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A Study on Neutrosophic Frontier and Neutrosophic Semi-frontier in Neutrosophic Topological Spaces

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ABSTRACT. In this paper neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topology are introduced and several of their properties, characterizations and examples are established.

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KEYWORDS : Neutrosophic frontier and Neutrosophic semi-frontier.

I. INTRODUCTION

Theory of Fuzzy sets [21], Theory of Intuitionistic fuzzy sets [2], Theory of Neutrosophic sets [10] and the theory of Interval Neutrosophic sets [13] can be considered as tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [10]. In 1965, Zadeh [21] introduced fuzzy set theory as a mathematical tool for dealing with uncertainties where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov [2] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. The neutrosophic set was introduced by Smarandache [10] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set. In 2012, Salama, Alblowi [18], introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of Intuitionistic fuzzy topological space and a Neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element.

The concepts of neutrosophic semi-open sets, neutrosophic semi-closed sets, neutrosophic semi-interior and neutrosophic semi-closure in neutrosophic topological spaces were introduced by P. Iswarya and Dr. K. Bageerathi [12] in 2016. Frontier and semifrontier in intuitionistic fuzzy topological spaces were introduced by Athar

Kharal [4] in 2014. In this paper, we are extending the above concepts to neutrosophic topological spaces. We study some of the basic properties of neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topological spaces with examples. Properties of neutrosophic semi-interior, neutrosophic semi-closure, neutrosophic frontier and neutrosophic semi-frontier have been obtained in neutrosophic product related spaces.

II. NEUTROSOPHIC FRONTIER

In this section, the concepts of the neutrosophic frontier in neutrosophic topological space are introduced and also discussed their characterizations with some related examples.

Definition 2.1 Let $\alpha, \beta, \lambda \in [0, 1]$ and $\alpha + \beta + \lambda \leq 1$. A neutrosophic point [*NP* for short] $x_{(\alpha, \beta, \lambda)}$ of X is a *NS* of X which is defined by

$$x_{(\alpha, \beta, \lambda)} = \begin{cases} (\alpha, \beta, \lambda), & y = x, \\ (0, 0, 1), & y \neq x. \end{cases}$$

In this case, x is called the support of $x_{(\alpha, \beta, \lambda)}$ and α, β and λ are called the value, intermediate value and the non-value of $x_{(\alpha, \beta, \lambda)}$, respectively. A *NP* $x_{(\alpha, \beta, \lambda)}$ is said to belong to a *NS* $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ in X , denoted by $x_{(\alpha, \beta, \lambda)} \in A$ if $\alpha \leq \mu_A(x)$, $\beta \leq \sigma_A(x)$ and $\lambda \geq \gamma_A(x)$. Clearly a neutrosophic point can be represented by an ordered triple of neutrosophic points as follows : $x_{(\alpha, \beta, \lambda)} = (x_\alpha, x_\beta, C(x_{C(\lambda)}))$. A class of all *NPs* in X is denoted as $NP(X)$.

Definition 2.2 Let X be a *NTS* and let $A \in NS(X)$. Then $x_{(\alpha, \beta, \lambda)} \in NP(X)$ is called a neutrosophic frontier point [*NFP* for short] of A if $x_{(\alpha, \beta, \lambda)} \in NCl(A) \cap NCl(C(A))$. The intersection of all the *NFPs* of A is called a neutrosophic frontier of A and is denoted by $NFr(A)$. That is, $NFr(A) = NCl(A) \cap NCl(C(A))$.

Proposition 2.3 For each $A \in NS(X)$, $A \cup NFr(A) \subseteq NCl(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,
 $A \cup NFr(A) = A \cup (NCl(A) \cap NCl(C(A)))$
 $= (A \cup NCl(A)) \cap (A \cup NCl(C(A)))$
 $\subseteq NCl(A) \cap NCl(C(A))$
 $\subseteq NCl(A)$

Hence $A \cup NFr(A) \subseteq NCl(A)$.

From the above proposition, the inclusion cannot be replaced by an equality as shown by the following example.

Example 2.4 Let $X = \{ a, b \}$ and $\tau = \{ 0_N, A, B, C, D, 1_N \}$. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are $C(\tau) = \{ 1_N, E, F, G, H, 0_N \}$ where

$$A = \langle (0.5, 1, 0.1), (0.9, 0.2, 0.5) \rangle,$$

$$B = \langle (0.2, 0.5, 0.9), (0, 0.5, 1) \rangle,$$

$$C = \langle (0.5, 1, 0.1), (0.9, 0.5, 0.5) \rangle,$$

$$D = \langle (0.2, 0.5, 0.9), (0, 0.2, 1) \rangle,$$

$$E = \langle (0.1, 0, 0.5), (0.5, 0.8, 0.9) \rangle,$$

$$F = \langle (0.9, 0.5, 0.2), (1, 0.5, 0) \rangle,$$

$$G = \langle (0.1, 0, 0.5), (0.5, 0.5, 0.9) \rangle \text{ and}$$

$$H = \langle (0.9, 0.5, 0.2), (1, 0.8, 0) \rangle.$$

Here $NCl(A) = 1_N$ and $NCl(C(A)) = NCl(E) = E$. Then by Definition 2.2, $NFr(A) = E$.

Also $A \cup NFr(A) = \langle (0.5, 1, 0.1), (0.9, 0.8, 0.5) \rangle \subseteq 1_N$. Therefore $NCl(A) = 1_N \not\subseteq \langle (0.5, 1, 0.1), (0.9, 0.8, 0.5) \rangle$.

Theorem 2.5 For a $NS A$ in the $NTS X$, $NFr(A) = NFr(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,

$$\begin{aligned} NFr(A) &= NCl(A) \cap NCl(C(A)) \\ &= NCl(C(A)) \cap NCl(A) \\ &= NCl(C(A)) \cap NCl(C(C(A))) \end{aligned}$$

Again by Definition 2.2,

$$= NFr(C(A))$$

Hence $NFr(A) = NFr(C(A))$.

Theorem 2.6 If a $NS A$ is a NCS , then $NFr(A) \subseteq A$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,

$$\begin{aligned} NFr(A) &= NCl(A) \cap NCl(C(A)) \\ &\subseteq NCl(A) \end{aligned}$$

By Definition 4.4 (a) [18],

$$= A$$

Hence $NFr(A) \subseteq A$, if A is NCS in X .

The converse of the above theorem needs not be true as shown by the following example.

Example 2.7 From Example 2.4, $NFr(C) = G \subseteq C$. But $C \not\subseteq C(\tau)$.

Theorem 2.8 If a $NS A$ is NOS , then $NFr(A) \subseteq C(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 4.3 [18], A is NOS implies $C(A)$ is NCS in X . By Theorem 2.6, $NFr(C(A)) \subseteq C(A)$ and by Theorem 2.5, we get $NFr(A) \subseteq C(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.9 From Example 2.4, $NFr(G) = G \subseteq C(G) = C$. But $G \not\subseteq \tau$.

Theorem 2.10 For a $NS A$ in the $NTS X$, $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,

$$C(NFr(A)) = C(NCl(A) \cap NCl(C(A)))$$

By Proposition 3.2 (1) [18],

$$= C(NCl(A)) \cup C(NCl(C(A)))$$

By Proposition 4.2 (b) [18],

$$= NInt(C(A)) \cup NInt(A)$$

Hence $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Theorem 2.11 Let $A \subseteq B$ and $B \in NC(X)$ (resp., $B \in NO(X)$). Then $NFr(A) \subseteq B$ (resp., $NFr(A) \subseteq C(B)$), where $NC(X)$ (resp., $NO(X)$) denotes the class of neutrosophic closed (resp., neutrosophic open) sets in X .

Proof : By Proposition 1.18 (d) [12], $A \subseteq B$, $NCl(A) \subseteq NCl(B)$ ----- (1).

By Definition 2.2,

$$\begin{aligned} NFr(A) &= NCl(A) \cap NCl(C(A)) \\ &\subseteq NCl(B) \cap NCl(C(A)) \text{ by (1)} \\ &\subseteq NCl(B) \end{aligned}$$

By Definition 4.4 (b) [18],

$$= B$$

Hence $NFr(A) \subseteq B$.

Theorem 2.12 Let A be the NS in the $NTS X$. Then $NFr(A) = NCl(A) - NInt(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . By Proposition 4.2 (b) [18], $C(NCl(C(A))) = NInt(A)$ and by Definition 2.2,

$$\begin{aligned} NFr(A) &= NCl(A) \cap NCl(C(A)) \\ &= NCl(A) - C(NCl(C(A))) \\ &\qquad \qquad \qquad \text{by using } A - B = A \cap C(B) \end{aligned}$$

By Proposition 4.2 (b) [18],

$$= NCl(A) - NInt(A)$$

Hence $NFr(A) = NCl(A) - NInt(A)$.

Theorem 2.13 For a NS A in the NTS X , $NFr(NInt(A)) \subseteq NFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,

$$NFr(NInt(A)) = NCl(NInt(A)) \cap NCl(C(NInt(A)))$$

By Proposition 4.2 (a) [18],

$$= NCl(NInt(A)) \cap NCl(NCl(C(A)))$$

By Definition 4.4 (b) [18],

$$= NCl(NInt(A)) \cap NCl(C(A))$$

By Definition 4.4 (a) [18],

$$\subseteq NCl(A) \cap NCl(C(A))$$

Again by Definition 2.2,

$$= NFr(A)$$

Hence $NFr(NInt(A)) \subseteq NFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.14 Let $X = \{ a, b \}$ and $\tau = \{ 0_N, A, B, C, D, 1_N \}$. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are $C(\tau) = \{ 1_N, E, F, G, H, 0_N \}$ where

$$A = \langle (0.5, 0.6, 0.7), (0.1, 0.9, 0.4) \rangle,$$

$$B = \langle (0.3, 0.9, 0.2), (0.4, 0.1, 0.6) \rangle,$$

$$C = \langle (0.5, 0.9, 0.2), (0.4, 0.9, 0.4) \rangle,$$

$$D = \langle (0.3, 0.6, 0.7), (0.1, 0.1, 0.6) \rangle,$$

$$E = \langle (0.7, 0.4, 0.5), (0.4, 0.1, 0.1) \rangle,$$

$$F = \langle (0.2, 0.1, 0.3), (0.6, 0.9, 0.4) \rangle,$$

$$G = \langle (0.2, 0.1, 0.5), (0.4, 0.1, 0.4) \rangle \text{ and}$$

$$H = \langle (0.7, 0.4, 0.3), (0.6, 0.9, 0.1) \rangle.$$

Define $A_1 = \langle (0.4, 0.2, 0.8), (0.4, 0.5, 0.1) \rangle$. Then $C(A_1) = \langle (0.8, 0.8, 0.4), (0.1, 0.5, 0.4) \rangle$.

Therefore by Definition 2.2, $NFr(A_1) = H \not\subseteq 0_N = NFr(NInt(A_1))$.

Theorem 2.15 For a NS A in the NTS X , $NFr(NCl(A)) \subseteq NFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,

$$NFr(NCl(A)) = NCl(NCl(A)) \cap NCl(C(NCl(A)))$$

By Proposition 1.18 (f) [12] and 4.2 (b) [18],

$$= NCl(A) \cap NCl(NInt(C(A)))$$

By Proposition 1.18 (a) [12],

$$\subseteq NCl(A) \cap NCl(C(A))$$

Again by Definition 2.2,

$$= NFr(A)$$

Hence $NFr(NCl(A)) \subseteq NFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.16 From Example 2.14, let $A_2 = \langle (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) \rangle$.

Then $C(A_2) = \langle (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) \rangle$. Then by Definition 2.2, $NFr(A_2) = G$.

Therefore $NFr(A_2) = G \not\subseteq 0_N = NFr(NCl(A_2))$.

Theorem 2.17 Let A be the NS in the NTS X . Then $NInt(A) \subseteq A - NFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Now by Definition 2.2,

$$A - NFr(A) = A - (NCl(A) \cap NCl(C(A)))$$

$$= (A - NCl(A)) \cup (A - NCl(C(A)))$$

$$= A - NCl(C(A))$$

$$\supseteq NInt(A).$$

Hence $NInt(A) \subseteq A - NFr(A)$.

Example 2.18 From Example 2.14, $A_1 - NFr(A_1) = \langle (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) \rangle$.

Therefore $A_1 - NFr(A_1) = \langle (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) \rangle \not\subseteq 0_N = NInt(A_1)$.

Remark 2.19 In general topology, the following conditions are hold :

$$NFr(A) \cap NInt(A) = 0_N,$$

$$NInt(A) \cup NFr(A) = NCl(A),$$

$$NInt(A) \cup NInt(C(A)) \cup NFr(A) = 1_N.$$

But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 2.20 From Example 2.14,

$$NFr(A_2) \cap NInt(A_2) = G \cap C = G \neq 0_N.$$

$$NInt(A_2) \cup NFr(A_2) = C \cup G = C \neq 1_N = NCl(A_2).$$

$$NInt(A_2) \cup NInt(C(A_2)) \cup NFr(A_2) = C \cup 0_N \cup G = C \neq 1_N.$$

Theorem 2.21 Let A and B be the two NSs in the NTS X . Then $NFr(A \cup B) \subseteq NFr(A) \cup NFr(B)$.

Proof : Let A and B be the two NSs in the NTS X . Then by Definition 2.2,

$$NFr(A \cup B) = NCl(A \cup B) \cap NCl(C(A \cup B))$$

By Proposition 3.2 (2) [18],

$$= NCl(A \cup B) \cap NCl(C(A) \cap C(B))$$

by Proposition 1.18 (h) and (o) [12],

$$\subseteq (NCl(A) \cup NCl(B)) \cap (NCl(C(A)) \cap NCl(C(B)))$$

$$= [(NCl(A) \cup NCl(B)) \cap NCl(C(A))] \cap [(NCl(A) \cup NCl(B)) \cap NCl(C(B))]$$

$$= [(NCl(A) \cap NCl(C(A))) \cup (NCl(B) \cap NCl(C(A)))] \cap [(NCl(A) \cap NCl(C(B))) \cup (NCl(B) \cap NCl(C(B)))]$$

$$\cap [(NCl(A) \cap NCl(C(B))) \cup (NCl(B) \cap NCl(C(B)))]$$

Again by Definition 2.2,

$$= [NFr(A) \cup (NCl(B) \cap NCl(C(A)))] \cap [(NCl(A) \cap NCl(C(B))) \cup NFr(B)]$$

$$= (NFr(A) \cup NFr(B)) \cap [(NCl(B) \cap NCl(C(A))) \cup (NCl(A) \cap NCl(C(B)))]$$

$$\cup (NCl(A) \cap NCl(C(B)))]$$

$\subseteq NFr(A) \cup NFr(B)$.
Hence $NFr(A \cup B) \subseteq NFr(A) \cup NFr(B)$.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.22 By Example 2.14, we define
 $A_1 = \langle (0.2, 0, 0.5), (0.4, 0.1, 0.1) \rangle$,
 $A_2 = \langle (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) \rangle$,
 $A_1 \cup A_2 = A_3 = \langle (0.7, 0.9, 0.2), (0.5, 0.9, 0.1) \rangle$ and
 $A_1 \cap A_2 = A_4 = \langle (0.2, 0, 0.5), (0.4, 0.1, 0.3) \rangle$. Then
 $C(A_1) = \langle (0.5, 1, 0.2), (0.1, 0.9, 0.4) \rangle$,
 $C(A_2) = \langle (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) \rangle$,
 $C(A_3) = \langle (0.2, 0.1, 0.7), (0.1, 0.1, 0.5) \rangle$ and
 $C(A_4) = \langle (0.5, 1, 0.2), (0.3, 0.9, 0.4) \rangle$.
Therefore $NFr(A_1) \cup NFr(A_2) = E \cup G = E \not\subseteq G = NFr(A_3) = NFr(A_1 \cup A_2)$.

Note 2.23 The following example shows that $NFr(A \cap B) \not\subseteq NFr(A) \cap NFr(B)$ and $NFr(A) \cap NFr(B) \not\subseteq NFr(A \cap B)$.

Example 2.24 From Example 2.22, $NFr(A_1 \cap A_2) = NFr(A_4) = E \not\subseteq G = NFr(A_1) \cap NFr(A_2)$.
From Example 2.14, We define $B_1 = \langle (0.4, 0.5, 0.1), (0.2, 0.9, 0.5) \rangle$,
 $B_2 = \langle (0.5, 0.2, 0.9), (0.8, 0.4, 0.7) \rangle$,
 $B_1 \cup B_2 = B_3 = \langle (0.5, 0.5, 0.1), (0.8, 0.9, 0.5) \rangle$ and
 $B_1 \cap B_2 = B_4 = \langle (0.4, 0.2, 0.9), (0.2, 0.4, 0.7) \rangle$.
Then
 $C(B_1) = \langle (0.1, 0.5, 0.4), (0.5, 0.1, 0.2) \rangle$,
 $C(B_2) = \langle (0.9, 0.8, 0.5), (0.7, 0.6, 0.8) \rangle$,
 $C(B_3) = \langle (0.1, 0.5, 0.5), (0.5, 0.1, 0.8) \rangle$ and
 $C(B_4) = \langle (0.9, 0.8, 0.4), (0.7, 0.6, 0.2) \rangle$.
Therefore $NFr(B_1) \cap NFr(B_2) = 1_N \cap 1_N = 1_N \not\subseteq H = NFr(B_4) = NFr(B_1 \cap B_2)$.

Theorem 2.25 For any NSs A and B in the NTS X , $NFr(A \cap B) \subseteq (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$.

Proof : Let A and B be the two NSs in the NTS X .
Then by Definition 2.2,
 $NFr(A \cap B) = NCl(A \cap B) \cap NCl(C(A \cap B))$
By Proposition 3.2 (1) [18],
 $= NCl(A \cap B) \cap NCl(C(A) \cup C(B))$
By Proposition 1.18 (n) and (h) [12],
 $\subseteq (NCl(A) \cap NCl(B)) \cap (NCl(C(A)) \cup NCl(C(B)))$
 $= [(NCl(A) \cap NCl(B)) \cap NCl(C(A))]$
 $\cup [(NCl(A) \cap NCl(B)) \cap NCl(C(B))]$
Again by Definition 2.2,
 $= (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$
Hence $NFr(A \cap B) \subseteq (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.26 From Example 2.24,
 $(NFr(B_1) \cap NCl(B_2)) \cup (NFr(B_2) \cap NCl(B_1)) = (1_N \cap 1_N) \cup (1_N \cap 1_N) = 1_N \cup 1_N = 1_N \not\subseteq H = NFr(B_1 \cap B_2)$.

Corollary 2.27 For any NSs A and B in the NTS X , $NFr(A \cap B) \subseteq NFr(A) \cup NFr(B)$.

Proof : Let A and B be the two NSs in the NTS X .
Then by Definition 2.2,
 $NFr(A \cap B) = NCl(A \cap B) \cap NCl(C(A \cap B))$
By Proposition 3.2 (1) [18],
 $= NCl(A \cap B) \cap NCl(C(A) \cup C(B))$
By Proposition 1.18 (n) and (h) [12],
 $\subseteq (NCl(A) \cap NCl(B)) \cap (NCl(C(A)) \cup NCl(C(B)))$
 $= (NCl(A) \cap NCl(B) \cap NCl(C(A)))$
 $\cup (NCl(A) \cap NCl(B) \cap NCl(C(B)))$
Again by Definition 2.2,
 $= (NFr(A) \cap NCl(B)) \cup (NCl(A) \cap NFr(B))$
 $\subseteq NFr(A) \cup NFr(B)$
Hence $NFr(A \cap B) \subseteq NFr(A) \cup NFr(B)$.

The equality in the above corollary may not hold as seen in the following example.

Example 2.28 From Example 2.24,
 $NFr(B_1) \cup NFr(B_2) = 1_N \cup 1_N = 1_N \not\subseteq H = NFr(B_4) = NFr(B_1 \cap B_2)$.

Theorem 2.29 For any NS A in the NTS X ,
(1) $NFr(NFr(A)) \subseteq NFr(A)$,
(2) $NFr(NFr(NFr(A))) \subseteq NFr(NFr(A))$.
Proof : (1) Let A be the NS in the neutrosophic topological space X . Then by Definition 2.2,
 $NFr(NFr(A)) = NCl(NFr(A)) \cap NCl(C(NFr(A)))$
Again by Definition 2.2,
 $= NCl(NCl(A) \cap NCl(C(A))) \cap NCl(C(NCl(A) \cap NCl(C(A))))$
By Proposition 1.18 (f) [12] and by 4.2 (b) [18],
 $\subseteq (NCl(NCl(A)) \cap NCl(NCl(C(A)))) \cap NCl(NInt(C(A)) \cup NInt(A))$
By Proposition 1.18 (f) [12],
 $= (NCl(A) \cap NCl(C(A))) \cap (NCl(NInt(C(A))) \cup NCl(NInt(A)))$
 $\subseteq NCl(A) \cap NCl(C(A))$
By Definition 2.2,
 $= NFr(A)$
Therefore $NFr(NFr(A)) \subseteq NFr(A)$.
(2) By Definition 2.2,
 $NFr(NFr(NFr(A))) = NCl(NFr(NFr(A))) \cap NCl(C(NFr(NFr(A))))$

By Proposition 1.18 (f) [12],
 $\subseteq (NFr(NFr(A))) \cap NCI(C(NFr(NFr(A))))$
 $\subseteq NFr(NFr(A))$.
Hence $NFr(NFr(NFr(A))) \subseteq NFr(NFr(A))$.

Remark 2.30 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 2.31 Let $X = \{a, b\}$ and $\tau = \{0_N, A, B, 1_N\}$. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are $C(\tau) = \{1_N, C, D, 0_N\}$ where
 $A = \langle (0.8, 0.4, 0.5), (0.4, 0.6, 0.7) \rangle$,
 $B = \langle (0.4, 0.2, 0.9), (0.1, 0.4, 0.9) \rangle$,
 $C = \langle (0.5, 0.6, 0.8), (0.7, 0.4, 0.4) \rangle$ and
 $D = \langle (0.9, 0.8, 0.4), (0.9, 0.6, 0.1) \rangle$. Define
 $A_1 = \langle (0.6, 0.7, 0.8), (0.5, 0.4, 0.5) \rangle$. Then
 $C(A_1) = \langle (0.8, 0.3, 0.6), (0.5, 0.6, 0.5) \rangle$.
Therefore by Definition 2.2, $NFr(A_1) = D \not\subseteq C = NFr(NFr(A_1))$.

Theorem 2.32 Let A, B, C and D be the NSs in the NTS X . Then $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$.

Proof : Let A, B, C and D be the NSs in the NTS X . Then by Definition 2.2 [12],

$$\begin{aligned} &\mu_{(A \cap B) \times (C \cap D)}(x, y) \\ &= \min \{ \mu_{(A \cap B)}(x), \mu_{(C \cap D)}(y) \} \\ &= \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_C(y), \mu_D(y) \} \} \\ &= \min \{ \min \{ \mu_A(x), \mu_D(y) \}, \min \{ \mu_B(x), \mu_C(y) \} \} \\ &= \min \{ \mu_{(A \times D)}(x, y), \mu_{(B \times C)}(x, y) \}. \end{aligned}$$

Thus $\mu_{(A \cap B) \times (C \cap D)}(x, y) = \mu_{(A \times D) \cap (B \times C)}(x, y)$. Similarly

$$\begin{aligned} &\sigma_{(A \cap B) \times (C \cap D)}(x, y) \\ &= \min \{ \sigma_{(A \cap B)}(x), \sigma_{(C \cap D)}(y) \} \\ &= \min \{ \min \{ \sigma_A(x), \sigma_B(x) \}, \min \{ \sigma_C(y), \sigma_D(y) \} \} \\ &= \min \{ \min \{ \sigma_A(x), \sigma_D(y) \}, \min \{ \sigma_B(x), \sigma_C(y) \} \} \\ &= \min \{ \sigma_{(A \times D)}(x, y), \sigma_{(B \times C)}(x, y) \}. \end{aligned}$$

Thus $\sigma_{(A \cap B) \times (C \cap D)}(x, y) = \sigma_{(A \times D) \cap (B \times C)}(x, y)$. And also

$$\begin{aligned} &\gamma_{(A \cap B) \times (C \cap D)}(x, y) \\ &= \max \{ \gamma_{(A \cap B)}(x), \gamma_{(C \cap D)}(y) \} \\ &= \max \{ \max \{ \gamma_A(x), \gamma_B(x) \}, \max \{ \gamma_C(y), \gamma_D(y) \} \} \\ &= \max \{ \max \{ \gamma_A(x), \gamma_D(y) \}, \max \{ \gamma_B(x), \gamma_C(y) \} \} \\ &= \max \{ \gamma_{(A \times D)}(x, y), \gamma_{(B \times C)}(x, y) \}. \end{aligned}$$

Thus $\gamma_{(A \cap B) \times (C \cap D)}(x, y) = \gamma_{(A \times D) \cap (B \times C)}(x, y)$. Hence $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$.

Theorem 2.33 Let $X_i, i = 1, 2, \dots, n$ be a family of neutrosophic product related NTSs. If each A_i is a NS in X_i . Then $NFr(\prod_{i=1}^n A_i) = [NFr(A_1) \times NCI(A_2) \times \dots \times NCI(A_n)] \cup [NCI(A_1) \times NFr(A_2) \times NCI(A_3)$

$$\times \dots \times NCI(A_n)] \cup \dots \cup [NCI(A_1) \times NCI(A_2) \times \dots \times NFr(A_n)].$$

Proof : It suffices to prove this for $n = 2$. Let A_i be the NS in the neutrosophic topological space X_i . Then by Definition 2.2,

$$\begin{aligned} NFr(A_1 \times A_2) &= NCI(A_1 \times A_2) \cap NCI(C(A_1 \times A_2)) \\ &\text{By Proposition 4.2 (a) [18],} \\ &= NCI(A_1 \times A_2) \cap C(NInt(A_1 \times A_2)) \end{aligned}$$

$$\begin{aligned} &\text{By Theorem 2.17 (1) and (2) [12],} \\ &= (NCI(A_1) \times NCI(A_2)) \cap C(NInt(A_1) \times NInt(A_2)) \\ &= (NCI(A_1) \times NCI(A_2)) \cap \end{aligned}$$

$$C[(NInt(A_1) \cap NSCI(A_1)) \times (NInt(A_2) \cap NCI(A_2))]$$

$$\begin{aligned} &\text{By Lemma 2.3 (iii) [12],} \\ &= (NCI(A_1) \times NCI(A_2)) \cap [C(NInt(A_1) \cap \end{aligned}$$

$$\begin{aligned} &NCI(A_1)) \times 1_N \cup 1_N \times C(NInt(A_2) \cap NCI(A_2))] \\ &= (NCI(A_1) \times NCI(A_2)) \cap [(NCI(C(A_1)) \cup NInt(C(A_1))) \times 1_N \cup 1_N \times (NCI(C(A_2)) \cup NInt(C(A_2)))] \\ &= (NCI(A_1) \times NCI(A_2)) \cap [(NCI(C(A_1)) \times 1_N) \cup \end{aligned}$$

$$(1_N \times NCI(C(A_2)))]$$

$$\begin{aligned} &= [(NCI(A_1) \times NCI(A_2)) \cap (NCI(C(A_1)) \times 1_N)] \\ &\cup [(NCI(A_1) \times NCI(A_2)) \cap (1_N \times NCI(C(A_2)))] \end{aligned}$$

$$\begin{aligned} &\text{By Theorem 2.32,} \\ &= [(NCI(A_1) \cap NCI(C(A_1))) \times (1_N \cap NCI(A_2))] \\ &\cup [(NCI(A_1) \cap 1_N) \times (NCI(A_2) \cap NCI(C(A_2)))] \end{aligned}$$

$$\begin{aligned} &= (NFr(A_1) \times NCI(A_2)) \cup (NCI(A_1) \times NFr(A_2)). \end{aligned}$$

Hence $NFr(A_1 \times A_2) = (NFr(A_1) \times NCI(A_2)) \cup (NCI(A_1) \times NFr(A_2))$.

III. NEUTROSOPHIC SEMI-FRONTIER

In this section, we introduce the neutrosophic semi-frontier and their properties in neutrosophic topological spaces.

Definition 3.1 Let A be a NS in the NTS X . Then the neutrosophic semi-frontier of A is defined as $NSFr(A) = NSCI(A) \cap NSCI(C(A))$. Obviously $NSFr(A)$ is a NSC set in X .

Theorem 3.2 Let A be a NS in the NTS X . Then the following conditions are holds :

(i) $NSCI(A) = A \cup NInt(NCI(A))$,

(ii) $NSInt(A) = A \cap NCI(NInt(A))$.

Proof : (i) Let A be a NS in X . Consider
 $NInt(NCI(A \cup NInt(NCI(A))))$
 $= NInt(NCI(A) \cup NCI(NInt(NCI(A))))$
 $= NInt(NCI(A))$
 $\subseteq A \cup NInt(NCI(A))$

It follows that $A \cup NInt(NCI(A))$ is a NSC set in X . Hence $NSCI(A) \subseteq A \cup NInt(NCI(A))$ ----- (1)

By Proposition 6.3 (ii) [12], $NSCI(A)$ is NSC set in X . We have $NInt(NCI(A)) \subseteq NInt(NCI(NSCI(A))) \subseteq NSCI(A)$.

Thus $A \cup NInt(NCI(A)) \subseteq NSCI(A)$ ----- (2).
From (1) and (2), $NSCI(A) = A \cup NInt(NCI(A))$.

(ii) This can be proved in a similar manner as (i).

Theorem 3.3 For a NS A in the NTS X , $NSFr(A) = NSFr(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1,

$$\begin{aligned} NSFr(A) &= NSCI(A) \cap NSCI(C(A)) \\ &= NSCI(C(A)) \cap NSCI(A) \\ &= NSCI(C(A)) \cap NSCI(C(C(A))) \end{aligned}$$

Again by Definition 3.1,

$$= NSFr(C(A))$$

Hence $NSFr(A) = NSFr(C(A))$.

Theorem 3.4 If A is NSC set in X , then $NSFr(A) \subseteq A$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1,

$$\begin{aligned} NSFr(A) &= NSCI(A) \cap NSCI(C(A)) \\ &\subseteq NSCI(A) \end{aligned}$$

By Proposition 6.3 (ii) [12],

$$= A$$

Hence $NSFr(A) \subseteq A$, if A is NSC in X .

The converse of the above theorem is not true as shown by the following example.

Example 3.5 Let $X = \{ a, b, c \}$ and $\tau = \{ 0_N, A, B, C, D, 1_N \}$. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are $C(\tau) = \{ 1_N, F, G, H, I, 0_N \}$ where

- $A = \langle (0.5, 0.6, 0.7), (0.1, 0.8, 0.4), (0.7, 0.2, 0.3) \rangle$,
 - $B = \langle (0.8, 0.8, 0.5), (0.5, 0.4, 0.2), (0.9, 0.6, 0.7) \rangle$,
 - $C = \langle (0.8, 0.8, 0.5), (0.5, 0.8, 0.2), (0.9, 0.6, 0.3) \rangle$,
 - $D = \langle (0.5, 0.6, 0.7), (0.1, 0.4, 0.4), (0.7, 0.2, 0.7) \rangle$,
 - $E = \langle (0.8, 0.8, 0.4), (0.5, 0.8, 0.1), (0.9, 0.7, 0.2) \rangle$,
 - $F = \langle (0.7, 0.4, 0.5), (0.4, 0.2, 0.1), (0.3, 0.8, 0.7) \rangle$,
 - $G = \langle (0.5, 0.2, 0.8), (0.2, 0.6, 0.5), (0.7, 0.4, 0.9) \rangle$,
 - $H = \langle (0.5, 0.2, 0.8), (0.2, 0.2, 0.5), (0.3, 0.4, 0.9) \rangle$,
 - $I = \langle (0.7, 0.4, 0.5), (0.4, 0.6, 0.1), (0.7, 0.8, 0.7) \rangle$
- and

$$J = \langle (0.4, 0.2, 0.8), (0.1, 0.2, 0.5), (0.2, 0.3, 0.9) \rangle.$$

Here E and J are neutrosophic semi-open and neutrosophic semi-closed set respectively. Therefore the neutrosophic semi-open and neutrosophic semi-closed set topologies are $\tau_{NSO} = 0_N, A, B, C, D, E, 1_N$ and $C(\tau)_{NSC} = 1_N, F, G, H, I, J, 0_N$. Therefore $NSFr(C) = H \subseteq C$. But $C \notin C(\tau)_{NSC}$.

Theorem 3.6 If A is NSO set in X , then $NSFr(A) \subseteq C(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Proposition 4.3 [12], A is NSO set implies $C(A)$ is NSC set in X . By Theorem 3.4, $NSFr(C(A)) \subseteq C(A)$ and by Theorem 3.3, we get $NSFr(A) \subseteq C(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.7 From Example 3.5, $NSFr(J) = J \subseteq C(J) = E$. But $J \notin \tau_{NSO}$.

Theorem 3.8 Let $A \subseteq B$ and $B \in NSC(X)$ (resp., $B \in NSO(X)$). Then $NSFr(A) \subseteq B$ (resp., $NSFr(A) \subseteq C(B)$), where $NSC(X)$ (resp., $NSO(X)$) denotes the class of neutrosophic semi-closed (resp., neutrosophic semi-open) sets in X .

Proof : By Proposition 6.3 (iv) [12], $A \subseteq B$, $NSCI(A) \subseteq NSCI(B)$ ----- (1).

By Definition 3.1,

$$\begin{aligned} NSFr(A) &= NSCI(A) \cap NSCI(C(A)) \\ &\subseteq NSCI(B) \cap NSCI(C(A)) \text{ by (1)} \\ &\subseteq NSCI(B) \end{aligned}$$

By Proposition 6.3 (ii) [12],

$$= B$$

Hence $NSFr(A) \subseteq B$.

Theorem 3.9 Let A be the NS in the NTS X . Then $C(NSFr(A)) = NSInt(A) \cup NSInt(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1,

$$C(NSFr(A)) = C(NSCI(A) \cap NSCI(C(A)))$$

By Proposition 3.2 (1) [18],

$$= C(NSCI(A)) \cup C(NSCI(C(A)))$$

By Proposition 6.2 (ii) [12],

$$= NSInt(C(A)) \cup NSInt(A)$$

Hence $C(NSFr(A)) = NSInt(A) \cup NSInt(C(A))$.

Theorem 3.10 For a NS A in the NTS X , then $NSFr(A) \subseteq NFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Proposition 6.4 [12], $NSCI(A) \subseteq NCI(A)$ and $NSCI(C(A)) \subseteq NCI(C(A))$. Now by Definition 3.1,

$$\begin{aligned} NSFr(A) &= NSCI(A) \cap NSCI(C(A)) \\ &\subseteq NCI(A) \cap NCI(C(A)) \end{aligned}$$

By Definition 2.2,

$$= NFr(A)$$

Hence $NSFr(A) \subseteq NFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.11 From Example 3.5, let $A_1 = \langle (0.4, 0.1, 0.9), (0.1, 0.2, 0.6), (0.1, 0.3, 0.9) \rangle$, then $C(A_1) = \langle (0.9, 0.9, 0.4), (0.6, 0.8, 0.1), (0.9, 0.7, 0.1) \rangle$. Therefore $NFr(A_1) = H \not\subseteq J = NSFr(A_1)$.

Theorem 3.12 For a NS A in the NTS X , then $NSCI(NSFr(A)) \subseteq NFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1, $NSCI(NSFr(A)) = NSCI(NSCI(A) \cap NSCI(C(A))) \subseteq NSCI(NSCI(A)) \cap NSCI(NSCI(C(A)))$
By Proposition 6.3 (iii) [12],
 $= NSCI(A) \cap NSCI(C(A))$
By Definition 3.1,
 $= NSFr(A)$
By Theorem 3.10,
 $\subseteq NFr(A)$
Hence $NSCI(NSFr(A)) \subseteq NFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.13 From Example 3.5, $NFr(A_1) = H \not\subseteq J = NSCI(NSFr(A_1))$.

Theorem 3.14 Let A be a NS in the NTS X . Then $NSFr(A) = NSCI(A) - NSInt(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . By Proposition 6.2 (ii) [12], $C(NSCI(C(A))) = NSInt(A)$ and by Definition 3.1, $NSFr(A) = NSCI(A) \cap NSCI(C(A)) = NSCI(A) - C(NSCI(C(A)))$
by using $A - B = A \cap C(B)$
By Proposition 6.2 (ii) [12],
 $= NSCI(A) - NSInt(A)$
Hence $NSFr(A) = NSCI(A) - NSInt(A)$.

Theorem 3.15 For a NS A in the NTS X , then $NSFr(NSInt(A)) \subseteq NSFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1, $NSFr(NSInt(A)) = NSCI(NSInt(A)) \cap NSCI(C(NSInt(A)))$
By Proposition 6.2 (i) [12],
 $= NSCI(NSInt(A)) \cap NSCI(NSCI(C(A)))$
By Proposition 6.3 (iii) [12],
 $= NSCI(NSInt(A)) \cap NSCI(C(A))$
By Proposition 5.2 (ii) [12],
 $\subseteq NSCI(A) \cap NSCI(C(A))$
By Definition 3.1,
 $= NSFr(A)$
Hence $NSFr(NSInt(A)) \subseteq NSFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.16 Let $X = \{a, b, c\}$ and $\tau_{NSO} = 0_N, A, B, C, D, E, 1_N$ and $C(\tau)_{NSC} = 1_N, F, G, H, I, J, 0_N$ where

$A = \langle (0.3, 0.4, 0.2), (0.5, 0.6, 0.7), (0.9, 0.5, 0.2) \rangle$,
 $B = \langle (0.3, 0.5, 0.1), (0.4, 0.3, 0.2), (0.8, 0.4, 0.6) \rangle$,
 $C = \langle (0.3, 0.5, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.2) \rangle$,
 $D = \langle (0.3, 0.4, 0.2), (0.4, 0.3, 0.7), (0.8, 0.4, 0.6) \rangle$,
 $E = \langle (0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2) \rangle$,
 $F = \langle (0.2, 0.6, 0.3), (0.7, 0.4, 0.5), (0.2, 0.5, 0.9) \rangle$,
 $G = \langle (0.1, 0.5, 0.3), (0.2, 0.7, 0.4), (0.6, 0.6, 0.8) \rangle$,
 $H = \langle (0.1, 0.5, 0.3), (0.2, 0.4, 0.5), (0.2, 0.5, 0.9) \rangle$,
 $I = \langle (0.2, 0.6, 0.3), (0.7, 0.7, 0.4), (0.6, 0.6, 0.8) \rangle$
and
 $J = \langle (0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9) \rangle$.
Define $A_1 = \langle (0.2, 0.3, 0.4), (0.4, 0.5, 0.6), (0.3, 0.4, 0.8) \rangle$.
Then $C(A_1) = \langle (0.4, 0.7, 0.2), (0.6, 0.5, 0.4), (0.8, 0.6, 0.3) \rangle$. Therefore $NSFr(A_1) = I \not\subseteq 0_N = NSFr(NSInt(A_1))$.

Theorem 3.17 For a NS A in the NTS X , then $NSFr(NSCI(A)) \subseteq NSFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1, $NSFr(NSCI(A)) = NSCI(NSCI(A)) \cap NSCI(C(NSCI(A)))$
By Proposition 6.3 (iii) and Proposition 6.2 (ii) [12],
 $= NSCI(A) \cap NSCI(NSInt(C(A)))$
By Proposition 5.2 (i) [12],
 $\subseteq NSCI(A) \cap NSCI(C(A))$
By Definition 3.1,
 $= NSFr(A)$
Hence $NSFr(NSCI(A)) \subseteq NSFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.18 From Example 3.16, let $A_2 = \langle (0.2, 0.6, 0.2), (0.3, 0.4, 0.6), (0.3, 0.4, 0.8) \rangle$. Then $C(A_2) = \langle (0.2, 0.4, 0.2), (0.6, 0.6, 0.3), (0.8, 0.6, 0.3) \rangle$. Therefore $NSFr(A_2) = 1_N \not\subseteq 0_N = NSFr(NSCI(A_2))$.

Theorem 3.19 Let A be the NS in the NTS X . Then $NSInt(A) \subseteq A - NSFr(A)$.

Proof : Let A be the NS in the neutrosophic topological space X . Now by Definition 3.1, $A - NSFr(A) = A - (NSCI(A) \cap NSCI(C(A))) = (A - NSCI(A)) \cup (A - NSCI(C(A))) = A - NSCI(C(A)) \supseteq NSInt(A)$.
Hence $NSInt(A) \subseteq A - NSFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.20 From Example 3.16, $A_1 - NSF_r(A_1) = \langle (0.2, 0.3, 0.4), (0.4, 0.3, 0.7), (0.3, 0.4, 0.8) \rangle \notin 0_N = NSInt(A_1)$.

Remark 3.21 In general topology, the following conditions are hold :

$$\begin{aligned} NSF_r(A) \cap NSInt(A) &= 0_N, \\ NSInt(A) \cup NSF_r(A) &= NSCl(A), \\ NSInt(A) \cup NSInt(C(A)) \cup NSF_r(A) &= 1_N. \end{aligned}$$

But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 3.22 From Example 3.16, define $A_1 = \langle (0.4, 0.6, 0.1), (0.5, 0.8, 0.3), (0.9, 0.6, 0.2) \rangle$. Then $C(A_1) = \langle (0.1, 0.4, 0.4), (0.3, 0.2, 0.5), (0.2, 0.4, 0.9) \rangle$. Therefore $NSF_r(A_1) \cap NSInt(A_1) = F \cap D = \langle (0.2, 0.4, 0.3), (0.4, 0.3, 0.7), (0.2, 0.4, 0.9) \rangle \neq 0_N$.

$$NSInt(A_1) \cup NSF_r(A_1) = D \cup F = \langle (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) \rangle \neq 1_N = NSCl(A_1).$$

$$NSInt(A_1) \cup NSInt(C(A_1)) \cup NSF_r(A_1) = D \cup 0_N \cup F = \langle (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) \rangle \neq 1_N.$$

Theorem 3.23 Let A and B be NSs in the $NTS X$. Then $NSF_r(A \cup B) \subseteq NSF_r(A) \cup NSF_r(B)$.

Proof : Let A and B be NSs in the $NTS X$. Then by Definition 3.1,

$$\begin{aligned} NSF_r(A \cup B) &= NSCl(A \cup B) \cap NSCl(C(A \cup B)) \\ \text{By Proposition 3.2 (2) [18],} \\ &= NSCl(A \cup B) \cap NSCl(C(A) \cap C(B)) \\ \text{By Proposition 6.5 (i) and (ii) [12],} \\ &\subseteq (NSCl(A) \cup NSCl(B)) \cap (NSCl(C(A)) \cap NSCl(C(B))) \\ &= [(NSCl(A) \cup NSCl(B)) \cap NSCl(C(A))] \cap [(NSCl(A) \cup NSCl(B)) \cap NSCl(C(B))] \\ &= [(NSCl(A) \cap NSCl(C(A))) \cup (NSCl(B) \cap NSCl(C(A)))] \cap [(NSCl(A) \cap NSCl(C(B))) \cup (NSCl(B) \cap NSCl(C(B)))] \\ \text{By Definition 3.1,} \\ &= [NSF_r(A) \cup (NSCl(B) \cap NSCl(C(A)))] \cap [(NSCl(A) \cap NSCl(C(B))) \cup NSF_r(B)] \\ &= (NSF_r(A) \cup NSF_r(B)) \cap [(NSCl(B) \cap NSCl(C(A))) \cup (NSCl(A) \cap NSCl(C(B)))] \\ &\subseteq NSF_r(A) \cup NSF_r(B). \end{aligned}$$

Hence $NSF_r(A \cup B) \subseteq NSF_r(A) \cup NSF_r(B)$.

The converse of the above theorem needs not be true as shown by the following example.

Example 3.24 Let $X = \{ a \}$ with $\tau_{NSO} = 0_N$, $A, B, C, D, 1_N$ and $C(\tau)_{NSC} = 1_N, E, F, G, H, 0_N$ where $A = \langle (0.6, 0.8, 0.4) \rangle$,

$B = \langle (0.4, 0.9, 0.7) \rangle$,
 $C = \langle (0.6, 0.9, 0.4) \rangle$,
 $D = \langle (0.4, 0.8, 0.7) \rangle$,
 $E = \langle (0.4, 0.2, 0.6) \rangle$,
 $F = \langle (0.7, 0.1, 0.4) \rangle$,
 $G = \langle (0.4, 0.1, 0.6) \rangle$ and
 $H = \langle (0.7, 0.2, 0.4) \rangle$. Now we define
 $B_1 = \langle (0.7, 0.6, 0.5) \rangle$,
 $B_2 = \langle (0.6, 0.8, 0.2) \rangle$,
 $B_1 \cup B_2 = B_3 = \langle (0.7, 0.8, 0.2) \rangle$ and
 $B_1 \cap B_2 = B_4 = \langle (0.6, 0.6, 0.5) \rangle$. Then
 $C(B_1) = \langle (0.5, 0.4, 0.7) \rangle$,
 $C(B_2) = \langle (0.2, 0.2, 0.6) \rangle$,
 $C(B_3) = \langle (0.2, 0.2, 0.7) \rangle$ and
 $C(B_4) = \langle (0.5, 0.4, 0.6) \rangle$.
 Therefore $NSF_r(B_1) \cup NSF_r(B_2) = 1_N \cup E = 1_N \notin E = NSF_r(B_3) = NSF_r(B_1 \cup B_2)$.

Note 3.25 The following example shows that $NSF_r(A \cap B) \not\subseteq NSF_r(A) \cap NSF_r(B)$ and $NSF_r(A) \cap NSF_r(B) \not\subseteq NSF_r(A \cap B)$.

Example 3.26 From Example 3.24, we define

$A_1 = \langle (0.5, 0.1, 0.9) \rangle$,
 $A_2 = \langle (0.3, 0.5, 0.6) \rangle$,
 $A_1 \cup A_2 = A_3 = \langle (0.5, 0.5, 0.6) \rangle$, and
 $A_1 \cap A_2 = A_4 = \langle (0.3, 0.1, 0.9) \rangle$. Then
 $C(A_1) = \langle (0.9, 0.9, 0.5) \rangle$,
 $C(A_2) = \langle (0.6, 0.5, 0.3) \rangle$,
 $C(A_3) = \langle (0.6, 0.5, 0.5) \rangle$ and
 $C(A_4) = \langle (0.9, 0.9, 0.3) \rangle$.
 Therefore $NSF_r(A_1) \cap NSF_r(A_2) = F \cap 1_N = F \notin G = NSF_r(A_4) = NSF_r(A_1 \cap A_2)$.

Also $NSF_r(B_1 \cap B_2) = NSF_r(B_4) = 1_N \notin E = 1_N \cap E = NSF_r(B_1) \cap NSF_r(B_2)$.

Theorem 3.27 For any $NSs A$ and B in the $NTS X$, $NSF_r(A \cap B) \subseteq (NSF_r(A) \cap NSCl(B)) \cup (NSF_r(B) \cap NSCl(A))$.

Proof : Let A and B be NSs in the $NTS X$. Then by Definition 3.1,

$$\begin{aligned} NSF_r(A \cap B) &= NSCl(A \cap B) \cap NSCl(C(A \cap B)) \\ \text{By Proposition 3.2 (1) [18],} \\ &= NSCl(A \cap B) \cap NSCl(C(A) \cup C(B)) \\ \text{By Proposition 6.5 (ii) and (i) [12],} \\ &\subseteq (NSCl(A) \cap NSCl(B)) \cap (NSCl(C(A)) \cup NSCl(C(B))) \\ &= [(NSCl(A) \cap NSCl(B)) \cap NSCl(C(A))] \cup [(NSCl(A) \cap NSCl(B)) \cap NSCl(C(B))] \\ \text{By Definition 3.1,} \\ &= (NSF_r(A) \cap NSCl(B)) \cup (NSF_r(B) \cap NSCl(A)) \\ \text{Hence } NSF_r(A \cap B) &\subseteq (NSF_r(A) \cap NSCl(B)) \cup (NSF_r(B) \cap NSCl(A)). \end{aligned}$$

The converse of the above theorem is not true as shown by the following example.

Example 3.28 From Example 3.24, $(NSFr(A_1) \cap NSCl(A_2)) \cup (NSFr(A_2) \cap NSCl(A_1)) = (F \cap 1_N) \cup (1_N \cap F) = F \cup F = F \not\subseteq G = NSFr(A_1 \cap A_2)$.

Corollary 3.29 For any NSs A and B in the NTS X , $NSFr(A \cap B) \subseteq NSFr(A) \cup NSFr(B)$.

Proof : Let A and B be NSs in the NTS X . Then by Definition 3.1,

$$NSFr(A \cap B) = NSCl(A \cap B) \cap NSCl(C(A \cap B))$$

By Proposition 3.2 (1) [18],

$$= NSCl(A \cap B) \cap NSCl(C(A) \cup C(B))$$

By Proposition 6.5 (ii) and (i) [12],

$$\subseteq (NSCl(A) \cap NSCl(B)) \cap (NSCl(C(A)) \cup NSCl(C(B)))$$

$$= (NSCl(A) \cap NSCl(B) \cap NSCl(C(A))) \cup$$

$$(NSCl(A) \cap NSCl(B) \cap NSCl(C(B)))$$

By Definition 3.1,

$$= (NSFr(A) \cap NSCl(B)) \cup (NSCl(A) \cap NSFr(B))$$

$$\subseteq NSFr(A) \cup NSFr(B).$$

Hence $NSFr(A \cap B) \subseteq NSFr(A) \cup NSFr(B)$.

The equality in the above theorem may not hold as seen in the following example.

Example 3.30 From Example 3.24, $NSFr(A_1) \cup NSFr(A_2) = F \cup 1_N = 1_N \not\subseteq G = NSFr(A_4) = NSFr(A_1 \cap A_2)$.

Theorem 3.31 For any NS A in the NTS X ,

$$(1) NSFr(NSFr(A)) \subseteq NSFr(A),$$

$$(2) NSFr(NSFr(NSFr(A))) \subseteq NSFr(NSFr(A)).$$

Proof : (1) Let A be the NS in the neutrosophic topological space X . Then by Definition 3.1,

$$NSFr(NSFr(A))$$

$$= NSCl(NSFr(A)) \cap NSCl(C(NSFr(A)))$$

By Definition 3.1,

$$= NSCl(NSCl(A) \cap NSCl(C(A))) \cap$$

$$NSCl(C(NSCl(A) \cap NSCl(C(A))))$$

By Proposition 6.3 (iii) and 6.2 (ii) [12],

$$\subseteq (NSCl(NSCl(A)) \cap NSCl(NSCl(C(A)))) \cap$$

$$NSCl(NSInt(C(A)) \cup NSInt(A))$$

By Proposition 6.3 (iii) [12],

$$= (NSCl(A) \cap NSCl(C(A))) \cap (NSCl(NSInt(C(A)))$$

$$\cup NSCl(NSInt(A)))$$

$$\subseteq NSCl(A) \cap NSCl(C(A))$$

By Definition 3.1,

$$= NSFr(A)$$

Therefore $NSFr(NSFr(A)) \subseteq NSFr(A)$.

(2) By Definition 3.1,

$$NSFr(NSFr(NSFr(A))) = NSCl(NSFr(NSFr(A)))$$

$$\cap NSCl(C(NSFr(NSFr(A))))$$

By Proposition 6.3 (iii) [12],

$$\subseteq (NSFr(NSFr(A))) \cap NSCl(C(NSFr(NSFr(A))))$$

$$\subseteq NSFr(NSFr(A)).$$

Hence $NSFr(NSFr(NSFr(A))) \subseteq NSFr(NSFr(A))$.

Remark 3.32 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 3.33 From Example 3.16, $NSFr(A_2) = 1_N \not\subseteq 0_N = NSFr(NSFr(A_2))$.

Theorem 3.34 Let $X_i, i = 1, 2, \dots, n$ be a family of neutrosophic product related NTSs. If each A_i is a NS in X_i , then $NSFr(\prod_{i=1}^n A_i) = [NSFr(A_1) \times NSCl(A_2) \times \dots \times NSCl(A_n)] \cup [NSCl(A_1) \times NSFr(A_2) \times NSCl(A_3) \times \dots \times NSCl(A_n)] \cup \dots \cup [NSCl(A_1) \times NSCl(A_2) \times \dots \times NSFr(A_n)]$.

Proof : It suffices to prove this for $n = 2$. Let A_i be the NS in the neutrosophic topological space X_i .

Then by Definition 3.1,

$$NSFr(A_1 \times A_2) = NSCl(A_1 \times A_2) \cap NSCl(C(A_1 \times A_2))$$

By Proposition 6.2 (i) [12],

$$= NSCl(A_1 \times A_2) \cap C(NSInt(A_1 \times A_2))$$

By Theorem 6.9 (i) and (ii) [12],

$$= (NSCl(A_1) \times NSCl(A_2)) \cap C(NSInt(A_1) \times NSInt(A_2))$$

$$= (NSCl(A_1) \times NSCl(A_2)) \cap C[(NSInt(A_1) \cap$$

$$NSCl(A_1)) \times (NSInt(A_2) \cap NSCl(A_2))]$$

By Lemma 2.3 (iii) [12],

$$= (NSCl(A_1) \times NSCl(A_2)) \cap [C(NSInt(A_1) \cap$$

$$NSCl(A_1)) \times 1_N \cup 1_N \times C(NSInt(A_2) \cap NSCl(A_2))]$$

$$= (NSCl(A_1) \times NSCl(A_2)) \cap [(NSCl(C(A_1)) \cup NSInt(C(A_1))$$

$$) \times 1_N \cup 1_N \times (NSCl(C(A_2)) \cup NSInt(C(A_2)))]$$

$$= (NSCl(A_1) \times NSCl(A_2)) \cap [(NSCl(C(A_1)) \times 1_N)$$

$$\cup (1_N \times NSCl(C(A_2)))]$$

$$= [(NSCl(A_1) \times NSCl(A_2)) \cap (NSCl(C(A_1)) \times 1_N)]$$

$$\cup [(NSCl(A_1) \times NSCl(A_2)) \cap (1_N \times NSCl(C(A_2)))]$$

By Theorem 2.32,

$$= [(NSCl(A_1) \cap NSCl(C(A_1))) \times (1_N \cap NSCl(A_2))] \cup$$

$$[(NSCl(A_1) \cap 1_N) \times (NSCl(A_2) \cap NSCl(C(A_2)))]$$

$$= (NSFr(A_1) \times NSCl(A_2)) \cup (NSCl(A_1) \times NSFr(A_2))$$

Hence $NSFr(A_1 \times A_2) = (NSFr(A_1) \times NSCl(A_2)) \cup$

$$(NSCl(A_1) \times NSFr(A_2)).$$

CONCLUSION

In this paper, we studied the concepts of frontier and semi-frontier in neutrosophic topological spaces. In future, we plan to extend this neutrosophic topology concepts by neutrosophic continuous, neutrosophic semi-continuous, neutrosophic almost continuous and neutrosophic weakly continuous in neutrosophic topological spaces, and also to expand this neutrosophic concepts by nets, filters and borders.

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