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DEFINITIONS, SOLVED AND UNSOLVED PROBLEMS, CONJECTURES, AND THEOREMS IN NUMBER THEORY AND GEOMETRY

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edited by M. L. Perez

$$\sum_{n\geq r} \frac{1}{S(n)\sqrt[\alpha]{(S(n)-1)!}}$$

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PREFACE

Florentin Smarandache, an American mathematician of Romanian descent has generated a vast variety of mathematical problems. Some problems are easy, others medium, but many are interesting or unsolved and this is the reason why the present book appears. Here, of course, there are problems from various types. Solving these problems is addictive like eating pumpkin seed: having once started, one cannot help doing it over and over again.

The Editor

Problem 1. The equation

$$p_{n+1}^x - p_n^x = 1,$$

where p_n is the n-th prime, has a unique solution between 0.5 and 1;
- the maximum solution occurs for n = 1, i.e.

$$3^x - 2^x = 1 \text{ when } x = 1;$$

- the minimum solution occurs for n = 31, i.e.

$$127^x - 113^x = 1 \text{ when } x = 0.567148... = a_0.$$

Thus, Andrica's Conjecture

$$A_n = p_{n+1}^{1/2} - p_n^{1/2} < 1$$

is generalized to:

• $B_n = p_{n+1}^a - p_n^a < 1$, where $a < a_0$.

• $C_n = p_{n+1}^{1/k} - p_n^{1/k} < 2/k$, where $k \geq 2$.

• $D_n = p_{n+1}^a - p_n^a < 1/n$, where $a < a_0$ and n big enough, $n = n(a)$, holds for infinitely many consecutive primes.

a) Is this still available for $a_0 < a < 1$?

b) Is there any rank n_0 depending on a and n so that (4) is verified for all $n \geq n_0$?

• $p_{n+1}/p_n \leq 5/3$, and the maximum occurs at $n = 2$.

References: [67, 68, 86, 94]

Problem 2.

2.1: Representations of the odd numbers

A) Any odd integer n can be expressed as a combination of three primes, as follows:

1) As a sum of two primes minus another prime ($k = 3, s = 1$): $n = p + q - r$, where p, q, r are all prime numbers. The trivial solution: $p = p + q - q$ where p is prime is not included. For example:

1=3+5-7 =5+7-11 =7+11-17 =11+13-23=... ;
3=5+5-7 =7+19-23=17+23-37=... ;
5=3+13-11 =... ;
7=11+13-17=... ;
9=5+7-3 =... ;
11=7+17-13 =... .

a) Is this conjecture equivalent to the Goldbach's Conjecture (any odd integer ≥ 9 is the sum of three primes)?

b) Is the conjecture true when all the three prime numbers are different?

c) In how many ways can each odd integer be expressed as above?

2) As a prime minus another prime and minus again another prime ($k = 3, s = 2$): $n = p - q - r$, where p, q, r are all prime numbers. For example:

$$\begin{array}{rcccccccc} 1 & = & 13\text{-}5\text{-}7 & = & 17\text{-}5\text{-}11 & = & 19\text{-}5\text{-}13 & = & \ldots; \\ 3 & = & 13\text{-}3\text{-}7 & = & 23\text{-}7\text{-}13 & = & \ldots; & & \\ 5 & = & 13\text{-}3\text{-}5 & = & \ldots; & & & & \\ 7 & = & 17\text{-}3\text{-}7 & = & \ldots; & & & & \\ 9 & = & 17\text{-}3\text{-}5 & = & \ldots; & & & & \\ 11 & = & 19\text{-}3\text{-}5 & = & \ldots. & & & & \end{array}$$

a) Is this conjecture equivalent to the Goldbach's Conjecture (any odd integer ≥ 9 is the sum of three primes)?

b) Is the conjecture true when all the three prime numbers are different?

c) In how many ways can each odd integer be expressed as above?

B) Any odd integer n can be expressed as a combination of five primes as follows:

3) $n = p + q + r + t - u$, where p, q, r, t, u are all prime numbers, and $t \neq u$ ($k = 5, s = 1$). For example:

$$\begin{array}{rcccccc} 1 & = & 3+3+3+5\text{-}13 & = & 3+5+5+17\text{-}29 & = & \ldots; \\ 3 & = & 3+5+11+13\text{-}29 & = & \ldots; & & \\ 5 & = & 3+7+11+13\text{-}29 & = & \ldots; & & \\ 7 & = & 5+7+11+13\text{-}29 & = & \ldots; & & \\ 9 & = & 7+7+11+13\text{-}29 & = & \ldots; & & \\ 11 & = & 5+7+11+17\text{-}29 & = & \ldots. & & \end{array}$$

a) Is the conjecture true when all the five prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

4) $n = p + q + r - t - u$, where p, q, r, t, u are all prime numbers, and $t, u \neq p, q, r$ ($k = 5, s = 2$). For example:

$$\begin{array}{rcccccc} 1 & = & 3+7+17\text{-}13\text{-}13 & = & 3+7+23\text{-}13\text{-}19 & = & \ldots; \\ 3 & = & 5+7+17\text{-}13\text{-}13 & = & \ldots; & & \\ 5 & = & 7+7+17\text{-}13\text{-}13 & = & \ldots; & & \\ 7 & = & 5+11+17\text{-}13\text{-}13 & = & \ldots; & & \\ 9 & = & 7+11+17\text{-}13\text{-}13 & = & \ldots; & & \\ 11 & = & 7+11+19\text{-}13\text{-}13 & = & \ldots. & & \end{array}$$

a) Is the conjecture true when all five prime numbers are different?
b) In how many ways can each odd integer be expressed as above?

5) $n = p + q - r - t - u$, where p, q, r, t, u are all prime numbers, and $r, t, u \neq p, q$ ($k = 5, s = 3$). For example:

$$\begin{array}{rcccl} 1 & = & 11+13\text{-}3\text{-}3\text{-}17 & = & \dots ; \\ 3 & = & 13+13\text{-}3\text{-}3\text{-}17 & = & \dots ; \\ 5 & = & 3+29\text{-}5\text{-}5\text{-}17 & = & \dots ; \\ 7 & = & 3+31\text{-}5\text{-}5\text{-}17 & = & \dots ; \\ 9 & = & 3+37\text{-}7\text{-}7\text{-}17 & = & \dots ; \\ 11 & = & 5+37\text{-}7\text{-}7\text{-}17 & = & \dots . \end{array}$$

a) Is the conjecture true when all the five prime numbers are different?
b) In how many ways can each odd integer be expressed as above?

6) $n = p - q - r - t - u$, where p, q, r, t, u are all prime numbers, and $q, r, t, u \neq p$ ($k = 5, s = 4$). For example:

$$\begin{array}{rcccl} 1 & = & 13\text{-}3\text{-}3\text{-}3\text{-}3 & = & \dots ; \\ 3 & = & 17\text{-}3\text{-}3\text{-}3\text{-}5 & = & \dots ; \\ 5 & = & 19\text{-}3\text{-}3\text{-}3\text{-}5 & = & \dots ; \\ 7 & = & 23\text{-}3\text{-}3\text{-}5\text{-}5 & = & \dots ; \\ 9 & = & 29\text{-}3\text{-}5\text{-}5\text{-}7 & = & \dots ; \\ 11 & = & 31\text{-}3\text{-}5\text{-}5\text{-}7 & = & \dots . \end{array}$$

a) Is the conjecture true when all the five prime numbers are different?
b) In how many ways can each odd integer be expressed as above?

2.2: Representations of the even numbers

A) Any even integer n can be expressed as a combination of two primes, as follows:

1) $n = p - q$, where p, q are both primes ($k = 2, s = 1$). For example:

$$\begin{array}{rcccccl} 2 & = & 7\text{-}5 & = & 13\text{-}11 & = & \dots ; \\ 4 & = & 11\text{-}7 & = & \dots ; \\ 6 & = & 13\text{-}7 & = & \dots ; \\ 8 & = & 13\text{-}5 & = & \dots \end{array}$$

In how many ways can each odd integer be expressed as above?

B) Any even integer n can be expressed as a combination of four primes as follows:

2) $n = p + q + r - t$, where p, q, r, t are all prime numbers ($k = 4, s = 1$). For example:

2	=	3+3+3-7	=	3+5+5-11	=	... ;
4	=	3+3+5-7	=	=		... ;
6	=	3+5+5-7	=	=		... ;
8	=	11+5+5-13	=	=		...

a) Is the conjecture true when all the four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

3) $n = p+q-r-t$, where p, q, r, t are all prime numbers ($k = 4, s = 1$). For example:

2	=	11+11-3-17	=	11+11-13-7	=	... ;
4	=	11+13-3-17	=	... ;		
6	=	13+13-3-17	=	... ;		
8	=	11+17-7-13	=	...		

a) Is the conjecture true when all the four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

4) $n = p-q-r-t$, where p, q, r, t are all prime numbers ($k = 4, s = 3$). For example:

2	=	11-3-3-3	=	13-3-3-5	=	... ;
4	=	13-3-3-3	=	... ;		
6	=	17-3-3-5	=	... ;		
8	=	23-3-5-7	=	...		

a) Is the conjecture true when all the four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

2.3: General conjecture:

Let $k \geq 3$, and $1 \leq s < k$, be integers. Then:

i) If k is odd, any odd integer can be expressed as a sum of $k-s$ primes (first set) minus a sum of s primes (second set) [so that the primes of the first set is different from the primes of the second set].

a) Is the conjecture true when all the k prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

ii) If k is even, any even integer can be expressed as a sum of $k-s$ primes (first set) minus a sum of s primes (second set) [so that the primes of the first set is different from the primes of the second set].

a) Is the conjecture true when all the k prime numbers are different?

b) In how many ways can each even integer be expressed as above?

References: [71], p. 190 from [82].

Definition 1. $E(n) = x_1^{x_2} + x_2^{x_3} + ... + x_n^{x_1}$, where $n \geq 2$, with $x_1, x_2, ..., x_n > 1$ and has the greatest common divisor 1.

Problem 3. Is the number of the primes of this form finite or infinite?
References: [20, 25, 50, 51].

Theorem 1. Let $S(n)$ be the Smarandache function: $S(n)$ is the smallest number, such that $S(n)!$ is divisible by n. Let p be an integer > 4. Then:

$$p \text{ is prime iff } S(p) = p.$$

Here and everywhere below "iff" denotes "if and only if".
References: [32, 72].

The following four statements are derived from the Wilson Theorem (p is prime iff $(p-1)!$ is congruent to $-1 \pmod{p}$), but they improve it because the factorial is reduced.

Theorem 2. Let p be an integer ≥ 3. Then:

$$p \text{ is prime iff } (p-3)! \text{ is congruent to } \frac{p-1}{2} \pmod{p}.$$

References: [73] and pp. 94-98 from [81].

Theorem 3. Let p be an integer ≥ 4. Then:

$$p \text{ is prime iff } (p-4)! \text{ is congruent to } (-1)^{\lfloor \frac{p}{3} \rfloor + 1} \lfloor \frac{p+1}{6} \rfloor \pmod{p}.$$

References: [73] and pp. 94-98 from [81].

Theorem 4. Let p be an integer ≥ 5. Then:

$$p \text{ is prime iff } (p-5)! \text{ is congruent to } rh + \frac{r^2-1}{24} \pmod{p},$$

with $h = \lfloor \frac{p}{24} \rfloor$ and $r = p - 24h$.

References: [73] and pp. 94-98 from [81].

Theorem 5. Let $p = (k-1)!h + 1$ be a positive integer $k > 5$, h being a natural number. Then:

$$p \text{ is prime iff } (p-k)! \text{ is congruent to } (-1)^t h \pmod{p},$$

with $t = h + \lfloor \frac{p}{h} \rfloor + 1$.

References: [73] and pp. 94-98 from [81].

Everywhere above $\lfloor x \rfloor$ means the inferior integer part of x, i.e., the smallest integer greater than or equal to x.

Theorem 6. Characterization of Twin Primes:
Let p and $p+2$ be positive odd integers. Then the following statements are equivalent:

a) p and $p+2$ are both primes;

b) $(p-1)!(3p+2)+2p+2$ is congruent to 0 (mod $p(p+2)$);

c) $(p-1)!(p-2)-2$ is congruent to 0 (mod $p(p+2)$);

d) $\frac{[(p-1)!+1]}{p} + \frac{[2(p-1)!+1]}{p+2}$ is an integer.

Theorem 7. Characterization of Pairs of Primes:
Let p and $p+k$ be positive integers, with $(p, p+k) = 1$. Then p and $p+k$ are both primes iff $(p-1)!(p+k)+(p+k-1)!p+2p+k$ is congruent to 0 (mod $p(p+k)$).

Theorem 8. Characterization of Triplets of Primes:
Let $p-2$, p and $p+4$ be positive integers, coprime two by two. Then $p-2$, p and $p+4$ are all primes iff

$$(p-1)! + \frac{p[(p-3)!+1]}{p-2} + \frac{p[(p+3)!+1]}{p+4} \equiv -1 \pmod{p}.$$

Theorem 9. Characterization of Quadruples of Primes:
Let p, $p+2$, $p+6$ and $p+8$ be positive integers, coprime two by two. Then p, $p+2$, $p+6$ and $p+8$ are all primes iff

$$\frac{p[(p-1)!+1]}{p} + \frac{2![(p-1)!+1]}{p+2} + \frac{6![(p-1)!+1]}{p+6} + \frac{8![(p-1)!+1]}{p+8}$$

is an integer.

Theorem 10. General Theorem of Characterization of n Prime Numbers simultaneously:

Let $p_1, p_2, \ldots, p_n$ be positive integers > 1, coprime two by two, and $1 \le k_i \le p_i$ for $1 \le i \le n$. Then the following statements are equivalent:

a) $p_1, p_2, \ldots, p_n$ are prime simultaneously;

b) $\sum_{i=1}^{n} [(p_i - k_i)!(k_i - 1)! - (-1)^{k_i}] \prod_{j \ne i} p_j \equiv 0 \pmod{p_1 p_2 \ldots p_n}$;

c) $\dfrac{\sum_{i=1}^{n} [(p_i - k_i)!(k_i - 1)! - (-1)^{k_i}] \prod_{j \ne i} p_j}{p_{s+1} \cdots p_n} \equiv 0 \pmod{p_1 p_2 \ldots p_s}$;

d) $\dfrac{\sum_{i=1}^{n} [(p_i - k_i)!(k_i - 1)! - (-1)^{k_i}] p_j}{p_i} \equiv 0 \pmod{p_j}$;

e) $\dfrac{\sum_{i=1}^{n} [(p_i - k_i)!(k_i - 1)! - (-1)^{k_i}]}{p_i}$ is an integer.

References: [78], [79], pp. 13-18 from [81].

Problem 4. Study Smarandache - Kurepa Function:
For p being prime, $SK(p)$ is the smallest integer, such that $!SK(p)$ is divisible by p, where $!SK(p) = 0! + 1! + 2! + \ldots + (p-1)!$. For example,

p	2	3	7	11	17	19	23	31	37	41	61	71	73	89
$SK(p)$	2	4	6	6	5	7	7	12	22	16	55	54	42	24

Reference: [3].

Problem 5. Study Smarandache - Wagstaff Function:
For p being prime, $SW(p)$ is the smallest integer, such that $W(SW(p))$ is divisible by p, where $W(p) = 1! + 2! + \ldots + p!$. For example,

p	3	11	17	23	29	37	41	43	53	67	73	79	97
$SW(p)$	2	4	5	12	19	24	32	19	20	20	7	57	6

Reference: [3].

Problem 6. Study Smarandache Ceil Functions of n–th Order:
$S_k(n)$ is the smallest integer, for which n divides $S_k(n)^k$. For example, for

$k=2$, we have:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$S_2(n)$	2	4	3	6	10	12	5	9	14	8	6	20	22	15	12	7

References: [13], pp. 27-30 from [40].

Problem 7. Study Pseudo-Smarandache Function:
$Z(n)$ is the smallest integer, such that $1+2+\ldots+Z(n)$ is divisible by n. For example,

n	1	2	3	4	5	6	7
$Z(n)$	1	3	2	3	4	3	6

Reference: [43].

Problem 8. Study Smarandache Near-To-Primordial Function:
$SNTP(n)$ is the smallest prime, such that either p^*-1, p^* or p^*+1 is divisible by n, where p^*, for a prime number p, is the product of all primes less than or equal to p. For example,

n	1	2	3	4	5	6	7	8	9	10	11	...
$SNTP(n)$	2	2	2	*	3	3	3	*	*	5	11	...

where "*" means "does not exist". $SNTP(n)$ is undefined for squareful integers: 4, 8, 9, 12, ...
References: [1, 54].

Problem 9. Study Smarandache double-factorial function:
$SDF(n)$ is the smallest number, such that $SDF(n)!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1.3.5.....m, & \text{if } m \text{ is odd} \\ 2.4.6.....m, & \text{if } m \text{ is even} \end{cases}$$

For example,

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$SDF(n)$	1	2	3	4	5	6	7	4	9	10	11	6	13	14	5	6

Reference: Section No. 54 from [33].

Theorem 11. Smarandache Primitive Functions:
Let p be prime. $S_p : \mathcal{N} \to \mathcal{N}$, having the property that $(S_p(n))!$ is divisible

by p^n, and it is the smallest integer with this property, where here and everywhere below $\mathcal{N}$ is the set of the natural numbers.

For example, $S_3(4) = 9$, because $9!$ is divisible by 3^4, and it is the smallest one with this property.

These functions help us compute the Smarandache Function.

Reference: [72].

The following three Smarandache's problems are solved by Ion Balacenoiu (see [12]).

Theorem 12. Smarandache Functions of the First Kind:
$S_n : \mathcal{N}^* \to \mathcal{N}^*, \mathcal{N}^* = \{1, 2, 3, ...\}$

i) If $n = u^r$ (with $u = 1$, or $u = p$ being a prime number), then $S_n(a) = k$, where k is the smallest positive integer such that $k!$ is a multiple of u^{ra};

ii) If $n = p_1^{r_1} . p_2^{r_2} ... p_t^{r_t}$, then $S_n(a) = \max\limits_{1 \le j \le t} \{S_{p_j^{r_j}}(a)\}$.

Theorem 13. Smarandache Functions of the Second Kind:
$S^k : \mathcal{N}^* \to \mathcal{N}^*$, $S^k(n) = S_n(k)$ for $k \in \mathcal{N}^*$, where S_n are the Smarandache Functions of the First Kind.

Theorem 14. Smarandache Functions of the Third Kind:
$S_a^b(n) = S_{a_n}(b_n)$, where S_{a_n} is the Smarandache Function of the First Kind, and the sequences (a_n) and (b_n) are different from the following situations:
i) $a_n = 1$ and $b_n = n$, for $n \in \mathcal{N}^*$;
ii) $a_n = n$ and $b_n = 1$, for $n \in \mathcal{N}^*$.

Reference: [12].

Definition 2. A function $f : \mathcal{N}^* \to \mathcal{N}^*$ is called Smarandache-multiplicative one if for any $(a, b) = 1, f(ab) = \max(f(a), f(b))$, i.e., if it reflects the main property of the Smarandache Function.

Reference: [89].

Definition 3. Let $f : \mathcal{N} \to \mathcal{N}$ be a strictly increasing function and x an element of $\mathcal{R}$ - the set of the real numbers. Then:
a) *Inferior* $f-$ *part of* x $ISf(x)$ is the smallest k, such that

$$f(k) \le x < f(k+1).$$

b) *Superior f– part of x* $SSf(x)$ is the smallest k, such that

$$f(k) < x \leq f(k+1).$$

Particular cases (they are discussed in [8, 10, 11, 61]):
a) Inferior prime part:
For any positive real number n $ISp(n)$ is defined as the largest prime number less than or equal to n. The first values of this function are:

$$2,3,3,5,5,7,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,23,23...$$

b) Superior prime part:
For any positive real number n $SSp(n)$ is defined as the smallest prime number greater than or equal to n. The first values of this function are:

$$2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23...$$

c) Inferior square part:
For any positive real number n $ISs(n)$ is defined as the largest square less than or equal to n. The first values of this function are:

$$0,1,1,1,4,4,4,4,4,9,9,9,9,9,9,9,16,16,16,16,16,16,16,16,16,25...$$

d) Superior square part:
For any positive real number n $SSs(n)$ is defined as the smallest square greater than or equal to n. The first values of this function are:

$$0,1,4,4,4,9,9,9,9,9,16,16,16,16,16,16,16,25,25,25,25,25,25,$$

$$25,25,25,36,36...$$

e) Inferior cube part:
For any positive real number n $ISc(n)$ is defined as the largest cube less than or equal to n. The first values of this function are:

$$0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,27,27...$$

f) Superior cube part:
For any positive real number n $SSc(n)$ is defined as the smallest cube greater than or equal to n. The first values of this function are:

$$0,1,8,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,27,27,27,27,27...$$

g) Inferior factorial part:
For any positive real number n $ISf(n)$ is defined as the largest factorial less than or equal to n. The first values of this function are:

$$1, 2, 2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 24, 24, 24, 24...$$

h) Superior factorial part: For any positive real number n $SSf(n)$ is defined as the largest factorial less than or equal to n. The first values of this function are:

$$1, 2, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24...$$

References: [69, 80]

Definition 4. Let $f : \mathcal{N} \to \mathcal{N}$ be a strictly increasing function and x be an element of $\mathcal{R}$. Then: *fractional* $f-$ *part of* x $FSf(x) = x - ISf(x)$, where $ISf(x)$ is the Inferior $f-$part of x, defined above.

Particular cases:
a) Fractional Prime Part:
$FSp(x) = x - ISp(x)$, where $ISp(x)$ is the Inferior Prime Part, defined above. Example: $FSp(12.501) = 12.501 - ISp(12.501) = 12.501 - 11 = 1.501$.
b) Fractional Square Part:
$FSs(x) = x - ISs(x)$, where $ISs(x)$ is the Inferior Square Part, defined above. Example: $FSs(12.501) = 12.501 - ISs(12.501) = 12.501 - 9 = 3.501$.
c) Fractional Cubic Part:
$FSc(x) = x - ISc(x)$, where $ISc(x)$ is the Inferior Cubic Part, defined above. Example: $FSc(12.501) = 12.501 - ISc(12.501) = 12.501 - 8 = 4.501$.
d) Fractional Factorial Part:
$FSf(x) = x - ISf(x)$, where $ISf(x)$ is the Inferior Factorial Part, defined above. Example: $FSf(12.501) = 12.501 - ISf(12.501) = 12.501 - 6 = 6.501$.
Remark 2.1 This is a generalization of the fractional part of a number.
Remark 2.2 In a similar way the following ones are defined:
- Inferior Fractional $f-$Part $IFSf(x) = x - ISf(x) = FSf(x)$;
- Superior Fractional $f-$Part $SFSf(x) = SSf(x) - x$; for example, Superior Fractional Cubic Part of 12.501 is $SFSc(12.501) = 27 - 12.501 = 14.499$.

Definition 5. Let $g : A \to A$ be a strictly increasing function, and let "$\sim$" be a given internal law on A. Then we say that $f : A \to A$ is *complementary with respect to the function g and the internal law "$\sim$"* if $f(x)$ is the smallest k, such that there exists $z \in A$ so that $x \sim k = g(z)$.

Particular cases (the first three cases are discused in [8, 9]):

a) Square Complementary Function: $f : \mathcal{N} \to \mathcal{N}$, $f(x)$ is the smallest k, such that xk is a perfect square. The first values of this function are:

$$1, 2, 3, 1, 5, 6, 7, 2, 1, 10, 11, 3, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3, 7 \ldots$$

b) Cubic Complementary Function: $f : \mathcal{N} \to \mathcal{N}$, $f(x)$ is the smallest k, such that xk is a perfect cube. The first values of this function are:

$$1, 4, 9, 2, 25, 36, 49, 1, 3, 100, 121, 18, 169, 196, 225, 4, 289, 12, 361, 50 \ldots$$

More general:

c) $m-$power Complementary Function: $f : \mathcal{N} \to \mathcal{N}$, $f(x)$ is the smallest k, such that xk is a perfect $m-$power.

d) Prime Complementary Function: $f : \mathcal{N} \to \mathcal{N}$, $f(x)$ is the smallest k, such that $x + k$ is a prime. The first values of this function are:

$$1, 0, 0, 1, 0, 1, 0, 3, 2, 1, 0, 1, 3, 2, 1, 0, 1, 3, 2, 1, 0, 1, 5, 4, 3, 2, 1, 0, 1, 5 \ldots$$

References: [24, 34, 58, 69, 80, 92]

Problem 10. Smarandache-Fibonacci Ttriples:

$$11, 121, 4902, 26245, 32112, 64010, 368140, 415664, 2091206, 2519648,$$

$$4573053, 7783364, 79269727, 136193976, 321022289, 445810543, 559199345,$$

$$670994143, 836250239, 893950202, 937203749, 10411478032, 1148788154.$$

Integer n is such one that $S(n) = S(n-1) + S(n-2)$, where $S(k)$ is the Smarandache function. It is not known whether this sequence has infinitely or finitely many terms. H. Ibstedt and C. Ashbacher independently conjectured that there are infinitely many terms. H. Ibsteat found the largest known number of this kind: 19448047080036.

References: [7, 40]

Problem 11. Smarandache-Radu Duplets:

$$224, 2057, 265225, 843637, 6530355, 24652435, 35558770, 40201975,$$

$$45388758, 46297822, 67697937, 138852445, 157906534, 171531580,$$

$$299441785, 551787925, 1223918824, 1276553470, 1655870629,$$
$$1853717287, 1994004499, 2256222280.$$

Integer n is such one that between $S(n)$ and $S(n+1)$ there is no prime ($S(n)$ and $S(n+1)$ included), where $S(k)$ is the Smarandache function. It is not known whether this sequence has infinitely or finitely many terms. H. Ibstedt conjectured that there are infinitely many terms. H. Ibsteat found the largest known number of this kind:

$$270329975921205253634707051822848570391313.$$

References: [40, 62]

Definition 6. Functional Iteration of First Kind:

Let $f : A \to A$ be a function, such that $f(x) \le x$ for all x, and $\min\limits_{x \in A} f(x) \ge m_0 \ne -\infty$. Let f have $p \ge 1$ fix points $m_0 \le x_1 < x_2 < \ldots < x_p$. [The point x is called a fix one if $f(x) = x$.]. Then $SI1_f(x)$ is the smallest number of iterations k, such that $\underbrace{f(f(\ldots f(}_{k \text{ times}} x \underbrace{) \ldots))}_{k \text{ times}} =$ constant.

Example: Let $n > 1$ be an integer, and $d(n)$ be the number of the positive divisors of n, $d : \mathcal{N} \to \mathcal{N}$. Then $SI1_d(n)$ is the smallest number of iterations k, such that $\underbrace{d(d(\ldots d(}_{k \text{ times}} x \underbrace{) \ldots))}_{k \text{ times}} = 2$; because $d(n) < n$ for $n > 2$, and the fix points of function d are 1 and 2. Thus $SI1_d(6) = 3$, because $d(d(d(6))) = d(d(4)) = d(3) = 2$; $SI1_d(5) = 1$, because $d(5) = 2$.

Definition 7. Functional Iteration of Second Kind:
Let $g : A \to A$ be a function such that $g(x) > x$ for all x, and let $b > x$. Then $SI2_g(x, b)$ is the smallest number of iterations k such that $\underbrace{g(g(\ldots g(}_{k \text{ times}} x \underbrace{) \ldots))}_{k \text{ times}} \ge b$.

Example: Let $n > 1$ be an integer, and $\sigma(n)$ be the sum of positive divisors of n, (1 and n included), $\sigma : \mathcal{N} \to \mathcal{N}$. Then $SI2_\sigma(n, b)$ is the smallest number of iterations k such that $\underbrace{\sigma(\sigma(\ldots \sigma(}_{k \text{ times}} x \underbrace{) \ldots))}_{k \text{ times}} \ge b$; because $\sigma(n) > n$ for $n > 1$. Thus $SI2_\sigma(4, 11) = 3$, because $\sigma(\sigma(\sigma(4))) = \sigma(\sigma(7)) = \sigma(8) = 15 \ge 11$.

Definition 8. Functional Iteration of Third Kind:
Let $h : A \to A$ be a function, such that $h(x) < x$ for all x, and let $b < x$. Then $SI3_h(x,b)$ is the smallest number of iterations k, such that $\underbrace{h(h(...h(}_{k \text{ times}}x\underbrace{)...))}_{k \text{ times}} \leq b$.

Example: Let n be an integer and $gd(n)$ be the greatest divisors of n, less than n, $gd : \mathcal{N}^* \to \mathcal{N}^*$. Then $gd(n) < n$ for $n > 1$ and $SI3_{gd}(60,3) = 4$, because $gd(gd(gd(gd(60)))) = gd(gd(gd(30))) = gd(gd(15)) = gd(5) = 1 \leq 3$.
References: [39], pp. 52-58 from [40], Problem 52 from [80].

The following sixteen theorems are proved by F. Smarandache [29, 64, 18, 35, 19]. More decimals for each constant have been computed by Weisstein [94].

Theorem 15. The First Constant of Smarandache's:

$$\sum_{n\geq 2} \frac{1}{S(n)!}$$

is convergent to a number s_1, which satisfies $0.000 \leq s_1 \leq 0.717$.
Reference: pp. 116-118 from [29].

Theorem 16. The Second Constant of Smarandache's:

$$\sum_{n\geq 2} \frac{S(n)}{n!}$$

is convergent to an irrational number s_2.
Reference: pp. 119-120 from [29].

Theorem 17. The Third Constant of Smarandache's:

$$\sum_{n\geq 2} \frac{1}{S(2)S(3)...S(n)}$$

is convergent to a number s_3, which satisfies $0.71 \leq s_1 \leq 1.01$.
Reference: pp. 121-126 from [29].

Theorem 18. The Fourth Constant of Smarandache:

$$\sum_{n\geq 2} \frac{n^\alpha}{S(2)S(3)...S(n)},$$

where $\alpha \geq 1$, is convergent to a number s_4.
Reference: pp. 121-126 from [29].

Theorem 19. The series

$$\sum_{n\geq 1} (-1)^{n-1}\frac{S(n)}{n!},$$

converges to an irrational number.
Reference: [64].

Theorem 20. The series

$$\sum_{n\geq 2} \frac{S(n)}{(n+1)!},$$

converges to a number s_6, where $e^{-\frac{3}{2}} < s_6 < \frac{1}{2}$.
References: [18, 35].

Theorem 21. The series

$$\sum_{n\geq r} \frac{S(n)}{(n+r)!},$$

where r is a natural number, converges to a number s_7.
Reference: [35].

Theorem 22. The series

$$\sum_{n\geq r} \frac{S(n)}{(n-r)!},$$

where $r \neq 0$ is a natural number, converges to a number s_8.
Reference: [35].

Theorem 23. The series

$$\sum_{n\geq r} \frac{1}{\sum\limits_{i=2}^{n} \frac{S(i)!}{i}}$$

converges to a number s_9.

Reference: [35].

Theorem 24. The series

$$\sum_{n\geq r} \frac{1}{S(n)\sqrt[\alpha]{S(n)!}},$$

where $\alpha > 1$, is convergent to a number s_{10}.

References: [19, 35].

Theorem 25. The series

$$\sum_{n\geq r} \frac{1}{S(n)\sqrt[\alpha]{(S(n)-1)!}},$$

where $\alpha > 1$, is convergent to a number s_{11}.

Reference: [19].

Theorem 26. Let $f : \mathcal{N}^* \rightarrow \mathcal{R}$ be a function which satisfies the condition

$$f(t) \leq \frac{c}{t^{\alpha}(d(t!)) - d((t-1)!)}$$

for $t \neq 0$ being a natural number, $d(x)$ being the number of divisors of x, and the given constants $\alpha, c > 1$. Then the series

$$\sum_{n\geq 1} f(S(n))$$

is convergent to a number s_{11_f}.

Reference: [19].

Theorem 27. The series

$$\sum_{n\geq 1} \frac{1}{(\prod_{k=2}^{n} S(k)!)^n}$$

is convergent to a number s_{13}.
Reference: [19].

Theorem 28. The series

$$\sum_{n\geq 1} \frac{1}{S(n)!\sqrt{S(n)!}(\log S(n))^p},$$

where $p > 1$, is convergent to a number s_{14}.
Reference: [19].

Theorem 29. The series

$$\sum_{n\geq 1} \frac{2^n}{S(2^n)!},$$

is convergent to a number s_{15}.
Reference: [19].

Theorem 30. The series

$$\sum_{n\geq 1} \frac{S(n)}{n^{1+p}},$$

where $p > 1$ is a real number, converges to a number s_{16}. For $0 \leq p \leq 2$ the series diverges.
Reference: [19].

Definition 9. An Algorithm of Multiplication of Two Integer Numbers, A and B:
- let $k \geq 2$ be an integer;
- write A and B in two different vertical columns: $c(A)$, respectively $c(B)$;
- multiply A by k, and write the product A_1 in column $c(A)$;
- derive B by k, and write the integer part of the quotient B_1 in column

$c(B)$;
... and so on with the new numbers A_1 and B_1, until we get $B_i < k$ in column $c(B)$;
- write another column $c(r)$, on the right side of $c(B)$, such that: for each number of column $c(B)$, which may be a multiple of k plus the rest r (where $r = 0, 1, ..., k-1$), the corresponding number in $c(r)$ will be r;
- multiply each number of column A by its corresponding r of $c(r)$, and put the new products in another column $c(P)$ on the right side of $c(r)$;
- finally, add all numbers of column $c(P)$.
$A \times B =$ the sum of all numbers of $c(P)$.

Remark that any multiplication of integer numbers can be done only by multiplication with $2, 3, ..., k$, divisions by k, and additions.

This is a generalization of Russian multiplication (when $k = 2$).

This multiplication is usefull when k is very small, the best values being for $k = 2$ (Russian multiplication – known from Egyptian times), or $k = 3$. If k is greater than or equal to $\min(10, B)$, this multiplication is trivial (the obvious multiplication).

Example 1 (if we choose $k = 3$): $73 \times 97 =?$

×3	/3		
c(A)	c(B)	c(r)	c(P)
73	97	1	73
219	32	2	438
657	10	1	657
1971	3	0	0
5913	1	1	5913
		total:	7081

Therefore: $73 \times 97 = 7081$.

Remark that any multiplication of integer numbers can be done only by multiplication with $2, 3$, divisions by 3, and additions.

Example 2 (if we choose $k = 4$): $73 \times 97 =?$

×4	/4		
c(A)	c(B)	c(r)	c(P)
73	97	1	73
292	24	0	0
1168	6	2	2336
4672	1	1	4672
		total:	7081

Therefore: $73 \times 97 = 7081$.

Remark that any multiplication of integer numbers can be done only by multiplication with $2, 3, 4$, divisions by 4, and additions.

Example 3 (if we choose $k = 5$): $73 \times 97 =?$

$\times 5$	$/5$		
c(A)	c(B)	c(r)	c(P)
73	97	2	146
365	19	4	1450
1825	3	3	5475

total: 7081

Therefore: $73 \times 97 = 7081$.

Remark that any multiplication of integer numbers can be done only by multiplication with $2, 3, 4, 5$, divisions by 5, and additions.

The above multiplication becomes less usefull when k increases. Look at example (4) to see what happens when $k = 10$:

$\times 10$	$/10$		
c(A)	c(B)	c(r)	c(P)
73	97	7	511
730	9	9	6570

total: 7081

Therefore: $73 \times 97 = 7081$.

Remark that any multiplication of integer numbers can be done only by multiplication with $2, 3, ..., 9, 10$, divisions by 10, and additions – hence we obtain just the obvious multiplication.

Reference: see Section 110 from [33]

Definition 10. An Algorithm of Division of an Integer Number, A by k^n, where $k, n \geq 2$ are integers:
- write A and k^n in two different vertical columns: $c(A)$, respectively $c(k^n)$;
- multiply A by k, and write the integer quotient A_1 in column $c(A)$;
- write the quotient $q_1 = k^{n-1}$ in column $c(k^n)$;

... and so on with the new numbers A_1 and q_1, until we get $q_n = 1 (= k^0)$ in column $c(k^n)$;
- write another column $c(r)$, in the left side of $c(A)$, such that: for each number of column $c(A)$, which may be a multiple of k plus the remainder r (where $r = 0, 1, ..., k-1$), the corresponding number in $c(r)$ will be r;
- write another column $c(P)$ by its corresponding r of $c(r)$, and put the new products in another column $c(R)$ on the left side of $c(P)$;
- finally, add all numbers of column $c(R)$ to get the final remainder R_n,

while the final quotient will be stated in front of $c(k^n)$'s 1. Therefore: $A/(k^n) = A_n$ and remainder R_n.

Remark that any division of an integer number by k^n can be done only by divisions to k, calculations of powers of k, multiplications with $1, 2, ..., k-1$, additions.

The above division becomes less usefull when k increases. Look at example (4) to see what happens when $k = 10$:

Example 1: $1357/(2^7) =?$

			/2	/2	
$c(R)$	$c(P)$	$c(r)$	$c(A)$	$c(2^7)$	
1	2^0	1	1357	2^7	$line_1$
9	2^1	0	678	2^6	$line_2$
4	2^2	1	339	2^5	$line_3$
8	2^3	1	169	2^4	$line_4$
0	2^4	0	84	2^3	$line_5$
0	2^5	0	42	2^2	$line_6$
64	2^6	1	21	2^1	$line_7$
			10	2^0	$last_{line}$
77					

Therefore: $1357?(2^7) = 10$ and the remainder is 77.

Remark that the division of an integer number by any power of 2 can be done only by divisions to 2, calculations of powers of 2, multiplications and additions.

Example 2: $19495/(3^8) =?$

			/3	/3	
$c(R)$	$c(P)$	$c(r)$	$c(A)$	$c(3^8)$	
1	3^0	1	10495	3^8	$line_1$
0	3^1	0	6498	3^7	$line_2$
0	3^2	0	2166	3^6	$line_3$
54	3^3	2	722	3^5	$line_4$
0	3^4	0	240	3^4	$line_5$
486	3^5	2	80	3^3	$line_6$
1458	3^6	2	26	3^2	$line_7$
4374	3^7	2	8	3^1	$line_8$
			2	3^0	$last_{line}$
6373					

Therefore: $19495?(3^8) = 2$ and the remainder is 6373.

Remark that the division of an integer number by any power of 3 can be done only by divisions to 3, calculations of powers of 3, multiplications and additions.

Reference: see Section 111 from [33]

Definition 11. A Prime Base:

$$0, 1, 10, 100, 101, 1000, 1001, 10000, 10001, 10010, 10100, 100000, 100001,$$

$$1000000, 1000001, 1000010, 1000100, 10000000, 10000001, 100000000,$$

$$100000001, 100000010, 100000100, 1000000000, 1000000001, 10000000100,$$

$$1000000101, ...$$

Smarandache defined over the set of natural numbers the following infinite base: $p_0 = 1$, and for $k \geq 1$ p_k is the $k-$th prime number. He proved that every positive integer A may be uniquely written in the prime base as:

$$A = (\overline{a_n...a_1a_0})_{SP} \stackrel{def}{=} \sum_{i=0}^{n} a_i p_i,$$

with all $a_i = 0$ or 1 (of course, $a_n = 1$), in the following way:
- if $p_n \leq A < p_{n+1}$ then $A = p_n + r_1$;
- if $p_m \leq r_1 < p_{m+1}$ then $r_1 = p_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

Therefore, any number may be written as a sum of prime numbers $+e$, where $e = 0$ or 1.

If we note by $p(A)$ the Smarandache superior part of A (i.e. the largest prime less than or equal to A), then A is written in the Smarandache prime base as:

$$A = p(A) + p(A - p(A)) + p(A - p(A) - p(A - p(A))) + ...$$

This base is important for partitions with primes.

References: [33, 37]

Definition 12. A Square Base:

$$0, 1, 2, 3, 10, 11, 12, 13, 20, 100, 101, 102, 103, 110, 111, 112, 1000, 1001, 1002,$$

$$1003, 1010, 1011, 1012, 1013, 1020, 10000, 10001, 10002, 10003, 10010,$$

$$10011, 10012, 10013, 10020, 10100, 10101, 100000, 100001, 100002, 100003,$$

$$100010, 100011, 100012, 100013, 100020, 100100, 100101, 100102, 100103,$$

$$100110, 100111, 100112, 101000, 101001, 101002, 101003, 101010, 101011,$$

$$101012, 101013, 101020, 101100, 101101, 101102, 1000000, \ldots$$

Smarandache defined over the set of natural numbers the following infinite base: for $k \geq 0$ $s_k = k^2$. He proved that every positive integer A may be written in a unique way in the Smarandache square base as:

$$A = (\overline{a_n \ldots a_1 a_0})_{S2} \stackrel{def}{=} \sum_{i=1}^{n} a_i s_i,$$

with all $a_i = 0$ or 1 for $i \geq 2$, $0 \leq a_0 \leq 3$, $0 \leq a_1 \leq 2$, and, of course, $a_n = 1$, in the following way:
- if $s_n \leq A < s_{n+1}$ then $A = s_n + r_1$;
- if $s_m \leq r_1 < s_{m+1}$ then $r_1 = s_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

Therefore, any number may be written as a sum of squares (1, being obvious, is not counted as a square) $+e$, where $e = 0, 1$ or 3.

If we note by $s(A)$ the Smarandache superior square part of A (i.e. the largest square less than or equal to A), then A is written in the Smarandache square base as:

$$A = s(A) + s(A - s(A)) + s(A - s(A) - s(A - s(A))) + \ldots$$

This base is important for partitions with squares.

References: [33, 37]

Definition 13. A m–Power Base (generalization):

Each number n written in m–power base, where ≥ 2 is an integer.

Smarandache defined over the set of natural numbers the following infinite m–power base: for $k \geq 0$ $t_k = k^m$. He proved that every positive integer A may be written in a unique way in the m–power base as:

$$A = (\overline{a_n \ldots a_1 a_0})_{Sm} \stackrel{def}{=} \sum_{i=0}^{n} a_i t_i,$$

with $a_i = 0$ or 1 for $i \geq m$, $0 \leq a_i \leq [\frac{(i+2)^m - 1}{(i+1)^m}]$ (integer part) for $i = 0, 1, \ldots, m-1$, and, of course, $a_n = 1$, in the following way:
- if $t_n \leq A < t_{n+1}$ then $A = t_n + r_1$;
- if $t_m \leq r_1 < t_{m+1}$ then $r_1 = t_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

Therefore, any number may be written as a sum of m–powers (1, being obvious, is not counted as a m–power) $+e$, where $e = 0, 1, 2, \ldots$ or $2^m - 1$.

If we note by $t(A)$ the Smarandache superior m-power part of A (i.e. the largest m-power less than or equal to A), then A is written in the Smarandache m-power base as:

$$A = t(A) + t(A - t(A)) + t(A - t(A) - t(A - t(A))) + \dots$$

This base is important for partitions with m-powers.

References: [33, 37]

Definition 14. A Factorial Base:

$$0, 1, 10, 11, 20, 21, 100, 101, 110, 111, 120, 121, 200, 201, 210, 211, 220, 221,$$

$$300, 301, 310, 311, 320, 321, 1000, 1001, 1010, 1011, 1020, 1021, 1100, 1101,$$

$$1110, 1111, 1120, 1121, 1200, \dots$$

Smarandache defined over the set of natural numbers the following infinite base: for $k \geq 1$ $f_k = k!$. He proved that every positive integer A may be written in a unique way in the factorial base as:

$$A = (\overline{a_n \dots a_1 a_0})_{(F)} \stackrel{def}{=} \sum_{i=1}^{n} a_i f_i,$$

with all $a_i = 0, 1, \dots, i$ for $i \geq 1$ in the following way:
- if $f_n \leq A < f_{n+1}$, then $A = f_n + r_1$;
- if $f_m \leq r_1 < f_{m+1}$, then $r_1 = f_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

What is very interesting is that: $a_1 = 0$ or 1; $a_2 = 0, 1$ or 2; $a_3 = 0, 1, 2$ or 3, and so on ...

If we note the Smarandache superior factorial part of A (i.e. the largest factorial less than or equal to A), by $f(A)$ then A is written in the Smarandache factorial base as:

$$A = f(A) + f(A - f(A)) + f(A - f(A) - f(A - f(A))) + \dots$$

Rules of addition and subtraction in Smarandache factorial base: for each digit a_i we add and substract in base $i + 1$, for $i \geq 1$. Here is an example, for an addition:

base	5	4	3	2	
		2	1	0	+
		2	2	1	
	1	1	0	1	

because: $0+1=1$ (in base 2); $1+2=10$ (in base 3), therefore, we write 0 and keep 1; $2+2+1=11$ (in base 4). Now a subtraction

base	5	4	3	2	
	1	0	0	1	-
		3	2	0	
			1	1	

because: $1-0=1$ (in base 2); $0-2=?$ it is not possible (in base 3), go to the next left unit, which is 0 again (in base 4), go again to the next left unit, which is 1 (in base 5), therefore, $1001 \to 0401 \to 0331$ and then $0331-320=11$.

Find some rules for multiplication and division.

In a general case, if we want to design a base such that any number

$$A = (\overline{a_n \ldots a_1 a_0})_{(B)} \stackrel{def}{=} \sum_{i=1}^{n} a_i b_i,$$

with all $a_i = 0, 1, \ldots, t_i$ for $i \geq 1$, where all $t_i \geq 1$, then this base should be $b_1 = 1, b_{i+1} = (t_i + 1).b_i$ for $i \geq 1$.

References: [33, 37]

Definition 15. A Generalized Base:

Smarandache defined over the set of natural numbers the following infinite generalized base: $1 = g_0 < g_1 < \ldots, g_k < \ldots$). He proved that every positive integer A may be written in a unique way in the Smarandache generalized base as:

$$A = (\overline{a_n \ldots a_1 a_0})_{SG} \stackrel{def}{=} \sum_{i=0}^{n} a_i g_i,$$

with $0 \leq a_i \leq [\frac{g_{i+1}-1}{g_i}]$ (integer part) for $i = 0, 1, \ldots, n$, and, of course, $a_n \geq 1$, in the following way:
- if $g_n \leq A < g_{n+1}$ then $A = g_n + r_1$;
- if $g_m \leq r_1 < g_{m+1}$ then $r_1 = g_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

If we note by the Smarandache superior generalized part of A (i.e. the largest g_i less than or equal to A), $g(A)$ then A is written in the Smarandache generalized base as:

$$A = g(A) + g(A - g(A)) + g(A - g(A) - g(A - g(A))) + \ldots$$

This base is important for partitions: a generalized base may be any infinite integer set (of primes, squares, cubes, any $m-$powers, Fibonacci/Lucas numbers, Bernouly numbers, Smarandache numbers, etc.) whose partitions are studied.

A particular case is when the base verifies: $2g_i \geq g_{i+1}$ for any i, and $g_0 = 1$, since all the coefficients of a written number in this base will be 0 or 1.

Remark: another particular case: if one takes $g_i = p^{i-1}$, $i = 1, 2, 3, ...$, p an integer ≥ 2, one gets the representation of a number in the numerical base p (p may be 10 (decimal), 2 (binary), 16 (hexadecimal), etc.).

References: [33, 37]

Definition 16. A Concatenated Natural Sequence:

$$1, 22, 333, 4444, 55555, 666666, 7777777, 88888888, 999999999,$$
$$10101010101010101010, 1111111111111111111111,$$
$$121212121212121212121212, 13131313131313131313131313,$$
$$1414141414141414141414141414, 151515151515151515151515151515, ...$$

Definition 17. A Concatenated Prime Sequence:

$$2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719,$$
$$23571113171923, ...$$

Back concatenated prime sequence:

$$2, 32, 532, 7532, 117532, 13117532, 1713117532, 191713117532,$$
$$2319171311753, ...$$

Conjecture: there are infinitely many primes among the numbers of the first sequence.

Definition 18. A Concatenated Odd Sequence:

$$1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517$$

Back concatenated odd sequence:

$$1, 31, 531, 7531, 97531, 1197531, 131197531, 15131197531, 1715131197531$$

Conjecture: there are infinitely many primes among these numbers.

Definition 19. A Concatenated Even Sequence:

$$2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, \ldots$$

Back concatenated even sequence:

$$2, 42, 642, 8642, 108642, 12108642, 1412108642, 161412108642, \ldots$$

Definition 20. A Concatenated S−sequence (generalization): Let $s_1, s_2, \ldots, s_n, \ldots$ be an infinite integer sequence (noted by S). Then:

$$s_1, \overline{s_1 s_2}, \overline{s_1 s_2 s_3}, \ldots \overline{s_1 s_2 s_3 \ldots s_n}, \ldots$$

is called Concatenated S−sequence, and

$$s_1, \overline{s_2 s_1}, \overline{s_3 s_2 s_1}, \ldots \overline{s_n \ldots s_3 s_2 s_1}, \ldots$$

is called Back concatenated S−sequence.

Problem 12. How many terms of the Concatenated S−Sequence belong to the initial S−sequence? Or, how many terms of the Concatenated S−Sequence verify the relation of other given sequence?

Look now at some other examples, where S is the sequence of squares, cubes, Fibonacci, respectively.

Definition 21. A Concatenated Square Sequence:

$$1, 14, 149, 14916, 1491625, 149162536, 14916253649, 1491625364964, \ldots$$

Back concatenated square sequence:

$$1, 41, 941, 16941, 2516941, 362516941, 49362516941, 6449362516941, \ldots$$

How many of them are prefect squares?

Definition 22. A Concatenated Cubic Sequence:

$$1, 18, 1827, 182764, 182764125, 182764125216, 182764125216343, \ldots$$

Back concatenated cubic sequence:

$$1, 81, 2781, 642781, 125642781, 216125642781, 343216125642781, \ldots$$

How many of them are prefect cubes?

Definition 23. A Concatenated Fibonacci Sequence:

$$1, 11, 112, 1123, 11235, 112358, 11235813, 1123581321, 112358132134, \ldots$$

Back concatenated Fibonacci sequence:

$$1, 11, 211, 3211, 53211, 853211, 13853211, 2113853211, 342113853211, \ldots$$

Does any of these numbers is a Fibonacci number?

Definition 24. A Power Function: $SP(n)$ is the smallest number m, such that m^m is divisible by n.

Remarks: If p is prime, then $SP(p) = p$. If r is a square free, then $SP(r) = r$. If $n = \prod_{i=1}^{k} p_i^{s_i}$ and all $s_i \leq p_i$, then $SP(n) = n$. If $n = p^s$, where p is prime, then:

$$SP(n) = \begin{cases} p, & \text{if } 1 \leq s \leq p \\ p^2, & \text{if } p+1 \leq s \leq 2p^2 \\ p^3, & \text{if } 2p^2+1 \leq s \leq 3p^3 \\ \cdot & \cdot \quad \cdot \\ p^t, & \text{if } (t-1)p^{t-1}+1 \leq s \leq tp^t \end{cases}$$

Reference: [83]

Definition 25. A Reverse Sequence:

$$1, 21, 321, 4321, 54321, 654321, 7654321, 87654321, 987654321,$$

$$10987654321, 1110987654321, 121110987654321, \ldots$$

Definition 26. A Multiplicative Sequence:

$$2, 3, 6, 12, 18, 24, 36, 48, 54, \ldots$$

General definition: if m_1, m_2, are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number equal to the product of two previous distinct terms. All terms of rank ≥ 3 are divisible by m_1, and m_2. In our case the first two terms are 2, respectively 3.

Definition 27. A Wrong Sequence: A number $n = \overline{a_1 a_2 ... a_k}$ consisted of at least two digits, with the property: the sequence $a_1, a_2, ..., a_k$, $b_{k+1}, b_{k+2}, ...$ (where b_{k+i} is the product of the previous k terms, for any $i \geq 1$) contains n as its term.

The author conjectured that no number is wrong. Therefore, this sequence is empty.

Definition 28. Impotent Numbers:

$$2, 3, 4, 5, 7, 9, 11, 12, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, ...$$

A number n those proper divisors product is less than n. This sequence contains terms with the forms of p and p^2, where p is a prime.

Definition 29. A Random Sieve:

$$1, 5, 6, 7, 11, 13, 17, 19, 23, 25, , 29, 31, 35, 37, 41, 43, 47, 53, 59, ...$$

General definition:
- choose at random a positive number u_1;
- delete all the multiples of all its divisors, except this number;
- choose another number u_2 being greater than n_1 among the remaining;
- delete all the multiples of all its divisors, except this second number;
... and so on.

The remaining numbers are all coprime two by two.

The obtained sequence u_k, $k \geq 1$, is less dense than the prime number sequence, but it tends to the prime number sequence and also tends to infinity. This sequence may be important in this way. In our case $u_1 = 6, u_2 = 19, u_3 = 35, ...$

Definition 30. A Cubic Base:

$$0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27,$$

$$30, 31, 32, 100, 101, 102, 103, 104, 105, 106, 107, 110, 111, 112, 113, 114, 115,$$

$$116, 117, 120, 121, 122, 123, ...$$

Smarandache defined over the set of natural numbers the following infinite base: for $k \geq 1$ $s_k = k^3$. He proved that every positive integer A may be written in a unique way in the cubic base as:

$$A = (\overline{a_n \ldots a_1 a_0})_{C3} \overset{def}{=} \sum_{i=1}^{n} a_i s_i,$$

with $0 \le a_1 \le 7$, $0 \le a_2 \le 3$, $0 \le a_3 \le 2$, and $0 \le a_i \le 1$, for $i \ge 4$, and, of course, $a_n = 1$, in the following way:
- if $c_n \le A < c_{n+1}$, then $A = c_n + r_1$;
- if $c_m \le r_1 < c_{m+1}$, then $r_1 = c_m + r_2$, $m < n$;
and so on until one obtains a remainder $r_j = 0$.

Therefore, any number may be written as a sum of cubes (1, being obvious, is not counted as a cube) $+e$, where $e = 0, 1, \ldots, 7$.

If we note the superior cubic part of A (i.e. the largest cube less than or equal to A) by $c(A)$, then A is written in the cube base as:

$$A = c(A) + c(A - c(A)) + c(A - c(A) - c(A - c(A))) + \ldots$$

This base is important for partitions with cubes.

Definition 31. A Triangular Base:

$$1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000,$$

$$10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010,$$

$$100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011,$$

$$1000012, 1000100, \ldots$$

Numbers, written in the triangular base, are defined as follows: $t(n) = \dfrac{n(n+1)}{2}$, for $n \ge 1$.

Definition 32. A Double Factorial Base:

$$1, 10, 100, 101, 110, 200, 201, 1000, 1001, 1010, 1100, 1101, 1110, 1200,$$

$$10000, 10001, 10010, 10100, 10101, 10110, 10200, 10201, 11000, 11001,$$

$$11010, 11100, 11101, 11110, 11200, 11201, 12000, \ldots$$

Numbers, written in the double factorial base, are defined as follows: $df(n) = n!!$.

Definition 33. A General Base: Let $1 = b_1 < b_2 < b_3 < \ldots < b_n < \ldots$ be a strictly increasing integer sequence. Then any positive integer can by written in a unique way in this infinite base (in the same way as the previous particular bases). If we note the superior square part of A (i.e., the largest bk less than or equal to A) by $b(A)$, then A is written in the general base as:

$$A = b(A) + b(A - b(A)) + b(A - b(A) - b(A - b(A))) + \dots$$

This base may be important for partitions with increasing sequences.

Reference: [70]

Definition 34. A Non-Multiplicative Sequence (General definition): Let $m_1, m_2, \dots, m_k$ be the first $k \geq 2$ terms of the sequence. Then m_i, for $i \geq k+1$, is the smallest number not equal to the product of k previous distinct terms.

Definition 35. A Non-Arithmetic Progression:

$$1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, \dots$$

General definition: if m_1, m_2 are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number, such that no 3-term arithmetic progression is in the sequence. In our case the first two terms are 1 and 2. Generalization: the same initial conditions, but with no i-term arithmetic progression in the sequence (for a given $i \geq 3$).

Definition 36. A Cubic Product Sequence:

$$2, 9, 217, 13825, 1728001, 373248001, 128024064001, 65548320768001, \dots$$

$C_n = 1 + s_1 s_2 \dots s_n$, where c_k is the k-th cubic number.

Problem 13. How many of them are prime?

Definition 37. A Factorial Product Sequence:

$$2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, \dots$$

$F_n = 1 + f_1 f_2 \dots f_n$, where f_k is the k-th factorial number.

Problem 14. How many of them are prime?

Definition 38. An $U-$ Product Sequence (generalization): Let u_n, $n \geq 1$, be a positive integer sequence. Then we define a sequence, as follows: $U_n = 1 + u_1 u_2 \ldots u_n$.

Reference: [70]

Definition 39. A S_2 function (numbers):

$$1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 12,$$
$$5, 26, 9, 14, 29, 30, 31, 8, 33, \ldots$$

$S_2(n)$ is the smallest integer m, such that m^2 is divisible by n.

Definition 40. A S_3 function (numbers):

$$1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 6,$$
$$5, 26, 3, 14, 29, 30, 31, 4, 33, \ldots$$

$S_3(n)$ is the smallest integer m, such that m^3 is divisible by n.

Definition 41. An Anti-Symmetric Sequence:

$$11, 1212, 123123, 12341234, 1234512345, 123456123456,$$
$$12345671234567, 1234567812345678, 123456789123456789,$$
$$12345678910123456678910, 1234567891011123456789 1011,$$
$$123456789101112123456789101112, \ldots$$

Definition 42. Recurrence Type Sequences:

A. $1, 2, 5, 26, 677, 701, 842, 866, 1517, 458330, 458333, 458354, \ldots$

$SS_2(n)$ is the smallest number, strictly greater than the previous one, which is the squares sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.

Recurrence definition:

1) number a belongs to SS_2;
2) If b, c belong to SS_2, then $b^2 + c^2$ belongs to SS_2, too;
3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to SS_2.

Sequence SS_2 is increasingly ordered. Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 2$, belong to SS_2.

B. $1, 1, 2, 4, 5, 6, 16, 17, 18, 20, 21, 22, 25, 26, 27, 29, 30, 31, 36, 37, 38,$

$40, 41, 42, 43, 45, 46, ...$

$SS_1(n)$ is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the squares sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.

Recurrence definition:

1) number a belongs to SS_1;
2) If $b_1, b_2, ..., b_k$ belong to SS_1, where $k \geq 1$, then $b_1^2 + b_2^2 + ... + b_k^2$ belongs to SS_1, too;
3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to SS_1.

The sequence SS_1 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 1$, belong to SS_1.

C. $1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, ...$

$NSS_2(n)$ is the smallest number, strictly greater than the previous one, which is not the squares sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.

Recurrence definition:

1) numbers $a \leq b$ belong to NSS_2;
2) If b, c belong to NSS_2, then $b^2 + c^2$ does not belong to NSS_2; any other numbers belong to NSS_2;
3) only numbers, obtained by rules 1) and /or 2) applied a finite number of times, belong to NSS_2.

The sequence NSS_2 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to NSS_2.

D. $1, 2, 3, 6, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28,$

$29, 30, 31, 32, 33, 34, 38, 39, 42, 43, 44, 47, ...$

$NSS_1(n)$ is the smallest number, strictly greater than the previous distinct terms of the sequence; in our particular case the first term is 1.

Recurrence definition:

1) number a belongs to NSS_1;
2) If $b_1, b_2, ..., b_k$ belong to NSS_1, where $k \geq 1$, then $b_1^2 + b_2^2 + ... + b_k^2$ do not belong to NSS_1;

3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to NSS_1.

Sequence NSS_1 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 1$, belong to NSS_1.

E. $1, 2, 9, 730, 737, 389017001, 389017008, 389017729, ...$

$CS_2(n)$ is the smallest number, strictly greater than the previous one, which is the cubes sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.

Recurrence definition:

1) numbers $a \leq b$ belongs to CS_2;
2) If c, d belong to CS_2, then $c^3 + d^3$ belongs to cS_2, too;
3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to CS_2.

Sequence CS_2 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to CS_2.

F. $1, 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731,$

$737, 738, 739, 1241, ...$

$CS_1(n)$ is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.

Recurrence definition:

1) numbers $a \leq b$ belongs to CS_1;
2) If $b_1, b_2, ..., b_k$ belong to CS_1, where $k \geq 1$, then $b_1^2 + b_2^3 + ... + b_k^3$ belongs to CS_1;
3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to CS_1.

Sequence CS_1 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 1$, belong to CS_1.

G. $1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$

$21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, ...$

$NCS_2(n)$ is the smallest number, strictly greater than the previous one, which is not the cubes sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.

Recurrence definition:

1) numbers $a \leq b$ belongs to NCS_2;
2) If c, d belong to NCS_2, then $c^3 + d^3$ does not belong to NCS_2; any

other numbers do belong to NCS_2;

3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to CS_2.

Sequence NCS_2 is increasingly ordered. Rule 1) may be changed by the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to NCS_2.

H. 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22,

23, 24, 25, 26, 27, 30, 31, 32, 33, 34, 37, 38, 39, ...

$NCS_1(n)$ is the smallest number, strictly greater than the previous one, which is not the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.

Recurrence definition:

1) number a belongs to NCS_1;

2) if $b_1, b_2, ..., b_k$ belong to NCS_1, where $k \geq 1$, then $b_1^2 + b_2^3 + ... + b_k^3$ do not belong to NCS_1;

3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to NCS_1.

Sequence NCS_1 is increasingly ordered. Rule 1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 1$, belong to NCS_1.

I. A General-Recurrence (Positive) Type Sequence:

General Positive recurrence definition:

Let $k \geq j$ be natural numbers, and $a_1, a_2, ..., a_k$ - given elements, and R - a j-relationship (relation among j elements). Then:

1) elements $a_1, a_2, ..., a_k$ belong to $SGPR$;

2) if $m_1, m_2, ..., m_j$ belong to $SGPR$, then $R(m_1, m_2, ..., m_j)$ belongs to $SGPR$, too;

3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to $SGPR$.

Sequence $SGPR$ is increasingly ordered.

Method of constructing the Smarandache (positive) general recurrence sequence:

- level 1: the given elements $a_1, a_2, ..., a_k$ belong to $SGPR$;
- level 2: apply the relationship R for all combinations of j-elements among $a_1, a_2, ..., a_k$; the results belong to SGRP, too; order all the elements of levels 1 and 2 together;

. . .

- level $i+1$: if $b_1, b_2, ..., b_m$ are all the elements of levels 1, 2, ..., $i-1$, and $c_1, c_2, ..., c_n$ are all the elements of levels i, then apply the relationship R for all combinations of j-elements among $b_1, b_2, ..., b_m, c_1, c_2, ..., c_n$, so

that at least one element is from level i; the results belong to SGRP, too; order all elements of levels i and $i+1$ together;
and so on ...

J. A General-Recurrence (Negative) Type Sequence:
General Negative recurrence definition:

Let $k \geq j$ be natural numbers, and $a_1, a_2, ..., a_k$ - given elements, and R - a j-relationship (relation among j elements). Then:
1) elements $a_1, a_2, ..., a_k$ belong to $SGNR$;
2) if $m_1, m_2, ..., m_j$ belong to $SGNR$, then $R(m_1, m_2, ..., m_j)$ does not belong to $SGNR$;
3) only numbers, obtained by rules 1) and /or 2), applied a finite number of times, belong to $SGNR$.

Sequence $SGNR$ is increasingly ordered.

Method of constructing the Smarandache (negative) general recurrence sequence:
- level 1: the given elements $a_1, a_2, ..., a_k$ belong to $SGNR$;
- level 2: apply the relationship R for all combinations of j elements among $a_1, a_2, ..., a_k$; none of the results belong to SGRP, too; the smallest element, strictly greater than all $a_1, a_2, ..., a_k$, and different from the previous results, belongs to SGNR; order all the elements of levels 1 and 2 together;

. . .

- level $i+1$: if $b_1, b_2, ..., b_m$ are all the elements of levels 1, 2, ..., $i-1$, and $c_1, c_2, ..., c_n$ are all elements of level i, then apply the relationship R for all combinations of j-elements among $b_1, b_2, ..., b_m, c_1, c_2, ..., c_n$, so that at least one element is from the level i; none of the results belong to SGNR; the smallest element, strictly greater than all the previous elements, and different from the previous results, belongs to SGNR; order all the elements of levels i and $i+1$ together;
and so on ...

Problem 15. Partition Type Sequences:

A. $1, 1, 1, 2, 2, 2, 2, 3, 4, 4, ...$

How many times is n written as a sum of non-null squares, disregarding the terms order? For example:

$$\begin{aligned} 9 &= 1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2 \\ &= 1^2+1^2+1^2+1^2+1^2+2^2 \\ &= 1^2+2^2+2^2 \\ &= 3^2, \end{aligned}$$

therefore $ns(9) = 4$.

B. $1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3,$

$4, 4, 4, 5, 5, 5, 5, 5, 6, 6 ...$

How many times is n written as a sum of non-null cubes, disregarding the terms order? For example:

$$\begin{aligned} 9 &= 1^3+1^3+1^3+1^3+1^3+1^3+1^3+1^3+1^3 \\ &= 1^3+2^3 \end{aligned}$$

therefore $nc(9) = 2$.

C. A General-Partition Type Sequence

How many times can n be written in the form of

$$n = R(f(n_1), f(n_2), ..., f(n_k))$$

for some k and $n_1, n_2, ..., n_k$ between 1 and n?

Particular cases: when $f(x) = x^2$, or x^3, or $x!$, or x^x, etc. and relation R is the obvious addition of numbers, or multiplication, etc.

Reference: [70].

Definition 43. A Non-Geometric Progression:

$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 26, 27, 29, 30,$

$31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 50, 51, 53, ...$

General definition: if m_1, m_2 are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number, such that no 3-term geometric progression is in the sequence. In our case the first two terms are 1, respectively 2.

Generalization: if $m_1, m_2, ..., m_{i-1}$ are the first $i-1$ terms of the sequence, $i \geq 3$, then m_k, for $k \geq u$, is the smallest number, such that no i-term geometric progression is in the sequence.

Definition 44. A Unary Sequence:

$11, 111, 11111, 1111111, 11111111111, 1111111111111, 11111111111111111,$

$1111111111111111111, 11111111111111111111111, 11111111111111111111111111111, ...$

$u(n) = \overline{11...1}$, p_n digits of "1", where p_n is the n-th prime.

Is there an infinite number of primes belonging to the sequence?

Definition 45. Let $V_1, V_2, ..., V_n$ be an n–sided polygon, $n \geq 3$, and S be a given set of m elements, $m \geq n$. In each vertex V_i, $1 \leq i \leq n$, of the polygon one puts at most one element of S, and on each side s_i, $1 \leq i \leq n$, of the polygon one puts $e_i \geq 0$ elements from S.

Let R be a given relationship of two or more elements of S. All elements of S should be put on the polygon's sides and vertices so that the repationship R on each side gives the same result. What connections can be found existing among the number of the sides of the polygon, set S (what elements and how many), and the relation R in order for this problem to have solution(s)?

Definition 46.

I) Smarandache Polyhedrons (With Edge Points): similar definition generalized in the 3-dimensional space: the points are put on the vertices, and edges only of the polyhedron (not on inside faces).

II) Smarandache Polyhedrons (With Face Points): similar definition generalized in the 3-dimensional space: the points are put on the vertices, and inside faces of the polyhedron (not on the edges).

III) Smarandache Polyhedrons (With Edge/Face Points): similar definition generalized in the 3-dimensional space: the points are put on the vertices, on edges, and on inside faces of the polyhedron.

IV) Smarandache Polyhedrons (With Edge Points): similar definition generalized in the 3-dimensional space: the points are put on the edges, and inside faces only of the polyhedron (not in verteces).

What connections can be found existing among the number of vertices/faces/edges of the polyhedrons, set S (what elements and how many), and the relation R in order for this problem to have solution(s)?

Examples: I

a) For an equilateral triangle, the set $\mathcal{N}_6 = \{1, 2, 3, 4, 5, 6\}$, the addition, and on each side to have three elements exactly, and in each vertex an element, one gets:

$$\begin{array}{ccccc} & & 1 & & \\ & 6 & & 5 & \\ 2 & & 4 & & 3 \end{array}$$

the sum of each of the three sides is 9 [the minimum elements 1,2,3 are put in the vertices];

$$\begin{array}{ccccc} & & 1 & & \\ & 6 & & 5 & \\ 2 & & 4 & & 3 \end{array}$$

the sum of each of the three sides is 12 [the maximal elements 6,5,4 are put in the vertices].

There are no other possible combinations to keep (the sum constant).

Therefore: The Triangular Index $SGI(3) = (9, 12; 2)$, means: the minimum value, the maximum value, the total number of combinations, respectively.

b) For a square, the set $\mathcal{N}_{12} = \{1, 2, 3, 4, 5, 6, ..., 12\}$, the addition, and on each side to have four elements exactly, and in each vertex an element. What is the Square Index $SGI(4)$?

c) For a regular pentagon, the set $\mathcal{N}_{20} = \{1, 2, 3, 4, 5, 6, ..., 20\}$, the addition, and on each side to have five elements exactly, and in each vertex an element. What is the Pentagonal Index $SGI(5)$?

d) For a regular hexagon, the set $\mathcal{N}_{30} = \{1, 2, 3, 4, 5, 6, ..., 30\}$, the addition, and on each side to have six elements exactly, and in each vertex an element. What is the Hexagonal Index $SGI(6)$?

e) Generally speaking: For a $n-$sided regular polygon, the set $\mathcal{N}_{n^2-n}$ $= \{1, 2, 3, 4, 5, 6, ..., n^2 - n\}$, the addition, and on each side to have n elements exactly. What is the $n-$Sided Regular Polygonal Index $SGI(n)$?

Examples: II

a) For a regular tetrahedron (4 vertices, 4 triangular faces, 6 edges), the set $\mathcal{N}_{10} = \{1, 2, 3, 4, 5, 6, ..., 10\}$, the addition, and on each edge to have three elements exactly (in each vertex one element) – to get the same sum.

b) For a regular hexahedron (8 vertices, 6 square faces, 12 edges), the set $\mathcal{N}_{32} = \{1, 2, 3, ..., 32\}$, the addition, and on each edge to have four elements exactly (in each vertex one element) – to get the same sum.

c) For a regular octahedron (6 vertices, 8 triangular faces, 12 edges), the set $\mathcal{N}_{18} = \{1, 2, 3, ..., 18\}$, the addition, and on each edge to have three elements exactly (in each vertex one element) – to get the same sum.

d) For a regular dodecahedron (14 vertices, 12 square faces, 24 edges), the set $\mathcal{N}_{62} = \{1, 2, 3, ..., 62\}$, the addition, and on each edge to have four elements exactly (in each vertex one element) – to get the same sum.

Compute the Polyhedron Index $SHI(v, e, f) = (m, M; nc)$ in each case, where v is the number of the polyhedron verteces, e is the number of edges, f is the number of faces, m is the minimum sum value and M is the maximum sum value, nc is the total number of combinations.

e) Or other versions for the previous particular polyhedrons:

- putting elements on the inside faces and vertices only (not on the edges);
- or putting elements on the inside faces, edges and vertices, too;

- or putting elements on the inside faces and edges only (not on the vertices).

References: [13, 15, 16, 38, 40, 42, 44, 52, 69, 92].

Definition 47. An Add-on Sequence:

Let $G = \{g_1, g_2, ..., g_k, ...\}$ be an ordered set of positive integers with a given property $\mathcal{G}$. Then the corresponding Smarandache G Add-On Sequence is defined through

$$SG = \{a_i \mid a_1 = g_1, g_k = a_{k-1} 10^{1+\log_{10}(g_k)} + g_k,\ k \geq 1\}.$$

This definition is a generalization of Definitions 17, 18 and 19.

Definition 48. Concatenation Type Sequences

Let $s_1, s_2, ..., s_n, ...$ be an infinite integer sequence (noted by S). Then the concatenation is defined as

$$s_1, \overline{s_1 s_2}, \overline{s_1 s_2 s_3}, ...$$

Problem 16. How many terms of this concatenated S-sequence belong to the initial s-sequence.

Definition 49. Partition Type Sequences

Let f be an arithmetic function, and R a k-relation among numbers. How many times can n be expressed in the form of;

$$n = R(f(n_1), f(n_2), ..., f(n_k)),$$

for some k and $n_1, n_2, ..., n_k$, so that $n_1 + n_2 + ... + n_k = n$?

Problem 17. How many times can n be expressed as a sum of non-null squares (or cubes, or m-powers)?

Definition 50. Construction of Elements of the Square-Partial-Digital Subsequence

The Square-partial-digital subsequence (SSPDS) is the sequence of square integers which admit a partition for which each segment is a square integer. An example is $506^2 = 256036$, which has partition 256 / 0 / 36. C. Ashbacher showed that SSPDS is infinite by exhibiting two infinite families of elements. We will extend his results by showing how to construct infinite families of elements of a SSPDS containing the desired patterns of digits.

Problem 18. 441 belongs to the SSPDS, and its square $441^2 = 194481$ also belongs to the SSPDS. Can another example of integers m, m^2, m^4 all belonging to SSPDS be found?

Problem 19. It is relatively easy to find two consequtive squares in SSPDS, e.g., $12^2 = 144$ and $13^2 = 169$. Does the SSPDS contain three or more consequtive squares as well? What is the maximum length?

Problem 20. How many primes are there in the expression: $x^y + y^x$, where $gcd(x, y) = 1$? K. Kashihara announced that there are only finitely many numbers of the above form, which are products of factorials. F. Luca proved the following conjecture:

Let a, b, and c be three integers with $ab \neq 0$. Then the equation $ax^y + by^x = cz^n$, with $x, y, z \geq 2$, and $gcd(x, y) = 1$, has finitely many solutions (x, y, z, n).

Problem 21. J. Castillo asked in [21] how many primes are there in the expression: $x_1^{x_2} + x_2^{x_3} + \ldots + x_n^{x_1}$, where $n > 1, x_1, x_2, \ldots, x_n > 1$, and $gcd(x_1, x_2, \ldots, x_n) = 1$?

This is a generalization of Problem 43 called "Smarandache expression" in [21]. F. Luca announced a lower bound for the size of the largest prime divisor of an expression of type $ax^y + by^x$, where $ab \neq 0$, $x, y \geq 2$, and $gcd(x, y) = 1$.

The following three problems are proved by Henry Ibstedt in [41]

Theorem 31. Subtraction Periodic Sequences

Let c be a positive integer. Start with the positive integer N and let N' be its digital reverse. Put $N_1 = |N' - c|$, and let N_1' be its digital reverse. Put $N_2 = |_1N' - c|$, and let N_2' be its digital reverse. And so on. We shall obtain eventually a repetition. For example, with $c = 1$ and $N = 52$ we obtain the sequence:

$$52, 24, 41, 13, 30, 02, 19, 90, 08, 79, 96, 68, 85, 57, 74, 46, 63, 35, 52, \ldots$$

Here a repetition occurs after 18 steps, and the length of the repeating cycle is 18.

First example: $c = 1$, $10 \leq N \leq 999$.

Every other member of this interval is an entry point into one of five cyclic periodic sequences (four of these are of the length of 18, and one of the length of 9). When N is of the form of $11k$ or $11k-1$, then the iteration process results in 0.

Second example: $1 \leq c \leq 9$, $100 \leq N \leq 999$. For $c = 1, 2$ or 5 all iterations result in the invariant 0 sometimes, after, a large number of iterations. For the other values of c there are only eight different possible values for the length of the loops, namely 11,22,33,50,100,167,189,200. For $c = 7$ and $N = 109$ we have an example of the longest loop obtained: it has 200 elements, and the loop closes after 286 iterations. (H. Ibstedt)

Theorem 32. Multiplication Periodic Sequences

Let $c > 1$ be a positive integer. Start with the positive integer N, multiply each digit x of N by c and replace that digit by the last digit of cx to give N_1. And so on. We shall obtain eventually a repetition. For example, with $c = 7$ and $N = 68$ we obtain the sequence:

$$68, 26, 42, 8468, \ldots$$

Integers which digits are all equal to 5 are invariant under the given operation after one iteration.

If $c = 2$, there are four term loops, starting with the first or second term. If $c = 3$, there are four term loops, starting with the first term. If $c = 4$, there are four term loops, starting with the first or second term. If $c = 4$ or 5, the sequence is invariant after one iteration. If $c = 7$, there are four term loops, starting with the first term. If $c = 8$, there are four term loops, starting with the second term. If $c = 9$, there are two loops, starting with the first term.

Theorem 33. Mixed Composition Periodic Sequences

Let N be a two-digit number. Add the digits, and if the sum is greater than 10 add them again. Also take the absolute value of their difference. These are the first and second digits of N_1. Now repeat this. For example, with $N = 75$ we obtain the sequence:

$$75, 32, 51, 64, 12, 31, 42, 62, 84, 34, 71, 86, 52, 73, 14, 53, 82, 16, 75, \ldots$$

There are no invariants in this case. Four numbers: 36, 90, 93, and 99 produce two-element loops. The longest loops have 18 elements. There are also loops of 4, 6, and 12 elements. (H. Ibstedt)

There will always be a periodic (invariant) sequence whenever we have a function $f : S \to S$, where S is a finite set, and we repeat the function

f more times than $card(S)$. Thus the General periodic sequence is defined as:
$a_1 = f(s)$, where s is an element of S;
$a_2 = f(a_1) = f(f(s))$; $a_3 = f(a_2) = f(f(a_1)) = f(f(f(s)))$

Definition 51. Erdős-Smarandache Numbers:

$$2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 20, 21, 22, 23, 26, 28, 29, 30,$$

$$31, 33, 34, 35, \ldots$$

Solutions of the diophantine equation $P(n) = S(n)$, where $P(n)$ is the largest prime factor which divides n, and $S(n)$ is the Smarandache function.

References: [36, 68]

Problem 22. A Square Product Sequence:

$$2, 5, 27, 577, 14401, 518401, 25401601, 1625702401, 131681894401,$$

$$13168189440001, 1593350922240001, \ldots$$

$S_n = 1 + s_1 s_2 \ldots s_n$, where s_k is the k-th square number. How many of them are prime?

Problem 23. A Cubic Product Sequence:

$$2, 9, 217, 13825, 1728001, 373248001, 128024064001, 65548320768001, \ldots$$

$C_n = 1 + c_1 c_2 \ldots c_n$, where c_k is the k-th cubic number. How many of them are prime?

Problem 24. A Factorial Product Sequence:

$$2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, \ldots$$

$F_n = 1 + f_1 f_2 \ldots f_n$, where f_k is the k-th factorial number. How many of them are prime?

Problem 25. How many solutions have the equalities:

$$S(n+1) + S(n+2) = S(n+3) + S(n+4)?$$

M. Bencze found the solutions $n = 5, 6, 27$;

$$S(n+1) - S(n+2) = S(n+3) - S(n+4)?$$

M. Bencze found the solutions $n = 0, 1, 48$;

$$S(n+1) + S(n+2) + S(n+3) = S(n+4) + S(n+5) + S(n+6)?$$

M. Bencze found the solution $n = 4$.

Problem 25. Continued Fractions
If we consider the consecutive sequence

$$1, 12, 123, 1234, 12345, \ldots$$

we form a simple continued fraction

$$1 + \cfrac{1}{12 + \cfrac{1}{123 + \cfrac{1}{1234 + \cfrac{1}{12345 + \ldots}}}}.$$

If we consider the reverse sequence

$$1, 21, 321, 4321, 54321, \ldots$$

to the previous one we get a general continued fraction

$$1 + \cfrac{1}{12 + \cfrac{21}{123 + \cfrac{321}{1234 + \cfrac{4321}{12345 + \ldots}}}}.$$

Calculate each of the above continued fractions. The previous example is convergent (Dodge [31]) but what about the second?
Reference: [22, 23, 31, 87]

Problem 27. Palindromic Numbers and Iterations: A number is called *palindromic* if it reads the same forwards and backwards. For example: 121, 34566543, 1111. The Pseudo-Smarandache function $Z(n)$ is defined for any $n \geq 1$ as the smallest integer m, such that n divides evenly $1 + 2 + \ldots + m$. There are some palindromic numbers n such that $Z(n)$ is also palindromic $Z(909) = 404, Z(2222) = 1111$. Let $Z^k(n) =$

$Z(Z(Z(...(n)...)))$, where function Z is executed k times. $Z^0(n)$ is, by convention, n.

Unsolved problem: What is the largest value of m, such that for some n, $Z^k(n)$s is a palindrome for all $k = 0, 1, 2, ..., m$?

Conjecture (Ashbacher): There is no largest value of m, such that for some n, $Z^k(n)$s is a palindrome for all $k = 0, 1, 2, ..., m$?

Reference: [43]

Problem 28. Some upper bounds for the Smarandache function average: Let $\overline{S} = \frac{1}{n}.(S(1) + S(2) + ... + S(n))$ be the Smarandache function average. S. Tabirca and T. Tabirca proved that

$$\overline{S} \leq \frac{3}{8}.n + \frac{1}{4} + \frac{2}{n}, \text{ for } n > 5;$$

$$\overline{S} \leq \frac{21}{72}.n + \frac{1}{12} - \frac{2}{n}, \text{ for } n > 23$$

They conjectured that $\overline{S} = \frac{2n}{\ln n}$, for $n > 1$.

The following three theorems are proved by F. Smarandache in [75].

Theorem 34. On Concurrent Lines: If a polygon with n sides ($n \geq 4$) is circumscribed to a circle, then there are at least three concurrent lines among the polygon's diagonals and the lines which join tangential points of two non-adjacent sides.

Theorem 35. On Cevians Theorem: Let AA', BB', CC' be three concurrent cevians (lines), in the point P, in the triangle ABC. Then

$$\frac{PA}{PA'} + \frac{PB}{PB'} + \frac{PC}{PC'} \geq 6,$$

and

$$\frac{PA}{PA'} \cdot \frac{PB}{PB'} \cdot \frac{PC}{PC'} = \frac{BA}{BA'} \cdot \frac{CB}{CB'} \cdot \frac{AC}{AC'} \geq 8.$$

Theorem 36. On Podaire Theorem: Let AA', BB', CC' be the altitudes of the triangle ABC. Note $AB = c, BC = a, CA = b$, and $A'B' = c', B'C' = a', C'A' = b'$. Then

$$a'b' + b'c' + c'a' \leq \frac{1}{4}(a^2 + b^2 + c^2).$$

Problem 29. Let p be a positive prime and $s \geq 2$ be an integer. Then $Z(p^s) = p^{s+1} - 1$, if p is even; or $Z(p^s) = p^{s-1} - 1$, if p is odd. The solution set of the diophantine equation $Z(x) = 8$ is $\{9, 12, 18, 36\}$. For any positive integer n the diophantine equation $Z(x) = n$ has solutions.
Unsolved Problem 1: The diophantine equation $Z(x) = Z(x+1)$ has no solutions.
Unsolved Problem 2: For any given positive number r there exists an integer s, such that the absolute value of $Z(s) - Z(s+1)$ is greater than r.

Reference: [43]

Definition 52. Smarandache similar triangles: Let us denote by $T(a, b, c)$ the triangle ABC with side lengths of a, b, c. Then the two similar triangles $T(a, b, c)$ and $T(a', b', c')$ are said to be Smarandache similar ones if $S(a) = S(a'), S(b) = S(b'), S(c) = S(c')$.

Reference: [65]

Definition 53. Smarandache Related Triangles: Unsolved problems 1), 2), 4) are solved by Charles Ashbacher in [5]. Being given a triangle $T(a, b, c)$, we say that a triangle $T(a', b', c')$ is Smarandache related to T if $S(a) = S(a'), S(b) = S(b'), S(c) = S(c')$. Note that triangles T and T' may or may not be similar.

The following four problems are interested and Problems 1), 2), 4) are solved by Charles Ashbacher in [5].
Problem 1: Are there two distinct dissimilar Phytagorean triangles that are Smarandache related, i.e., both $T(a, b, c)$ and $T(a', b', c')$ are Phytagorean, so that $S(a) = S(a'), S(b) = S(b'), S(c) = S(c')$, but not being similar.
Problem 2: Are there two distinct and dissimilar 60^o–triangles or 120^o–triangles that are Smarandache related?
Problem 3: Let triangle $T(a, b, c)$ be given. Is it possible to give either an exact formula or an upper bound for the total number of triangles (without actually determining all of them) that are Smarandache related to T?
Problem 4: Consider other ways of relating two triangles in the Smarandache number sense. For example, are three two triples of natural numbers (a, b, c) and (a', b', c'), such that $a + b + c = a' + b' + c' = 180^o$ and $S(a) = S(a'), S(b) = S(b'), S(c) = S(c')$. If such distinct triples exist, the two triangles would be Smarandache related via their angles. Of course, in this relationship the side lengths of the triangles may not be natural numbers.

Reference: [65]

The following four definitions are introduced in F. Smarandache's paper "Paradoxist Mathematics" (see pages 5-28 from [82]).

Definition 54. Paradoxist Geometry: A new type of geometry was constructed by F. Smarandache (see [57]) in 1969 simultaneously in a partially Euclidean and partially non-Euclidean space by replacing the Euclid's fifth postulate (axiom of parallels) with the following five-statement propositions:
a) there are at least one straight line and one point exterior to it in the space, for which only one line passes through the point and does not intersect the initial line [1 parallel];
b) there are at least one straight line and one point exterior to it in the space, for which only a finite number of lines $l_1, l_2, ..., l_k$ ($k \geq 2$) pass through the point and do not intersect the initial line [2 or more (in a finite number) parallels];
c) there are at least one straight line and one point exterior to it in the space, for which any line that passes through the point intersects the initial line [0 parallel];
d) there are at least one straight line and one point exterior to it in the space, for which an infinite number of lines that pass through the point (but not all of them) do not intersect the initial line [an infinite number of parallels, but not all lines passing through];
e) there are at least one straight line and one point exterior to it in the space, for which any line that passes through the point does not intersect the initial line [an infinite number of parallels, all lines passing throught the point].

Definition 55. Non-Geometry: A curious geometry was constructed by F. Smarandache (see [26]) in 1969 with the following "axioms":
1. It is not always possible to draw a line from an arbitrary point to another arbitrary point.
2. It is not always possible to extend by continuity a finite line to an infinite line.
3. It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.
4. Not all the right angles are congruent.
5. If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angles, then not always the two lines extended towards infinity cut each other in the side where the angles are strictly less than two right angles.

Definition 56. Counter-Projective Geometry This type of geometry has been constructed by F. Smarandache (see [17]) in 1969.

Let P, L be two sets, and r a relation included in $P \times L$. The elements of P are called points, and those of L lines. When (p, l) belongs to r, we say that line l contains point p. For these, one imposes the following counter-axioms:

1) There exist either at least two lines, or no line, that contains two given distinct points.

2) Let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1, q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ and collinear triples. Then the line containing p_1, p_2, and the line containing q_1, q_2 are not intersecting each other.

3) Every line contains at most two distinct points.

Definition 57. Anti-Geometry

It is possible to deformalize entirely Hilbert's groups of axioms of the Euclidean Geometry, and to construct a model, such that none of its fixed axioms holds. Let us consider the following things:

- a set of <points>: $A, B, C, \ldots$
- a set of <lines>: $h, k, l, \ldots$
- a set of <planes>: α, β, γ, ... and
- a set of relationships among these elements: "are situated", "between", "parallel", "congruent", "continuous", etc. Then, we can deny all Hilbert's twenty axioms [see David Hilbert, "Foundations of Geometry", translated by J.Townsend, 1950; and Roberto Bolona, "Non-Euclidean Geometry",1938]. There exist cases, within a geometric model, whem the same axiom is verified by certain points/lines/planes and denied by others.

GROUP I. ANTI-AXIOMS OF CONNECTION:

I.1. Two distinct points of A and B do not always completely determine a line.

Let us consider the following model **MD:**

Get an ordinary plane δ, but with an infinite hole in being of the following shape:

```
                            P
                            .
                           . .
p                         .   .
l semi-plane delta 1     .     .   curve f1 (frontier)
a ...................            .....................
n
e                                   . I       . J

d
e                                              . K
l ...................            .....................
t semi-plane delta 2     .     .   curve f2 (frontier)
a                         .   .
                           . .
                            .
                            Q
```

Plane δ is a reunion of two disjoint planar semi-planes; $f1$ lies in MD, but f2 does not; P, Q are two extreme points on f that belong to MD.

One defines a LINE l as a geodesic curve if two points A, B that belong to MD lie also in l. If a line passes two times through the same point, then it is called a double point (KNOT).

One defines a PLANE α as a surface, such that for any two points A, B that lie in α and belong to MD there is a geodesic curve which passes through A, B and lies also in α.

Now, let us have two strings of the same length: one ties P and Q with the first string s_1, so that curve s_1 is folded in two or more different planes and s_1 is the plane δ. Next, do the same with string s_2, tie Q with P, but over plane δ, and so that s_2 has a different form from s_1; and a third string s_3, from P to Q, much longer than s_1. Then, s_1, s_2, s_3 belong to MD.

Let I, J, K be three isolated points – as some islands, i.e. not joined with any other point of MD, exterior to plane δ.

Unsolved problem: Of course, this model is not perfect, and it is far from the best. Readers are asked to improve it, or to make up a new one that is better.

Let A, B be two distinct points in δ_{1-f_1}. P and Q are two points on s_1, but they do not completely determine a line, referring to the first axiom of Hilbert, because $A - P - s_1 - Q$ are different from $B - P - s_1 - Q$.

I.2. There is at least one line l and at least two distinct points A and B of l, such that A and B do not completely determine line l.

Line $A - P - s_1 - Q$ is not completely determined by P and Q in the previous construction, because $B - P - s_1 - Q$ is another line passing through P and Q.

I.3. Three points A, B, C not situated in the same line do not always completely determine a plane α.

Let A, B be two distinct points in δ_{1-f_1}, such that A, B, P are not colinear. There are many planes containing these three points: δ_1 extended with any surface s containing s_1, but not cutting s_2 in between P and Q, for example.

I.4. There is at least one plane, α, and at least three points A, B, C in it not lying in the same line, such that A, B, C do not completely determine the plane α. (See the previous example.)

I.5. If two points A, B of a line l lie in a plane α, it does not mean that every point of l lies in α.

Let A be a point in δ_{1-f_1}, and B - another point on s_1 between P and Q. Let α be the following plane: δ_1 extended with a surface s containing s_1, but not cutting s_2 between P and Q, and tangent to δ_2 on a line QC, where C is a point in δ_{2-f_2}. Let D be a point in δ_{2-f_2}, not lying on the line QC. Now, A, B, D are lying on the same line $A - P - s_1 - Q - D$, A, B are in plane α, but D is not.

I.6. If two planes α, β have a point A in common, does not mean that they have at least one second point in common.

Construct the following plane α: a closed surface containing s_1 and s_2, and intersecting δ_1 in one point only, P. Then α and δ_1 have a single point in common.

I.7. There exist lines, where only one point lies, or planes, where only two points lie, or space where only three points lie.

Hilbert's I.7 axiom may be contradicted if the model has discontinuities. Let us consider the isolated points area.

Point I may be regarded as a line, because it is not possible to add any new point to I to form a line.

One constructs a surface that intersects the model only in points I and J.

GROUP II. ANTI-AXIOMS OF ORDER

II.1 If A, B, C are points of a line and B lies between A and C, it does not mean that B always lies between C and A, as well.

Let T lies in s_1, and V lies in s_2, both of them closer to Q, but different from it. Then: P, V, T are points on line $P - s_1 - Q - s_2 - P$, i.e. the closed curve that starts from point P and lies in s_1 and passes through point Q and lies back to s_2 and ends in P), and T lies between P and V

– because PT and TV are both geodesic curves, but T does not lie between V and P
– because from V the line goes to P and then to T, therefore, P lies between V and T.

By definition: segment AB is a system of points lying upon a line between A and B the extremes are included.

Warning: AB may be different from BA; for example: the segment PQ formed by the system of points starting with P, ending with Q, and lying in s_1, is different from the segment QP formed by the system of points starting with Q, ending with P, but belonging to s_2.

Moreover, AB may sometimes be different from AB; for example: the segment PQ formed by the system of points starting with P, ending with Q, and lying in s_1, is different from the segment PQ formed by the system of points starting with P, ending with Q, but belonging to s_2.

II.2. If A and C are two points of a line, then: there does not always exist a point B lying between A and C, or there does not always exist a point D such that C lies between A and D.

For example: let F be a point on f_1, F being different from P, and G a point in δ_1, G does not belong to f_1; draw the line l which passes through G and F; then: there exists a point B lying between G and F
– because GF is an obvious segment, but there is no point D, such that F lies between G and D;
– because GF is right bounded in F (GF may not be extended to the other side of F, because otherwise the line will not remain a geodesic curve one anymore).

II.3. There exist at least three points situated on a line, so that: one point lies between the other two, and another point lies between the other two, as well.

For example: let R, T be two distinct points, different from P and Q, situated on line $P - s_1 - Q - s_2 - P$, such that the lengths of PR, RT, TP are all equal; then: R lies between P and T, and T lies between R and P, and P lies between T and R, as well.

II.4. Four points A, B, C, D of a line cannot always be arranged: so that B lies between A and C and also A between D, and such that C lies between A and D and also between B and D.

For example: let T and R are two dictinct points, different from P and Q, situated on line $P - s_1 - Q - s_2 - P$, so that the lengths of PR, RQ, QT and TP are all equal, therefore, R belongs to s_1, and T belongs to s_2; then P, R, Q, T are situated on the same line:

so that R lies between P and Q, but not between P and T
– because the geodesic curve PT does not pass through R,

and so that Q does not lie between P and T
– because the geodesic curve PT does not pass through Q,
but lies between R and T;

- let A and B be two points in δ_{2-f_2}, such that A, Q , B are colinear, and let C, D be two points on s_1, s_2 respectively, all of the four points being different from P and Q; then A, B, C, D are points situated on the same line as $A{-}Q{-}s_1{-}P{-}s_2{-}B$, which is the same $A{-}Q{-}s_2{-}P{-}s_1{-}B$, therefore, we may have two different orders of these four points in the same time: A, C, D, B and A, D, C, B.

II.5. Let A, B, C be three points not lying in the same line, and let l be a line lying in the same plane ABC, not passing through any of the points A, B, C. Then, if line l passes through a point of segment AB, it does not mean that line l will always pass either through a point of segment BC, or through a point of segment AC.

For example: let AB be a segment passing through P in the semi-plane δ_1, and let C be a point lying in δ_1, too, on the left side of line AB; thus A, B, C do not lie on the same line; now consider line $Q-s_2-P-s_1-Q-D$, where D is a point, lying in semi-plane δ_1 but not on f_2: therefore, this line passes through point P of the segment AB, but it passes neuther through a point of the segment BC, nor through a point of the segment AC.

GROUP III. ANTI-AXIOM OF PARALLELS.

In a plane α through a point A, lying outside of line l, either no line, or only one line, or a finite number of lines can be drawn (At least two of these situations should occur.) The line(s) is (are) called parallel(s) to l through the given point a.

For example:

- let l_0 be the line $N - P - s_1 - Q - R$, where N is a point lying in δ_1 not on f_1, and R is a similar point lying in δ_2 not on f_2, and let A be a point lying on s_2, then: no parallel to l_0 can be drawn through A (since any line passing through A, hence, through s_2, will intersect s_1, hence, l_0 in P and Q);

- if line l_1 lies in δ_1, so that l_1 does not intersect frontier f_1, then: through any point lying on the left side of l_1 one and only one parallel will pass;

- let B be a point lying in f_1, different from P, and another point C lying in δ_1, not on f_1; let A be a point lying in δ_1 outside of BC; then: an infinite number of parallels to line BC can be drawn through point A.

Theorem: There are at least two lines l_1, l_2 of a plane, which do not meet a third line l_3 of the same plane, but they meet each other, (i.e. if l_1 is parallel to l_3, and all of them are in the same plane it is not necessary that l_1 is parallel to l_2).

For example: consider three points A, B, C lying in f_1, and different from P, and D a point in δ_1 not on f_1; draw lines AD, BE and CE, so that E is a point in δ_1 not on f_1 and both BE and CE do not intersect AD; then: BE is parallel to AD, CE is also parallel to AD, but BE is not parallel to CE because point E belongs to both of them.]

GROUP IV. ANTI-AXIOMS OF CONGRUENCE

IV.1. If A, B are two points on a line l, and A' is a point upon the same or another line l', then: upon a given side of A' on the line l', we cannot always find only one point B' so that segment AB is congruent to segment $A'B'$.

For example:

- let AB be a segment lying in δ_1 and having no point in common with f_1, and let us construct line $C - P - s_1 - Q - s_2 - P$ (noted by l' which is the same with $C - P - s_2 - Q - s_1 - P$, where C is a point lying in δ_1 neither in f_1, nor on AB; take a point A' on l', between C and P, so that $A'P$ is smaller than AB; now, there exist two distinct points B' on s_2, such that $A'B1'$ is congruent to AB, with $A'B1$ different from $A'B2$;

- but if we consider a line l' lying in δ_1 and limited by the frontier f_1 on the right side (the limit point being noted by N), and take a point A' on l', close to M, such that $A'M$ is less than $A'B'$, then: there is no point B' on the right side of l', so that $A'B'$ is congruent to AB.

A segment may not be congruent to itself! For example: let A be a point on s_1, closer to P also; A and B are lying on the same line $A - Q - B - P - A$, which is the same as line $A - P - B - Q - A$, but AB measured on the first representation of the line is strictly greater than AB measured on the second representation of their line.

IV.2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A"B"$, then segment $A'B'$ is not always congruent to $A"B"$.

For example: let AB be a segment lying in δ_{1-f_1}, and consider the line $C - P - s_1 - Q - s_2 - P - D$, where C, D are two distinct points in δ_{1-f-1}, such that C, P, D are colinear. Suppose that segment AB is congruent to segment CD (i.e. $C - P - s_1 - Q - s_2 - P - D$). Get also an obvious segment $A'B'$ in δ_{1-f_1}, different from the preceding ones, but congruent to AB. Then, segment $A'B'$ is not congruent to segment CD (considered as $C - P - D$, i.e. not passing through Q.)

IV.3. If AB, BC are two segments of the same line l, which have no points in common aside from point B, and $A'B'$, $B'C'$ are two segments of the same line or of another line l', having no point in common aside from B', so that AB is congruent to $A'B'$ and BC is congruent to $B'C'$, then segment AC is not always congruent to $A'C'$.

For example: let l be a line lying in δ_1, not on f_1, and let A, B, C be three distinct points on l, such that AC is greater than s_1; let l' be the following line: $A' - P - s_1 - Q - s_2 - P$, where A' lies in δ_1, not on f_1, and get B' on s_1 so that $A'B'$ is congruent to AB, get C' on s_2, so that BC is congruent to $B'C'$ (points A, B, C are thus chosen); then: segment $A'C'$ which is firstly seen as $A' - P - B' - Q - C'$ is not congruent to AC, because $A'C'$ is the geodesic curve $A' - P - C'$ (the shortest way from A' to C' does not pass through B') which is strictly less than AC.

Definitions: Let h, k be two lines having a point O in common. Then system $(h.O, k)$ is called the angle of lines h and k in point O. (Since some of our lines are curves, we take the angle of the tangents to the curves in their common point.)

The angle, formed by lines h and k situated in the same plane, noted by $\angle(h, k)$, is equal to the arithmetic mean of the angles formed by h and k in all their common points.

IV.4. Let an angle (h, k) be given in the plane α, and let a line h' be given in the plane β. Suppose that in plane β a definite side of line h' be assigned, and a point O'. Then in plane β there are one, or more, or even no half-line(s) k' emanating from point O', so that the angle (h, k) is congruent to angle (h', k'), and at the same time the interior points of angle (h', k') lie upon one or the both sides of h'.

Examples:

- Let A be a point in δ_{1-f_1}, and B, C two distinct points in δ_{2-f_2}; let h be the line $A - P - s_1 - Q - B$, and k be the line $A - P - s_2 - Q - C$; since h and k intersect in an infinite number of points (the segment AP), where they normally coincide – i.e. in each point at this kind their angle is congruent to zero, angle (h, k) is congruent to zero. Now, let A' be a point in δ_{1-f_1}, different from A and B' a point in δ_{2-f_2}, different from B, and draw the line h' as $A' - P - s_1 - Q - B$; there exist an infinite number of lines k', of the form $A' - P - s_2 - Q - C'$ (where C' is any point in δ_{2-f_2}, not in the line QB'), such that angle (h, k) is congruent to (h', k'), because (h', k') is also congruent to zero, and the line $A' - P - s_2 - Q - D'$ if D' is not on the line QC'.
- if h, k, and h' are three lines in δ_{1-P}, which intersect the frontier f_1 in one point at most, then there exists only one line k' on a given part of h', such that angle (h, k) is congruent to the angle (h', k').
- Is there any case when, with these hypotheses, no k' exists?
- Not every angle is congruent to itself; for example: $\angle(s_1, s_2)$ is not congruent to $\angle(s_1, s_2)$ [because one can construct two distinct lines: $P - s_1 - Q - A$ and $P - s_2 - Q - A$, where A is a point in δ_{2-f_2}, for the first angle, which

becomes equal to zero; and $P - s_1 - Q - A$ and $P - s_2 - Q - B$, where B is another point in δ_{2-f_2}, B different from A, for the second angle, which becomes strictly greater than zero!

IV.5. If angle (h, k) is congruent to angle (h', k') and angle $(h", k")$, then angle (h', k') is not always congruent to angle $(h", k")$. (A similar construction to the previous one.)

IV.6. Let ABC and $A'B'C'$ be two different triangles, so that
AB is congruent to $A'B'$,
AC is congruent to $A'C'$,
$\angle BAC$ is congruent to $\angle B'A'C'$.

Then, not always $\angle ABC$ is congruent to $\angle A'B'C'$ and $\angle ACB$ is congruent to $\angle A'C'B'$.

For example: let M, N be two distinct points in δ_{2-f_2}, thus obtaining the triangle PMN; Now take three points R, M', N' in δ_{1-f_1}, such that RM' is congruent to PM, RN' is congruent to RN, and angle (RM', RN') is congruent to angle (PM, PN). $RM'N'$ is an obvious triangle. Of course both the triangles are not congruent, because for instance, PM and PN cut each other twice - in P and Q - while RM' and RN' only once - in R. (These are geodesical triangles.)

Definitions: Two angles are called supplementary if they have the same vertex, one side in common, and the other not common sides form a line.

A right angle is the angle, congruent to its supplementary angle.

Two triangles are congruent if their angles are congruent two by two, and their sides are congruent two by two.

Propositions: A right angle is not always congruent to another right angle.

For example: let $A - P - s_1 - Q$ be a line, with A lying in δ_{1-f_1}, and $B - P - s_1 - Q$ another line, with B lying in δ_{1-f_1} and B not lying on line AP; we consider the tangent t at s_1 in P, and B chosen in a way that $\angle(AP, t)$ is not congruent to $\angle(BP, t)$; let A', B' be other points lying in δ_{1-f_1} so that $\angle APA'$ is congruent to $\angle A'P - s_1 - Q$, and $\angle BPB'$ is congruent to $\angle B'P - s_1 - Q$. Then:
- angle APA' is right, because it is congruent to its supplementary (by construction);
- angle BPB' is right, because it is congruent to its supplementary (by construction);
- but $\angle APA'$ is not congruent to $\angle BPB'$, because the first one is a half of angle $A - P - s_1 - Q$, i.e. a half of $\angle(AP, t)$, while the second is a half of $B - P - s_1 - Q$, i.e. a half of $\angle(BP, t)$.

The theorems of congruence for triangles [two sides and an angle in between; two angles and a common side; three side] may not hold even in the Critical Zone (s_1, s_2, f_1, f_2) of the Model.

Property: The sum of the angles of a triangle can be:
- 180 degrees, if all of its vertices A, B, C are lying, for example, in δ_{1-f_1};
- strictly less than 180 degrees [any value in the interval $(0, 180)$], for example: let R, T be two points in δ_{2-f_2}, so that Q does not lie in RT, and S another point on s_2;

then triangle SRT has $\angle(SR, RT)$ congruent to 0, because SR and ST have an infinite number of common points (the segment SQ), and $\angle QTR + \angle TRQ$ congruent to $180 - \angle TQR$ [by construction we may vary $\angle TQR$ in the interval $(0, 180)$];
- even 0 degree: let A be a point in δ_{1-f_1}, B a point in δ_{2-f_2}, and C a point on s_3, very close to P; then ABS is a non-degenerate triangle (because its vertices are non-colinear), but

$$\angle(A - P - s_1 - Q - B, A - P - s_3 - C)$$

$$= \angle(B - Q - s_1 - P - A, B - Q - s_3 - P - s_3 - C)$$

$$= \angle(C - s_3 - P - A.C - s_3 - P - s_1 - Q - B) = 0$$

(one considers the length of $C - s_3 - P - s_1 - Q - B$ strictly less than the length of $C - s_3 - B$; the area of this triangle is also 0;
- more than 180 degrees, for example: let A, B be two points in δ_{1-f_1}, so that $\angle PAB + \angle PBA + \angle(s_1, s_2; inQ)$ is strictly greater than 180 degrees; then triangle ABQ, formed by the intersection of lines $A - P - s_2 - Q, Q - s_1 - P - B, AB$ will have the sum of its angles strictly greater than 180 degrees.

Definition: A circle of center M is a totality of all points A for which the segments MA are congruent to one another.

For example, if the center is Q, and the length of the segments MA is chosen to be greater than the length of s_1, then the circle is formed by the arc of circle centered in Q, of radius MA, and lying in δ_2, plus another arc of circle centered in P, of radius $MA-$length of $s1$, lying in δ_1

GROUP V. ANTI-AXIOMS OF CONTINUITY (ANTI-ARCHIMEDIAN AXIOM)

Let A, B be two points. Take the points $A_1, A_2, A_3, A_4, \ldots$, so that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4, etc. and so that segments $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ are congruent to one another. Then, among this series of points, there not always exists a certain point A_n, such that B lies between A and A_n.

For example: let A be a point in δ_{1-f_1}, and B a point on f_1, B different from P; let us consider on line AB points $A_1, A_2, A_3, A_4, \ldots$ in between A and B, such that $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ etc. are congruent to each other; then, we find that there is no point behind B (considering the direction from A to B), because B is a limit point (line AB ends in B).

The Bolzano's (intermediate value) theorem may not hold in the Critical Zone of the Model.

Unsolved problem: Is there a better model for this anti-geometry?

Definition 58. A Class of Paradoxes

Let $< A >$ be an attribute, and let $< Non - A >$ be its negation. Then:

Paradox 1. $ALL\ IS\ < A >,\ THE\ < Non - A >\ TOO.$

Examples:

E11: All is possible, the impossible too.

E12: All is present, the absents too.

E13: All is finite, the infinite too.

Paradox 2. $ALL\ IS\ < Non - A >,\ THE\ < A >\ TOO.$

Examples:

E21: All is impossible, the possible too.

E22: All is absent, the presents too.

E23: All is infinite, the finite too.

Paradox 3. $NOTHING\ IS\ < A >,\ NOT\ EVEN\ < A >.$

Examples:

E31: Nothing is perfect, not even the perfect.

E32: Nothing is absolute, not even the absolute.

E33: Nothing is finite, not even the finite.

Remark: The three kinds of paradoxes are equivalent.

More general paradox: *ALL (Verb)* $< A >$, *THE* $< Non - A >$ *TOO.*

Replacing $< A >$ by an attribute, we find a paradox.

Let us analyse the first one (E11): *"All is possible, the impossible too".*

If this sentence is true, then we get that *"The impossible is possible too"*, which is a contradiction; therefore the sentence is false (object language).

But the sentence may be true, because *"All is possible"* involves that *"the impossible is possible"*, i.e., *"it is possible to have impossible things"*, which is correct (meta-language).

Of course, among these ones, there are unsuccessful paradoxes, but the proposed method obtains beautiful others.

References: [2, 14, 44, 45, 46, 48, 49, 53, 55, 56, 57, 63, 77, 91, 93, 96, 97]

Definition 59. Linguistic Paradoxes

Let $< N >$, $< V >$, $< A >$ be some noun, verb, and attribute, respectively, and $< Non - N >$, $< Non - V >$, $< Non - A >$ be their antonyms, respectively. For example, if $< A >$ is *small*, then $< Non-A >$ is *big* or *large*, etc.

Also let $< N' >$, $< N'' >$, etc. represent synonyms of $< N >$ or even $< N >$, and so $< V' >$, $< V'' >$, etc. or $< A' >$, $< A'' >$, etc.

Let $< NV >$ represent a noun-ed verb, and $< NV' >$ a synonym, etc.

Then, one defines the following classes of linguistic paradoxes and semi-paradoxes.

1. $< Non - N >$ is a better $< N >$.
   $< Non - A >$ is a better $< A >$.
   $< Non - V >$ is a better $< V >$.

Examples:

Not to speak is sometimes a better speech.
Not to complain is a better complaint.
Unattractive is sometimes better than attractive.
Slow is sometimes better than fast.
No government is sometimes a better government.
A non-ruler is sometimes a better ruler.
No news is a good news.
Not to stare is sometimes a better look.
Not to love is a better love.
Not to move is sometimes a better move.
Impoliteness is a better politeness.
Not to hear is better than not listening.
No reaction is sometimes the best reaction.
Not to show kindness is a better kindness [welfare].
She is better than herself.
No fight is a better fight [i.e., to fight by non-violent means].

2. Only $< N >$ is truly a $< Non - N >$.
   Only $< A >$ is truly a $< Non - A >$.

Examples:

Only a rumor is truly a gossip.
Only a fiction is truly a fact.
Only normal is truly not normal.

Nobody is truly a 'somebody'.
Only fiction is truly real.
The friend is the most dangerous hidden enemy.
Only you are truly not you [= you act strangely].
Only mercy can be truly merciless.
If you spit at the sky, it will fall in your face.
Only gentleness is truly wildness.

3. This is so $< A >$, that it looks $< Non - A >$.
Examples:
This is so true, that it looks false!
This is so ripe, that it looks spoilt.
This is so friendly, that it looks hostile.
He seems so trustworthy, that he looks untrustworthy.
This is so fake, that it looks real!
This is so proper, that it looks improper.
This is so beautiful, that it looks unreal.
This is so simple, that it looks difficult.
The story was so real, that it looked fiction.
Can't see the trees for the forest.

4. There is some $< N >$, which is $< A >$ and $< Non - A >$ at the same time.
Examples:
There are events which are good and bad at the same time.
There are lows which are good and bad at the same time.
There are some news which are real and wrong at the same time.
There are some insects which are helpful and dangerous at the same time [like the spider].
There are men who are handsome and ugly at the same time.
There are classes that are fun and boring at the same time.
There are some ministers who are bevievers and mis-believers at the same time.
There are moments that are sweet and sour.
There are games which are challenged and not competitive at the same time.
Food which are simultaneously hot and cold.
The game was excluding, yet boring [because we were losing].
People are smart and foolish at the same time [i.e., smart at something, and foolish at other thing].

5. There is some $< N >$, which $< V >$ and really $< Non - V >$ at the

same time.

Examples:

There are people who trick and do not really trick at the same time.
There are some people who play and do not play at the same time.
Some of life's experiences are punishments and rewards as the same time.
Exercise is exhausting but also invigorating.
There are children who listen and do not really listen at the same time.
There are teachers who teach and do not teach at the same time.
There are people who spell and misspell at the same time.
Nice and rough people concomitantly.
Politicians who lie and tell the truth at the time!

6. To $< V >$, even when $< Non - V >$.

Examples:

A sage thinks even when he does not think.
I exist even when I do not exist.
A clown is funny even when he is not being funny.
To die of thirst surrounded by water [saltwater].
To be poet and not to know it.
A mother worries even when she does not worry.
To believe even when you do not believe.
Is matching even when not-matching.
I sleep even when I am awake.
Always running round.
To dream, even when not sleeping.

7. This $< N >$ is enough $< Non - N >$.

Examples:

This silence is enough noise.
This vacation from work is hard work.
The superiority brings enough inferiority [= listlessness].
This day is my night.
This diary is enough non-diary.
This sleep is enough awake.
This sweet truthfulness is enough sarcasm.
This table of four is enough for six people.
I had enough.
This job is enough recreation [when enjoying the job].

8. $< Non - V >$ sometimes means $< V >$.

Examples:

Not to speak sometimes means to speak.

Not to touch sometimes means to touch.
To preserve peace sometimes means going to war.
To destroy life (as in viruses) sometimes means to preserve life.
Not to listen sometimes means to listen.
Two feet forward sometimes means standing still.
Not to litter sometimes means to litter.
Speeding is sometimes not speeding [in case of emergency].
Not to show anger is sometimes to show anger.

9. $< N >$ without $< N >$.
Examples:
Hell without hell.
The style without style.
The rule applied: there were no rule!
Our culture is our lack of culture.
Live without living.
Some people are so afraid of death, that they do not live.
Work without work.
Cannot live with them, cannot live without them.
Death without death [for a Christian dying is going on the eternal life].
Guilty without guilt [sometimes is guilty but does bot feel guilty].

10.a. $< N >$ inside/within the $< Non - N >$.
Examples:
Movement inside the immobility.
Silence within the noise.
Slavery within the freedom.
Loneliness within a crowd.
A circle within a circle.
The wrestling ring inside a squared section.
To find wealth in poverty [i.e., happiness and love].

10.b. $< Non - N >$ in the $< N >$.
Examples:
Immobility inside the movement.
Noise inside the silence.
The eye of the storm.
Government. Bureaucracies.
Inequality inside the equality.
Single inside the marriage.
Anger inside the happiness.

Warmth in the cold.
Cold in the heat.
Laughing without being happy.
Has not gotten anywhere.
Poverty in wealth [no poverty or love in a wealth family].

11. The $< A >$ of the $< Non - A >$.
Examples:

The shadow of the light.
Music of silence.
Relaxing of exercise effect.
The restrictions of the free.
Life through death.
The sound/loudness of the silence.
I can see the light at the end of the tunnel.
The slave of freedom [someone who could not give up his freedom, not even in marriage].

12. $< V >$ what one $< Non - V >$.
Examples:

To see what one cannot see.
To hear what one cannot hear.
To taste what one cannot taste.
To accept what one cannot understand.
To say what one cannot say [to tell a secret].
To wait patiently when one does not know how to wait.
To breathe what one cannot breathe.
To feel what one cannot feel.
To appreciate what one dis-appreciates.
To believe what one cannot believe [faith].
To smell what one cannot smell.

13. Let us $< V >$ by $< Non - V >$.
Examples:

Let us strike by not striking [= Japanese strike].
Let us talk by not talking; [means to think].
To vote by not voting all.
To help someone by not helping [using experience as a teacher].
Let us justify by not justifying.
Let us win by not winning.
Let us strip by not stripping [to make bare or clear].
Let us fight by not fighting [Ghandi's motto].

14. $< N >$ of the $< Non - N >$.

Examples:

The benefits we get from non-benefits.
The smoke we got from non-smokers.
The rewards we get from hard work.
The service we get from non-service.
The good that comes from bad.
The pleasure we get from the pain.

15. $< Non - A >$ is $< A >$.

Examples:

The bad is good [because makes you try harder].
The good is bad [because does not leave any room for improvement].
Work is a blessing.
The poor is spiritually rich.
Sometimes ugly is beautiful [because beauty is in the eyes of the beholden].
You have to kiss a lot of frogs before you find a prince.
Hurt is healing.
"There is no absolute" is an absolute.
Not to commit any error is an error.

16. A $< Non - N > < N >$.

Examples:

A positive negative [which means: a failure enforces you to do better].
A sad happiness.
An impossible possibility.
Genuine imitation leather.
A loud whisper.
A beautiful disaster [which means beauty can be found anywhere].
A bitter sweet.
A harst gentleness [a gentleness that is very firm with you].
A guiltless sinner [someone who does not regret sinning].

17. Everything has an $< A >$ and a $< Non - A >$.

Examples:

Everything has a sense and a non-sense.
Everything has a truthful side and a wrong side.
Everything has a beginning and an end.
Everything has a birth and a death.
Everything has its time and a non-time.
Everything has an appearance and non-appearance.
Everybody has a good side and a bad side.

Everyone has a right and a wrong.

18. $< V >$ what $< Non - V >$.

Examples:

To be what you are not.
One needs what one does not need.
Expect the unexpected!
Culture exists by its non-existence.
No matter how rich we are, we never make enough money.
One purchases what one does not purchase.
To work when we are not working.
To die might mean to live forever [= for an artist].
One wants because one does not want [sometimes one wants something only because someone else likes it].

Look at this Funny Law example:

A Paradoxist Government: Suppose you have two cows. Then the government kills them and milks you!

The list of such invented linguistic paradoxes can be indifinitely extended. It is specific to each language, and it is based on language expressions and types of sentence and phrase constructions and structures.

One can also play with antonymic adverbs, prepositions, etc to construct other categories of linguistic paradoxes.

References: [2, 14, 44, 45, 46, 48, 49, 53, 55, 56, 57, 63, 77, 91, 93, 94, 96, 97]

Definition 60. Sorites Paradoxes

These sorites paradoxes are associated with Eubulides of Miletus (fourth century B.C.):

1) Invisible paradox:

Our visible world is composed of a totality of invisible particles.

a) An invisible particle does not form a visible object, nor do two invisible particles, three invisible particles, etc. However, at some point, the collection of invisible particles becomes large enough to form a visible object, but there is apparently no definite point where this occurs.

b) A similar paradox is developed in an opposite direction. It is always possible to remove an atom from an object in such a way that what is left is still a visible object. However, repeating and repeating this procedure, at some point, the visible object is decomposed so that the left part becomes invisible, but there is no definite point where this occurs.

Between A and $Non-A$ there is no clear distinction, no exact frontier. Where does A really end and $Non - A$ begin? We extend Zadeh's fuzzy set term to neutrosophic concept.

2) Mass paradox:

Things with mass result from atoms with quasi-null mass.

3) Infinite paradox:

A finite line pegment is formed by infinitely many points.

References: [85]

Definition 61. Linguistic Tautologies

The tautology is a redundancy, a pleonasm, a needless repetition of an idea, according to the "Webster's New World Dictionary", Third College Edition, 1988.

However, the following classes of tautologies - using repetition - go to a deeper meaning, and even changes the sense. A double assertion reverses to a negation.

One also may play with the synonyms.

Let $< N >$, $< V >$, $< A >$ be some noun, verb, and attribute, respectively. Also let $< N' >$, $< N'' >$, etc. represent synonyms of $< N >$ (or just $< N >$), and so $< V' >$, $< V'' >$, etc. or $< A' >$, $< A'' >$, etc.

Let $< NV >$ represent a noun-ed verb, and $< NV' >$ a synonym, etc.

Then, one defines the following classes of linguistic tautologies and semi-tautologies.

1. Mirror semi-paradox:

   $< N >$ of the $< N' >$.

   $< N >$ of the $< N' >$ of the $< N'' >$.

Examples:

Best of the best.

Worst of the worst of the worst.

A mother of the mother of the mother... [the maternal grandmother].

A follower of the followers.

The row of the rows. [lots of rows.]

2. This is not an $< N >$, this is an $< N' >$.

Examples:

This is not a teacher, this is a professor.

This is not a car, this is a Volkswagen.

This is not a truck, this is a Chevy.

This is not noise, this is music.

This is not music, this is noise.

This is not a cedar tree, this is a <gad>. [gad = Navajo name for cedar

tree]
This is not a sword, this is a sabre.
This is not a problem, this is a exercise. [= easier]
Practice makes you practice.
This is not a girl, this is Katie.
This is not a horse, this is a pony.

3. $< N >$ is not enough $< N' >$
$< A >$ is not enough $< A' >$.
Examples:
Sufficient is not enough sufficient [which means: to do more than "sufficient"].
Punishment is not enough punishment.
Health is not enough wealth.
Clean is not enough clean.
Studying is not enough studying [which means to do more than just getting by, i.e. to do research].
Extravagant is not enough extravagant.
Time is not enough time.
The more you have, the more you want.
Attention is not enough attention [some people need action, too].

4. More $< A >$ than $< A' >$.
Examples:
Better than better [= perfection].
Worst than worst [= evil].
Sweeter than sweeter [= honey].
More life than life [= spirituality].
More depressed than depressed.
Faster than faster.
More beautiful than pretty.
Uglier than ugly [= really ugly].
Smarter than smart [= like a genius].

5. How $< A >$ is an $< A' > < N >$?
Examples:
How democratic is a so called democratic society?
How republican is a so called republican society?
How civilized is a so called civilized person?
How free is a so called free country?
How commanding is a so called commanding officer?
How Pop Culture is a so called Pop Culture?

How strong is a strong man?
How lone is a lone ranger? [not very, he has tanto].

6. No $< A >$ is really $< A' >$.
Examples:
No friend is really a friend [s/he betrays you when you do not even expect!].
No luck is really a luck.
No original is really original.
No husband is really a husband [you learn to depend on yourself!]
No tomboy is really a tomboy [girl considered boyish].
No work is really less work.
No true Marxist is really a true Marxist [they contradict their own beliefs].
No magic is really magic [all is only a trick].

7. I would rather prefer $< A >$, than $< A' >$.
Examples:
I would rather prefer pretty, than prettier.
I want this, not that.
I would rather prefer this, to that.
I would rather be old, than old.
I would rather prefer great than big.
I would rather be crazy than crazy [crazy like foolish, than crazy like insane].

8. More $< A >$, than $< A' >$.
Examples:
Prettier than pretty.
More real than real.
More advantage than advantage.
More help than help.
More smiles than smiles [she did not psychically smile, but there were smiles all over her face].
More cries than cries.
More meters than kilometers.
Make everyday a rainbow day.
He earns more than himself.
More suspictions than suspected.

9. $< V >$ those who $< V' >$ you.
Examples:
Ignore those who ignore you.
Criticize those who criticize you.

Defend those who defend you.

10. $< V >$ because $< V' >$.
Examples:
I want because I want.
I think because I think.
I hear because I listen.
I see because I look.
I need because I need.
I know because I know.
I live because I live.
I believe what is unbelievable [faith].
I am happy because I am happy [there is no reason for my happiness].

11. $< V >$ the $< NV' >$.
Examples:
I hate the haters (therefore, I hate myself!)
I envy the evniers (therefore, I envy myself).
I am strange to strangers.
I cheat the cheaters (therefore, I cheat myself).
I lie to liers (therefore, I lie myself).
I kick the kickers (therefore, I kick myself).
I love the lovers.

The list of the so invented linguistic tautologies can be indefinitely extended. It is specific to each language, and it is based on language expressions and types of sentence and phrase constructions and structures. One can also play with synonimic adverbs, prepositions, etc. to construct other categories of linguistic tautologies.

References: [2, 14, 44, 45, 46, 48, 49, 53, 55, 56, 57, 63, 77, 91, 93, 94, 96, 97]

Definition 62. Hypothesis: "There is no Speed Barrier in the Universe"

F. Smarandache extended an experiment of the University of Innsbruck - Austria researchers, which have proved that the speed of light can be overpassed, contradicting Einstein's prediction. The Smarandache's hypothesis is that speed may by even infinite, which looks fantastic! At this speed, a being would not age and life would have no death.

References: [47, 84, 30]

Definition 63. Smarandache Primes, Squares, Cubes, $M-$Powers, and More General "$T-$"Numbers

Smarandache Prime is a term of any of the Smarandache sequences, or a value of any of the Smarandache-type functions, which is a prime number.

For example: Stephan [88] proved that in the Smarandache Reverse Sequence

$$1, 21, 321, 4321, ..., 121110987654321, ...$$

the only Smarandache Reverse Prime is:

$$RSm(82) = 8281807978...121110987654321$$

among the first 750 terms, while Weisstein extended the search up to the first 2,739 terms. One conjectures that this is the only prime in the whole sequence.

Similarly one defines:

A Smarandache Square is a term of any of the Smarandache sequences (except, of course, the sequence involving squares), or a value of any of the Smarandache-type functions, which is a perfect square number.

A Smarandache Cube is a term of any of the Smarandache sequences (except, of course, the sequence involving cubes), or a value of any of the Smarandache-type functions, which is a perfect cube number.

And so on: A Smarandache $m-$Power is a term of any of the Smarandache sequences (except, of course, the sequence involving $m-$powers), or a value of any of the Smarandache-type functions, which is a perfect $m-$power number.

A Smarandache Palindromic Number is a term of any of the Smarandache sequences, or a value of any of the Smarandache-type functions, which is palindromic.

Generalization: Let "$T-$" be a specific defined number (for example: perfect number, or Bell number, or at most a prime number, etc.). Then: A Smarandache "$T-$"Number is a term of any of the Smarandache sequences, or a value of any of the Smarandache-type functions, which is "T".

References: [88], pp. 310-311 and 1661-1663 from [94].

Definition 64. Let the set $Z_p = \{0, 1, 2, ..., p-1\}$. The elements of Z_p can be written in a unique way as $m-$adic numbers.

Let $r = (a_{n-1}a_{n-2}...a_1a_0)_m$ and $s = (b_{n-1}b_{n-2}...b_1b_0)_m$ be two elements from Z_p. Then, one can introduce the operation:

$$r \wr - \wr s = (|a_{n-1} - b_{n_1}||a_{n-2} - b_{n-2}|...|a_1 - b_1||a_0 - b_0|)_m,$$

where $|x-y|$ is the absolute value of $x-y$.

The set $(Z_p, \wr - \wr)$ is known as the Smarandache Groupoid.

The complement of r, $C(r) = r \wr - \wr Sup(Z_p)$, where $Sup(Z_p)$ is the maximal element of Z_p.

We say that rRs, iff $r \wr - \wr C(r) = s \wr - \wr C(s)$. R is a relation of equivalence, which partitions Z_p into equivalence classes. The equivalence class below is defined as D–form:

$$D_{Sup(Z_p)} = \{r \in Z_p \mid r \wr - \wr C(r) = Sup(Z_p)\}.$$

Let us introduce two more operations on Z_p:

$$r \wr + \wr s = (\{a_{n-1} + b_{n_1}\}\{a_{n-2} + b_{n-2}|\}...\{a_1 + b_1\}\{a_0 + b_0\})_m,$$

where $\{x+y\}$ is the remainder of $x+y$ modulo m.

$$r \wr . \wr s = (\{a_{n-1}.b_{n_1}\}\{a_{n-2}.b_{n-2}|\}...\{a_1.b_1\}\{a_0.b_0\})_m,$$

where $\{x.y\}$ is the remainder of $x.y$ modulo m.

Then the set $(Z_p, \{x+y\}, \{x.y\})$ is known as the Smarandache Ring. If m is prime, then the set $(Z_p, \{x+y\}, \{x.y\})$ is known as the Smarandache Field.

Unsolved problems:

1) Study the Smarandache Ring, and the Smarandache Field.

2) If one introduces other operations on Z_p:

$$r \wr \wedge \wr s = (\{a_{n-1} \wedge b_{n_1}\}\{a_{n-2} \wedge b_{n-2}\}...\{a_1 \wedge b_1\}\{a_0 \wedge b_0\})_m,$$

where $\{x \wedge y\}$ is the remainder of $x \wedge y$ (x to the power y) modulo m.

Study $(Z_p, \wr \wedge \wr)$.

3) And in a similar way for any introduced digit operation on Z_p:

$$r \wr F \wr s = (\{a_{n-1}Fb_{n_1}\}\{a_{n-2}Fb_{n-2}|\}...\{a_1Fb_1\}\{a_0Fb_0\})_m,$$

where $\{xFy\}$ is the remainder of xFy (i.e., $F(x,y)$, for F being a given function, $F : \mathcal{N} \to \mathcal{N}$) modulo m.

References: [90]

[1] Ashbacher, C., A note on the Smarandache Near-To-Primordial function. *Smarandache Notions Journal,* Vol. 7, No. 1-2-3, Aug. 1996, 46-49.

[2] Ashbacher, C., " 'The most paradoxist mathematician of the world' by Charles T. Le", review in *Journal of Recreational Mathematics,* USA, Vol. 28 (2), 1996-7, 130.

[3] Ashbacher, C., Some properties of the Smarandache-Kupera and Smarandache-Wagstaff functions. *Mathematics and Informatics Quarterly* , Vol. 7, No. 3, Sept. 1997, 114-116.

[4] Ashbacher, C., Smarandache geometries. *Smarandache Notions Journal,* Vol. 8, No. 1-2-3, 1997, 212-215.

[5] Ashbacher C., Solutions to Some Sastry Problems on Smarandache Number Related Triangles, *Smarandache Notions Journal,* Vol. 11, No. 1-2-3, 2000, 110-115.

[6] Ashbacher, C., Smarandache series convergence. to appear.

[7] Ashbacher, C., M. Mudge, Personal Computer World, London, Oct. 1995, 302.

[8] Atanassov K., On Some of the Smarandache's Problems. American Research Press, Lupton, 1999.

[9] Atanassov K., On the 22-nd, the 23-rd, and the 24-th Smarandache's problems. Notes on Number Theory and Discrete Mathematics, Vol. 5 (1999), No. 2, 80-82.

[10] Atanassov K., On the 37-th and the 38-th Smarandache's problems. Notes on Number Theory and Discrete Mathematics, Vol. 5 (1999), No. 2, 83-85.

[11] Atanassova V., K. Atanassov, On the 43-rd and the 44-th Smarandache's problems. Notes on Number Theory and Discrete Mathematics, Vol. 5 (1999), No. 2, 86-88.

[12] Balacenoiu, I., Smarandache numerical functions. *Bulletin of Pure and Applied Sciences,* Vol. 14E, No. 2, 1995, 95-100.

[13] Begay, A., Smarandache ceil functions. *Bulletin of Pure and Applied Sciences,* Vol. 16E, No. 2, 1997, 227-229.

[14] Begay, A., The Smarandache semantic paradox. *Humanistic Mathematics Network Journal,* Harvey Mudd College, Claremont, CA, USA, Issue No. 17, May 1998, 48.

[15] Bencze, M., Smarandache recurrence type sequences. *Bulletin of Pure and Applied Sciences,* Bombay, Vol. 16E, No. 2, 1997, 231-236.

[16] Bencze, M., Smarandache relationships and sub-sequences. *Bulletin of Pure and Applied Sciences,* Bombay, Vol. 17E, No. 1, 1998, 89-95.

[17] Brown, J., The Smarandache counter-projective geometry. *Abstracts of Papers Presented to the American Mathematical Society Meetings,* Vol. 17, No. 3, Issue 105, 1996, 595.

[18] Burton, E., On some series involving the Smarandache function. *Smarandache Notions Journal,* Vol. 6, No. 1, June 1995, 13-15.

[19] Burton, E., On some convergent series. *Smarandache Notions Journal,* Vol. 7, No. 1-2-3, Aug. 1996, 7-9.

[20] Castillo, J., Primes of the form $x^y + y^x$, *Mathematical Spectrum,* Vol. 29, No. 1, 1996/7, 21.

[21] Castillo, J., letter to the Editor, *Mathematical Spectrum,* Vol. 29, No. 1, 1997/8, 21.

[22] Castillo, J., Smarandache continued fractions. *Bulletin of Pure and Applied Sciences,* Delhi, Vol. 17E, No. 1, 1998, 149-151.

[23] Castillo, J., Smarandache continued fractions. *Smarandache Notions Journal,* Vol. 9, No. 1-2, 1998, 40-42.

[24] Castillo, J., Other Smarandache type functions.
http: //www.gallup.unm.edu/ ~ smarandache/funct2.txt

[25] Castini, P., Letter to the Editor, *Mathematical Spectrum,* Vol. 28, No. 3, 1995/6, 68.

[26] Chimienti, S., M. Bencze, Smarandache non-geometry. *Smarandache Notion Journal,* Vol. 9, No. 1-2, 1998, 45-46.

[27] Chimienti, S., M. Bencze, Smarandache anti-geometry. *Bulletin of Pure and Applied Sciences,* Delhi, Vol. 17E, No. 1, 1998, 103-114.

[28] Chimienti, S., M. Bencze, Smarandache anti-geometry. *Smarandache Notions Journal,* Vol. 9, No. 1-2, 1998, 49-63.

[29] Cojocaru, I., S. Cojocaru, The First Constant of Smarandache. Smarandache Notions Journal, Vol. 7, No. 1-2-3, Aug. 1996.

[30] Dennis Jay Gordon, On Smarandache Number in Quantum Physics, *Smarandache Notions Journal,* Vol. 11, No. 1-2-3, 2000, p. 320.

[31] Dodge, C., Letter to the Author, August 4, 1998.

[32] Dumitrescu, C., A brief history of the Smarandache function. *Octagon,* Vol. 2, No. 1, 1994, 15-16.

[33] Dumitrescu, C., V. Seleacu, Some Notions and Questions in Number Theory. Erhus University Press, Glendale, 1994.

[34] Dumitrescu, C., V. Seleacu, Some Notions and Questions in Number Theory. Xiquan Publ. Hse., Phoenix-Chicago, 1994.

[35] Dumitrescu, C., V. Seleacu, The Smarandache Function, Erhus University Press, Vail, USA, 1994, pp. 48-61.

[36] Erdös, P., C. Ashbacher, Thoughts of Pal Erdös on Some Smarandache Notions, *Smarandache Notion Journal,* Vol. 8, No. 1-2-3, 1997, 220-224.

[37] Grebenikova, O., Some bases of numerations, *Abstracts of Papers Presented to the American Mathematical Society,* Vol. 17, No. 3, Issue 105, 1996, 588.

[38] Iacobescu, F., Smarandache partition type and other sequences. *Bulletin of Pure and Applied Sciences,* Bombay, Vol. 16E, No. 2, 1997, 237-240.

[39] Ibstedt, H., Smarandache iterations of first and second kinds. *Abstracts of Papers Presented to the American Mathematical Society,* Vol. 17, No. 4, Issue 106, 1996, 680.

[40] Ibstedt, H., Surfing on the Ocean of Numbers – a few Smarandache Notions and Similar Topics. Erhus University Press, Vail, AZ, 1997.

[41] Ibstedt H., On Smarandache Periodic Sequences, *Smarandache Notions Journal,* Vol. 8, No. 1-2-3, 1997, 126-135.

[42] Ibstedt, H., Computer Analysis of sequences. American research Press, Lupton, 1998.

[43] Kashihara, K., Comments and Topics on Smarandache Notions and Problems. Erhus University Press, Vail, AZ, 1996.

[44] Le, C., The Smarandache class of paradoxes. *Journal of Indian Academy of Sciences*, Vol. 18, No. 1, 1996, 53-55; *Abracadabra*, Salinas, CA, 1993; *Tempus*, Bucharest, No. 2, 1994.

[45] Le, C., The Smarandache class of paradoxes. *Bulletin of the Transilvania University of Brasov*, Vol. 1 (36) New Series, Series B, 1994, 7-8.

[46] Le, C., The Smarandache class of paradoxes. *Bulletin of Pure and Applied Sciences*, Delhi, Vol. 14 E, No. 2, 1995, 109-110.

[47] Le, C., J. Gilbert, Book review, The Internet Pilot To Physics, http://physicsweb.org/TIPTOP/FORUM/BOOKS/describebook, phtml?entry_id=115.

[48] Le, C., The most paradoxist mathematician of the world: Florentin Smarandache. *Bulletin of Pure and Applied Sciences*, Delhi, Vol. 15 E, No. 1, 1996, 81-100.

[49] Le, C., The Smarandache class of paradoxes. *Journal of Indian Academy of Mathematics*, Indore, Vol. 18, No. 1, 1996, 53-55.

[50] Luca, F., Prodicts of factorials in Smarandache type expressions, *Smarandache Notions Journal*, Vol. 8, No. 1-2-3, 1997, 29-41.

[51] Luca, F., Perfect powers in Smarandache type expressions, *Smarandache Notions Journal*, Vol. 8, No. 1-2-3, 1997, 72-89.

[52] Marimutha, H., Smarandache concatenate type sequences. *Bulletin of Pure and Applied Sciences*, Bombay, Vol. 16E, No. 2, 1997, 225-226.

[53] Mitroiescu, I., The Smarandache class of paradoxes applied in computer sciences. *Abstracts of Papers Presented to the American Mathematical Society*, New Jersey, Vol. 16, No. 3, Issue 101, 1995, 651.

[54] Mudle, M., The Smarandache Near-To-Primordial (S.N.T.P) function. *Smarandache Notions Journal*, Vol. 7, No. 1-2-3, Aug. 1996, 45.

[55] Mudge, M., A paradoxist mathematician: his function, paradoxist geometry and class of paradoxes. *Smarandache Notion Journal*, Vol. 7, No. 1-2-3, 1996, 127-129.

[56] Popescu, T., Estetica paradoxismului. *Editura Tempus*, Bucarest, Vol. 26, 1995, 27-28.

[57] Popescu, M., A model for the Smarandache paradoxist geometry. *Abstracts of Papers Presented to the American Mathematical Society Meetings,* Vol. 17, No. 1, Issue 103, 1996, 265.

[58] Popescu, M., V. Seleacu, About the Smarandache complementary prime function. *Smarandache Notions Journal,* Vol. 7, No. 1-2-3, Aug. 1996, 12-22.

[59] Porubsky, S., On Smarandache's form of the individual Fermat-Euler theorem. *Smarandache Notions Journal,* Vol. 8, No. 1-2-3, Fall 1997, 5-20.

[60] Porubsky, S., On Smarandache's form of the individual Fermat-Euler theorem. *Proceedings on the First International Conference on Smarandache Type Notions in Number Theory,* University of Craiova, Aug. 21-24, 1997, 163-178.

[61] Radeva V., Atanassov K., On the 40-th and 41-th Smarandache's problems. Notes on Number Theory and Discrete Mathematics, Vol. 4 (1998), No. 3, 101-104.

[62] Radu, I., *Mathematical Spectrum,* Vol. 27, No. 2, 1994/5, 43.

[63] Rotaru, I., Din nou despre Florentin Smarandache. *Vatra,* Tg. Mures, Romania, Nr. 2 (299), 1996, 93-94.

[64] Sandor, J., On the irrationality of certain alternative Smarandache series. *Smarandache Notions Journal,* Vol. 8, No. 1-2-3, Fall 1997, 143-144.

[65] Sastry, K. R. S., Smarandache Number Related Triangles, *Smarandache Notions Journal,* Vol. 11, No. 1-2-3, 2000, 107-109.

[66] Seagull, L., The Smarandache function and the number of primes up to x. *Mathematical Spectrum,* University of Shielfield, Vol. 28, No. 3, 1995/6, 53.

[67] Sloane, N.J.A., Sequence A001223/M0296 ("Andrica's Conjecture") in *An On-Line Version of the Encyclopedia of Integer Sequences* .

[68] Sloane, N. J. A., Sequence A038458 ("Smarandache Constant" = 0. 5671481302020177146468468755334825645867902493886382068402 8522182...) in *An On-Line Version of the Encyclopedia of Integer Sequences. http: www.research.att.com/cgi-bin/access.cgi/as/njas* /sequences/eishis. cgi.

[69] Sloane, N. J. A., S. Plouffe, The Encyclopedia of Integer Sequences. online, email:superseeker@ research.att.com (SUPERSEEKER by N. J. A. Sloane, S. Plouffe, B. Salvy, ATT Bell Labs, Murray Hill, NJ 07974, USA).

[70] Smarandache, F., Properties of the Numbers, University of Craiova, Archivesm, 1975; Arizona State University, Special Collections, Tempe, AZ, USA.

[71] Smarandache, F., Conjectures on Primes' Summation Arizona State University, Special Collections, Hayden Library, Tempe, AZ, 1979.

[72] Smarandache, F., A function in the number theory, *Analete Univ. Timisoara,* Fascicle 1, Vol. XVIII, 1980, 79-88.

[73] Smarandache, F., Criteria for a positive integer to be prime. *Gazeta Matematica*, Bucharest, No. 2, 1981, 49-52.

[74] Smarandache, F., Une generalization de theoreme d'Euler. *Buletinul Universitatii Brasov* Seria C, Vol. XXIII, 1981, 7-12.

[75] Smarandache, F., Problemes avec and sans problemes!, Ed. Somipress, Fes, Morocco, 1983, Problem & Solution, No. 5.36.

[76] Smarandache, F., Paradoxist mathematics. (lecture), Bloomsburg University, Mathematical department, PA, USA, Nov. 1985.

[77] Smarandache, F., Mathematical Fancies & Paradoxes. *The Eugene Strens Memorial on Intuitive and ecreational Mathematics and its History*. University of Calgary, Alberta, Canada, 27 July – 2 August 1986.

[78] Smarandache, F., Proposed problem # 328 ("Prime pairs and Wilson's theorem"). *College Mathematics Journal,* USA, March 1988, 191-192.

[79] Smarandache, F., Characterization of n prime numbers simultaneously. *Libertas Mathematica*, University of Texas at Arlington, Vol. XI, 1991, 151-155.

[80] Smarandache, F. Only Problems, Not Solutions!. Xiquan Publ. House, Chicago, 1993.

[81] Smarandache, F., Collected Papers, Vol. 1, Ed. Tempus, Bucharest, 1996.

[82] Smarandache, F., Collected Papers, Vol. II, Kishinev University Press, Kishinev, 1997.

[83] Smarandache, F., Collected Papers, Vol. III. Tempus Publ. House, Bucharest, 1998.

[84] Smarandache, F., There is no speed barrier in the universe, *Bulletin of Pure and Applied Sciences*, Delhi, India, Vol. 17D (Physics), No. 1, 1998, p. 61.

[85] Smarandache, F., Neutrosophy, Neutrosophic Probability, Set, and Logic. American Research Press, Rehoboth, 1998.

[86] Smarandache, F., Conjectures which Generalize Andrica's Conjecture. Arizona State University, Hayden Library, Special Collections, Tempe, AZ, USA.

[87] Smarandoiu, S., Convergence of Smarandache continued fractions., *Abstracts of Papers Presented to the American Mathematical Society,* Vol. 17, No. 4, Issue 106, 1996, 680.

[88] Stephan, R.,, Factors and Primes in Two Smarandache Sequences. *Smarandache Notions Journal,* Vol. 9, No. 1-2, 1998, 4-10.

[89] Tabirca, S., About S-multiplicative functions. *Octagon,* Brasov, Vol. 7, 1999, No. 1, 169-170.

[90] Talukdar, D., Smarandache Groupoid, Ring, and Field. http://www.gallvp.unm.edu/ smarandache/talvkdar.txt.

[91] Tilton, H., Smarandache's paradoxes. *Math. Power,* Tucson, AZ, USA, Vol. 2, No. 9, 1996, 1-2.

[92] "The Florentin Smarandache papers". Special Collection, Arizona State University, Hayden Library, Tempe, Box 981006, AZ 85287-1006, USA; Carol Moore & Marilyn Wurzburger: librarians).

[93] Vasiliu, F., Paradoxism's Main Roots. Xiquan Publishing House, Phoenix, USA, 1994.

[94] Weisstein, E., CRC Concise Encyclopedia of Mathematics", *CRC Press,* Boca Raton, FL, USA, 1998, p. 44.

[95] Zhong, C., On Smarandache continued fractions., *Smarandache Notions Journal,* Vol. 9, No. 1-2, June 1998, 40-42.

[96] Zitarelli, D., 'Le, Charles T. / The Most Paradoxist Mathematician of the World'. *Historia Mathematica,* PA, USA, Vol. 22, No. 4, 1995, 460.

[97] Zitarelli, D., 'Mudge, Michael R. / A paradoxist mathematician: his function, paradoxist geometry and class of paradoxes'. *Historia Mathematica,* PA. USA, Vol. 24, No. 1, 1997, 114.

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