state is a triplet in spin.

The $\vec{\sigma}'_s$ behave like vectors whose components are Pauli spin

$$I- \quad \bar{K}_2^m K_2^m = 0.28 \cdot 0.28 = 0.0784$$

$$II- \quad Q_{11} K_2^m = 0.24 K_2^m = K_2^m$$

$$III- \quad \bar{K}_2^m K_2^m = 0.28 + 0.28 = 0.56$$

$$IV- \quad 2 \bar{K}_2^m K_2^m = 2 \cdot 0.28 = 0.56$$

In (1.32), $\bar{K}_2^m$ and K_2^m are the vectors

$$\bar{K}_2^m = (\bar{K}_1^m + \bar{K}_2^m + \bar{K}_3^m + \bar{K}_4^m)$$

$$K_2^m = K_1^m + K_2^m + K_3^m + K_4^m$$

where

$$\bar{K}_1^m = \frac{1}{\sqrt{2}} \quad \text{and} \quad K_1^m = \frac{1}{\sqrt{2}} \cdot \sqrt{2}$$

Symmetry with respect to Q_{11} and Q_{22} (the only 12) follows from the fact that the direction around which is a cylinder is given. The $\bar{V}_2^m$ behave like vectors whose components are still given

matrices,^{5,7} i.e.,

$$\vec{\sigma}_i = (\sigma_{x_i}, \sigma_{y_i}, \sigma_{z_i}),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in units of $\hbar$.

Using the basic α 's and β 's where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which satisfy the relations⁵

$$\bar{\alpha}_i \alpha_j = \bar{\beta}_i \beta_j = \delta_{ij}, \quad \bar{\alpha}_i \beta_j = \bar{\beta}_i \alpha_j = 0$$

we can build up such a set of spin wave functions. Letting the position determine the subscript (i.e., $\alpha \alpha \beta \alpha = \alpha_1 \alpha_2 \beta_3 \alpha_4$, etc.), and suppressing the magnetic quantum number m , for convenience of notation, and writing χ_s^m in the form χ_i , we obtain the following set of nine spin wave functions satisfying the properties of (1.32).

Quintet;

$$\begin{aligned} \chi_1 &= \chi_2^2 = \alpha \alpha \alpha \alpha, \\ \chi_2 &= \chi_2^{-2} = \beta \beta \beta \beta, \end{aligned}$$

matrices, $\bar{a}, \bar{b}, \bar{c}, \bar{d}$.

$$(\bar{a}, \bar{b}, \bar{c}, \bar{d}) = \bar{a}$$

where

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \bar{a}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \bar{b}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \bar{c}$$

in units of $\hbar$

Using the basis α_i and β_i where

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \beta, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha$$

which satisfy the relations

$$\alpha_i \alpha_j = \bar{a} \delta_{ij}, \quad \beta_i \beta_j = \bar{b} \delta_{ij}, \quad \alpha_i \beta_j = \bar{c} \delta_{ij} = 0$$

we can build up such a set of spin wave functions. Letting the Pauli

tion determine the subscript (i.e., $\alpha_i \beta_i = \alpha_i \beta_i = \alpha_i \beta_i$),

and suppressing the magnetic quantum number M , for con-

venience of notation, and writing χ_s in the form χ_s ,

we obtain the following set of nine spin wave functions satisfying

the properties of (1.2).

Quintet

$$\begin{aligned} \chi_1 &= \chi_2^2 = \alpha \alpha \alpha \alpha \alpha \\ \chi_2 &= \chi_2^2 = \alpha \alpha \alpha \beta \alpha \end{aligned}$$

$$\chi_3 = \chi_2^1 = \frac{1}{2} (\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha),$$

$$\chi_4 = \chi_2^{-1} = \frac{1}{2} (\beta\beta\beta\alpha + \beta\beta\alpha\beta + \beta\alpha\beta\beta + \alpha\beta\beta\beta),$$

$$\chi_5 = \chi_2^0 = \frac{1}{\sqrt{6}} (\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta + \beta\alpha\beta\alpha + \alpha\alpha\beta\beta + \beta\beta\alpha\alpha),$$

(1.34)

Triplet;

$$\chi_6 = \chi_1^1 = \frac{1}{2} (\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha - \alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha),$$

$$\chi_7 = \chi_1^{-1} = \frac{1}{2} (\beta\beta\beta\alpha + \beta\beta\alpha\beta - \beta\alpha\beta\beta - \alpha\beta\beta\beta),$$

(1.35)

$$\chi_8 = \chi_1^0 = \frac{1}{\sqrt{2}} (\alpha\alpha\beta\beta - \beta\beta\alpha\alpha),$$

and Singlet;

$$\chi_9 = \chi_0^0 = \frac{1}{\sqrt{12}} (\alpha\beta\alpha\beta + \beta\alpha\beta\alpha + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2(\alpha\alpha\beta\beta + \beta\beta\alpha\alpha))$$

If we denote by z_1^1 , z_1^{-1} , z_1^0 , and z_0^0 , the spin wave functions

$$\begin{aligned} z_1^1 &= \alpha\alpha, & z_1^0 &= \alpha\beta + \beta\alpha, \\ z_1^{-1} &= \beta\beta, & z_0^0 &= \alpha\beta - \beta\alpha, \end{aligned}$$

(1.36)

$$X^3 = X_5^1 = \frac{1}{5} (\alpha \alpha \alpha \beta + \alpha \alpha \beta \alpha + \alpha \beta \alpha \alpha + \beta \alpha \alpha \alpha)$$

$$X^4 = X_5^{-1} = \frac{1}{5} (\beta \beta \beta \alpha + \beta \beta \alpha \beta + \beta \alpha \beta \beta + \alpha \beta \beta \beta)$$

$$X^2 = X_5^0 = \frac{1}{\sqrt{5}} (\alpha \beta \alpha \beta + \alpha \beta \beta \alpha + \beta \alpha \alpha \beta + \beta \alpha \beta \alpha + \alpha \alpha \beta \beta + \beta \beta \alpha \alpha)$$

(1.31)

Triplets

$$X^0 = X_1^1 = \frac{1}{5} (\alpha \alpha \alpha \beta + \alpha \alpha \beta \alpha - \alpha \beta \alpha \alpha - \beta \alpha \alpha \alpha)$$

$$X_1^{-1} = \frac{1}{5} (\beta \beta \beta \alpha + \beta \beta \alpha \beta - \beta \alpha \beta \beta - \alpha \beta \beta \beta)$$

(1.32)

$$X_8^0 = X_1^0 = \frac{1}{\sqrt{5}} (\alpha \alpha \beta \beta - \beta \beta \alpha \alpha)$$

and doublets

$$X^4 = X_0^0 = \frac{1}{\sqrt{5}} (\alpha \beta \alpha \beta + \beta \alpha \alpha \beta + \alpha \beta \beta \alpha + \beta \alpha \beta \alpha - 2(\alpha \alpha \beta \beta + \beta \beta \alpha \alpha))$$

If we denote by ξ_1^1 , ξ_1^{-1} , ξ_1^0 , and ξ_0^0 , the spin wave

functions

$$\xi_1^1 = \alpha \alpha, \quad \xi_1^0 = \alpha \beta + \beta \alpha,$$

(1.33)

$$\xi_1^{-1} = \beta \beta, \quad \xi_0^0 = \alpha \beta - \beta \alpha$$

then $\chi_1 \dots \chi_q$ can be written in the form

$$\begin{aligned}
 \chi_1 &= z_1' z_1' , \\
 \chi_3 &= \frac{1}{2} (z_1' z_1^0 + z_1^0 z_1') , \\
 \chi_5 &= \frac{1}{\sqrt{6}} (z_1' z_1^{-1} + z_1^{-1} z_1' + z_1^0 z_1^0) , \\
 \chi_4 &= \frac{1}{2} (z_1^{-1} z_1^0 + z_1^0 z_1^{-1}) , \\
 \chi_2 &= z_1^{-1} z_1^{-1} , \\
 \chi_6 &= \frac{1}{2} (z_1' z_1^0 - z_1^0 z_1') , \\
 \chi_8 &= \frac{1}{\sqrt{2}} (z_1' z_1^{-1} - z_1^{-1} z_1') , \\
 \chi_7 &= \frac{1}{2} (z_1^{-1} z_1^0 - z_1^0 z_1^{-1}) , \\
 \chi_9 &= \frac{1}{\sqrt{12}} (z_1^0 z_1^0 - 2(z_1' z_1^{-1} + z_1^{-1} z_1')) ,
 \end{aligned} \tag{1.37}$$

which are symmetric in φ_{12} and φ_{34} . In addition, we construct the set of spin wave functions

$$\begin{aligned}
 &z_0^0 z_0^0 , & \frac{1}{2} (z_1' z_0^0 + z_0^0 z_1') , \\
 &\frac{1}{2} (z_0^0 z_1^0 + z_1^0 z_0^0) , & \frac{1}{2} (z_1^{-1} z_0^0 + z_0^0 z_1^{-1}) , \\
 &\frac{1}{2} (z_0^0 z_1^0 - z_1^0 z_0^0) , & \frac{1}{2} (z_1' z_0^0 - z_0^0 z_1') , \\
 &\frac{1}{2} (z_0^0 z_1^{-1} - z_1^{-1} z_0^0) , &
 \end{aligned}$$

which are antisymmetric in φ_{12} and φ_{34} and which are a normalized orthogonal set. In addition these spin wave functions are orthogonal to $\chi_1 \dots \chi_q$. These antisymmetric spin wave

than $X_1, \dots, X_p$ can be written in the form

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.37)$$

$$X_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_7 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_9 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which are symmetric in $\mathcal{Q}_{12}$ and $\mathcal{Q}_{13}$. In addition, we constructed the set of spin wave functions

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which are antisymmetric in $\mathcal{Q}_{12}$ and $\mathcal{Q}_{13}$ and which are a normal-

ized orthogonal set. In addition these spin wave functions are orthogonal to $X_1, \dots, X_p$. These antisymmetric spin wave

functions will not concern us in our calculation.

Inspection of the expressions (1.34) ... (1.36) or (1.37) shows the orthogonality and normality, thus verifying I. Property II is obvious on inspection. From the fact that

$$\begin{aligned} (\sigma_z)_i \alpha_i &= \alpha_i, & (\sigma_z)_i \beta_i &= -\beta_i, \\ (\sigma_z)_j \alpha_i &= \alpha_i (\sigma_z)_j & (\sigma_z)_j \beta_i &= \beta_i (\sigma_z)_j \\ & (i \neq j), & (i \neq j), \end{aligned}$$

one can calculate the eigenvalues of $\chi_1, \dots, \chi_9$ with respect to S_z and justify the notation used in (1.37). One obtains

$$\begin{aligned} S_z \chi_1 &= 2 \chi_1 = 2 \chi_2^2, & S_z \chi_6 &= \chi_6 = \chi_1', \\ S_z \chi_2 &= -2 \chi_2 = -2 \chi_2^{-2}, & S_z \chi_7 &= -\chi_7 = -\chi_1^{-1}, \\ S_z \chi_3 &= \chi_3 = \chi_2', & S_z \chi_8 &= 0 \chi_1^0, \\ S_z \chi_4 &= -\chi_4 = -\chi_2^{-1}, & S_z \chi_9 &= 0 \chi_0^0, \\ S_z \chi_5 &= 0 = 0 \chi_2^0, \end{aligned} \quad (1.38)$$

In order to calculate the eigenvalues of $\vec{S}^2$, one can write (1.33) as

$$\begin{aligned} \vec{S}^2 &= \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + \vec{S}_4^2 \\ &+ 2 (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_1 \cdot \vec{S}_4 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_4 + \vec{S}_3 \cdot \vec{S}_4) \end{aligned} \quad (1.39)$$

functions will not concern us here.

Inspection of the expansion (1.31) or (1.32)

shows the orthogonality and norm of ψ_n with respect to $\rho(x)$.

It is obvious on inspection that the last two

$$(\psi_2, \psi_1) = 0, \quad (\psi_2, \psi_0) = 0$$

$$(\psi_3, \psi_1) = 0, \quad (\psi_3, \psi_0) = 0, \quad (\psi_3, \psi_2) = 0$$

one can estimate the eigenvalues of $X_1, \dots, X_n$ with respect

to ψ_n and finally the relation (1.33). One obtains

$$2\psi_1 X_1 = X_1^2 = X_1^2$$

$$2\psi_2 X_2 = -2X_2^2 = -X_2^2$$

$$2\psi_3 X_3 = X_3^2 = X_3^2$$

$$2\psi_4 X_4 = -4X_4 = -X_4$$

$$2\psi_5 X_5 = 0 = 0$$

$$2\psi_0 X_0 = X_0^2 = X_0^2$$

$$2\psi_1 X_1 = -X_1^2 = -X_1^2$$

$$2\psi_2 X_2 = 0 = 0$$

$$2\psi_3 X_3 = 0 = 0$$

$$2\psi_4 X_4 = 0 = 0$$

In order to calculate the eigenvalues of $X_1, \dots, X_n$ one can write

(1.33) as

$$2\psi_n = 2\psi_{n-1} + 2\psi_{n-2} + \dots + 2\psi_0$$

$$+ 2(2\psi_{n-1} + 2\psi_{n-2} + \dots + 2\psi_0 + 2\psi_{n-2} + \dots + 2\psi_0)$$

(1.34)

Since $\vec{S}_i = \frac{1}{2} \vec{\sigma}_i$, one obtains $\vec{S}_i \cdot \vec{S}_j = \frac{1}{4} \vec{\sigma}_i \cdot \vec{\sigma}_j$
and using the relation⁷

$$Q_{ij} = \frac{1}{2} (1 + \vec{\sigma}_i \cdot \vec{\sigma}_j) ,$$

one then obtains,

$$2 \vec{S}_i \cdot \vec{S}_j = Q_{ij} - \frac{1}{2} .$$

Now $\vec{\sigma}_i^2 = (\sigma_x)_i^2 + (\sigma_y)_i^2 + (\sigma_z)_i^2 = 3$, hence $\vec{S}_i^2 = 3/4$
and $\vec{S}^2$ become

$$\vec{S}^2 = Q_{12} + Q_{13} + Q_{14} + Q_{23} + Q_{24} + Q_{34} ,$$

(1.40)

in $\hbar^2$ units.

Observing that the quintet wave functions are completely symmetric,
one obtains

$$\vec{S}^2 \chi_1 \cdots \chi_5 = 6 \chi_1 \cdots \chi_5 .$$

Calculation of eigenvalues for the triplet yields

$$\vec{S}^2 \chi_6 \cdots \chi_8 = 2 \chi_6 \cdots \chi_8 ,$$

Since $\vec{z}_1 = \frac{1}{2}\vec{v}_1$, one obtains
and using the relation

$$\phi_{10} = \frac{1}{2}(1 + \sqrt{0.6})$$

one then obtains

$$2\vec{z}_1 \cdot \vec{z}_2 = \phi_{10} - \frac{1}{2}$$

Now $\vec{v}_1^2 = (\vec{v}_1)^2 + (\vec{v}_2)^2 + (\vec{v}_3)^2 = 3$, hence $\vec{z}_1^2 = 3/4$
and $\vec{z}_2^2$ becomes

$$\vec{z}_2^2 = \phi_{12} + \phi_{13} + \phi_{14} + \phi_{23} + \phi_{24} + \phi_{34}$$

(2.40)

in K^3 units.

Observing that the pointed wave functions are completely symmetric,

one obtains

$$2\vec{z}_1 \cdot \vec{z}_2 = \phi_{11} \dots \phi_{11} = \phi_{11} \dots \phi_{11}$$

Calculation of eigenvalues for the triplet yields

$$2\vec{z}_1 \cdot \vec{z}_2 = \phi_{11} \dots \phi_{11} = \phi_{11} \dots \phi_{11}$$

and for the singlet gives

$$\vec{S}^2 \chi_q = 0 ,$$

thus verifying

$$\vec{S}^2 \chi_s^m = s(s+1) \chi_s^m ,$$

and establishing III. As an example, the singlet calculations will be given.

We have first,

$$Q_{12} \chi_q = \chi_q , \quad Q_{34} \chi_q = \chi_q ,$$

and

$$Q_{14} \chi_q = \frac{1}{\sqrt{12}} (\beta\beta\alpha\alpha + \alpha\alpha\beta\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2(\beta\alpha\beta\alpha + \alpha\beta\alpha\beta)) ,$$

$$Q_{13} \chi_q = \frac{1}{\sqrt{12}} (\alpha\beta\alpha\beta + \beta\alpha\beta\alpha + \beta\beta\alpha\alpha + \alpha\alpha\beta\beta - 2(\beta\alpha\alpha\beta + \alpha\beta\beta\alpha)) ,$$

$$Q_{23} \chi_q = \frac{1}{\sqrt{12}} (\alpha\alpha\beta\beta + \beta\beta\alpha\alpha + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2(\alpha\beta\alpha\beta + \beta\alpha\beta\alpha)) ,$$

$$Q_{24} \chi_q = \frac{1}{\sqrt{12}} (\alpha\beta\alpha\beta + \beta\alpha\beta\alpha + \alpha\alpha\beta\beta + \beta\beta\alpha\alpha - 2(\alpha\beta\beta\alpha + \beta\alpha\alpha\beta)) .$$

Inspection of these terms shows that

and for the second five

$$x_1 = 1$$

thus verifying

$$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1$$

and substituting $x_1 = 1$ in the first equation we get

we have

we have

$$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1$$

and

$$Q_1(x) = \frac{1}{12} (x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1)$$

$$Q_2(x) = \frac{1}{12} (x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1)$$

$$Q_3(x) = \frac{1}{12} (x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1)$$

$$Q_4(x) = \frac{1}{12} (x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1)$$

Inspection of these cases shows that

$$\begin{aligned}
 & (\varphi_{13} + \varphi_{14} + \varphi_{23} + \varphi_{24}) \chi_q = \\
 & \frac{2}{\sqrt{12}} (-\alpha\beta\alpha\beta - \beta\alpha\beta\alpha - \beta\alpha\alpha\beta - \alpha\beta\beta\alpha + 2(\beta\beta\alpha\alpha + \alpha\alpha\beta\beta)) \\
 & = -2\chi_q,
 \end{aligned}$$

so that one obtains finally

$$\vec{S}^2 \chi_q = 2\chi_q - 2\chi_q = 0.$$

The triplet calculations are performed in the same fashion.

5. Spin Sums. The Born Approximation.

Equation (1.30) contains an expression

$$\bar{\chi}_i \iint \bar{\psi}_0(1) \bar{\psi}_0(2) \text{ SPIN OPERATORS } \psi_j d\vec{p}_1 d\vec{p}_2, \quad (1.41)$$

where ψ_j ($j = 1, \dots, 9$) involves the χ'_s in (1.34...36) which are the spin wave functions adequate to describe the elastic scattering. The operators in (1.41) come from the term $V_{int} K$ in (1.30). Since V_{int} is of the form $\sum F_{ij} J(r_{ij})$ where F_{ij} consists of linear and quadratic combinations of p_{ij} and Q_{ij} then by calculating the values of $\chi_1 \dots \chi_9$

$$(\varphi_{12} + \varphi_{11} + \varphi_{22} + \varphi_{21}) X_p =$$

$$\left(-\frac{2}{15} \right) (-\kappa \alpha \alpha \alpha - \alpha \kappa \alpha \alpha - \alpha \kappa \alpha \alpha - \alpha \kappa \alpha \alpha + 2(\alpha \alpha \alpha \alpha + \alpha \alpha \alpha \alpha))$$

$$= -\frac{2}{15} X_p$$

so that one obtains finally

$$\frac{1}{2} X_p = 2 X_p - 2 X_p = 0.$$

The triplet calculations are performed in the same fashion.

B. Spin Sums. The Born Approximation.

Equation (1.30) contains an expression

$$\bar{\chi}_i \left\{ \bar{\psi}_0(1) \bar{\psi}_0(2) \text{ SPIN OPERATORS } \psi_0^{\dagger} \psi_0^{\dagger} \right\} \quad (1.41)$$

where $\psi_0^{\dagger} (j = 1, \dots, 9)$ involves the χ_j in (1.31...33) which are the spin wave functions adequate to describe the elastic scattering. The operators in (1.41) come from the term $V_{00} \tau K$ in (1.30). Since $V_{00} \tau$ is of the form $\sum F_{ij} T(\tau_{ij})$ where F_{ij} consists of linear and quadratic combinations of ρ_i and ρ_j then by calculating the values of $\chi_1 \dots \chi_9$

under operators $Q_{13} \cdot Q_{14} \cdot Q_{13} \cdot Q_{24} \cdot Q_{13} Q_{14} \cdot Q_{13} Q_{23}$
 $Q_{13} Q_{24} \cdot \dots \cdot Q_{13} Q_{14} Q_{24}$, etc., one finds that terms such as

$$\bar{\chi}_i \text{ OPERATORS } \chi_j$$

will always vanish unless $i = j$ (or equivalently $s = s'$, $m = m'$ for the χ_s^m notation). This justifies equation (1.31) for our problem, which was

$$-\frac{\hbar^2}{2m} \nabla_r^2 \phi_c + \bar{\chi}_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{INT} K \psi_i d\vec{p}_1 d\vec{p}_2 = E_0 \phi_c \quad (1.31)$$

We make the usual assumption⁶ that the asymptotic form of the wave function $\phi_c(\vec{r})$ can be expressed as an incident plane wave plus and outgoing scattered wave, i.e.,

$$\phi_c(\vec{r}) = e^{ikz} + \frac{e^{ikr}}{r} f_c(\theta) \quad , \quad (1.42)$$

where

$$k = \sqrt{2mE_0}/\hbar$$

and $f_c(\theta)$ is the angular part of the scattered component of ϕ_c . The differential cross section for the scattering in the case of an unpolarized incident beam of deuterons is given

under operators $\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_N$ and $\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_N$ are defined as

$$\hat{Q}_i = \text{OPERATORS } \hat{Q}_i$$

will always vanish unless $i = j$ for any $i, j = 1, 2, \dots, N$.
for the $\hat{Q}_i$ (operator), the position operator (1.1) for
our problem, which are

$$\hat{Q}_i = \frac{1}{2m} \nabla^2 \psi_i + \hat{V}_i \psi_i \quad (1.1)$$

We take the usual convention that the asymptotic form of the wave
function $\psi(\vec{r})$ can be represented as an incident plane wave plus
and outgoing scattered wave, i.e.,

$$\psi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} + f(\theta) \frac{e^{ikr}}{r} \quad (1.2)$$

$$k = \sqrt{2mE} \quad \text{where}$$

and $f(\theta)$ is the angular part of the scattered component of
 ψ . The differential cross section for the scattering in
the case of an unpolarized incident beam of particles is given

by $d\sigma$ where

$$d\sigma = I(\theta) d\Omega = \frac{1}{q} \sum_{l=1}^q |F_l(\theta)|^2 d\Omega, \quad (1.43)$$

and where $d\Omega = \sin\theta d\theta d\phi$.

An expression for ϕ_l of (1.31) in terms of an integral equation can be written as ^{5,6}

$$\begin{aligned} \phi_l = & -\frac{m}{2\pi\hbar^2} \int \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} e^{i\vec{k}(\vec{r}-\vec{r}')} \bar{\chi}_l \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{12} K \psi_l d\vec{p}_1 d\vec{p}_2 \\ & + e^{i\vec{k} \cdot \vec{z}}. \end{aligned} \quad (1.44)$$

Now let $\vec{n}$ be a unit vector in the $\vec{r}$ direction (the direction of scattering) and let $\vec{n}_0$ be a unit vector in the $+z$ direction (incident direction). If r is large compared to the deuteron systems, we can write ⁶

$$|\vec{r}-\vec{r}'| \doteq r - \vec{n} \cdot \vec{r}', \quad \frac{1}{|\vec{r}-\vec{r}'|} \doteq \frac{1}{r}, \quad (1.45)$$

and (1.44) becomes

$$\phi_l = e^{i\vec{k} \cdot \vec{z}} \quad (1.46)$$

$$-\frac{m}{2\pi\hbar^2} \frac{e^{i\vec{k} \cdot \vec{r}}}{r} \bar{\chi}_l \int d\vec{r}' e^{-i\vec{k} \cdot \vec{n} \cdot \vec{r}'} \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{12} K \psi_l d\vec{p}_1 d\vec{p}_2.$$

Hence identifying $\mathcal{F}_i(\theta)$ in (1.46) from (1.42) one obtains

$$\mathcal{F}_i(\theta) = -\frac{m}{2\pi\hbar^2} \bar{\chi}_i \int d\vec{r}' e^{-ik\vec{n}\cdot\vec{r}'} \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{int} K \psi_i d\vec{p}_1 d\vec{p}_2 \quad (1.47)$$

Born's approximation states that if the scattered component is small in comparison to the incident wave at the scatterer, then the wave function ψ_i in the integral of (1.47) can be replaced by the incident plane wave,⁶ i.e.,

$$\psi_i \rightarrow e^{ik\vec{n}_0\cdot\vec{r}'} \psi_0(1) \psi_0(2) \chi_i \quad (1.48)$$

Making use of the Born approximation, the expression for $\mathcal{F}_i(\theta)$ then becomes

$$\mathcal{F}_i(\theta) = -\frac{m}{2\pi\hbar^2} \bar{\chi}_i \int d\vec{r}' e^{-ik\vec{n}\cdot\vec{r}'} \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_{int} K e^{ik\vec{n}_0\cdot\vec{r}'} \psi_0(1) \psi_0(2) \chi_i d\vec{p}_1 d\vec{p}_2 \quad (1.49)$$

In order to calculate the $\mathcal{F}_i$'s from (1.49), we will have to calculate expressions of the type $\bar{\chi}_i$ Q-operators χ_i , the spin terms. We will also have to know the value of terms such as P_{ij} (functions of $\vec{r}, \vec{p}_1, \vec{p}_2$). This is because such

Hence identifying $\bar{\psi}(0)$ in (1.15) with $\bar{\psi}(0)$ on the left

$$\bar{\psi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\psi}(k) e^{-ikx} dk \quad (1.16)$$

(1.16)

Bohr's approximation holds for $\bar{\psi}(0)$ if the function $\bar{\psi}(k)$ is

in comparison to the function $\bar{\psi}(k)$ of the order of the

function $\bar{\psi}(k)$ in the interval of k can be replaced by

incident plane wave, i.e.,

$$\bar{\psi}(k) = \bar{\psi}(0) e^{-ikx} \quad (1.17)$$

Making use of the form specified, the expansion for $\bar{\psi}(0)$

then becomes

$$\bar{\psi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\psi}(k) e^{-ikx} dk \quad (1.18)$$

(1.18)

In order to calculate the $\bar{\psi}(0)$ from (1.18), we will have

to calculate expressions of the form $\bar{\psi}(k)$ in the interval

the spin term. We will have to know the value of terms such

as $\bar{\psi}(k)$ (function of k) in the interval of k . This is because such

terms come in through the operators in $V_{INT} K$.

We can designate by A_i any terms of the form

$$A_i = \bar{\chi}_i Q_{ab} \chi_i , \quad (1.50)$$

by B_i any term of the form

$$B_i = \bar{\chi}_i Q_{ab} Q_{cd} \chi_i \quad (\text{where } a, b, c, d \text{ exhaust } 1, 2, 3, 4), \quad (1.51)$$

by C_i any term of the form

$$C_i = \bar{\chi}_i Q_{ab} Q_{cd} \chi_i \quad (\text{where } a, b, c, d \text{ do not exhaust } 1, 2, 3, 4) \quad (1.52)$$

and by D_i either of the terms

$$D_i = \begin{aligned} &\bar{\chi}_i Q_{14} Q_{13} Q_{24} \chi_i , \\ &\bar{\chi}_i Q_{23} Q_{13} Q_{24} \chi_i . \end{aligned} \quad (1.53)$$

The operators Q_{ab} and Q_{cd} in the above expressions (1.50...53)

are restricted to the set Q_{12} , Q_{14} , Q_{23} and Q_{24} .

They are the only ones coming in through $V_{INT} K$.

For the quintet, $i = 1...5$, we see by inspection that

$A_i = B_i = C_i = D_i = 1$. If the calculation is carried out for the triplet and singlet wave functions, one finds that $A_i = C_i = D_i$.

The final results of these calculations are contained in Table 1.

terms come in through the operators in $\bar{K}$.
We can deal with A_1 and B_1 of the form

$$A_1 = \bar{K}_1 \psi_{12} \psi_{13} \quad (1.20)$$

by B_1 and C_1 of the form

$$B_1 = \bar{K}_1 \psi_{12} \psi_{13} \psi_{14} \quad (1.21)$$

by C_1 and D_1 of the form

$$C_1 = \bar{K}_1 \psi_{12} \psi_{13} \psi_{14} \psi_{15} \quad (1.22)$$

and by D_1 and E_1 of the form

$$D_1 = \bar{K}_1 \psi_{12} \psi_{13} \psi_{14} \psi_{15} \psi_{16} \quad (1.23)$$

The operators ψ_{12} and ψ_{13} in the above expressions (1.20...1.23)

are restricted to the set $\psi_{12}, \psi_{13}, \psi_{14}, \psi_{15}$ and ψ_{16} .

They are the only ones occurring in $\bar{K}_1$.

For the distinct, $i = 1, \dots, 6$, we use by inspection that

$$A_i = B_i = C_i = D_i = 1 \quad (1.24)$$

the trivial and algebraic relations, we have that $A_i = C_i = D_i$.

The final result of these calculations are collected in Table 1.

Table 1. Spin Terms.

1	$A_1 = C_1 = D_1$	B_1
1	1	1
2	1	1
3	1	1
4	1	1
5	1	1
6	0	-1
7	0	-1
8	0	-1
9	$-\frac{1}{2}$	1

A sample calculation of $B_1 = B_9$ for the singlet will be given.

From (1.36) one obtains

$$Q_{13} Q_{24} K_9$$

$$= \frac{1}{\sqrt{12}} (\alpha\beta\alpha\beta + \beta\alpha\beta\alpha + \beta\alpha\alpha\beta + \alpha\beta\beta\alpha - 2(\beta\beta\alpha\alpha + \alpha\alpha\beta\beta)),$$

hence multiplying on the left by $\overline{K}_9 = \frac{1}{\sqrt{12}} (\overline{\alpha\beta\alpha\beta} + \overline{\beta\alpha\beta\alpha} + \overline{\alpha\beta\beta\alpha} + \overline{\beta\alpha\alpha\beta} - 2(\overline{\alpha\alpha\beta\beta} + \overline{\beta\beta\alpha\alpha}))$ one obtains

$$\frac{1}{12} (1+1+1+1+4(1+1)) = 1.$$

Likewise one obtains for $Q_{14} Q_{23} K_9$,

$$Q_{14} Q_{23} K_9 = Q_{13} Q_{24} K_9 =$$

$$\frac{1}{\sqrt{12}} (\beta\alpha\beta\alpha + \alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2(\beta\beta\alpha\alpha + \alpha\alpha\beta\beta)) = K_9.$$

Q13 Q24 Q3

$\frac{1}{10} = \frac{1}{10}$

$$((x \times 0) + 0 \times x) \cdot 0 = 0 \times 0 + 0 \times 0$$

$$1 = (1+1)^2 + 1 + 1 + 1 + 1) \frac{1}{81}$$

$$Q_{12} \otimes Q_{23} \otimes Q_{34} = Q_{12} \otimes Q_{23} \otimes K_0 =$$

Since $Q_{13} Q_{24} = Q_{24} Q_{13}$ and $Q_{14} Q_{23} = Q_{23} Q_{14}$, we can conclude that $B_q = 1$. The remaining singlet and the triplet calculations are performed in the same fashion.

6. Coordinate Exchanges.

The spatial part of the wave function will depend on the three variables $\vec{r}$, $\vec{p}_1$ and $\vec{p}_2$. We can define a new set $\vec{r}$, $\vec{x}$ and $\vec{y}$ for convenience, i.e.,

$$\begin{aligned} \vec{r} &= \vec{r}, \\ \vec{x} &= \frac{\vec{p}_1 + \vec{p}_2}{2}, \quad \vec{y} = \frac{\vec{p}_1 - \vec{p}_2}{2}. \end{aligned} \quad (1.54)$$

The following relations will then hold,

$$\begin{aligned} \vec{r}_{13} &= \vec{r} + \vec{y}, & \vec{p}_1 &= \vec{x} + \vec{y}, \\ \vec{r}_{14} &= \vec{r} + \vec{x}, & \vec{p}_2 &= \vec{x} - \vec{y}, \\ \vec{r}_{23} &= \vec{r} - \vec{x}, \\ \vec{r}_{24} &= \vec{r} - \vec{y}, \end{aligned} \quad (1.55)$$

We can now write out a table, Table 2, indicating the effect of the P_{ij} operators on $\vec{r}$, $\vec{p}_1$ and $\vec{p}_2$. We express the results in terms of $\vec{r}$, $\vec{x}$ and $\vec{y}$.

Integrals in $d\vec{p}_1$ and $d\vec{p}_2$ originally, will be transformed into integrals in $d\vec{x}$ and $d\vec{y}$. The Jacobian of this transformation is,

Since $\phi_{12}\phi_{23} = \phi_{13}\phi_{23}$ and $\phi_{12}\phi_{23} = \phi_{13}\phi_{23}$, we can conclude that $\phi_{12} = \phi_{13}$. The remaining relations are satisfied in the same fashion.

6. Coordinate Transformations.

The essential part of the new coordinate will depend on the three variables x , y , and z . We can define a new set $\bar{x}$, $\bar{y}$, and $\bar{z}$ for convenience, i.e.,

$$\bar{x} = \frac{x + y + z}{3}, \quad \bar{y} = \frac{x + y - z}{2}, \quad \bar{z} = \frac{x - y - z}{2}.$$

The following relations will then hold:

$$\begin{aligned} \bar{x} &= \frac{x + y + z}{3}, & \bar{y} &= \frac{x + y - z}{2}, & \bar{z} &= \frac{x - y - z}{2}, \\ \bar{x} &= \frac{x + y + z}{3}, & \bar{y} &= \frac{x + y - z}{2}, & \bar{z} &= \frac{x - y - z}{2}, \\ \bar{x} &= \frac{x + y + z}{3}, & \bar{y} &= \frac{x + y - z}{2}, & \bar{z} &= \frac{x - y - z}{2}, \\ \bar{x} &= \frac{x + y + z}{3}, & \bar{y} &= \frac{x + y - z}{2}, & \bar{z} &= \frac{x - y - z}{2}, \end{aligned}$$

We can now write out a table, Table 6, indicating the effect of the P operator on $\bar{x}$, $\bar{y}$, and $\bar{z}$. We express the results in terms of $\bar{x}$, $\bar{y}$, and $\bar{z}$. Integrals in $\bar{x}$, $\bar{y}$, and $\bar{z}$ will be transformed into integrals in x , y , and z . The function of this transformation is

$$J \left(\frac{\vec{p}_1, \vec{p}_2}{\vec{x}, \vec{y}} \right) = \begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{vmatrix} = 8.$$

(1.56)

So that in the integrals, $d\vec{p}_1 d\vec{p}_2$ is replaced by $8 d\vec{x} d\vec{y}$.

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$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(0.0)

So that in the inverse, 49.44 is replaced by 4.44 .

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Table 2. Coordinate Exchanges.

	on $\vec{r}$	on $\vec{P}_1$	on $\vec{P}_2$
P_{13}	$-\vec{y}$	$\vec{x} - \vec{r}$	$\vec{x} + \vec{r}$
P_{14}	$-\vec{x}$	$\vec{y} - \vec{r}$	$-\vec{y} - \vec{r}$
P_{23}	$\vec{x}$	$\vec{y} + \vec{r}$	$-\vec{y} + \vec{r}$
P_{24}	$\vec{y}$	$\vec{x} + \vec{r}$	$\vec{x} - \vec{r}$
$P_{13} P_{14}$	$-\vec{x}$	$\vec{y} - \vec{r}$	$\vec{r} + \vec{y}$
$P_{14} P_{13}$	$-\vec{y}$	$\vec{x} - \vec{r}$	$-\vec{r} - \vec{x}$
$P_{13} P_{23}$	$\vec{x}$	$-\vec{r} - \vec{y}$	$\vec{r} - \vec{y}$
$P_{23} P_{13}$	$-\vec{y}$	$\vec{r} - \vec{x}$	$\vec{r} + \vec{x}$
$P_{13} P_{24}$	$-\vec{r}$	$\vec{x} - \vec{y}$	$\vec{x} + \vec{y}$
$P_{24} P_{13}$	$-\vec{r}$	$-\vec{y} + \vec{x}$	$\vec{x} + \vec{y}$
$P_{14} P_{23}$	$-\vec{r}$	$\vec{y} - \vec{x}$	$-\vec{x} - \vec{y}$
$P_{23} P_{14}$	$-\vec{r}$	$+\vec{y} - \vec{x}$	$-\vec{x} - \vec{y}$
$P_{14} P_{24}$	$\vec{y}$	$-\vec{r} - \vec{y}$	$\vec{x} - \vec{r}$
$P_{24} P_{14}$	$-\vec{x}$	$\vec{r} - \vec{y}$	$-\vec{y} - \vec{r}$
$P_{23} P_{24}$	$\vec{y}$	$\vec{r} + \vec{x}$	$\vec{r} - \vec{x}$
$P_{24} P_{23}$	$\vec{x}$	$\vec{r} + \vec{y}$	$\vec{y} - \vec{r}$
$P_{14} P_{13} P_{24}$	$\vec{x}$	$-\vec{r} - \vec{y}$	$\vec{y} - \vec{r}$
$P_{23} P_{13} P_{24}$	$-\vec{x}$	$\vec{r} - \vec{y}$	$\vec{y} + \vec{r}$

CHAPTER II

SOLUTION FOR THE INTERACTION

$$V_{ij} = (g_1 + M P_{ij} + g_2 Q_{ij} + g P_{ij} Q_{ij}) J(r_{ij}).$$

1. The $\vec{p}$ Integrations.

We shall now consider an interaction of the type ^{1,3}

$$V_{ij} = (g_1 + M P_{ij} + g_2 Q_{ij} + g P_{ij} Q_{ij}) J(r_{ij}), \quad (2.0)$$

as being the interaction between nucleon pairs. In (2.0), g_1 ,

M , g_2 and g are parameters whose values have been discussed by several authors.^{1,2,3} $J(r_{ij})$ is the radial part of the interaction depending only on $|\vec{r}_{ij}|$. In the case of the two protons we will have to write

$$V_{24} = (g_1 + M P_{24} + g_2 Q_{24} + g P_{24} Q_{24}) J(r_{24}) + \frac{e^2}{r_{24}}, \quad (2.1)$$

in order to include the Coulomb repulsion. In (2.0) the term in

g_1 represents the non-exchange (Wigner) interaction, in M the coordinate exchange (Majorana) interaction, in g_2 the spin exchange (Bartlett) interaction, and in g the spin-coordinate exchange (Heisenberg) interaction.¹⁰ Since $P_{ij} = 1$ and $Q_{ij} = 1$ for the deuteron, we normalize (2.0) by requiring that

$$M = 1 - g - g_1 - g_2, \quad (2.2)$$

so that V_{12} goes over into $J(r_{12})$ for the deuteron ground state. The term $V_{INT} K$ in (1.49) is then written as

$$\begin{aligned}
 V_{INT} K = & \\
 & \left[(g_1 + m P_{13} + g_2 Q_{13} + g P_{13} Q_{13}) J(r_{13}) \right. \\
 & + (g_1 + m P_{14} + g_2 Q_{14} + g P_{14} Q_{14}) J(r_{14}) \\
 & + (g_1 + m P_{23} + g_2 Q_{23} + g P_{23} Q_{23}) J(r_{23}) \\
 & + (g_1 + m P_{24} + g_2 Q_{24} + g P_{24} Q_{24}) J(r_{24}) \\
 & \left. + e^2 / r_{24} \right] (1 - P_{13} Q_{13} - P_{24} Q_{24} + P_{13} Q_{13} P_{24} Q_{24}). \quad (2.3)
 \end{aligned}$$

This gives a total of 64 nuclear terms and 4 Coulomb terms in the integral of (1.49). The expression (2.3) is inserted into (1.49) where we set $e^{i \mathbf{k} \cdot \vec{r}_0} = f(\vec{r})$ to abbreviate. One then obtains a sum of 68 integrals. The integrands will involve A space and B spin operators (where A, B = 0, 1, 2, 3), which we shall denote by (A, B). These integrands are as follows:

no limit N_2 goes over into $\delta(r, z)$ for the constant ground state. The term $V_{int} K$ in (1.10) is then written as

$$V_{int} K = \left[(g_1 + m p_{12} + g_2 Q_{12} + g_3 Q_{13}) \delta(r_{12}) \right. \\ + (g_1 + m p_{14} + g_2 Q_{14} + g_3 Q_{15}) \delta(r_{14}) \\ + (g_1 + m p_{23} + g_2 Q_{23} + g_3 Q_{25}) \delta(r_{23}) \\ + (g_1 + m p_{24} + g_2 Q_{24} + g_3 Q_{25}) \delta(r_{24}) \\ \left. + \frac{g_3^2}{4} \left[(1 - p_{12} Q_{12} - p_{14} Q_{14} + p_{13} Q_{15} + p_{23} Q_{23} + p_{24} Q_{24} + p_{35} Q_{35}) \right] \right] \delta(r, z). \quad (1.11)$$

This gives a total of 18 integrals and 18 integrals in the integral of (1.10). The expression (1.11) is inserted into (1.10) where we set $\delta(r, z) = \delta(r, z) + \delta(r, z)$ and then obtain a sum of 36 integrals. The integrals with two δ terms and δ again operators (where $\delta = \delta, 1, 2, 3$), which we shall denote by (1.12). These integrals are as follows:

I. Nuclear terms;

Type-

$$(0,0) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g_1 J_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad g_1 J_{14} \quad "$$

$$" \quad g_1 J_{23} \quad "$$

$$" \quad g_1 J_{24} \quad "$$

$$" \quad (-g_1) J_{13} \quad "$$

$$" \quad (-g_1) J_{24} \quad "$$

$$(1,0) \bar{\Psi}_0(1) \bar{\Psi}_0(2) m J_{13} P_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad m J_{14} P_{14} \quad "$$

$$" \quad m J_{23} P_{23} \quad "$$

$$" \quad m J_{24} P_{24} \quad "$$

$$" \quad (-g_2) J_{13} P_{13} \quad "$$

$$" \quad (-g_2) J_{24} P_{24} \quad "$$

$$(0,1) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g_2 J_{13} Q_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad g_2 J_{14} Q_{14} \quad "$$

$$" \quad g_2 J_{23} Q_{23} \quad "$$

$$" \quad g_2 J_{24} Q_{24} \quad "$$

$$" \quad (-m) J_{13} Q_{13} \quad "$$

$$" \quad (-m) J_{24} Q_{24} \quad "$$

$$(1,1) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g J_{13} P_{13} Q_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad g J_{14} P_{14} Q_{14} \quad "$$

$$" \quad g J_{23} P_{23} Q_{23} \quad "$$

$$" \quad g J_{24} P_{24} Q_{24} \quad "$$

$$" \quad g J_{13} P_{24} Q_{24} \quad "$$

$$" \quad g J_{24} P_{13} Q_{13} \quad "$$

$$\bar{\Psi}_0(1) \bar{\Psi}_0(2) (-g_1) J_{13} P_{13} Q_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c$$

$$" (-g_1) J_{14} P_{13} Q_{13} "$$

$$" (-g_1) J_{23} P_{13} Q_{13} "$$

$$" (-g_1) J_{24} P_{13} Q_{13} "$$

$$" (-g_1) J_{13} P_{24} Q_{24} "$$

$$" (-g_1) J_{14} P_{24} Q_{24} "$$

$$" (-g_1) J_{23} P_{24} Q_{24} "$$

$$" (-g_1) J_{24} P_{24} Q_{24} "$$

$$(2,1) \bar{\Psi}_0(1) \bar{\Psi}_0(2) (-M) J_{13} P_{13} P_{24} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c$$

$$" (-M) J_{14} P_{14} P_{13} Q_{13} "$$

$$" (-M) J_{14} P_{14} P_{24} Q_{24} "$$

$$" (-M) J_{23} P_{23} P_{13} Q_{13} "$$

$$" (-M) J_{23} P_{23} P_{24} Q_{24} "$$

$$" (-M) J_{24} P_{24} P_{13} Q_{13} "$$

$$" g_2 J_{13} P_{13} P_{24} Q_{24} "$$

$$" g_2 J_{24} P_{13} P_{24} Q_{13} "$$

$$(1,2) \bar{\Psi}_0(1) \bar{\Psi}_0(2) M J_{13} P_{24} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c$$

$$" M J_{24} P_{13} Q_{13} Q_{24} "$$

$$" (-g_2) J_{13} P_{24} Q_{13} Q_{24} "$$

$$" (-g_2) J_{14} P_{13} Q_{14} Q_{13} "$$

$$\begin{array}{l}
 \bar{\Psi}_0(1) \bar{\Psi}_0(2) (-g_2) J_{14} P_{24} Q_{14} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c \\
 " \quad \quad \quad (-g_2) J_{23} P_{13} Q_{23} Q_{13} \quad \quad \quad " \\
 " \quad \quad \quad (-g_2) J_{23} P_{24} Q_{23} Q_{24} \quad \quad \quad " \\
 " \quad \quad \quad (-g_2) J_{24} P_{13} Q_{24} Q_{13} \quad \quad \quad "
 \end{array}$$

$$\begin{array}{l}
 (2,2) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g_1 J_{13} P_{13} P_{24} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c \\
 " \quad \quad \quad g_1 J_{14} \quad \quad \quad " \quad \quad \quad " \\
 " \quad \quad \quad g_1 J_{23} \quad \quad \quad " \quad \quad \quad " \\
 " \quad \quad \quad g_1 J_{24} \quad \quad \quad " \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{13} \quad \quad \quad " \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{24} P_{24} P_{13} Q_{24} Q_{13} \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{14} P_{14} P_{13} Q_{14} Q_{13} \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{14} P_{14} P_{24} Q_{14} Q_{24} \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{23} P_{23} P_{13} Q_{23} Q_{13} \quad \quad \quad " \\
 " \quad \quad \quad (-g) J_{23} P_{23} P_{24} Q_{23} Q_{24} \quad \quad \quad "
 \end{array}$$

$$\begin{array}{l}
 (3,2) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g_1 J_{14} P_{14} P_{13} P_{24} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_c \\
 " \quad \quad \quad g_1 J_{23} P_{23} P_{13} P_{24} Q_{13} Q_{24} \quad \quad \quad "
 \end{array}$$

$$(2,3) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g_2 J_{14} P_{13} P_{24} Q_{14} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad g_2 J_{23} P_{13} P_{24} Q_{23} Q_{13} Q_{24} \quad "$$

$$(3,3) \bar{\Psi}_0(1) \bar{\Psi}_0(2) g J_{14} P_{14} P_{13} P_{24} Q_{14} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad g J_{23} P_{23} P_{13} P_{24} Q_{23} Q_{13} Q_{24} \quad "$$

II. Coulomb terms;

Type-
 $(0,0) \bar{\Psi}_0(1) \bar{\Psi}_0(2) \frac{e^2}{r_{24}} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$

$$(1,1) \bar{\Psi}_0(1) \bar{\Psi}_0(2) \left(-\frac{e^2}{r_{24}}\right) P_{13} Q_{13} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

$$" \quad " \quad P_{24} Q_{24} \quad "$$

$$(2,2) \bar{\Psi}_0(1) \bar{\Psi}_0(2) \frac{e^2}{r_{24}} P_{13} P_{24} Q_{13} Q_{24} \Psi_0(1) \Psi_0(2) f(\vec{r}) \chi_1$$

Multiplying on the left of these expressions by $\bar{\chi}_1$ and operating on the functions with the P_{ij} operators, one obtains, in view

of (1.50...5) and tables 1 and 2, the following integrands:

Nuclear;

Integral type (u_i)

(0,0)

					$i =$
$\bar{\Psi}_0(1)\bar{\Psi}_0(2)$	g_1	$f(\vec{r})$	$J(\vec{r}+\vec{y})$	$\Psi_0(\vec{x}+\vec{y})\Psi_0(\vec{x}-\vec{y})$	1
"	g_1	"	$J(\vec{r}+\vec{x})$	"	1
"	g_1	"	$J(\vec{r}-\vec{x})$	"	1
"	g_1	"	$J(\vec{r}-\vec{y})$	"	1
"	$(-g_1)$	"	$J(\vec{r}+\vec{y})$	"	1
"	$(-g_1)$	"	$J(\vec{r}-\vec{y})$	"	1

(1,0)

$\bar{\Psi}_0(1)\bar{\Psi}_0(2)$	m	$J(\vec{r}+\vec{y})$	$f(-\vec{y})$	$\Psi_0(\vec{x}-\vec{r})\Psi_0(\vec{x}+\vec{r})$	2
"	m	$J(\vec{r}+\vec{x})$	$f(-\vec{x})$	$\Psi_0(\vec{y}-\vec{r})\Psi_0(-\vec{y}-\vec{r})$	2
"	m	$J(\vec{r}-\vec{x})$	$f(\vec{x})$	$\Psi_0(\vec{y}+\vec{r})\Psi_0(\vec{r}-\vec{y})$	2
"	m	$J(\vec{r}-\vec{y})$	$f(\vec{y})$	$\Psi_0(\vec{x}+\vec{r})\Psi_0(\vec{x}-\vec{r})$	2
"	$(-g_2)$	$J(\vec{r}+\vec{y})$	$f(-\vec{y})$	$\Psi_0(\vec{x}-\vec{r})\Psi_0(\vec{x}+\vec{r})$	2
"	$(-g_2)$	$J(\vec{r}-\vec{y})$	$f(\vec{y})$	$\Psi_0(\vec{x}+\vec{r})\Psi_0(\vec{x}-\vec{r})$	2

(0,1)

$\bar{\Psi}_0(1)\bar{\Psi}_0(2)$	g_2	$f(\vec{r})$	A_1	$J(\vec{r}+\vec{y})\Psi_0(\vec{x}+\vec{y})\Psi_0(\vec{x}-\vec{y})$	1
"	g_2	"	"	$J(\vec{r}+\vec{x})$	1
"	g_2	"	"	$J(\vec{r}-\vec{x})$	1
"	g_2	"	"	$J(\vec{r}-\vec{y})$	1
"	$(-m)$	"	"	$J(\vec{r}+\vec{y})$	1
"	$(-m)$	"	"	$J(\vec{r}-\vec{y})$	1

(1,1)

$\bar{\psi}_0(1)$	$\bar{\psi}_0(2)$	A_L	g	$J(\vec{r}+\vec{y})$	$f(-\vec{y})$	$\psi_0(\vec{x}-\vec{r})$	$\psi_0(\vec{x}+\vec{r})$	2
"	"		g	$J(\vec{r}+\vec{x})$	$f(-\vec{x})$	$\psi_0(\vec{y}-\vec{r})$	$\psi_0(-\vec{y}-\vec{r})$	2
"	"		g	$J(\vec{r}-\vec{x})$	$f(\vec{x})$	$\psi_0(\vec{y}+\vec{r})$	$\psi_0(\vec{r}-\vec{y})$	2
"	"		g	$J(\vec{r}-\vec{y})$	$f(\vec{y})$	$\psi_0(\vec{x}+\vec{r})$	$\psi_0(\vec{x}-\vec{r})$	2
"	"		g	$J(\vec{r}+\vec{y})$	$f(\vec{y})$	$\psi_0(\vec{x}-\vec{r})$	$\psi_0(\vec{x}+\vec{r})$	3
"	"		g	$J(\vec{r}-\vec{y})$	$f(-\vec{y})$	$\psi_0(\vec{x}+\vec{r})$	$\psi_0(\vec{x}-\vec{r})$	3
"	"	$(-g_1)$	$J(\vec{r}+\vec{y})$	$f(-\vec{y})$	$\psi_0(\vec{x}-\vec{r})$	$\psi_0(\vec{x}+\vec{r})$		2
"	"	$(-g_1)$	$J(\vec{r}+\vec{x})$		"			5
"	"	$(-g_1)$	$J(\vec{r}-\vec{x})$		"			5
"	"	$(-g_1)$	$J(\vec{r}-\vec{y})$		"			3
"	"	$(-g_1)$	$J(\vec{r}+\vec{y})$		"			3
"	"	$(-g_1)$	$J(\vec{r}+\vec{x})$		"			5
"	"	$(-g_1)$	$J(\vec{r}-\vec{x})$		"			5
"	"	$(-g_1)$	$J(\vec{r}-\vec{y})$		"			2

(1,2)

$\bar{\psi}_0(1)$	$\bar{\psi}_0(2)$	$(-g_2)$	B_L	$J(\vec{r}+\vec{y})$	$f(\vec{y})$	$\psi_0(\vec{x}+\vec{r})$	$\psi_0(\vec{x}-\vec{r})$	3
"	"	$(-g_2)$	A_L	$J(\vec{r}+\vec{x})$	$f(-\vec{y})$	"		5
"	"	$(-g_2)$	A_L	$J(\vec{r}+\vec{x})$	$f(\vec{y})$	"		5
"	"	$(-g_2)$	A_L	$J(\vec{r}-\vec{x})$	$f(-\vec{y})$	"		5
"	"	$(-g_2)$	A_L	$J(\vec{r}-\vec{x})$	$f(\vec{y})$	"		5
"	"	$(-g_2)$	B_L	$J(\vec{r}-\vec{y})$	$f(-\vec{y})$	"		3
"	"	$\mathcal{M}$	B_L	$J(\vec{r}+\vec{y})$	$f(\vec{y})$	"		3
"	"	$\mathcal{M}$	B_L	$J(\vec{r}-\vec{y})$	$f(-\vec{y})$	"		3

(2,1)

$\bar{\Psi}_0(1) \bar{\Psi}_0(2)$	$(-m)$	A_i	$J(\vec{r}+\vec{y}) + (-\vec{r})$	$\Psi_0(\vec{x}-\vec{y}) \Psi_0(\vec{x}+\vec{y})$	4
"	$(-m)$	"	$J(\vec{r}+\vec{x}) + (-\vec{y})$	$\Psi_0(\vec{x}-\vec{r}) \Psi_0(\vec{r}-\vec{x})$	5
"	$(-m)$	"	$J(\vec{r}+\vec{x}) + (\vec{y})$	$\Psi_0(\vec{r}-\vec{x}) \Psi_0(\vec{x}-\vec{r})$	5
"	$(-m)$	"	$J(\vec{r}-\vec{x}) + (-\vec{y})$	$\Psi_0(\vec{r}-\vec{x}) \Psi_0(\vec{r}+\vec{x})$	5
"	$(-m)$	"	$J(\vec{r}-\vec{x}) + (\vec{y})$	$\Psi_0(\vec{r}+\vec{x}) \Psi_0(\vec{r}-\vec{x})$	5
"	$(-m)$	"	$J(\vec{r}-\vec{y}) + (-\vec{r})$	$\Psi_0(\vec{x}-\vec{y}) \Psi_0(\vec{x}+\vec{y})$	4
"	g_2	"	$J(\vec{r}+\vec{y}) + (-\vec{r})$	"	4
"	g_2	"	$J(\vec{r}-\vec{y}) + (-\vec{r})$	"	4

(2,2)

$\bar{\Psi}_0(1) \bar{\Psi}_0(2)$	$(-g)$	B_i	$J(\vec{r}+\vec{y}) + (-\vec{r})$	$\Psi_0(\vec{y}-\vec{x}) \Psi_0(\vec{x}+\vec{y})$	4
"	$(-g)$	B_i	$J(\vec{r}-\vec{y}) + (-\vec{r})$	$\Psi_0(\vec{x}-\vec{y}) \Psi_0(\vec{x}+\vec{y})$	4
"	$(-g)$	A_i	$J(\vec{r}+\vec{x}) + (-\vec{y})$	$\Psi_0(\vec{x}-\vec{r}) \Psi_0(\vec{r}-\vec{x})$	5
"	$(-g)$	A_i	$J(\vec{r}+\vec{x}) + (\vec{y})$	$\Psi_0(\vec{r}-\vec{x}) \Psi_0(\vec{x}-\vec{r})$	5
"	$(-g)$	A_i	$J(\vec{r}-\vec{x}) + (-\vec{y})$	$\Psi_0(\vec{r}-\vec{x}) \Psi_0(\vec{r}+\vec{x})$	5
"	$(-g)$	A_i	$J(\vec{r}-\vec{x}) + (\vec{y})$	$\Psi_0(\vec{r}+\vec{x}) \Psi_0(\vec{r}-\vec{x})$	5
"	g_1	B_i	$J(\vec{r}+\vec{y}) + (-\vec{r})$	$\Psi_0(\vec{x}-\vec{y}) \Psi_0(\vec{x}+\vec{y})$	4
"	g_1	B_i	$J(\vec{r}+\vec{x})$	"	4
"	g_1	B_i	$J(\vec{r}-\vec{x})$	"	4
"	g_1	B_i	$J(\vec{r}-\vec{y})$	"	4

(3,2)

$\bar{\Psi}_0(1) \bar{\Psi}_0(2)$	m	B_i	$J(\vec{r}+\vec{x}) + (\vec{x})$	$\Psi_0(\vec{r}-\vec{y}) \Psi_0(\vec{y}-\vec{r})$	3
"	m	B_i	$J(\vec{r}-\vec{x}) + (-\vec{x})$	$\Psi_0(\vec{r}-\vec{y}) \Psi_0(\vec{r}+\vec{y})$	3

(2,3)

$\bar{\Psi}_0(1) \bar{\Psi}_0(2)$	g_2	A_i	$J(\vec{r}+\vec{x}) + (-\vec{r})$	$\Psi_0(\vec{x}-\vec{y}) \Psi_0(\vec{x}+\vec{y})$	4
"	g_2	A_i	$J(\vec{r}-\vec{x})$	"	4

Integral type

$$\begin{array}{lcl}
 (3,3) & \bar{\psi}_0(1) \bar{\psi}_0(2) \int A_c J(\vec{r}+\vec{x}) f(\vec{x}) \psi_0(-\vec{r}-\vec{y}) \psi_0(\vec{y}-\vec{r}) & 3 \\
 & " \int A_c J(\vec{r}-\vec{x}) f(-\vec{x}) & "
 \end{array}$$

Coulomb;

$$\begin{array}{lcl}
 (0,0) & \bar{\psi}_0(1) \bar{\psi}_0(2) \frac{e^2}{|\vec{r}-\vec{y}|} f(\vec{r}) \psi_0(\vec{x}+\vec{y}) \psi_0(\vec{x}-\vec{y}) & (C_1) \\
 & & 1
 \end{array}$$

$$\begin{array}{lcl}
 (1,1) & \bar{\psi}_0(1) \bar{\psi}_0(2) \frac{(-e^2)}{|\vec{r}-\vec{y}|} A_c f(-\vec{y}) \psi_0(\vec{x}-\vec{r}) \psi_0(\vec{x}+\vec{r}) & 2 \\
 & " " f(\vec{y}) & " 3
 \end{array}$$

$$\begin{array}{lcl}
 (2,2) & \bar{\psi}_0(1) \bar{\psi}_0(2) \frac{e^2}{|\vec{r}-\vec{y}|} B_c f(-\vec{r}) \psi_0(\vec{x}-\vec{y}) \psi_0(\vec{x}+\vec{y}) & 4
 \end{array}$$

In these expressions $\bar{\psi}_0(1) \bar{\psi}_0(2) = \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y})$. When the integration is performed on these integrands, we obtain nine distinct integrals, $u_1, \dots, u_5$ and $c_1, \dots, c_4$. The nine integrals are found to be

$$u_1 = \iint d\vec{x} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) J(\vec{r}+\vec{y}) f(\vec{r}) \psi_0(\vec{x}+\vec{y}) \psi_0(\vec{x}-\vec{y}), \quad (2.4)$$

$$u_2 = \iint d\vec{x} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) J(\vec{r}+\vec{y}) f(-\vec{y}) \psi_0(\vec{x}+\vec{r}) \psi_0(\vec{x}-\vec{r}), \quad (2.5)$$

(2.3)

$$\psi_0(u) \psi_0(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt$$

Consequently,

(2.4)

$$\psi_0(u) \psi_0(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt$$

(2.5)

$$\psi_0(u) \psi_0(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt$$

(2.6)

$$\psi_0(u) \psi_0(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt$$

In these circumstances, $\psi_0(u) \psi_0(v)$ is a function of u and v which is integrable in the sense of Lebesgue, and the integral $\int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt$ is found to be

$$U_1 = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt \right]$$

$$U_2 = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(u) \psi_0(v) e^{-\frac{1}{2}t^2} dt \right]$$

$$u_3 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) J(\vec{r}+\vec{y}) f(\vec{y}) \psi_0(\vec{x}+\vec{r}) \psi_0(\vec{x}-\vec{r}), \quad (2.6)$$

$$u_4 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) J(\vec{r}+\vec{y}) f(-\vec{r}) \psi_0(\vec{r}+\vec{y}) \psi_0(\vec{x}-\vec{y}), \quad (2.8)$$

$$u_5 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) J(\vec{r}+\vec{x}) f(-\vec{y}) \psi_0(\vec{x}+\vec{r}) \psi_0(\vec{x}-\vec{r}) \quad (2.9)$$

and the Coulomb integrals,

$$c_1 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) \frac{e^2}{|\vec{r}-\vec{y}|} f(\vec{r}) \psi_0(\vec{x}+\vec{y}) \psi_0(\vec{x}-\vec{y}), \quad (2.10)$$

$$c_2 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) \frac{e^2}{|\vec{r}-\vec{y}|} f(-\vec{y}) \psi_0(\vec{x}-\vec{r}) \psi_0(\vec{r}+\vec{y}), \quad (2.11)$$

$$c_3 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) \frac{e^2}{|\vec{r}-\vec{y}|} f(\vec{y}) \psi_0(\vec{x}-\vec{r}) \psi_0(\vec{x}+\vec{r}), \quad (2.12)$$

$$c_4 = \iint d\vec{r} d\vec{y} \bar{\psi}_0(\vec{x}+\vec{y}) \bar{\psi}_0(\vec{x}-\vec{y}) \frac{e^2}{|\vec{r}-\vec{y}|} f(-\vec{r}) \psi_0(\vec{x}+\vec{y}) \psi_0(\vec{x}-\vec{y}). \quad (2.13)$$

If we now set

$$\begin{aligned}
 a_1 &= (4g_1 - 2g) + (4g_2 - 2m) A_c, \\
 a_2 &= (4m - 2g_2) + (4g - 2g_1) A_c, \\
 a_3 &= (4g - 2g_1) A_c + (4m - 2g_2) B_c, \\
 a_4 &= (4g_2 - 2m) A_c + (4g_1 - 2g) B_c, \\
 a_5 &= -4A_c(m + g + g_1 + g_2) = -4A_c,
 \end{aligned}
 \tag{2.14}$$

one can represent the sum of the 68 integrals as,

$$\sum_{j=1}^5 a_j u_j + c_1 - A_c(c_2 + c_3) + B_c c_4. \tag{2.15}$$

The equation (1.49) for $\mathcal{F}_c(\theta)$ thus becomes for this interaction, remembering that the Jacobian equals 8 in (1.56),

$$\mathcal{F}_c(\theta) = -\frac{4m}{\pi \hbar^2} \int d\vec{r} e^{-i(\vec{k} \cdot \vec{n} + \vec{r} \cdot \vec{r})} \left(\sum_{j=1}^5 a_j u_j + c_1 - A_c(c_2 + c_3) + B_c c_4 \right). \tag{2.16}$$

If we denote by U_c the integral

$$U_c = \int d\vec{r} e^{-i(\vec{k} \cdot \vec{n} + \vec{r} \cdot \vec{r})} u_c, \tag{2.17}$$

and by V_c the integral

$$V_c = \int d\vec{r} e^{-i(\vec{k} \cdot \vec{n} + \vec{r} \cdot \vec{r})} c_c, \tag{2.18}$$

If we now set

$$A_1 = (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = 1$$

$$A_2 = (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = 1$$

(1.1)

$$A_3 = (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = 1$$

$$A_4 = (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = 1$$

$$A_5 = (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = 1$$

one can represent the sum of the two

$$\sum_{j=1}^n (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = n$$

The equation (1.1) for A_j that follows from (1.1) remembering that the domain is $0 \leq A_j \leq 1$.

(1.2)

$$\sum_{j=1}^n (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) = n$$

If we denote by U_j the integral

(1.3)

$$U_j = \int_0^1 (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) dx$$

and by V_j the integral

(1.4)

$$V_j = \int_0^1 (1/2)(1 - \sqrt{3}) + (1/2)(1 + \sqrt{3}) dx$$

then $\mathcal{I}_L(\theta)$ becomes

(2.19)

$$\mathcal{I}_L(\theta) = -\frac{4m}{\pi \hbar^2} \left(\sum_{j=1}^5 a_j U_j + V_1 - A_L(V_2 + V_3) + B_L V_4 \right).$$

2. Solution with Gaussian Functions.

We shall now calculate the U's and V's by setting J and as follows,²

$$J(r) = -w_0 e^{-\mu^2 r^2}, \quad (2.20)$$

$$\psi_0(r) = a e^{-\lambda^2 r^2}, \quad (2.21)$$

with $f(\vec{r}) = e^{i\vec{k} \cdot \vec{r}}$. The unit vector $\vec{n}_0$ is in the $+z$ (incident) direction; μ is the reciprocal of the range of the nuclear interaction; $|w_0|$ is the depth of the nuclear potential, and a is a normalizing constant whose value is

$$a^2 = \frac{2^{3/2} \lambda^3}{\pi^{3/2}}, \quad (2.22)$$

where λ is a constant characteristic of the deuteron ground state wave function. 1,2 Values for these parameters will be chosen in later computations (Chapter III).

then $\bar{E}(e)$ becomes

(2.13)

$$\bar{E}(e) = \frac{1}{\pi} \int_0^\pi \left(\frac{1}{2} (V_1 + V_2) + \frac{1}{2} (V_1 - V_2) \cos \theta \right) d\theta$$

2. Solution with boundary conditions

It shall be assumed that the boundary conditions are

as follows

(2.14)

$$V_1 = V_2 = 0$$

(2.15)

$$\frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial x} = 0$$

with $\psi(r) = 0$ at the boundary of the region. The boundary conditions of the region of the problem are that the potential is zero at the boundary of the region, and is a constant value at the boundary of the region.

(2.16)

$$\frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial x} = 0$$

where λ is a constant determined by the boundary conditions. The boundary conditions of the region are that the potential is zero at the boundary of the region, and is a constant value at the boundary of the region.

As a sample calculation, we shall integrate u_1 of (2.4).

Making the above substitutions into (2.4), we obtain

$$u_1 = -w_0 a^4 e^{i k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2} \iint d\vec{x} d\vec{y} e^{-4\lambda^2 x^2 - 4\lambda^2 y^2 - 2\mu^2 \vec{r} \cdot \vec{y} - \mu^2 y^2}$$

Setting

$$X(\lambda) = \int d\vec{x} e^{-4\lambda^2 x^2} = \frac{\pi^{3/2}}{8\lambda^3},$$

u_1 becomes

$$u_1 = -w_0 a^4 X(\lambda) e^{i k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2} \int d\vec{y} e^{-(4\lambda^2 + \mu^2)y^2 - 2\mu^2 \vec{r} \cdot \vec{y}}$$

Let $\delta = \cos(\vec{r}, \vec{y})$ and the integral is then

$$u_1 = -w_0 a^4 X(\lambda) e^{i k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2} \frac{1}{2\pi} \int_0^\infty y^2 dy e^{-(4\lambda^2 + \mu^2)y^2} \int_{-1}^1 e^{-2\mu^2 r y \delta} d\delta,$$

which becomes

$$u_1 = -\frac{w_0 \pi a^4 X(\lambda)}{\mu^2 r} e^{i k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2} \int_0^\infty y dy e^{-(4\lambda^2 + \mu^2)y^2} (e^{2\mu^2 r y} - e^{-2\mu^2 r y}),$$

as a result of the...

holding the...

...

...

$$X(t) = \dots$$

...

...

...

...

...

$$H = \dots$$

and then becomes

$$u_1 = -\frac{w_0 a^4 \chi(\lambda) \pi}{\mu^2 r} e^{(k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2)} \int_{-\infty}^{+\infty} y dy e^{-(4\lambda^2 + \mu^2) y^2 + 2\mu^2 r y}$$

since

$$-\int_0^{\infty} y dy e^{-A^2 y^2 - By} = \int_{-\infty}^0 y dy e^{-A^2 y^2 + By}$$

Now setting $z = Ly + t$ where $L^2 = 4\lambda^2 + \mu^2$ and

$$t = -\frac{\mu^2 r}{L}, \text{ we obtain for } u_1,$$

$$u_1 = -\frac{w_0 a^4 \chi(\lambda) \pi}{\mu^2 r} e^{(k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2)} \int_{-\infty}^{+\infty} e^{t^2} \frac{(z-t)}{L^2} e^{-z^2} dz.$$

The integral whose integrand is ze^{-z^2} vanishes since the integrand is odd. One thus obtains

$$u_1 = -\frac{w_0 a^4 \chi(\lambda) \pi^{3/2}}{(4\lambda^2 + \mu^2)^{3/2}} e^{(k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2 + \frac{\mu^4 r^2}{4\lambda^2 + \mu^2})}$$

since $\int_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}.$

and then becomes

$$u_1 = -w_0 \frac{\pi(x) x^u}{x^2} = -w_0 \pi(x) x^{u-2}$$

since

$$-\int_0^\infty x^u \pi(x) dx = -\int_0^\infty x^{u-2} \pi(x) dx = -\int_0^\infty x^{u-2} \pi(x) dx$$

Now setting $x = t^2$, we have

$$t = \frac{x}{2} \quad \text{so that} \quad dx = 2t dt$$

$$u_1 = -w_0 \frac{\pi(x) x^u}{x^2} = -w_0 \pi(x) x^{u-2}$$

The integral whose integrand is u_1 becomes

integrating is odd. The new integrand

$$u_1 = -w_0 \frac{\pi(x) x^u}{x^2} = -w_0 \pi(x) x^{u-2}$$

since

$$\int_0^\infty x^u \pi(x) dx = \int_0^\infty x^{u-2} \pi(x) dx$$

Putting in $\pi^{3/2}/8\lambda^3$ for $X(\lambda)$, we obtain

(2.23)

$$U_1 = -\frac{w_0 a^4 \pi^3}{8\lambda^3 (4\lambda^2 + \mu^2)^{3/2}} e^{(k \vec{n}_0 \cdot \vec{r} - \mu^2 r^2 + \frac{\mu^4 r^2}{4\lambda^2 + \mu^2})}$$

From (2.17), the value of U_1 will then be

$$U_1 = -\frac{w_0 a^4 \pi^3}{8\lambda^3 (4\lambda^2 + \mu^2)^{3/2}} \int d\vec{r} e^{(k(\vec{n}_0 - \vec{n}) \cdot \vec{r} + (\frac{\mu^4}{4\lambda^2 + \mu^2} - \mu^2) r^2)}$$

Now $\frac{\mu^4}{4\lambda^2 + \mu^2} - \mu^2 = -\frac{4\lambda^2 \mu^2}{4\lambda^2 + \mu^2} = -\theta_0^2$, so that we obtain

$$U_1 = -\frac{w_0 a^4 2\pi^4}{8\lambda^3 (4\lambda^2 + \mu^2)^{3/2}} \int_0^\infty e^{-\theta_0^2 r^2} r^2 dr \int_0^\pi e^{(k|\vec{n}_0 - \vec{n}| r \cos \bar{\theta})} \sin \bar{\theta} d\bar{\theta}.$$

Integrating over $\bar{\theta}$ yields

$$U_1 = -\frac{2w_0 a^4 \pi^4}{8\lambda^3 (4\lambda^2 + \mu^2)^{3/2}} \int_0^\infty \frac{e^{-\theta_0^2 r^2}}{(k|\vec{n}_0 - \vec{n}| r)} (e^{(k|\vec{n}_0 - \vec{n}| r)} - e^{-(k|\vec{n}_0 - \vec{n}| r)})$$

Integrating in π for λ and π at $\lambda = 0$

$$U_1 = -W_0 \frac{\pi^2}{8\lambda^2 (1 + \lambda^2)} = -W_0 \frac{\pi^2}{8\lambda^2} \left(1 - \lambda^2 + \lambda^4 - \lambda^6 + \dots \right)$$

from (2.17), the value of U_1 will then be

$$U_1 = -W_0 \frac{\pi^2}{8\lambda^2} \left(1 - \lambda^2 + \lambda^4 - \lambda^6 + \dots \right)$$

$$\text{Now } U_1 = -W_0 \frac{\pi^2}{8\lambda^2} \left(1 - \lambda^2 + \lambda^4 - \lambda^6 + \dots \right)$$

$$U_1 = -W_0 \frac{\pi^2}{8\lambda^2} \left(1 - \lambda^2 + \lambda^4 - \lambda^6 + \dots \right)$$

Integrating over θ yields

$$U_1 = -W_0 \frac{\pi^2}{8\lambda^2} \left(1 - \lambda^2 + \lambda^4 - \lambda^6 + \dots \right)$$

which becomes as before

$$U_1 = \frac{-2w_0 a^4 \pi^4 \int_{-\infty}^{+\infty} e^{-B_0^2 r^2 + i k |\vec{n}_0 - \vec{n}| r} r dr}{8 \lambda^3 i k |\vec{n}_0 - \vec{n}| (4 \lambda^2 + \mu^2)^{3/2}} \cdot$$

The procedure for further integration is the same as for u_1 , the result being

$$U_1 = \frac{-2w_0 a^4 \pi^4 \sqrt{\pi} e^{-\frac{k^2 |\vec{n}_0 - \vec{n}|^2}{4 B_0^2}}}{8 \lambda^3 (4 \lambda^2 + \mu^2)^{3/2} 2 B_0^3} \cdot$$

Now the scattering angle θ is the angle between $\vec{n}$ and $\vec{n}_0$, hence

$$|\vec{n}_0 - \vec{n}|^2 = 2 - 2 \vec{n}_0 \cdot \vec{n} = 2(1 - \cos \theta) = 4 \sin^2 \frac{\theta}{2},$$

so that U_1 becomes, setting $B_0^2 = 4 \lambda^2 \mu^2 / (4 \lambda^2 + \mu^2)$,

$$U_1 = \frac{-w_0 a^4 \pi^{7/2}}{64 \lambda^6 \mu^3} e^{-\frac{k^2 (4 \lambda^2 + \mu^2)}{4 \lambda^2 \mu^2} \sin^2 \frac{\theta}{2}} \cdot$$

which becomes as before

$$U_1 = - \frac{1}{2} \omega_0^2 \frac{1}{\omega_0^2 - \omega^2} \frac{1}{\omega_0^2 - \omega^2}$$

The procedure for further integration is the same as for

the result being

$$U_1 = - \frac{1}{2} \omega_0^2 \frac{1}{\omega_0^2 - \omega^2} \frac{1}{\omega_0^2 - \omega^2}$$

Now the remaining integral is in the same form as the first

hence

$$U_1 = - \frac{1}{2} \omega_0^2 \frac{1}{\omega_0^2 - \omega^2} \frac{1}{\omega_0^2 - \omega^2}$$

so that U_1 becomes

$$U_1 = - \frac{1}{2} \omega_0^2 \frac{1}{\omega_0^2 - \omega^2} \frac{1}{\omega_0^2 - \omega^2}$$

The procedure for the remaining integrals is much the same, the Coulomb integrals being somewhat more complicated. These integrations, when carried out, yield the following

(2.24)

$$U_1 = \frac{-w_0 \pi^{1/2} a^4}{64 \mu^3 \lambda^6} e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2}} \sin^2 \theta/2, \quad (2.25)$$

$$U_2 = \frac{-w_0 \pi^{1/2} a^4}{64 \lambda^6 (\mu^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4\lambda^2(\mu^2 + \lambda^2)} (\mu^2 \sin^2 \theta/2 + \lambda^2)} \quad (2.26)$$

$$U_3 = \frac{-w_0 \pi^{1/2} a^4}{64 \lambda^6 (\mu^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4\lambda^2(\mu^2 + \lambda^2)} (\mu^2 \cos^2 \theta/2 + \lambda^2)} \quad (2.27)$$

$$U_4 = \frac{-w_0 \pi^{1/2} a^4}{64 \mu^3 \lambda^6} e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2}} \cos^2 \theta/2 \quad (2.28)$$

$$U_5 = \frac{-w_0 \pi^{1/2} a^4}{16 \lambda^6 (3\mu^2 + 4\lambda^2)^{3/2}} e^{-\frac{k^2(\mu^2 + 2\lambda^2)}{2\lambda^2(3\mu^2 + 4\lambda^2)}} \quad (2.29)$$

$$V_1 = \frac{\pi^4 e^2 a^4}{64 \lambda^6 k^2} \cos^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \sin^2 \theta/2} \quad (2.30)$$

$$V_4 = \frac{\pi^4 e^2 a^4}{64 \lambda^6 k^2} \sec^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \cos^2 \theta/2}$$

The procedure for the remaining integrals is much the same, the Coulomb integrals being somewhat more complicated. These integrations when carried out, yield the following

$$U_1 = \frac{W_0 \pi^2 a^3}{6 \lambda^2 \mu \nu} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.2)$$

$$U_2 = \frac{W_0 \pi^2 a^3}{6 \lambda^2 \mu^2 \nu} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.3)$$

$$U_3 = \frac{W_0 \pi^2 a^3}{6 \lambda^2 \mu \nu^2} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.4)$$

$$U_4 = \frac{W_0 \pi^2 a^3}{6 \lambda^2 \mu^2 \nu^2} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.5)$$

$$U_5 = \frac{W_0 \pi^2 a^3}{6 \lambda^2 \mu^2 \nu^2} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.6)$$

$$V = \frac{\pi^2 a^3}{6 \lambda^2 \mu \nu} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.7)$$

$$V_H = \frac{\pi^2 a^3}{6 \lambda^2 \mu \nu} \left[\frac{1}{2} \frac{(1 + \lambda^2 \mu^2)}{\lambda^2 \mu^2} - \frac{1}{2} \frac{(1 + \lambda^2 \nu^2)}{\lambda^2 \nu^2} \right] \quad (2.8)$$

(2.31)

$$V_2 = \frac{\pi^4 e^2 a^4}{16 \lambda^7 k} \cos \theta/2 e^{-\frac{k^2}{4 \lambda^2}} F\left(\frac{k}{2 \lambda} \sin \theta/2\right),$$

(2.32)

$$V_3 = \frac{\pi^4 e^2 a^4}{16 \lambda^7 k} \sec \theta/2 e^{-\frac{k^2}{4 \lambda^2}} F\left(\frac{k}{2 \lambda} \cos \theta/2\right).$$

In these equations $\vec{n} \cdot \vec{n}_0 = \cos \theta$ and in (2.31...32) $F(x)$ is the function

$$F(x) = \int_0^x e^{-z^2} dz \quad . \quad (2.33)$$

All the integrals show angular dependence except U_5 which is part of the S wave scattering. For convenience, we set

$$K_0 = \frac{W_0 \pi^{7/2} a^4 m}{16 \lambda^6 \hbar^2}, \quad (2.34)$$

$$K_1 = \frac{W_0 \pi^{5/2} a^4 m}{4 \lambda^6 (3\mu^2 + 4\lambda^2)^{3/2} \hbar^2}, \quad K_2 = -\frac{\pi^2 e^2 a^4 m}{4 \lambda^6 \hbar^2}.$$

Then inserting equations (2.24...32) into (2.19), we obtain a formula for $\mathcal{F}_i(\theta)$,

(2.31)

$$V_2 = \frac{\pi^2 \epsilon^2 \omega^2}{16 \lambda^2 \epsilon^2} \int_0^x \epsilon^2 \lambda^2 F(\lambda) d\lambda$$

(2.32)

$$V_3 = \frac{\pi^2 \epsilon^2 \omega^2}{16 \lambda^2 \epsilon^2} \int_0^x \epsilon^2 \lambda^2 F(\lambda) d\lambda$$

In these equations $N \cdot M_0 = \cos \theta$ and $M_0 = \cos \theta$ and $M_0 = \cos \theta$ in the function

$$F(x) = \int_0^x \epsilon^2 \lambda^2 d\lambda$$

All the integrals show angular dependence except U_0 which is part of the δ wave scattering. For convenience, we set

$$K_0 = \frac{\omega_0 \pi^2 \epsilon^2 \omega^2}{16 \lambda^2 \epsilon^2}$$

$$K_1 = \frac{\omega_0 \pi^2 \epsilon^2 \omega^2}{16 \lambda^2 \epsilon^2}$$

$$K_2 = \frac{\omega_0 \pi^2 \epsilon^2 \omega^2}{16 \lambda^2 \epsilon^2}$$

Then inserting equations (2.31...32) into (2.30), we obtain a formula for $\frac{d}{dt} \epsilon^2$.

$$J_1(\theta) = \frac{a_1 K_0}{u^3} e^{-\frac{k^2(4\lambda^2 + u^2)}{4\lambda^2 u^2}} \sin^2 \theta/2$$

$$+ \frac{a_2 K_0}{(u^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4\lambda^2(u^2 + \lambda^2)} (\lambda^2 + u^2 \sin^2 \theta/2)}$$

$$+ \frac{a_3 K_0}{(u^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4\lambda^2(u^2 + \lambda^2)} (\lambda^2 + u^2 \cos^2 \theta/2)}$$

$$+ \frac{a_4 K_0}{u^3} e^{-\frac{k^2(4\lambda^2 + u^2)}{4\lambda^2 u^2}} \cos^2 \theta/2$$

$$+ a_5 K_1 e^{-\frac{k^2(u^2 + 2\lambda^2)}{2\lambda^2(3u^2 + 4\lambda^2)}} \quad (2.35)$$

$$+ \frac{K_2}{4k^2} \csc^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \sin^2 \theta/2}$$

$$+ \frac{B_1 K_2}{4k^2} \sec^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \cos^2 \theta/2}$$

$$- \frac{A_1 K_2}{\lambda k} e^{-\frac{k^2}{4\lambda^2}} \csc \theta/2 F\left(\frac{k}{2\lambda} \sin \theta/2\right)$$

$$- \frac{A_1 K_2}{\lambda k} e^{-\frac{k^2}{4\lambda^2}} \sec \theta/2 F\left(\frac{k}{2\lambda} \cos \theta/2\right).$$

$$g'(0) = \frac{d}{dx} \left(\frac{1}{x} \right) \bigg|_{x=0} = -\frac{1}{x^2} \bigg|_{x=0} = -\infty$$

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The values of $q_j(i)$, A_i and B_i can be obtained from (2.14), (2.2) and Table 1, and are found to be;

Quintet- $i = 1 \dots 5$,

$$\begin{aligned} a_1 &= 6g_1 + 6g_2 - 2 = a_4, \\ a_2 &= 4 - 6g_1 - 6g_2 = a_3, \\ a_5 &= -4, \quad A_i = 1, \quad B_i = 1 \end{aligned} \quad (2.36)$$

Triplet- $i = 6 \dots 8$,

$$\begin{aligned} a_1 &= 4g_1 - 2g = -a_4, \\ a_2 &= 4 - 4g - 4g_1 - 6g_2 = -a_3, \\ a_6 &= 0, \quad A_i = 0, \quad B_i = -1, \end{aligned} \quad (2.37)$$

Singlet- $i = 9$,

$$\begin{aligned} a_1 &= 1 + 3g_1 - 3g - 3g_2 = a_4, \\ a_2 &= 4 - 6g - 3g_1 - 6g_2 = a_3, \\ a_5 &= 2, \quad A_i = -\frac{1}{2}, \quad B_i = 1. \end{aligned} \quad (2.38)$$

When these values are used in (2.35), we obtain expressions $\mathcal{F}_Q$,

$\mathcal{F}_T$ and $\mathcal{F}_S$ for the quintet, triplet and singlet scattering, namely ;

The values of $\hat{\beta}_i(t)$, $\hat{A}_i$ and $\hat{B}_i$ can be obtained from

(S.II), (S.S) and Table 1, and are found to be

Quintet-1 = 1...5

$$d_1 = 0.81 + 0.82 - 2.83 = 0$$

$$d_2 = 1.4 - 0.81 - 0.82 - 0.1 = 0$$

$$d_3 = -1, \quad A_1 = 1, \quad B_1 = 1$$

Triplet-1 = 6...8

$$d_1 = 1.81 - 1.82 - 0.1 = 0$$

$$d_2 = 1.4 - 1.81 - 1.82 - 0.1 = -0.33$$

$$d_3 = 0, \quad A_1 = 0, \quad B_1 = 0$$

Single-1 = 9

$$d_1 = 1 + 0.81 - 0.82 - 0.1 = 0$$

$$d_2 = 1.4 - 0.81 - 0.82 - 0.1 = 0$$

$$d_3 = 1, \quad A_1 = 1, \quad B_1 = 1$$

When these values are used in (2.5), we obtain expressions for

$\hat{\beta}_i$ and $\hat{A}_i$ for the dataset, which are listed in Table 1.

namely:

Quintet-

(2.39)

$$J_a(\theta) =$$

$$\frac{a_1 K_0}{u^3} \left[e^{-\frac{k^2(4\lambda^2+u^2)}{4\lambda^2 u^2}} \sin^2 \theta_2 + e^{-\frac{k^2(4\lambda^2+u^2)}{4\lambda^2 u^2}} \cos^2 \theta_2 \right]$$

$$+ \frac{a_2 K_0}{(u^2+\lambda^2)^{3/2}} e^{-\frac{k^2}{4(u^2+\lambda^2)}} \left[e^{-\frac{k^2 u^2 \sin^2 \theta_2}{4\lambda^2(u^2+\lambda^2)}} + e^{-\frac{k^2 u^2 \cos^2 \theta_2}{4\lambda^2(u^2+\lambda^2)}} \right]$$

$$- 4K_1 e^{-\frac{k^2(u^2+2\lambda^2)}{2\lambda^2(3u^2+4\lambda^2)}}$$

$$+ \frac{K_2}{4k^2} \left[\cos^2 \theta_2 e^{-\frac{k^2}{4\lambda^2} \sin^2 \theta_2} + \sec^2 \theta_2 e^{-\frac{k^2}{4\lambda^2} \cos^2 \theta_2} \right]$$

$$- \frac{K_2}{\lambda k} e^{-\frac{k^2}{4\lambda^2}} \csc \theta_2 F\left(\frac{k}{2\lambda} \sin \theta_2\right)$$

$$- \frac{K_2}{\lambda k} e^{-\frac{k^2}{4\lambda^2}} \sec \theta_2 F\left(\frac{k}{2\lambda} \cos \theta_2\right),$$

$$\frac{1}{2} \alpha(\theta) =$$

$$\frac{1}{\pi^2} \left[\frac{1}{6} \frac{K_0}{\sqrt{s}} - \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} + \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} \right]$$

$$\left[\frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} - \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} + \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} \right]$$

$$- \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} - \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}}$$

$$+ \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} + \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}}$$

$$- \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} - \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}}$$

$$- \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}} - \frac{1}{6} \frac{K_0}{\sqrt{s}} \frac{(s+K_0^2)}{2\sqrt{s}}$$

Triplet-

(2.40)

$$\mathbb{F}_T(\theta) =$$

$$\frac{a_1 K_0}{\mu^3} \left[e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2} \sin^2 \theta/2} - e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2} \cos^2 \theta/2} \right]$$

$$+ \frac{a_2 K_0}{(\mu^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4(\mu^2 + \lambda^2)}} \left[e^{-\frac{k^2 \mu^2 \sin^2 \theta/2}{4\lambda^2(\mu^2 + \lambda^2)}} - e^{-\frac{k^2 \mu^2 \cos^2 \theta/2}{4\lambda^2(\mu^2 + \lambda^2)}} \right]$$

$$+ \frac{K_2}{4k^2} \left[\cos^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \sin^2 \theta/2} - \sec^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \cos^2 \theta/2} \right],$$

$$= (8) \frac{1}{2}$$

$$\frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{2} \right) \right] = \frac{1}{8}$$

$$\frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{2} \right) \right] = \frac{1}{8}$$

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$$\frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{2} \right) \right] = \frac{1}{8}$$

Singlet-

(2.41)

$$I_s(0) =$$

$$\frac{a_1 K_0}{\mu^3} \left[e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2} \sin^2 \theta/2} + e^{-\frac{k^2(4\lambda^2 + \mu^2)}{4\lambda^2 \mu^2} \cos^2 \theta/2} \right]$$

$$+ \frac{a_2 K_0}{(\mu^2 + \lambda^2)^{3/2}} e^{-\frac{k^2}{4(\mu^2 + \lambda^2)}} \left[e^{-\frac{k^2 \mu^2 \sin^2 \theta/2}{4\lambda^2(\mu^2 + \lambda^2)}} + e^{-\frac{k^2 \mu^2 \cos^2 \theta/2}{4\lambda^2(\mu^2 + \lambda^2)}} \right]$$

$$+ 2K_1 e^{-\frac{k^2(\mu^2 + 2\lambda^2)}{2\lambda^2(3\mu^2 + 4\lambda^2)}}$$

$$+ \frac{K_2}{4k^2} \left[\csc^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \sin^2 \theta/2} + \sec^2 \theta/2 e^{-\frac{k^2}{4\lambda^2} \cos^2 \theta/2} \right]$$

$$+ \frac{K_2}{2\lambda k} e^{-\frac{k^2}{4\lambda^2}} \csc \theta/2 F\left(\frac{k}{2\lambda} \sin \theta/2\right)$$

$$+ \frac{K_2}{2\lambda k} e^{-\frac{k^2}{4\lambda^2}} \sec \theta/2 F\left(\frac{k}{2\lambda} \cos \theta/2\right).$$

Using these formulae, we can compute the differential cross section, $d\sigma$ of (1.43), averaged over the initial states and summed over the final states which is

$$\begin{aligned} d\sigma &= I(\theta) d\Omega = \frac{1}{9} \sum_{l=1}^9 |\mathbb{F}_l|^2 d\Omega. \\ &= \left[\frac{5}{9} |\mathbb{F}_Q|^2 + \frac{3}{9} |\mathbb{F}_T|^2 + \frac{1}{9} |\mathbb{F}_S|^2 \right] d\Omega. \end{aligned} \quad (2.42)$$

Since

and

$$\cos \frac{(\pi - \theta)}{2} = \sin \frac{\theta}{2}$$

$$\sin \frac{(\pi - \theta)}{2} = \cos \frac{\theta}{2},$$

inspection of $\mathbb{F}_Q$, $\mathbb{F}_T$ and $\mathbb{F}_S$ for symmetry and antisymmetry shows that

$$\mathbb{F}_Q(\theta) = \mathbb{F}_Q(\pi - \theta),$$

$$\mathbb{F}_T(\theta) = -\mathbb{F}_T(\pi - \theta), \quad (2.43)$$

$$\mathbb{F}_S(\theta) = \mathbb{F}_S(\pi - \theta).$$

In this respect then, we can say that $\mathbb{F}_Q$ and $\mathbb{F}_S$ behave like even Legendre polynomials $P_{2n}(\theta)$ while $\mathbb{F}_T$ behaves like an odd Legendre polynomial for it is easy to show that

Using these formulas, we can compute the differential cross section,

$d\sigma/d\Omega$ of (1.4), averaged over the initial states and summed over

the final states which is

$$(2.2) \quad \frac{d\sigma}{d\Omega} = \frac{1}{4} \sum_{\substack{\text{initial} \\ \text{states}}} \sum_{\substack{\text{final} \\ \text{states}}} |\mathcal{M}|^2 = \frac{1}{4} \sum_{\substack{\text{initial} \\ \text{states}}} |\mathcal{M}|^2$$

$$= \frac{1}{4} \left[|\mathcal{M}_{++}|^2 + |\mathcal{M}_{+-}|^2 + |\mathcal{M}_{-+}|^2 + |\mathcal{M}_{--}|^2 \right]$$

and

Since

$$\cos(\pi - \theta) = -\cos \theta$$

$$\sin(\pi - \theta) = \sin \theta$$

inspection of $\mathcal{M}_{\pm\pm}$, $\mathcal{M}_{\pm\mp}$ and $\mathcal{M}_{\mp\pm}$, for symmetry and anti-symmetry

shows that

$$\mathcal{M}_{00}(\theta) = \mathcal{M}_{00}(\pi - \theta)$$

(2.3)

$$\mathcal{M}_{\pm\mp}(\theta) = -\mathcal{M}_{\pm\mp}(\pi - \theta)$$

$$\mathcal{M}_{\pm\pm}(\theta) = \mathcal{M}_{\pm\pm}(\pi - \theta)$$

In this respect then, we can say that $\mathcal{M}_{00}$ and $\mathcal{M}_{\pm\pm}$ behave like

even Legendre polynomials $P_n(\cos \theta)$ while $\mathcal{M}_{\pm\mp}$ behaves like

an odd Legendre polynomial for it is easy to show that

$$P_{2n}(\theta) = P_{2n}(\pi - \theta) ,$$

while

$$P_{2n+1}(\theta) = - P_{2n+1}(\pi - \theta) .$$

The character of the scattering will be determined by $V_{INT} K \psi_c$ of (1.47). Now $P_{13} Q_{13} P_{24} Q_{24}$ commutes with V_{INT} and in addition we have

$$P_{13} Q_{13} K \psi_c = - K \psi_c , \quad P_{24} Q_{24} K \psi_c = - K \psi_c ,$$

since $K \psi_c$ is symmetrized. So that we have a symmetry condition on $V_{INT} K \psi_c$ namely

$$P_{13} Q_{13} P_{24} Q_{24} V_{INT} K \psi_c = V_{INT} K \psi_c . \quad (2.44)$$

Now by inspecting (1.34...36), one can see that for the quintet and singlet states one has

$$Q_{13} Q_{24} \chi_Q = \chi_Q , \quad Q_{13} Q_{24} \chi_S = \chi_S ,$$

so that (1.44) becomes

$$P_{13} P_{24} V_{INT} K \psi_c = V_{INT} K \psi_c ,$$

$$P_n(\theta) = P_n(\pi - \theta)$$

with

$$P_{n+1}(\theta) = -P_{n+1}(\pi - \theta)$$

The character of the alternating will be determined by $\sin \theta$ and $\cos \theta$. Now $P_0 \cos \theta = P_0 \cos \theta$ and in addition we have

$$P_1 \sin \theta = -P_1 \cos \theta, \quad P_2 \cos \theta = -P_2 \sin \theta$$

since $K \psi$ is symmetrical, so that we have a symmetrical relation on $\sin \theta$ and $\cos \theta$.

$$P_3 \sin \theta = P_3 \cos \theta, \quad P_4 \cos \theta = P_4 \sin \theta \quad (2.11)$$

Now by inspection (2.11...2.14), one can see that for the general and single steps one has

$$P_{2k} \cos \theta = P_{2k} \cos \theta, \quad P_{2k+1} \sin \theta = P_{2k+1} \sin \theta$$

so that (2.11) becomes

$$P_{2k} \cos \theta = P_{2k} \cos \theta, \quad P_{2k+1} \sin \theta = P_{2k+1} \sin \theta$$

The effect of $P_{13}P_{24}$ on $V_{INT} K \psi_L$ is to change Θ into $\pi - \Theta$, so that the symmetry exhibited for the quintet and singlet implies that the contribution to the integral of (1.47) will be the same at Θ and at $\pi - \Theta$, in agreement with (2.43).

On the other hand, the triplet spin wave functions have the property

$$Q_{13} Q_{24} \chi_T = -\chi_T,$$

so that (2.44) becomes in this instance

$$P_{13} P_{24} V_{INT} K \psi_L = -V_{INT} K \psi_L.$$

The antisymmetry, in this case, implies that the contribution from $\pi - \Theta$ will have to have the opposite sign to that at Θ , in agreement with (2.43). This also explains why terms with even l values do not enter into the triplet scattering formula (2.40). In the theory of scattering which employs the method of partial waves, the scattered wave is expressed as a series of Legendre polynomials ^{5,6}

$$f(\theta) = \frac{1}{2ik} \sum_{n=0}^{\infty} (2n+1) (e^{2i\eta_n} - 1) P_n(\cos\theta),$$

where the η_n are the phases. Thus the terms in this expansion consisting of odd Legendre polynomials will correspond to the triplet scattering in the deuteron-deuteron problem while the terms in even Legendre polynomials will correspond to the quintet and singlet scattering. A consequence of this antisymmetry for the triplet scattering is that it vanishes at 90° . The quintet and singlet scattering includes the S, D, G, etc. waves while the triplet includes the P, F, etc. waves. Inspection of the scattering formulae (2.39...41) shows the variation of the quintet and singlet over the range of θ values for which the greatest contribution is nuclear (that is, for angles greater than a few degrees and not too near 180 degrees) will be greater for higher energies, and hence the D and G contribution is greater for these energies. For smaller k values, the scattering in this region is chiefly of the S wave type, however our theory is not applicable at low energies (see Chapter III). There will be a phase cancellation between nuclear and Coulomb scattering at small angles. This will be discussed in Chapter III.

In order to perform a calculation of the scattering, one has to choose values for η_n and for the g 's of the interaction. Following Buckingham and Massey³, we take the relations,

$$\eta_n = 2g_2 = \frac{1}{3}(1 + 3g),$$

$$g = 2g_1 = \frac{1}{3}(1 - 3g), \quad (2.45)$$

$$\text{with } g = .6$$

to hold. These are consistent with $\eta_n = 1 - g - g_1 - g_2$.

where the $\bar{N}$ are the phases. Thus the terms in this expansion consisting of odd Legendre polynomials will correspond to the scattering in the forward-scattered region while the terms in even Legendre polynomials will correspond to the backward and side scattering. A consequence of this expansion is that the scattering is that it varies as θ^2 . The angles and angles scattering includes the $\theta, \theta', \theta''$, etc. waves while the angles includes the $\theta, \theta', \theta'', \theta''', \theta''''$, etc. waves. Inspection of the scattering pattern shows the variation of the points and angles over the range of θ values for which the greatest scattering is observed (that is, for angles greater than a few degrees and less than 100 degrees) will be greater for higher angles, and since the $\theta, \theta', \theta''$, etc. angles is greater for these angles, for angles $\theta, \theta', \theta'', \theta''', \theta''''$, etc. scattering in this region is chiefly of the $\theta, \theta', \theta'', \theta''', \theta''''$, etc. order theory is not applicable at low angles (see Chapter III). There will be a phase cancellation between incident and scattered waves at small angles. This will be discussed in Chapter III. In order to perform a calculation of the scattering, one has to choose values for $\bar{N}$ and for the θ 's of the interaction. With this Buckingham and Mauey⁵, we take the relations,

$$\bar{N} = 2\theta = \frac{1}{2}(1 + 3\theta)$$

(2.12)

$$\theta = 2\theta' = \frac{1}{2}(1 - 3\theta')$$

$$\theta'' = \theta'$$

to hold. These are consistent with $\theta'' = 1 - \theta' - \theta' = 1 - 2\theta'$.

One then obtains for the a 's,

Quintet-

$$a_1 = a_4 = 0, \quad (2.46)$$

$$a_2 = a_3 = 2, \quad a_5 = -4,$$

Triplet-

$$a_1 = a_4 = 0, \quad (2.47)$$

$$a_2 = -a_3 = 2.8, \quad a_5 = 0,$$

Singlet-

$$a_1 = a_4 = 0, \quad (2.48)$$

$$a_2 = a_3 = 3.2, \quad a_5 = 2.$$

These values are then introduced into equations (2.39...41). The choice (2.45) is not arbitrary, but is in agreement with previous results and follows from the theory of the neutron-proton scattering and the analysis of saturation requirements for nuclei.^{3,8}

Values of the function, $F(x)$, of (2.33) were obtained by graphical integration (trapezoid rule) for the range, $0 \leq x \leq 1.6$, and are listed in table 3. An enlarged graph was used to interpolate values of $F(x)$.

One then obtains for the a_i

Quartet-

(2.16)

$$a_1 = a_4 = 0,$$

$$a_2 = a_3 = x, \quad a_5 = -x,$$

Triplet-

(2.17)

$$a_1 = a_4 = 0,$$

$$a_2 = -a_3 = x, \quad a_5 = 0,$$

Singlet-

(2.18)

$$a_1 = a_4 = 0,$$

$$a_2 = a_3 = x, \quad a_5 = x.$$

These values are then introduced into equations (2.32...41). The choice (2.18) is not arbitrary, but is in agreement with previous results and follows from the theory of the neutron-proton scattering and the analysis of saturation requirements for model 3.8.

Values of the function $F(x)$, of (2.33) were obtained by graphical integration (trapezoid rule) for the range $0 \leq x \leq 1.6$, and are listed in table 3. An enlarged graph was used to interpolate values of $F(x)$.

Table 3. $F(x)$ values.

x	$F(x)$
0.0	0.00000
0.1	0.10050
0.2	0.20304
0.3	0.30978
0.4	0.42310
0.5	0.54590
0.6	0.68170
0.7	0.83491
0.8	1.01135
0.9	1.21850
1.0	1.46680
1.1	1.77040
1.2	2.14900
1.3	2.63100
1.4	3.25690
1.5	4.08620
1.6	5.27030

3. Criterion for the Validity of Born's Approximation.

Born's approximation ⁶ is the replacement of the wave function ψ_c in (1.47) by the plane wave $e^{ikz} \psi_0(1) \psi_0(2) \chi_c$. This procedure is valid only if the scattered wave is small in comparison to the incident wave in the vicinity of the scatterer. We have for the scattered wave, (1.44),

$$\phi_c(\vec{r}) = -\frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \chi_c \iint \bar{\psi}_0(1) \bar{\psi}_0(2) V_1 V_2 K e^{ikz'} \psi_0(1) \psi_0(2) \chi_c d\vec{r}_1 d\vec{r}_2 \quad (2.49)$$

so that as $\vec{r} \rightarrow 0$ one obtains for ϕ_c the expression,

• 2000 (x) 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842,

$F(x)$	x
0.0000	0.0
0.0001	0.1
0.0002	0.2
0.0003	0.3
0.0004	0.4
0.0005	0.5
0.0006	0.6
0.0007	0.7
0.0008	0.8
0.0009	0.9
0.0010	1.0
0.0011	1.1
0.0012	1.2
0.0013	1.3
0.0014	1.4
0.0015	1.5
0.0016	1.6
0.0017	1.7
0.0018	1.8
0.0019	1.9
0.0020	2.0

$$\left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\eta^2} d\eta \right] = 1$$

$$\phi_c(0) = -\frac{m}{2\pi\hbar^2} \int \frac{e^{i\vec{k}\cdot\vec{r}}}{r} d\vec{r} \int \bar{\psi}_0(1) \bar{\psi}_0(2) V_{int} K e^{i\vec{k}\cdot\vec{z}} \psi_0(1) \psi_0(2) \chi_c d\vec{p}_1 d\vec{p}_2 \quad (2.50)$$

The requirement is then that

$$|\phi_c(0)| \ll 1$$

In (2.50), the terms involving no exchange will be at least as large as those involving exchange, hence a calculation of $\phi_c(0)$ for these terms will set a lower limit to the energy for which the Born approximation is valid. For the choice (2.45...48), these non-exchange terms will not enter into the scattering, so that the scattered wave will be even smaller in the vicinity of the scatterer than the value we shall calculate. However, to be safe we seek a conservative lower limit for the energy in order that the Born approximation be valid. One can write the non-exchange part of ϕ_c (setting $q_1 = 1$) as,

$$\phi_c(0) = -\frac{4m}{\pi\hbar^2} \int \frac{e^{i\vec{k}\cdot\vec{r}}}{r} u_1 d\vec{r},$$

where u_1 is given by (2.23) as

$$u_1 = -\frac{w_0 a^4 \pi^3}{8\lambda^3 (4\lambda^2 + \mu^2)^{3/2}} e^{i\vec{k}\cdot\vec{n}_0\cdot\vec{r}} - \frac{4\lambda^2 \mu^2 r^2}{4\lambda^2 + \mu^2}$$

One then obtains for $\phi_c(0)$,

$$\phi_c(0) = \frac{m w_0 a^4 \pi^3}{\hbar^2 \lambda^3 (4\lambda^2 + \mu^2)^{3/2}} \frac{1}{k} \int_0^\infty e^{-\frac{4\lambda^2 \mu^2 r^2}{4\lambda^2 + \mu^2}} (e^{2i\vec{k}\cdot\vec{r}} - 1) dr,$$

(2.50)

$$\Phi(\omega) = -\frac{m}{2\pi\hbar^2} \left[\frac{1}{\omega} \int_0^\infty \psi(r) \psi(r) dr + K \int_0^\infty \psi(r) \psi(r) dr \right]$$

The requirement is that $\Phi(\omega) < 0$. In (2.50), the terms involving exchange will be at least as large as those involving exchange, hence a calculation of $\Phi(\omega)$ for these terms will not a lower limit for the energy for which the approximation is valid. For the choice (2.48...49), these non-exchange terms will not enter into the calculation, so that the non-exchange wave will be even smaller in the vicinity of the scatterer than the value we shall calculate. However, to be safe we seek a conservative lower limit for the energy in order that the Born approximation be valid. One can write the non-exchange part of Φ (setting $\psi = 1$)

as

$$\Phi(\omega) = -\frac{m}{2\pi\hbar^2} \left[\frac{1}{\omega} \int_0^\infty \psi(r) \psi(r) dr + K \int_0^\infty \psi(r) \psi(r) dr \right]$$

where U is given by (2.52) as

$$U = \frac{2\pi^2 \hbar^2 \omega^2}{8\pi^2 (\nu^2 + \omega^2)^2} - \frac{2\pi^2 \hbar^2 \omega^2}{8\pi^2 (\nu^2 + \omega^2)^2}$$

One then obtains for $\Phi(\omega)$

$$\Phi(\omega) = -\frac{m}{2\pi\hbar^2} \left[\frac{1}{\omega} \int_0^\infty \psi(r) \psi(r) dr + K \int_0^\infty \psi(r) \psi(r) dr \right]$$

which, after setting

$$B_0^2 = \frac{4\lambda^2\mu^2}{4\lambda^2 + \mu^2}$$

and

$$B = \frac{4W_0 m \lambda^3 \sqrt{\pi}}{\hbar^2 (4\lambda^2 + \mu^2)^{3/2}},$$

becomes

$$\phi_c(0) = \frac{B}{\hbar^2} \left(1 + \frac{\hbar^2}{B_0^2}\right) (1 - e^{-\hbar^2/B_0^2})$$

Hence one obtains for $|\phi_c(0)|^2$,

$$|\phi_c(0)|^2 = \frac{B^2}{\hbar^4} \left(1 + \frac{\hbar^2}{B_0^2}\right) (1 - e^{-\hbar^2/B_0^2})^2$$

Setting $x = \hbar^2/B_0^2$, then $|\phi_c(0)|^2$ can be written,

$$|\phi_c(0)|^2 = \left(\frac{B}{B_0^2}\right)^2 \left(\frac{1+x}{x^2}\right) (1 - e^{-x})^2. \quad (2.51)$$

For any choice of nuclear constants W_0 , μ and λ and for any energy E_0 one can determine the size of $|\phi_c(0)|^2$ from (2.51) and then compare this value with unity.

which, after setting

$$B_0^2 = \frac{4V^2 m^2}{4V^2 + m^2}$$

and

$$B = \frac{4V^2 m^2}{4V^2 + m^2} \sqrt{B_0^2}$$

$$B_0^2 = \frac{4V^2 m^2}{4V^2 + m^2} \sqrt{B_0^2}$$

becomes

$$\Phi(\omega) = \frac{B}{k^2} \left(1 + \sqrt{B_0^2} \right) (1 - e^{-k^2})$$

Hence one obtains for $|\Phi(\omega)|^2$

$$|\Phi(\omega)|^2 = \frac{B^2}{k^4} \left(1 + \sqrt{B_0^2} \right)^2 (1 - e^{-k^2})^2$$

Setting $k = \sqrt{B_0^2}$, then $|\Phi(\omega)|^2$ can be written

$$|\Phi(\omega)|^2 = \left(\frac{B}{B_0^2} \right)^2 \left(1 + \frac{1}{\sqrt{B_0^2}} \right)^2 (1 - e^{-\sqrt{B_0^2}})^2 \quad (5.21)$$

For any choice of nuclear constants W_0 , μ , and λ and for any energy E_0 one can determine the size of $|\Phi(\omega)|^2$ from (5.21) and then compare this value with unity.

CHAPTER III

CONCLUSION

1. Values of the Nuclear Constants.

In this chapter, we calculate the differential cross section for the scattering as given by the formulae (2.39...41), using values for the a 's given by (2.46...48). For nuclear constants, we take those used by Ashkin and Wu,² namely,

$$\lambda^2 = 7.16 \times 10^{24} \text{ cm.}^{-2}, \quad \mu^2 = 26.6 \times 10^{24} \text{ cm.}^{-2}. \quad (3.1)$$

These values have been selected by various authors² to be consistent with the binding energy of the deuteron when Gaussian wave functions of the type (2.21) and a Gaussian potential of the type (2.20) are used to describe the deuteron and the interaction between nucleons. The choice of a well depth W_0 for such functions has been obtained by Bethe¹⁰ for the case where $\mu^2 = 26.6 \times 10^{24}$, his value is $W_0 \approx 45$ MEV. Calculations of this well depth using tensor forces¹¹ yield a smaller value of $W_0 \approx 22$ MEV. In several of our calculations both values were used. The results of these calculations are tabulated along with a graph of $I(\theta)$ at the end of this chapter. One of the general features of these curves is the symmetry about 90 degrees. Inspection of Graph 3 shows that for I_θ , I_r and I_s there is phase

cancellation between nuclear and Coulomb contributions at very small and very large angles. The Coulomb contribution predominates at these end values of θ , due to the $\cos \theta/2$ and $\sin \theta/2$ terms. However, due to background, measurements in these regions are indefinite. For values of θ in the range, $40^\circ \leq \theta \leq 140^\circ$, the Coulomb contribution is small compared to the nuclear, especially at higher energies. A calculation of the differential cross section for the low value of 10.8 MEV. $= 2E_0$ was made, since Allred⁹ and others had published experimental results for the deuteron-deuteron elastic scattering at this energy. These same authors investigated the scattering at 3.4 MEV. $= 2E_0$, however we made no attempt to use these results, the energy being so low. As can be seen from Graphs 1 and 2, the agreement with the 10.8 MEV. data is extremely poor for $W_0 = 45$ MEV., being off by a factor of ten. A well depth $W_0 = 15.7$ MEV. gives agreement with experiment for $\theta \approx 90$ degrees, but the experimental values rise too rapidly as θ decreases to give good agreement. Using the criterion for the validity of Born's approximation, (2.51), it is shown in the next section that 10.8 MEV. is too low an energy to expect good results.

2. Values of $|\phi_c(\theta)|^2$.

For our choice of nuclear constants (3.1) and from equation (2.51), we obtain values of $|\phi_c(\theta)|^2$ for $W_0 = 45$ MEV.,

cancellation between nuclear and Coulomb contributions at very small and very large angles. The Coulomb contribution predominates at these and values of θ , due to the $\cos^2 \theta$ and $\sin^2 \theta$ terms. However, due to uncertainty, measurement in these regions are indistinct. For values of θ in the range, $10^\circ \leq \theta \leq 110^\circ$, the Coulomb contribution is small compared to the nuclear, especially at higher energies. A calculation of the differential cross section for the low value of 10.8 MeV, ≈ 2.5 , was made, since Alford and others had published experimental results for the deuteron-deuteron elastic scattering at this energy. These same authors investigated the scattering at 1.1 MeV, ≈ 2.5 , however we made no attempt to do these calculations, the energy being so low. As can be seen from Graphs 1 and 2, the agreement with the 10.8 MeV data is extremely poor for $W_0 = 7$ MeV, being off by a factor of four. A well depth $W_0 = 15.7$ MeV gives agreement with experiment for $\theta = 90$ degrees, but the experimental values rise too rapidly as θ decreases to give good agreement. Using the criterion for the validity of Born's approximation, (2.21), it is shown in the next section that 10.8 MeV is too low an energy to expect good results.

2. Values of $|\Phi(r)|^2$

For our choice of nuclear constants (2.1) and from equation (2.21), we obtain values of $|\Phi(r)|^2$ for $W_0 = 15.7$ MeV.

$W_0 = 25$ MEV., $2E_0 = 10.8$ MEV., and $2E_0 = 60$ MEV. as follows:

$2E_0 = 10.8$ MEV.	
$W_0 = 45$ MEV.	$ \phi_c(0) ^2 = 3.1$
$W_0 = 25$ MEV.	$ \phi_c(0) ^2 = .96$

$2E_0 = 60$ MEV.	
$W_0 = 45$ MEV.	$ \phi_c(0) ^2 = .74$
$W_0 = 25$ MEV.	$ \phi_c(0) ^2 = .23$

Inspection of the $|\phi_c(0)|^2$ values for $2E_0 = 10.8$ MEV. shows distinctly that the criterion $|\phi_c(0)|^2 < 1$ is not satisfied for either well depth, hence we cannot expect the scattering formulas based upon Born's approximation to be in agreement with experiment at this low energy. The 60 MEV. results are more encouraging, and one might anticipate fairly good agreement at energies above this. It might, however, be necessary to include a tensor interaction for higher energies. Graphs of $\bar{I}(\theta)$ as well as a few of the $\bar{J}(\theta)$ are included below.

The author wishes to thank the members of the faculty of the Physics Department of the University of New Mexico for their assistance and advice in matters pertaining to the work of this dissertation. He is particularly indebted to Dr. Roy Thomas, who proposed the dissertation topic to him and was very helpful in his suggestions concerning the work.

$W_0 = 25 \text{ MeV}$, $E_0 = 10.5 \text{ MeV}$, and $E_1 = 10.5 \text{ MeV}$, as follows:

$E_0 = 10.5 \text{ MeV}$		$E_1 = 10.5 \text{ MeV}$	
$W_0 = 25 \text{ MeV}$	$ \psi(\omega) ^2 = 1.2$	$W_0 = 25 \text{ MeV}$	$ \psi(\omega) ^2 = 1.2$
$W_0 = 15 \text{ MeV}$	$ \psi(\omega) ^2 = 1.2$	$W_0 = 15 \text{ MeV}$	$ \psi(\omega) ^2 = 1.2$

Inspection of the $|\psi(\omega)|^2$ values for $E_0 = 10.5 \text{ MeV}$, shows that by this the criterion $|\psi(\omega)|^2 < 1$ is not satisfied for values well below, hence we cannot expect the constant formula based upon Born's approximation to be in agreement with experiment at this low energy. The 50 MeV values are also very low, and one might expect that a fair good agreement to experiment should also be obtained. However, it is necessary to furnish a better foundation for higher energies. Graphs of $\bar{I}(\omega)$ as well as a table of $\bar{I}(\omega)$ are included below.

The author wishes to thank the members of the staff of the Physics Department of the University of New Mexico for their assistance and advice in matters pertaining to the work of this dissertation. He is particularly indebted to Mr. J. H. Thomas, who reviewed the dissertation topic to him and was very helpful in the suggestions concerning the work.

Graph 1. Experimental values of $I(\theta)$ vs $\theta/2$ in the center of mass system for $2E_0 = 10.8 \text{ MEV.}$, Allred, et al.
The results were symmetric about $\theta/2 = 45^\circ$.

Graph 2. Theoretical values of $I(\theta)$ vs $\theta/2$ for $W_0 = 45$ MEV. and $2E_0 = 10.8$ MEV.

Graph 3. Theoretical values of J_0 , J_T and J_s vs $\theta/2$ for $W_0 = 45 \text{ MEV.}$ and $2E_0 = 10.8 \text{ MEV.}$

Graph 4. Theoretical values of $I(\theta)$ vs $\theta/2$ for $w_0 = 15.7 \text{ MEV.}$
and $2E_0 = 10.8 \text{ MEV.}$

Graph 5. Theoretical values of $I(\theta)$ vs $\theta/2$ for $W_0 = 25$ MEV. and $2E_0 = 60$ MEV.

Graph 6. Theoretical values of $I(\theta)$ vs $\theta/2$ for $W_0 = 45 \text{ MEV}$. and $2E_0 = 60 \text{ MEV}$.

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