The Obata First Eigenvalue Theorem on a Seven Dimensional Quaternionic Contact Manifold

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In loving memory of Anthony David Bruno.

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Abstract

We prove an Obata-type rigidity result for the first eigenvalue of the sub-Laplacian on a compact seven dimensional quaternionic contact (QC) manifold which satisfies a Lichnerowicz-type bound on its QC-Ricci tensor, and has a non-negative Paneitz P-function. In particular, under the stated conditions, the lowest possible eigenvalue of the sub-Laplacian is achieved if and only if the manifold is QC-equivalent to the standard 3-Sasakian sphere.

Contents

Conventions

A, B, C	Capital letters from the beginning of the alphabet denote any section of the tangent bundle TM.
X, Y, Z	Capital letters from the end of the alphabet denote a section of the horizontal space H.
α, β, γ	Denotes any element of the set $\{0, 1, 2, 3\}$.
$(i\,j\,k)$	Denotes any cyclic permutation of (1, 2, 3).
$\sum_{(i\,j\,k)}$	Denotes a cyclic sum. For example,

$$\sum_{(i\,j\,k)} [T_{ii}^2 + 2T_{ij}^2] = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}^2 + 2T_{23}^2 + 2T_{31}^2. \tag{0.0.1}$$

$\{e_\gamma\}_{\gamma=0}^3$	Denotes a local orthonormal frame for the horizontal space H.
$P(e_\beta, e_\gamma, e_\gamma, e_\beta)$	Summation over repeated indices is implied. For example,

$$P(e_\beta, e_\gamma, e_\gamma, e_\beta) \overset{def}{=} \sum_{\beta,\gamma=0}^{3} P(e_\beta, e_\gamma, e_\gamma, e_\beta). \tag{0.0.2}$$

$\nabla_\alpha^* T^0$	For $\alpha = 0, 1, 2, 3$, and $I_0 \overset{def}{=} \mathrm{id}_H$, the divergences of T^0 are denoted:

$$\nabla^* T^0(X) \overset{def}{=} \nabla T^0(e_\gamma, e_\gamma, X), \quad \nabla_\alpha^* T^0(X) \overset{def}{=} \nabla T^0(e_\gamma, I_\alpha e_\gamma, X). \tag{0.0.3}$$

Chapter 1

Introduction

Of continued interest in geometric analysis is the relationship between the geometry of a Riemannian manifold (M^n, g) and the spectrum of the Laplace-Beltrami (Laplacian) operator Δ. A theorem of Lichnerowicz states that if M is compact and the Ricci tensor of g satisfies $\mathrm{Ric} \geq (n-1)g$ then the first positive eigenvalue of the Laplacian satisfies $\lambda \geq n$. The Ricci tensor of the round metric $\mathring{g}$ on the unit sphere is $\mathrm{Ric} = (n-1)\mathring{g}$ and a theorem of Obata states that if such a manifold supports a function which achieves the eigenvalue $\lambda = n$, then (M^n, g) is isometric to $(\mathbb{S}^n, \mathring{g})$. In Chapter 2.1 we detail these theorems and describe their consequences for the eigenfunction f and the Riemannian metric g.

A Cauchy-Riemann (CR)-manifold is an abstract model of a real hypersurface in a complex vector space $\mathbb{C}^{n+1}$, and a Quaternionic Contact (QC)-manifold models a real hypersurface in a quaternionic vector space $\mathbb{H}^{n+1}$, cf. Chapters 2.2 and 3.3 respectively. The geometric properties of a Riemannian manifold are determined by its metric g, the sub-Riemannian CR and QC-manifolds have their geometric properties determined instead by a 1-form η. The spheres $\mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ and $\mathbb{S}^{4n+3} \hookrightarrow \mathbb{H}^{n+1}$ are important examples of such hypersurfaces; the former can be given a special type of CR-structure called Sasakian (cf. Chapter 2.2.2), and the latter can be given a corresponding QC-structure called 3-Sasakian (cf. Chapter 3.5). Analogous results to those of Lichnerowicz and Obata in the Riemannian setting hold on these manifolds as well, with the Sasakian and 3-Sasakian spheres playing the role of the round sphere in the CR and QC Obata-type Theorems, and a Lichnerowicz-type bound $\mathcal{L} \geq 4g$ playing the role of the lower Ricci bound $\mathrm{Ric} \geq (n-1)g$.

In the Riemannian setting, an eigenfunction f achieving the lowest possible eigenvalue necessarily satisfies the Hessian equation (2.1.5), and this phenomenon also occurs in the CR (2.2.8) and QC (3.6.3) cases. From Theorem 2.1.3 we see that a Riemannian manifold meeting the hypotheses of Obata's theorem is necessarily an Einstein manifold, i.e. the trace-free part of the Ricci tensor vanishes. In the QC-Obata type Theorem 3.7.1 the corresponding QC-Einstein property was proven for the $n > 1$ case (cf. Chapter 3.7), but had to be taken as an assumption when $n = 1$.

The QC-Einstein condition is equivalent to the vanishing of two symmetric, trace-free tensors T^0 and U defined in (3.3.8). However, when $n = 1$ the tensor U vanishes identically and the identities used to prove $T^0 = 0$ when $n > 1$ become trivial when $n = 1$ (cf. Chapter 3.7.1). It is the Main Theorem in this dissertation that the conclusions of the QC Obata-type Theorem 3.7.1 hold in dimension 7 without the QC-Einstein assumption. In Chapter 4 we prove that such a QC-manifold is necessarily QC-Einstein, then the second part of Theorem 3.7.1 will allow us to obtain the main result.

Theorem 1.0.1 (Main Theorem)**.** *Let (M, η) be a closed, compact, QC-manifold of dimension seven and let g be the horizontal metric. Suppose the following QC-Ricci curvature lower-bound holds true*

$$\mathcal{L}(X,X) \stackrel{def}{=} 2Sg(X,X) + \frac{10}{3}T^0(X,X) \geq 4g(X,X), \qquad X \in \Gamma(H), \tag{1.0.1}$$

and the P-function of any eigenfunction associated to the first non-zero eigenvalue *of the sub-Laplacian is non-negative. If the (lowest) eigenvalue of the sub-Laplacian is* 4*, then* (M, η) *is QC-equivalent to the standard* 3*-Sasakian sphere.*

To prove the Main Theorem, we begin by using the compactness of M to obtain the vanishing of the P-form (cf. Chapter 3.4) of any eigenfunction f of lowest possible eigenvalue $\lambda = 4$. We intrepret the Lichnerowicz-type bound (1.0.1) as the non-negativity of a certain quadratic form $\mathcal{P}$ (4.2.6) that is related to the P-form of f. This, together with the assumed positivity of P-function (cf. Definition 3.4.1), will imply that the horizontal gradient ∇f belongs to the kernel of $\mathcal{P}$. This allows one to see that certain components of T^0 vanish, and the vanishing of $\mathcal{P}$ is equivalent to the vanishing of T^0. The fact that $\mathcal{P}(X, \nabla f) = 0$ yields an important identity for $T^0(X, \nabla f)$, and the covariant derivative $\nabla T^0(X, Y, \nabla f)$ contains further components of T^0 we wish to show vanish.

The significance of the identity (4.2.8) for $T^0(X, \nabla f)$ comes from the fact that in the 7-dimensional QC-case we can frame the 4-dimensional horizontal space by the gradient of the extremal first eigenfunction, obeying $\Delta f = 4f$, and its images under the almost-complex structures; i.e. the vector fields $\{\nabla f, I_1 \nabla f, I_2 \nabla f, I_3 \nabla f\}$, suitably normalized, form an orthonormal basis for H_p at points where $\nabla f|_p \neq 0$. That this basis extends a.e. to a global orthonormal frame for H is a consequence of the fact that f satisfies an elliptic PDE (4.2.25) involving the Laplacian Δ^h associated to the extended Riemannian metric h of (3.3.6). Since f is an eigenfunction it cannot vanish identically, the unique continuation result of Aronszajn then implies that f cannot vanish on any open set either. It follows that ∇f cannot vanish on any open set, otherwise $f = \text{const}$ and this would imply that $0 = \Delta f = 4f$, a contradiction.

The properties of the canonical connection ∇ imply that we can find the components $\nabla T^0(X, \nabla f, I_1 \nabla f)$ by finding $\nabla T^0(X, I_1 \nabla f, \nabla f)$ instead through (4.2.10), but we cannot find formulas for the "mixed" components $\nabla T^0(X, I_1 \nabla f, I_2 \nabla f)$ in this way. To find them, we first show that ∇T^0 satisfies the "$5-3$" formula from Lemma

4.2.3 relating the torsion T^0, the normalized QC-Scalar curvature S, the eigenfunction f, and their derivatives. From here, we use our special frame to systematically determine the components $\nabla T^0(I_i\nabla f, I_j\nabla f, I_k\nabla f)$ and in the process obtain formulas for the QC-Ricci 2-forms $\rho_k(I_j\nabla f, \xi_j)$ and the vertical Hessian $\nabla^2 f(\xi_i, \xi_j)$.

With these formulas and several lemmas we proceed to show that the normalized QC-Scalar curvature is constant in Chapter 4.4, in fact $S = 2$. Once it is known that $S = 2$ many of the previous identities simplify considerably and in Lemma 4.5.2 we obtain the following relationships between the components of T^0 and the eigenfunction f and its derivatives:

$$fT_{jk} = \frac{1}{4}\left[f_iT_{kk} - f_kT_{ki}\right] = \frac{1}{4}\left[f_jT_{ij} - f_iT_{jj}\right], \quad fT_{ii} = \frac{1}{4}\left[f_kT_{ij} - f_jT_{ki}\right].$$

At this point, we no longer need to show any further components of T^0 vanish because the identities above are used to show that $|T^0| = 0$ directly in (4.6.1). This proves that M is in fact QC-Einstein. The final conclusion that (M, η) is QC-equivalent to the 3-Sasakian sphere follows from part 2 of Theorem 3.7.1.

Chapter 2

Background

2.1 The Riemannian Case

On a Riemannian manifold (M^n, g) the Levi-Civita connection ∇ is determined by the metric g, and so the geometry of the manifold determines the spectrum of the Laplacian $\Delta = -\operatorname{tr}^g \circ \nabla^2$. On the other hand, if M is compact and g satisfies the lower bound on its Ricci tensor

$$\operatorname{Ric}(X, X) \geq (n-1)g(X, X)$$

then a theorem of Lichnerowicz [44] places the lower bound $\lambda \geq n$ on the positive eigenvalues of the Laplacian. A result of Obata [45] subsequently showed that if there is a function satisfying $\Delta f = nf$ then the manifold is in fact isometric to the round unit sphere $(\mathbb{S}^n, \mathring{g})$, and so the spectrum of the Laplacian can help determine the geometry as well. Explicitly, we have

Theorem 2.1.1. *Suppose (M^n, g) is a compact Riemannian manifold satisfying the lower Ricci bound*

$$Ric(X, X) \geq (n-1)g(X, X). \tag{2.1.1}$$

Then,

- *(Lichnerowicz) if M admits a function f such that $\Delta f = \lambda f$ and $\lambda \neq 0$, then $\lambda \geq n$.*
- *(Obata) if M admits a function f such that $\Delta f = nf$, then M is isometric to the round unit sphere.*

This characterization of a compact manifold satisfying the lower Ricci-bound (2.1.1), and admitting a function that achieves the eigenvalue n, through an isometry with the round sphere is Obata's rigidity result. However, the assumptions of the theorem yield even more information about the eigenfunction f and the metric g, and in anticipation of the Main Theorem 1.0.1 in this dissertation we describe this additional data in the following section.

2.1.1 The Hessian Equation

Let f be any eigenfunction of the Laplacian and the hypotheses of Theorem 2.1.1 hold. Decompose the Hessian of f into a trace-free part plus a multiple of the metric

$$\nabla^2 f = \left(\nabla^2 f\right)_{[0]} + \frac{1}{n}\langle \nabla^2 f, g\rangle g. \tag{2.1.2}$$

Recall Bochner's formula

$$-\frac{1}{2}\Delta(|\nabla f|^2) = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + \mathrm{Ric}(\nabla f, \nabla f) \tag{2.1.3}$$

and integrate this over the compact M using the divergence theorem to find

$$0 = \int_M \left[|(\nabla^2 f)_{[0]}|^2 + \frac{1}{n}(\Delta f)^2 - g(\nabla(\Delta f), \nabla f) + \mathrm{Ric}(\nabla f, \nabla f) \right] \mathrm{Vol}_g. \tag{2.1.4}$$

Then, since $\Delta f = \lambda f$ and $\mathrm{Ric}(\nabla f, \nabla f) \geq (n-1)|\nabla f|^2$, we have

$$0 \geq \int_M |(\nabla^2 f)_{[0]}|^2 \,\mathrm{Vol}_g + \frac{n-1}{n}(n - \lambda) \int_M |\nabla f|^2 \,\mathrm{Vol}_g.$$

The first term above is non-negative, hence this inequality can only hold when $\lambda \geq n$. Consequently, if $\lambda = n$ then the trace-free part $(\nabla^2 f)_{[0]}$ of the Hessian of f must vanish and therefore (2.1.2) shows that

$$\nabla^2 f = \frac{1}{n}\langle \nabla^2 f, g\rangle\, g = \frac{1}{n}\,(-\Delta f)\, g.$$

Thus, if a compact Riemannian manifold satisfies $\mathrm{Ric} \geq (n-1)g$ then

$$\Delta f = nf \qquad \Longleftrightarrow \qquad \nabla^2 f = -fg$$

since the reverse implication holds by definition $\Delta f = -\mathrm{tr}^g(\nabla^2 f)$. In fact, Obata showed in [45] a more general result concerning *complete* manifolds which admit functions satisfying a Hessian equation instead.

Theorem 2.1.2. *A complete Riemannian manifold (M^n, g) admits a non-constant f such that*

$$\nabla^2 f = -fg \tag{2.1.5}$$

if and only if it is isometric to the round unit sphere.

Assuming that M is complete and satisfies the Ricci-bound (2.1.1), the Bonnet-Myers and Hopf-Rinow theorems imply that M is compact. M being compact and supporting a function satisfying the *Riemannian Hessian equation* (2.1.5), along with the generalized Toponogov Theorem, can be used to show that M is isometric to the round unit sphere [17, Chapter 3.4]. Although Theorem 2.1.2 and equation (2.1.5) have their counterparts in the CR and QC cases, and in some cases these sub-Riemannian manifolds have a corresponding Bonnet-Myers theorem (cf. [3, 4]), a sub-Riemannian Toponogov theorem is (currently) an open problem. However, we shall see in subsequent sections that if these sub-Riemannian manifolds are complete with respect to their associated Riemannian metrics then a reduction to Theorem 2.1.2 is possible to prove rigidity.

2.1.2 The Einstein Metric

Other than yielding the Hessian equation by showing that the trace-free part of $\nabla^2 f$ vanishes, the hypotheses of Obata's portion of Theorem 2.1.1 also imply an important property of the metric g, namely that the trace-free part $\mathrm{Ric}_{[0]}$ of its Ricci tensor must also vanish.

Theorem 2.1.3 (S. Ivanov & D. Vassilev, [38]). *Suppose (M, g) is a compact Riemannian manifold of dimension n which satisfies the lower Ricci bound (2.1.1). If the lowest possible eigenvalue is achieved, $\Delta f = nf$ for some function f, then (M, g) is an Einstein space.*

The Ricci tensor of the round sphere is $\mathrm{Ric} = (n-1)\mathring{g}$ and so it is an Einstein manifold, but this theorem shows that you can conclude the Einstein property from the assumptions of Obata's theorem alone, not just from its conclusion. As shown in the previous section, an f as in Theorem 2.1.3 must satisfy the Hessian equation. Taking the covariant derivative of (2.1.5) and employing a Ricci identity leads to a simple formula for the curvature tensor

$$R(X, Y, Z, \nabla f) = df(X)g(Y, Z) - df(Y)g(X, Z)$$

from which the expression $\mathrm{Ric}(X, \nabla f) = (n-1)df(X)$ and a divergence formula

$$\nabla^* R(X, Y, \nabla f) = f\,\mathrm{Ric}(X, Y) - (n-1)fg(X, Y)$$

both follow. These equations are used to show that the Lie derivative $L_{\nabla f}|\mathrm{Ric}_0|^{2k} = 4kf|\mathrm{Ric}_0|^{2k}$ whereupon integration over the compact M and the divergence theorem yield

$$(n-4k)\int_M |\mathrm{Ric}_0|^{2k} f^2 \,\mathrm{Vol}_g = \int_M |\mathrm{Ric}_0|^{2k} |\nabla f|^2 \,\mathrm{Vol}_g.$$

Hence, choosing k large enough shows that $\mathrm{Ric}_0 = 0$ and so g is an *Einstein metric.*

The Main Theorem in this dissertation is an analogue of Theorem 2.1.3 in the 7-dimensional QC case and its proof will deviate significantly from this approach. We will instead exploit the fact that the Ricci tensor of a QC-manifold can be expressed in terms of the torsion of the canonical connection, and the torsion enjoys some very useful properties especially in dimension 7.

2.2 Cauchy-Riemann Manifolds

Let M be a real, oriented $(2n+1)$-dimensional smooth manifold and $TM_{\mathbb{C}} \overset{def}{=} TM \otimes_{\mathbb{R}} \mathbb{C}$ be its complexified tangent bundle. Fix a smooth sub-bundle $T^{1,0}(M) := \mathrm{span}_{\mathbb{C}}\{Z_1, \ldots, Z_n\} \subset TM_{\mathbb{C}}$, of complex dimension n, such that $T^{1,0}(M) \cap \overline{T^{1,0}(M)} = \{0\}$ and $T^{1,0}(M)$ is *formally integrable*:

$$[T^{1,0}(M), T^{1,0}(M)] \subset T^{1,0}(M). \tag{2.2.1}$$

The sub-bundle $T^{1,0}(M)$ is a CR-structure on M, and the pair $(M, T^{1,0}(M))$ is a CR-manifold. Let $H := \mathrm{Re}(T^{1,0}(M) \oplus \overline{T^{1,0}(M)})$ be the real part of the sub-bundle, hereafter referred to as the *horizontal space*. Then, H has an *almost-complex structure*, $J \in \mathrm{End}(H)$ and $J^2 = -\mathrm{id}_H$, given by

$$J(Z + \overline{Z}) := i(Z - \overline{Z}) \qquad \forall Z \in T^{1,0}(M).$$

Let $X_\alpha := \frac{1}{2}(Z_\alpha + \overline{Z}_\alpha)$ then by the above $JX_\alpha = \frac{i}{2}(Z_\alpha - \overline{Z}_\alpha)$, therefore $Z_\alpha = X_\alpha - iJX_\alpha$ and $iZ_\alpha = iX_\alpha + JX_\alpha$. Hence, we can realize each n-dimensional $\mathbb{C}$-vector space $T^{1,0}(M)$ as a $2n$-dimensional $\mathbb{R}$-vector space $H = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ with the endomorphism J playing the role of multiplication by i. In terms of this $\mathbb{R}$-basis for H, the formal integrability condition (2.2.1) amounts to the requirement that $[JX_\alpha, X_\beta] + [X_\alpha, JX_\beta] \in H$ and that the Nijenhuis tensor vanishes: $\forall X, Y \in H$

$$N^J(X,Y) \overset{def}{=} [JX, JY] - [X,Y] - J\left([JX,Y] + [X,JY]\right) \equiv 0. \tag{2.2.2}$$

Finally, suppose H is defined globally by a compatible contact form η, i.e. $H = \ker(\eta)$ and $\mathrm{Vol}_\eta = \eta \wedge (d\eta)^{2n}$ is a volume form on M. As η is a contact form, $d\eta$ is non-degenerate on H, and its compatibility with J means that

$$2g(X,Y) \stackrel{def}{=} -d\eta(JX,Y)$$

is a non-degenerate symmetric bilinear form. If g is also positive definite on H then the CR-structure is said to be *strictly pseudoconvex* and g will be referred to as the *horizontal metric*. The form η is not unique and a choice of one such η is called a *pseudohermitian structure* on M, and in this case $2\omega := d\eta|_H$ is called the *fundamental* 2*-form*. Hereafter an integrable, strictly pseudoconvex, pseudohermitian CR-manifold (M^{2n+1}, H, η) will be referred to as just a CR-manifold. Let V be the complementary to H sub-bundle in TM, the *vertical space*. The extended (Riemannian) metric on $TM = H \oplus V$ is

$$h := g + \eta \otimes \eta, \tag{2.2.3}$$

and we extend the complex structure $J \in \mathrm{End}(H)$ to TM by declaring that $J|_V \equiv 0$. The *Reeb vector field* ξ, uniquely determined by the equations $\eta(\xi) = 1$ and $\xi \lrcorner\, d\eta = 0$, spans the vertical space and $H \perp V$ with respect to the Riemannian metric (2.2.3).

2.2.1 The Tanaka-Webster Connection

On a CR-manifold (M^{2n+1}, H, η) there is a unique linear connection ∇, with torsion T, that preserves the given pseudohermitian structure, the Tanaka-Webster connection [48], [49]. Concretely, this connection satisfies

- The horizontal space is preserved: $\nabla_A \Gamma(H) \subset \Gamma(H)$ for any $A \in \Gamma(TM)$.
- The almost-complex structure J and η are parallel, hence

$$\nabla\xi = \nabla J = \nabla\eta = \nabla g = 0.$$

- For $X, Y \in \Gamma(H)$ the torsion satisfies:

$$T(X,Y) = d\eta(X,Y)\xi = 2\omega(X,Y)\xi = 2g(JX,Y)\xi.$$

For $\xi \in \Gamma(V)$ the torsion $T(\xi, \cdot\,) : H \to H$ is an endomorphism of the horizontal space which satisfies $g(T(\xi,X),Y) = g(T(\xi,Y),X) = -g(T(\xi,JX),JY)$. From the torsion endomorphism we can form the *Webster torsion* (or *pseudohermitian torsion*) $A(X,Y) \overset{def}{=} g(T(\xi,X),Y)$ which therefore satisfies $A(X,Y) = A(Y,X) = -A(JX,JY)$. The Webster torsion is completely trace-free $A(e_\alpha, e_\alpha) = A(e_\alpha, Je_\alpha) = 0$ (the trace here is taken over a local ON-frame $\{e_\alpha\}_{\alpha=1}^{2n}$ for H), and its vanishing is sometimes taken as the definition of a Sasakian manifold, which we now describe.

2.2.2 Kähler and Sasakian Manifolds

A complex manifold, of $\mathbb{C}$-dimension n, is a topological manifold whose coordinate charts are homeomorphisms onto neighborhoods of $\mathbb{C}^n$, such that the transition functions are bi-holomorphic. If $(z^1, \ldots, z^n)$ are the coordinates coming from a complex chart, then the correspondence $z^\alpha = x^\alpha + iy^\alpha$ allows us to also us to also view this as a chart in $\mathbb{R}^{2n}$ giving rise to coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ and thus we can view a complex manifold as a real manifold. The coordinate vector fields define a *canonical* almost-complex structure, $J \in \mathrm{End}(\mathbb{R}^{2n})$ and $J^2 = -\mathrm{id}$, by

$$J\frac{\partial}{\partial x^\alpha} := \frac{\partial}{\partial y^\alpha} \quad \text{and} \quad J\frac{\partial}{\partial y^\alpha} := -\frac{\partial}{\partial x^\alpha}.$$

and this definition is independent of the particular chart since the transition functions are holomorphic. An almost-complex structure is a *complex structure* if it comes from a complex chart as above, and the Newlander-Nirenberg theorem states that this occurs precisely when the Nijenhuis tensor (2.2.2) vanishes.

If g is a Riemannian metric on the complex manifold M, with complex structure J, then g is compatible with J if $g(JX,JY) = g(X,Y)$, in which case the compatible triple (M, J, g) is a *Hermitian manifold.* Let $\omega(X, Y) \overset{def}{=} g(JX, Y)$, then from $J^2 = -\mathrm{id}$ and the compatibility of g, it follows that $\omega(X, Y) = -\omega(Y, X)$. The 2-form ω is called the Kähler form and if ω is closed, $d\omega = 0$, then (M, ω) is a *Kähler manifold.*

Returning to the case of a CR-manifold (M^{2n+1}, J, η), recall that $H = \ker(\eta)$ and $\eta \wedge (d\eta)^n \neq 0$ and so M is necessarily a contact manifold. Let $\mathcal{C}(M) = M^{2n+1} \times \mathbb{R}^+$ be its Riemannian cone, t the coordinate on $\mathbb{R}^+$, h the Riemannian metric on M from (2.2.3), and $\hat{g} := t^2 h + dt \otimes dt$ the cone metric. Let $A \in \Gamma(TM)$ denote any section, $f \in C^\infty(\mathcal{C}(M))$, and define an endomorphism $\hat{J}$ of $T\mathcal{C}(M)$ by

$$\hat{J}\left(A, f\frac{d}{dt}\right) = \left(JA - f\xi, \eta(A)\frac{d}{dt}\right)$$

then $\hat{J}^2 = -\mathrm{id}$, and so $\hat{J}$ is almost-complex structure on $\mathcal{C}(M)$. If $(\mathcal{C}(M), \hat{J}, \hat{g})$ is Kähler then the contact manifold (M, J, η) is said to be *Sasakian* [7, 9].

2.2.3 The CR-Paneitz Operator

In the 3-dimensional CR-case, the analogue of Theorem 2.1.1 has an additional restriction not present in the Riemannian case, which is closely related to the embeddability problem for a CR-manifold (M^3, η) [15]. In particular, if M^3 is Sasakian then it is known that M is embeddable into a complex space [39], and the non-negativity condition (2.2.4) holds [18]. Let $\{e_\gamma\}_{\gamma=1}^{2n}$ be a local ON-frame for H and the summation convention hold. Given a smooth function f define a 1-form, called the *P-form of* f, by

$$P_f(X) := \nabla^3 f(X, e_\gamma, e_\gamma) + \nabla^3 f(JX, e_\gamma, Je_\gamma) + 4nA(X, J\nabla f),$$

and let $P_f(\nabla f)$ be the *P-function of f*. Then, the 4^{th}-order differential operator $f \mapsto \mathcal{C}f$ given by

$$\mathcal{C}f = \nabla^4 f(e_\alpha, e_\alpha, e_\gamma, e_\gamma) + \nabla^4 f(e_\alpha, Je_\alpha, e_\gamma, Je_\gamma) - 4n\nabla^* A(J\nabla f) - 4ng(\nabla^2 f, JA)$$

is the *CR-Paneitz operator*, in fact $\mathcal{C}f = -\nabla^* P_f$. In general, this operator is called non-negative, $\mathcal{C} \geq 0$, if for any $f \in C_0^\infty(M)$ we have

$$\int_M f\,\mathcal{C}(f)\,\mathrm{Vol}_\eta = -\int_M P_f(\nabla f)\,\mathrm{Vol}_\eta \geq 0. \tag{2.2.4}$$

When $n = 1$ the condition that $\mathcal{C} \geq 0$ is a CR-invariant since the CR-Paneitz operator is a conformal invariant [23]. When $n = 1$ and the Webster torsion vanishes, $A \equiv 0$, we also have that the CR-Paneitz operator is non-negative since (up to a constant) $\mathcal{C} = \Box_b \overline{\Box}_b$, where $\Box_b$ is the Kohn-Laplacian [16]. In the CR-analogues of the Obata and Lichnerowicz theorems we will take the non-negativity of the CR-Paneitz operator as an assumption when $n = 1$.

When M is compact and $n > 1$, we always have $\mathcal{C} \geq 0$. To see this let

$$B(X,Y) = (\nabla^2 f)_{[1]}(X,Y) = \frac{1}{2}\left(\nabla^2 f(X,Y) + \nabla^2 f(JX,JY)\right)$$

be the $(1,1)$-component of the horizontal Hessian of f, and let

$$B_0(X,Y) = B(X,Y) - \frac{1}{2n}\langle \nabla^2 f, g\rangle\, g(X,Y) - \frac{1}{2n}\langle \nabla^2 f, \omega\rangle\, \omega(X,Y)$$

be the completely trace-free part of B. Then, we have

Lemma 2.2.1 (C. Graham & J. Lee, [21])**.** *On a compact, strictly pseudoconvex, pseudohermitian CR manifold of dimension* $(2n+1)$*,* $n \geq 1$*, the following identities hold true*

$$-\nabla^* B_0(X) = \frac{n-1}{2n} P_f(X),$$

$$\int_M |B_0|^2\,\mathrm{Vol}_\eta = -\frac{n-1}{2n}\int_M P_f(\nabla f)\,\mathrm{Vol}_\eta = \frac{n-1}{2n}\int_M f\,\mathcal{C}f\,\mathrm{Vol}_\eta$$

In particular, if $n > 1$ *the CR-Paneitz operator is non-negative.*

Aside from its geometric significance, the CR-Paneitz operator appears as an additional term when we perform the integration-by-parts argument (2.1.4) on the analogous CR Bochner formula (cf. Section 2.2.4). Similarly, the QC-Paneitz operator appears when intergrating the QC-Bochner formula, and we these reserve the details for Chapter 3.6.

2.2.4 The CR Lichnerowicz-type Theorem

The CR analogue of Lichnerowicz' contribution to Theorem 2.1.1 first began with Greenleaf in [22] for $n > 2$, continued with $n = 2$ in [40], and finally $n = 1$ in [18]. The proof is similar to how Bochner's formula (2.1.3), along with the compactness of M and the lower Ricci-bound (2.1.1), were used to give a lower bound on the spectrum of the Laplacian and show that the eigenfunction $\Delta f = nf$ must satisfy a Hessian equation (2.1.5).

For some positive constant $k_0 > 0$, the CR lower Ricci-bound (or Lichnerowicz-type bound) is

$$\mathrm{Ric}(X,X) + 4A(X,JX) \geq k_0 g(X,X). \tag{2.2.5}$$

and the CR-Bochner formula, in real coordinates from [36], is given by:

$$\begin{aligned}\frac{1}{2}\Delta\left(|\nabla f|^2\right) = -g\left(\nabla(\Delta f), \nabla f\right) + \mathrm{Ric}(\nabla f, \nabla f)& \\ + 2A(J\nabla f, \nabla f) + |\nabla^2 f|^2 + 4\nabla^2 f(\xi, J\nabla f).&\end{aligned} \tag{2.2.6}$$

After several lemmas and integration over the compact M, one arrives at the following identity [36, (8.15)]:

$$\begin{aligned}0 = &\int_M \left[\mathrm{Ric}(\nabla f, \nabla f) + 4A(J\nabla f, \nabla f) - \frac{n+1}{n}(\Delta f)^2\right] \mathrm{Vol}_\eta \\ &+ \int_M \left[|(\nabla^2 f)_{[1]}|^2 - \frac{1}{2n}(\Delta f)^2 - \frac{1}{2n} g(\nabla^2 f, \omega) + |(\nabla^2 f)_{[-1]}|^2 - \frac{3}{2n} P_f(\nabla f)\right] \mathrm{Vol}_\eta.\end{aligned}$$

Now suppose f is an eigenfunction $\Delta f = \lambda f$, and use the Lichernowicz-type bound (2.2.5) with $X = \nabla f$ into the above to find

$$0 \geq \int_M \left[\left(-\frac{n+1}{n}\lambda + k_0 \right) |\nabla f|^2 + |(\nabla^2 f)_{[-1]}|^2 - \frac{3}{2n} P_f(\nabla f) \right] \mathrm{Vol}_\eta \,.$$

To control the additional term involving the P-function of f we invoke condition (2.2.4) and thus arrive at the CR-Lichnerowicz theorem.

Theorem 2.2.2 (A. Greenleaf [22], S.-Y Li & H.-S. Luk [40], H.-L. Chiu [18])**.** *Let (M, η) be a compact, strictly pseudoconvex, pseudohermitian manifold of dimension $(2n+1)$ such that for some $k_0 =$ constant > 0 we have the Lichnerowicz-type bound*

$$\mathrm{Ric}(X,X) + 4A(X,JX) \geq k_0 g(X,X) \qquad \forall X \in \Gamma(H). \tag{2.2.7}$$

- *If $n = 1$, then any eigenvalue λ of the sub-Laplacian Δ satisfies the inequality*
$$\lambda \geq \frac{n}{n+1} k_0.$$
- *If $n = 1$ and the CR-Paneitz operator is non-negative (cf. (2.2.4)), $\mathcal{C} \geq 0$, then*
$$\lambda \geq \frac{1}{2} k_0.$$

As described in Chapter 2.1.1 for the Riemannian case , an "*extremal first eigenfunction*", i.e. an f which realizes the bottom of the spectrum of Δ, has a special form for its Hessian $\nabla^2 f$. Similarly, in the CR-case an extremal first eigenfunction has a special form for its horizontal Hessian (cf. (3.3.3)):

Theorem 2.2.3 (S. Ivanov & D. Vassilev, [36])**.** *Let M be a compact, strictly pseudoconvex CR manifold of dimension $(2n+1)$, $n \geq 1$, satisfying*

$$\mathrm{Ric}(X,X) + 4A(X,JX) \geq k_0 g(X,X)$$

while if $n = 1$ assume, further, that the CR-Paneitz operator is non-negative on f, i.e. (2.2.4) holds true. If $\frac{n}{n+1}k_0$ is an eigenvalue of the sub-Laplacian, then the corresponding eigenfunctions satisfy the identity

$$\nabla^2 f(X,Y) = -\frac{k_0}{2(n+1)} f g(X,Y) - df(\xi)\omega(X,Y).$$

Using a homothety, one can reduce to the case that the smallest eigenvalue $\lambda_1 = 2n$ and $k_0 = 2(n+1)$, and therefore the *horizontal Hessian equation* becomes

$$\nabla^2 f(X,Y) = -fg(X,Y) - df(\xi)\omega(X,Y), \qquad X,Y \in \Gamma(H). \tag{2.2.8}$$

In anticipation of the QC-case, we mention [36, Remark 8.2] which states if f is extremal first eigenfunction then equality is achieved in the Lichnerowicz-type bound and the integral of its P-form vanishes:

$$\mathrm{Ric}(\nabla f, \nabla f) + 4A(J\nabla f, \nabla f) = k_0|\nabla f|^2 \quad \text{and} \quad \int_M P_f(\nabla f)\mathrm{Vol}_\eta = 0.$$

2.2.5 The CR Obata-type Theorem

In the case of a complete Riemannian manifold (M^n, g), Theorem 2.1.2 characterized the round unit sphere through the Hessian equation (2.1.5). Unlike the Riemannian case, the best known Obata-type Theorem for a complete *non-compact* CR-manifold (M^{2n+1}, η) uses the vanishing of the Webster torsion. When $n = 1$ this is taken as an assumption, when $n > 1$ one can show that Webster torsion vanishes if it was divergence-free.

Theorem 2.2.4 (S. Ivanov & D. Vassilev, [36])**.** *Let (M, η) be a strictly pseudoconvex, psuedohermitian CR-manifold of dimension $2n+1 \geq 5$ with a divergence-free pseudohermitian torsion, $\nabla^* A = 0$. Assume, further, that M is complete with respect to the Riemannian metric*

$$h = g + \eta \otimes \eta.$$

If there is a smooth function $f \not\equiv 0$ whose Hessian with respect to the Tanaka-Webster connection satisfies

$$\nabla^2 f(X,Y) = -fg(X,Y) - df(\xi)\omega(X,Y) \qquad X,Y \in H = \ker(\eta)$$

then, up to a scaling of η by a positive constant, (M, η) is the standard (Sasakian) CR-structure on the unit sphere in $\mathbb{C}^{n+1}$. In dimension three the above result holds provided the pseudohermitian torsion vanishes, $A = 0$.

In the proof of the $n > 1$ case, the authors arrive at the equation $|\nabla f|^2 A = 0$, and in order to conclude that A vanishes they must show that f cannot be a local constant. To see this, they prove that if f satisfies (2.2.8) then f also satisfies an elliptic equation involving the Riemannian Laplacian $\Delta^h f = \Delta f - \nabla^2 f(\xi, \xi)$ (i.e. the Laplacian with respect to the Riemannian metric (2.2.3)). In particular, we have from [36, Corollary 4.5 & Lemma 5.1]:

$$\begin{aligned} \Delta^h f &= (2n+1)f - \frac{1}{n}(\nabla^* A)(J\nabla f), && \text{if } n > 1 \\ \Delta^h f &= \left(2 + \frac{S-2}{6}\right) f - \frac{1}{12} g(\nabla f, \nabla S) + \frac{1}{3}(\nabla^* A)(J\nabla f), && \text{if } n = 1. \end{aligned}$$

Therefore, by a unique continuation result f is not a local constant and we must have that $A = 0$. In order to reduce to the Riemannian Obata Theorem 2.1.2, it's then shown that f satisfies the Riemannian Hessian Equation (2.1.5) with respect to the Levi-Civita connection of h, $(\nabla^h)^2 f = -fh$, hence (M^{2n+1}, h) is isometric to the round unit sphere and $A = 0$, i.e. M is Sasakian. With these observations one can apply [13, 14] to conclude that (M^{2n+1}, H, η) is (up to a scaling of η) the standard Sasakian structure on the unit sphere in $\mathbb{C}^{n+1}$.

The compact $n = 1$ case of the Obata-type theorem was completed in [37]. In the compact $n > 1$ case, the assumption of divergence-free Webster torsion in [36, 37] was able to be removed by [42, 43]. In the case of a compact Sasakian manifold Theorem 2.2.4 in fact characterizes the unit Sasakian sphere by the horizontal Hessian equation (2.2.8). If f satisfies the horizontal Hessian equation then it's also an extremal eigenfunction, $\Delta f = 2nf$, this observation and several others lead to the CR Obata-type theorem.

Theorem 2.2.5 (S. Ivanov & D. Vassilev [37], S.-Y Li & X. Wang [42, 43])**.** *Suppose* (M, J, η)*,* $dim(M) = 2n + 1$*, is a compact strictly pseudoconvex, psuedohermitian manifold which satisfies the Lichnerowicz-type bound* (2.2.7)*. If* $n \geq 2$*, then* $\lambda = \frac{n}{n+1} k_0$ *is an eigenvalue if and only if up to a scaling* (M, J, η) *is the standard pseudohermitian CR-structure on the unit sphere in* $\mathbb{C}^{n+1}$*. If* $n = 1$ *the same conclusion holds assuming in addition that the CR-Paneitz operator is non-negative,* $\mathcal{C} \geq 0$*.*

Chapter 3

Quaternionic Contact Manifolds

3.1 The Quaternions

As an $\mathbb{R}$-vector space the quaternions $\mathbb{H}$ are isomorphic to $\mathbb{R}^4$ and so each quaternion can be identified with an ordered quadruple of real numbers; $\mathbb{H} = \{(t, x, y, z) : t, x, y, z \in \mathbb{R}\}$ with addition and multiplication by scalars done as in $\mathbb{R}^4$. However, since the quaternions are also a division ring over $\mathbb{R}$ we adopt a notation that naturally describes how multplication works on $\mathbb{H}$, just like we do with the complex numbers $\mathbb{C}$.

First, we use the symbols $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$, and $k = (0, 0, 0, 1)$ for the standard basis of $\mathbb{R}^4$, so that we may write

$$\mathbb{H} = \{t + xi + yj + zk : t, x, y, z \in \mathbb{R}\}.$$

Then, we define the associative multiplication of $\mathbb{H}$ in terms of these symbols

$$i^2 = j^2 = k^2 = ijk = (-1, 0, 0, 0) = -1$$
$$ij = (0, 0, 0, 1) = k, \quad jk = (0, 1, 0, 0) = i, \quad ki = (0, 0, 1, 0) = j,$$

and the standard distributive laws that a ring over $\mathbb{R}$ enjoys. The identies above can be used to show that $ji = -ij$ and so, unlike multiplication in $\mathbb{C}$, quaternionic multiplication is not commutative. In general, if $q = t + ix + jy + kz$ is a quaternion then we can form its *conjugate* $\bar{q} = t - ix - jy - kz$ and this allows us to identify the real and imaginary parts of q:

$$\mathrm{Re}(q) = \frac{1}{2}(q + \bar{q}), \qquad \mathrm{Im}(q) = \frac{1}{2}(q - \bar{q}),$$

and the corresponding real and imaginary subspaces of $\mathbb{H}$

$$\mathrm{Re}(\mathbb{H}) = \{t : t \in \mathbb{R}\}, \qquad \mathrm{Im}(\mathbb{H}) = \{xi + yj + zk : x, y, z \in \mathbb{R}\}.$$

Conjugation also allows us to write the Hermitian inner product on $\mathbb{H}$ as $(q, p) = q\bar{p}$ and from this we can recover the Euclidean inner product

$$\langle q, p\rangle := \mathrm{Re}(q, p) = \frac{1}{2}(q\bar{p} + p\bar{q}),$$

and the *modulus* of a quaternion $|q|^2 = \langle q, q\rangle = t^2 + x^2 + y^2 + z^2$. Finally, if $q \neq 0$ then q is invertible, with its inverse given by

$$q^{-1} = \frac{1}{|q|^2}\bar{q}.$$

3.2 The Compact Symplectic Group

The Cartesian product $\mathbb{H}^n = \{(q^1, \ldots, q^n) : q^i \in \mathbb{H}\}$ is isomorphic to $\mathbb{R}^{4n}$ as an $\mathbb{R}$-module, but since we can multiply the quaternions we can also view $\mathbb{H}^n$ as a module over $\mathbb{H}$. However, because quaternionic multiplication is not commutative there is a distinction between left and right $\mathbb{H}$-modules, and our convention is that multiplication acts on the right

$$\mathbb{H} \times \mathbb{H}^n \to \mathbb{H}^n : (q, (q^1, \ldots, q^n)) \mapsto (q^1 q, \ldots, q^n q).$$

We also have Hermitian and Euclidean inner-products, as well as a norm on $\mathbb{H}^n$:

$$(q,p) = \sum_{i=1}^{n} q^i \bar{p}^i, \quad \langle q,p\rangle = \mathrm{Re}(q,p), \quad |q|^2 = \langle q,q\rangle. \tag{3.2.1}$$

The $\mathbb{H}$-linear transformations of $\mathbb{H}^n$ can be identified with the $n \times n$ matrices with quaternions as entires, $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) \cong M_{n\times n}(\mathbb{H})$, and those linear transformations which are also invertible are then $\mathrm{Aut}(\mathbb{H}^n) \cong \mathrm{GL}_n(\mathbb{H})$. Those automorphisms of $\mathbb{H}^n$ which preserve the Hermitian inner-product in (3.2.1) form the *compact Symplectic group*, a Lie group also known as the *quaternionic unitary group* $U(n,\mathbb{H})$, which can thus be described as

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}_n(\mathbb{H}) : A^\dagger A = AA^\dagger = \mathrm{id}_{\mathbb{H}^n}\}.$$

where $A^\dagger$ is the quaternionic conjugate transpose. In particular, since every non-zero quaternion is invertible we have $\mathrm{GL}_1(\mathbb{H}) \cong \mathbb{H}^*$, and we can identify the unit quaternions with the unit sphere in $\mathbb{R}^4$, i.e. $\mathrm{Sp}(1) = \{q \in \mathbb{H}^* : |q| = 1\} \cong \mathbb{S}^3$.

Let $v \in \mathbb{H} \cong \mathbb{R}^4$ and $p,q \in \mathrm{Sp}(1)$ be unit quaternions. Every map of the form $v \mapsto qvp^{-1}$ defines an element of $\mathrm{SO}(4)$. Therefore, we have a surjective homomorphism $\mathrm{Sp}(1)\times\mathrm{Sp}(1) \to \mathrm{SO}(4)$ whose kernel is $\langle(1,1),(-1,-1)\rangle \cong \mathbb{Z}_2$ since multiplication by real scalars is commutative in $\mathbb{H}$. We denote the isomorphic image of $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ in $\mathrm{SO}(4)$ by $\mathrm{Sp}(1)\mathrm{Sp}(1) \cong (\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\mathbb{Z}_2$. In general, let $\mathbb{H}$ act on $\mathbb{H}^n \cong \mathbb{R}^{4n}$ on the right $(q,(q^1,\dots,q^n)) \mapsto (q^1q^{-1},\dots,q^nq^{-1})$, this identifies the unit quaternions $\mathrm{Sp}(1)$ with a subgroup of $\mathrm{SO}(4n)$ (which we regard as endomorphisms of $\mathbb{H}^n$ acting on the *left*) and gives us another description of the compact Symplectic group

$$\mathrm{Sp}(n) = \{A \in \mathrm{SO}(4n) : AB = BA \quad \forall B \in \mathrm{Sp}(1)\}.$$

Once again, we obtain a surjective homomorphism $\mathrm{Sp}(n)\times\mathrm{Sp}(1) \to \mathrm{SO}(4n)$ and the isomorphic image of this map in $\mathrm{SO}(4n)$ is $\mathrm{Sp}(n)\mathrm{Sp}(1) \cong (\mathrm{Sp}(n)\times\mathrm{Sp}(1))/\mathbb{Z}_2$.

The Lie algebra of $\mathrm{Sp}(n)$ consists of the $n\times n$ skew-Hermitian matrices with quaternions as entries $\mathfrak{sp}(n) = \{A \in \mathfrak{gl}_n(\mathbb{H}) : A + A^\dagger = 0\}$ with the usual bracket. In particular, if $I, J, K \in \mathrm{SO}(4n)$ represent the images of $i, j, k \in \mathrm{Sp}(1)$, then $\mathfrak{sp}(1) = \mathrm{span}_{\mathbb{R}}\{I, J, K\}$. The Lie algebra of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. Just like the action of $\mathrm{Sp}(n)$ on $\mathbb{H}^n$ preserves its Hermitian inner-product, the action of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{H}^n$ preserves the Euclidean inner-product in (3.2.1).

3.3 Quaternionic Contact Structures

The notion of a *Quaternionic Contact Structure* was first introduced by O. Biquard [5, 6], but we follow the explicit description from [26]. A $(4n+3)$-dimensional smooth manifold M with a co-dimension 3 distribution H is a QC-manifold if

1. H has a conformal $\mathrm{Sp}(n)\mathrm{Sp}(1)$-structure:
   - a conformal class of metrics $[g]$,
   - a 2-sphere bundle $\mathbb{Q} \subset \mathrm{End}(H)$ locally generated by three almost-complex structures $\{I_s\}_{s=1}^3$ such that $I_s^2 = -\mathrm{id}_H$, and $I_1 I_2 = -I_2 I_1 = I_3$,
   - each I_s is Hermitian compatible with each $g \in [g]$: $g(I_s\cdot\,, I_s\cdot) = g(\cdot\,,\cdot)$.
2. There is an $\mathbb{R}^3$-valued 1-form $\eta := (\eta_1, \eta_2, \eta_3)$ such that locally $H = \bigcap_s \ker(\eta_s)$ and $2g(I_s\cdot\,,\cdot) = d\eta_s(\cdot\,,\cdot)$ for some $g \in [g]$ and any $s \in \{1, 2, 3\}$.

Hereafter we shall refer to H as the "horizontal space." As we shall see in the following section, Biquard showed that there is a unique sub-bundle V, the "vertical space," such that $TM = H \oplus V$. The three *fundamental* 2*-forms* of the QC-structure are determined by

$$2\omega_s|_H = d\eta_s|_H \quad \text{and} \quad \xi \lrcorner\, \omega_s = 0, \quad \forall \xi \in V. \tag{3.3.1}$$

The choice of η, I_s and g is not unique: if locally $\bar{\eta}$ is another 1-form generating H, with corresponding horizontal metric $\bar{g} \in [g]$ and almost-complex structures $\{\bar{I}_s\}_{s=1}^3$, then $\bar{\eta} = \mu\Psi\eta$ for some $\mathrm{SO}(3)$-valued function Ψ, and a smooth $\mu > 0$. However, if

η is fixed then I_s and g are unique if they exist. If instead H and g are fixed, then there is at most one family of associated 1-forms η and 2-sphere bundle $\mathbb{Q}$:

Lemma 3.3.1 (O.Biquard, [5], [35]). *Let $(M, H, [g], \mathbb{Q})$ be a QC-manifold, then*

1. *If (η, I_s, g) and $(\eta, \bar{I}_s, \bar{g})$ are two QC-structures on M, then $I_s = \bar{I}_s$, $s = 1, 2, 3$, and $g = \bar{g}$.*
2. *If (η, g) and $(\bar{\eta}, g)$ are two QC-structures on M with $\ker(\eta) = \ker(\bar{\eta}) = H$, then $\mathbb{Q} = \bar{\mathbb{Q}}$ and $\bar{\eta} = \Psi\eta$ for some matrix $\Psi \in SO(3)$ with smooth functions as entries.*

In general, we will let $(M, g, \mathbb{Q})$ be a QC-manifold with a fixed horizontal metric g and sphere bundle $\mathbb{Q}$, then H indeed has an $\mathrm{Sp}(n)\mathrm{Sp}(1)$-structure, and we denote with η any locally defined associated 1-form.

3.3.1 Invariant Decompositions

Any endomorphism $\Psi \in \mathrm{End}(H)$ can be decomposed with respect to the quaternionic structure $(g, \mathbb{Q})$ uniquely into four $\mathrm{Sp}(n)$-invariant parts:

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_s, Ψ^{+--} commutes with I_1 and anti-commutes with I_2 and I_3, etc. Explicitly,

$$4\Psi^{+++} = \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3, \quad 4\Psi^{+--} = \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3$$
$$4\Psi^{-+-} = \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3, \quad 4\Psi^{--+} = \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.$$

Each $\Psi \in \mathrm{End}(H)$ determines a bilinear form on H through the horizontal metric g, for example the fundamental 2-forms (3.3.1) ω_s are the bilinear forms determined by the almost-complex structures, $\omega_s(X, Y) = g(I_s X, Y)$, and the horizontal metric itself is $g(X,Y) = g(\mathrm{id}_H X, Y)$. Other than these two exceptions, we will usually denote the endomorphism of H and the bilinear form it determines by the same letter $\Psi(X,Y) := g(\Psi X, Y)$.

Let $\Upsilon := I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$ be the Casimir operator, it acts on $\mathrm{End}(H)$ by $(\Upsilon\Psi)(X,Y) = \Psi(I_1X, I_1Y) + \Psi(I_2X, I_2Y) + \Psi(I_3X, I_3Y)$ and satisfies $(\Upsilon - 3I)(\Upsilon + I) = 0$. The two $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant components of Ψ

$$\Psi_{[3]}(X,Y) = \frac{1}{4}\left[\Psi(X,Y) + \Psi(I_1X, I_1Y) + \Psi(I_2X, I_2Y) + \Psi(I_3X, I_3Y)\right]$$
$$\Psi_{[-1]}(X,Y) = \frac{1}{4}\left[3\Psi(X,Y) - \Psi(I_1X, I_1Y) - \Psi(I_2X, I_2Y) - \Psi(I_3X, I_3Y)\right]$$

are projections onto the eigenspaces of Υ corresponding to the eigenvalues 3 and -1, respectively [11]. From the formulas above we see that $g_{[3]} = g$ and $\omega_{s[-1]} = \omega_s$, and when $n = 1$ we have from [30, Lemma 2.1]:

Lemma 3.3.2. *The space $\Psi_{[3]}$ is four dimensional and the symmetric tensors in it are proportional to the metric. The space $\Psi_{[-1]}$ is twelve dimensional, in which lies the three dimensional space of the 2-forms ω_i. The latter determines the anti-symmetric part of the $\Psi_{[-1]}$-component.*

The horizontal metric g induces an inner-product on $\mathrm{End}(H)$ which we will denote with angle-brackets $\langle\cdot\,,\cdot\rangle$. If $A, B \in \mathrm{End}(H)$, and $\{e_\alpha\}_{\alpha=1}^{4n}$ is a local orthonormal frame for H, then

$$\langle A, B\rangle \stackrel{def}{=} \sum_{\alpha=1}^{4n} g\left(Ae_\alpha, Be_\alpha\right)$$

and every $\Psi \in \mathrm{End}(H)$ has an $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant orthogonal decomposition $\Psi = \Psi_{[3]} \oplus \Psi_{[-1]}$ with respect to this inner-product. With this inner-product, we can also realize each trace of an endomorphism A as a projection onto a subspace of $\mathrm{End}(H)$:

$$\langle A, g\rangle = \sum_{\alpha=1}^{4n} A\left(e_\alpha, e_\alpha\right), \qquad \langle A, \omega_s\rangle = \sum_{\alpha=1}^{4n} A\left(e_\alpha, I_s e_\alpha\right). \tag{3.3.2}$$

In particular, for a smooth function f and the canonical connection ∇ on a QC-manifold (cf. section 3.3.2), its *horizontal gradient* $\nabla f \in \Gamma(H)$ is defined through the equation $df(X) = g(\nabla f, X)$ for $X \in \Gamma(H)$. Since this connection preserves the splitting $TM = H \oplus V$, we can define an endomorphism of H by $X \mapsto \nabla_X(\nabla f)$. This endomorphism determines the bilinear form we call the *horizontal Hessian* $\nabla^2 f$:

$$\nabla^2 f(X,Y) \overset{def}{=} g(\nabla_X(\nabla f), Y), \qquad X, Y \in \Gamma(H). \tag{3.3.3}$$

Following (3.3.2), we will write the following traces as

$$\langle \nabla^2 f, g\rangle = \sum_{\alpha=1}^{4n} \nabla^2 f(e_\alpha, e_\alpha), \qquad \langle \nabla^2 f, \omega_s\rangle = \sum_{\alpha=1}^{4n} \nabla^2 f(e_\alpha, I_s e_\alpha).$$

3.3.2 The Canonical Connection

The canonical connection ∇ on a QC-manifold $(M^{4n+3}, g, \mathbb{Q})$ was discovered by O. Biquard [5] when $n > 1$ and by D. Duchemin [20] in the $n = 1$ case, and we will follow the conventions adopted by Biquard.

Theorem 3.3.3 (O. Biquard, [5])**.** *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n+3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then, there exists a unique connection ∇ with torsion T on M^{4n+3}, and a unique supplementary subspace V to H in TM, such that:*

- *∇ preserves the decomposition $H \oplus V$ and the metric g;*
- *for $X, Y \in H$, one has $T(X,Y) = -[X,Y]|_V$;*
- *∇ preserves the Sp(n)Sp(1)-structure on H, i.e. $\nabla g = 0$ and $\nabla_X \Gamma(\mathbb{Q}) \subset \Gamma(\mathbb{Q})$;*
- *for $\xi \in V$, the torsion endomorphism $T(\xi, \cdot)|_H$ of H lies in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^\perp \subset \mathfrak{gl}(4n)$;*
- *the connection on V is induced by the natural identification φ of V with the subspace $\mathfrak{sp}(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.*

When $n > 1$, Biquard also described the supplementary distribution V, which is generated locally by three *Reeb vector fields* $\{\xi_1, \xi_2, \xi_3\}$ determined through:

$$\eta_s(\xi_t) = \delta_{st}, \quad (\xi_s \lrcorner d\eta_s)\,|_H = 0, \quad (\xi_s \lrcorner d\eta_t)\,|_H = -\,(\xi_t \lrcorner d\eta_s)\,|_H . \tag{3.3.4}$$

Given a QC-structure $(M^{4n+3}, g, \mathbb{Q})$ it may not be possible to find three vector fields $\{\xi_1, \xi_2, \xi_3\}$ that satisfy the relations in (3.3.4) when $n = 1$. However, Duchemin showed in [20] that if we assume the existence of Reeb fields satisfying (3.3.4), then Theorem 3.3.3 holds. Henceforth, by a seven-dimensional QC-structure on M we will always mean a structure satisfying (3.3.4).

The equations in (3.3.4) also describe the $\mathfrak{sp}(1)$-connection 1-forms α_s for ∇ on the bundle $\mathbb{Q}$ (c.f. [25, Proposition 3.5, Corollary 3.6]). The isomorphism $\varphi : V \to \mathfrak{sp}(1)$, $\nabla\varphi = 0$, then yields the corresponding connection forms on V as well:

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j \quad \text{and} \quad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j \tag{3.3.5}$$

for any cyclic permutation $(i\,j\,k)$ of $(1\,2\,3)$. Using the Reeb fields we can extend the horizontal metric g on H to a Riemannian metric h on M by requiring $V \perp H$ and the Reeb fields are orthonormal. In fact,

$$h \overset{def}{=} g + \sum_{s=1}^{3} \eta_s \otimes \eta_s \tag{3.3.6}$$

achieves exactly this. Since g is parallel from Theorem 3.3.3, the 1-forms η_s satisfy the same identites in (3.3.5), h is symmetric and the Biquard connection preserves the type of a tensor, it follows that $\nabla h = 0$. Indeed, using the convention (0.0.1):

$$\begin{aligned} \nabla h &= \nabla g + \nabla \left(\sum_{s=1}^{3} \eta_s \otimes \eta_s \right) \\ &= \sum_{(i\,j\,k)} \left[(-\alpha_j \otimes \eta_k + \alpha_k \otimes \eta_j) \otimes \eta_i + \eta_i \otimes (-\alpha_j \otimes \eta_k + \alpha_k \otimes \eta_j) \right] = 0. \end{aligned}$$

We also extend the almost-complex structures to all of TM by setting $I_s|_V = 0$. The Riemannian metric does not depend on the action of SO(3) on V, but if η

undergoes a conformal change then h is multiplied by the same conformal factor (c.f. [25, Chapter 5]). The QC-analogue of a normal neighborhood of a point $p \in M$ is supplied by [25, Lemma 4.5]:

Theorem 3.3.4 ([25]). *In a neighborhood of any point $p \in M^{4n+3}$ and a $\mathbb{Q}$ -orthonormal basis*

$$\{X_1(p), X_2(p) = I_1X_1(p), \dots, X_{4n}(p) = I_3X_{4n-3}(p), \xi_1(p), \xi_2(p), \xi_3(p)\}$$

of the tangent space at p, there exists a $\mathbb{Q}$-orthonormal frame field

$$\{X_1, X_2 = I_1X_1, \dots, X_{4n} = I_3X_{4n-3}, \xi_1, \xi_2, \xi_3\}, \quad X_\alpha|_p = X_\alpha(p),\ \xi_s|_p = \xi_s(p),$$

for $\alpha = 1, \dots, 4n$ and $s = 1, 2, 3$, such that the connection 1-forms of the Biquard connection are all zero at the point p, i.e. we have

$$(\nabla_{X_\alpha}X_\beta)|_p = (\nabla_{\xi_i}X_\beta)|_p = (\nabla_{X_\alpha}\xi_t)|_p = (\nabla_{\xi_t}\xi_s)|_p = 0,$$

for $\alpha, \beta = 1, \dots, 4n$ and $s, t, r = 1, 2, 3$. In particular,

$$((\nabla_{X_\alpha}I_s)X_\beta)|_p = ((\nabla_{X_\alpha}I_s)\xi_t)|_p = ((\nabla_{\xi_t}I_s)X_\beta)|_p = ((\nabla_{\xi_t}I_s)\xi_r)|_p = 0.$$

Finally, for a local orthonormal frame $\{e_\gamma\}_{\gamma=1}^{4n}$ of H, the (horizontal) divergence of a horizontal vector field/1-form $\sigma \in \Lambda^1(H)$, is defined by

$$\nabla^*\sigma = -\mathrm{tr}|_H(\nabla\sigma) = -\nabla\sigma(e_\gamma, e_\gamma).$$

This yields the "integration by parts" formula on a compact M [25, 47]:

$$\int_M (\nabla^*\sigma)\mathrm{Vol}_\eta = 0. \tag{3.3.7}$$

3.3.3 Torsion and Curvature

From Theorem 3.3.3, the torsion $T(A, B) = \nabla_AB - \nabla_BA - [A, B]$ of the Biquard connection restricted to the horizontal space is given by $T(X, Y) = -[X, Y]|_V=$

$2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s$, but when one of its arguments is vertical we obtain the *torsion endomorphism*: $T_\xi \overset{def}{=} T(\xi,\cdot)|_H \in \mathrm{End}(H)$. Decomposing the torsion endomorphism $T_\xi \in (\mathfrak{sp}(n)\oplus\mathfrak{sp}(1))^\perp$ into its symmetric T^0_ξ and skew-symmetric b_ξ parts

$$T_\xi = T^0_\xi + b_\xi, \qquad \xi \in V,$$

we have the following theorem as stated in [25, Proposition 2.5]:

Theorem 3.3.5 (O. Biquard, [5])**.** *The torsion T_ξ is completely trace-free,*

$$tr(T_\xi) = \sum_{\alpha=1}^{4n} g(T_\xi(e_\alpha), e_\alpha) = 0, \quad tr(T_\xi \circ I) = \sum_{\alpha=1}^{4n} g(T_\xi(e_\alpha), Ie_\alpha) = 0, \ \ I \in \mathbb{Q},$$

where $e_1, \ldots, e_{4n}$ is an orthonormal basis of H. The symmetric part of the torsion has the properties:

$$T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \qquad i = 1,2,3.$$

In addition, we have

$$I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}.$$

The skew-symmetric part can be represented in the following way

$$b_{\xi_i} = I_i u, \qquad i = 1,2,3,$$

where u is a traceless symmetric $(1,1)$-tensor on H which commutes with I_1, I_2, I_3. If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor $T_\xi = T^0_\xi$.

Therefore, we can write $T_{\xi_i} = T^0_{\xi_i} + I_i u$ and from this decomposition the authors [25] define two $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant, symmetric, trace-free tensors on H

$$T^0(X,Y) \overset{def}{=} g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X, Y), \qquad U(X,Y) \overset{def}{=} g(uX,Y) \tag{3.3.8}$$

which enjoy the properties that

$$T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0, \tag{3.3.9}$$
$$3U(X,Y) - U(I_1X, I_1Y) - U(I_2X, I_2Y) - U(I_3X, I_3Y) = 0.$$

According to Theorem 3.3.5 in dimension 7 the tensor U vanishes identically, $U \equiv 0$. In addition, letting $T^0(\xi_s, I_sX, Y) := g(T^0(\xi_s, I_sX), Y)$, we can recover the torsion endomorphism and its symmetric part from T^0 [33, Proposition 2.3]:

$$4T^0(\xi_s, I_sX, Y) = T^0(X,Y) - T^0(I_sX, I_sY) \tag{3.3.10}$$
$$T(\xi_s, I_sX, Y) = \frac{1}{4}\left[T^0(X,Y) - T^0(I_sX, I_sY)\right] - U(X,Y)$$

Since the Biquard connection preserves the splitting $TM = H \oplus V$ and the bundle $\mathbb{Q}$, the *curvature operator* $R(A,B) = [\nabla_A, \nabla_B] - \nabla_{[A,B]}$ also preserves them. In particular, we may regard $R(A,B)|_H$ as an endomorphism and we have $R(A,B) \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$.Let $R(A,B,C,D) := h(R(A,B)C,D)$ for $A,B,C,D \in \Gamma(TM)$ and $\{e_\gamma\}_{\gamma=1}^{4n}$ be a local ON-frame for H. The QC-Ricci tensor and QC-Scalar curvature are defined similar to the Riemannian setting, except all traces are taken on H:

$$\mathrm{Ric}(B,C) := \sum_{\gamma=1}^{4n} R(e_\gamma, B, C, e_\gamma), \qquad \mathrm{Scal} := \sum_{\gamma=1}^{4n} \mathrm{Ric}(e_\gamma, e_\gamma) \tag{3.3.11}$$

The *normalized QC-Scalar curvature* is $S := \mathrm{Scal}/8n(n+2)$. Through the almost-complex structures, we define six additional QC Ricci-type tensors:

$$\rho_s(A,B) := \frac{1}{4n}\sum_{\gamma=1}^{4n} R(A,B,e_\gamma, I_se_\gamma), \quad \zeta_s(B,C) := \frac{1}{4n}\sum_{\gamma=1}^{4n} R(e_\gamma, B, C, I_se_\gamma)$$

where the ρ_s are called the *QC-Ricci 2-forms*. From [26, Lemma 4.3.2] the curvature of the Biquard connection on V is determined by

$$2\rho_k(A,B) = R(A,B,\xi_i,\xi_j), \quad A,B \in \Gamma(TM), \;\; \xi_s \in \Gamma(V).$$

When restricted to H, the tensors above can be expressed in terms of the torsion of the Biquard connection [25], see also [26, 33]. In particular, from [33, Theorem 2.4] we have the following identities:

$$
\begin{aligned}
&\mathrm{Ric}(X,Y) = (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + 2(n+2)Sg(X,Y)\\
&\rho_s(X,I_sY) = -\frac{1}{2}\left[T^0(X,Y) + T^0(I_sX,I_sY)\right] - 2U(X,Y) - Sg(X,Y)\\
&\zeta_s(X,I_sY) = \frac{2n+1}{4n}T^0(X,Y) + \frac{1}{4n}T^0(I_sX,I_sY) + \frac{2n+1}{2n}U(X,Y) + \frac{S}{2}g(X,Y)\\
&T(\xi_i,\xi_j) = -S\xi_k - [\xi_i,\xi_j]|_H, \qquad S = -h(T(\xi_1,\xi_2),\xi_3)\\
&g(T(\xi_i,\xi_j),X) = -\rho_k(I_iX,\xi_i) = -\rho_k(I_jX,\xi_j) = -h([\xi_i,\xi_j],X).
\end{aligned}
\tag{3.3.12}
$$

When $n=1$ the above formulas hold with $U=0$. A QC-structure is called *QC-Einstein* if the trace-free part of the QC-Ricci tensor vanishes, i.e. Ric is a scalar multiple of the metric,

$$\mathrm{Ric}(X,Y) = 2(n+2)Sg(X,Y). \tag{3.3.13}$$

Thus, when $n=1$ the structure is QC-Einstein if and only if $T^0=0$, which is equivalent to the vanishing of the torsion endomorphism [25, Proposition 4.2]:

Lemma 3.3.6. *A quaternionic contact manifold $(M,g,\mathbb{Q})$ is QC-Einstein if and only if the quaternionic contact torsion vanishes identically, $T_\xi = 0$, $\xi \in V$.*

3.4 The QC-Paneitz Operator and the Hessian inequality

In the lowest 7-dimensional QC-case, just like the lowest dimensional CR-case, the analogue of Theorem 2.1.1 has an additional assumption. For a compact QC-manifold $(M^{4n+3},g,\mathbb{Q})$ and $f \in C^\infty(M)$ we have from [30, Definition 3.1]:

Definition 3.4.1. *For a fixed f we a define a 1-form $P_f \equiv P[f]$ on M, which we call the P-form of f, by the following equation*

$$
\begin{aligned}
P_f(X) &= \nabla^3 f(X, e_\gamma, e_\gamma) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_\gamma, I_t e_\gamma) - 4nSdf(X) + 4nT^0(X, \nabla f) \\
&\qquad - \frac{8n(n-2)}{n-1} U(X, \nabla f), \qquad \text{if } n > 1, \\
P_f(X) &= \nabla^3 f(X, e_\gamma, e_\gamma) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_\gamma, I_t e_\gamma) - 4Sdf(X) + 4T^0(X, \nabla f), \ \text{if } n = 1.
\end{aligned}
\tag{3.4.1}
$$

The P-function of f is the function $P_f(\nabla f)$. Finally, the $\mathcal{C}$-operator is the fourth-order differential operator on M (independent of f!) defined by

$$
\mathcal{C}f = -\nabla^* P_f = (\nabla_{e_\alpha} P_f)(e_\alpha).
$$

We say that the P-function of f is non-negative if its integral exists and is non-positive

$$
\int_M f \cdot \mathcal{C}f \, \mathrm{Vol}_\eta = -\int_M P_f(\nabla f) \mathrm{Vol}_\eta \geq 0. \tag{3.4.2}
$$

If (3.4.2) holds for any smooth function of compact support we say that the $\mathcal{C}$-operator is non-negative.

Following Section 3.3.1, the horizontal Hessian $\nabla^2 f$ has an orthogonal decomposition into its [3] and [−1]-components:

$$
\nabla^2 f(X, Y) = (\nabla^2 f)_{[3]}(X, Y) + (\nabla^2 f)_{[-1]}(X, Y), \qquad X, Y \in \Gamma(H).
$$

Let $(\nabla^2 f)_{[3][0]}$ be the orthogonal complement of the projection of the Hessian $\nabla^2 f$ onto $\mathrm{span}\{\frac{1}{2\sqrt{n}} g\}$, i.e. the trace-free part of $(\nabla^2 f)_{[3]}$. Then, we have the orthogonal decomposition:

$$
(\nabla^2 f)_{[3]} = (\nabla^2 f)_{[3][0]} + \frac{1}{4n} \langle \nabla^2 f, g \rangle \, g. \tag{3.4.3}
$$

Similarly, using that span$\{\frac{1}{2\sqrt{n}}\omega_s\}$ is an orthonormal set in the $[-1]$-space, we have

$$(\nabla^2 f)_{[-1]} = (\nabla^2 f)_{[-1][0]} + \frac{1}{4n}\sum_{s=1}^{3}\langle \nabla^2 f, \omega_s\rangle\, \omega_s. \tag{3.4.4}$$

where $(\nabla^2 f)_{[-1][0]}(e_\alpha, I_s e_\alpha) = 0$ for any $s = 1, 2, 3$. Now, when $n > 1$ and M is compact, the $\mathcal{C}$-operator is indeed non-negative:

Theorem 3.4.2 (S. Ivanov, A. Petkov, & D. Vassilev, [30])**.** *On a QC-manifold of dimension $4n+3$ we have the formula*

$$4(\nabla_{e_\gamma}(\nabla^2 f)_{[3][0]})(e_\gamma, X) = \frac{n-1}{n} P_f(X).$$

In particular, if the manifold is compact then the $\mathcal{C}$-operator is non-negative for any dimension bigger than seven. In this case for any function f the function $\mathcal{C}f$ vanishes exactly when the trace-free part of the $[3]$-component of a function vanishes. In this case the P-form of f vanishes as well.

This conclusion of the above theorem follows from an application of the horizontal divergence theorem (3.3.7):

$$\frac{n-1}{4n}\int_M f\mathcal{C}f \mathrm{Vol}_\eta = -\frac{n-1}{4n}\int_M P_f(\nabla f)\mathrm{Vol}_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2 \mathrm{Vol}_\eta.$$

In the lowest 7-dimensional QC Lichnerowicz and Obata theorems, we will take non-negativity of the $\mathcal{C}$-operator as an assumption. Furthermore, the orthogonal decomposition of $\nabla^2 f$, along with equations (3.4.3), (3.4.4), yield the *Hessian inequality* for an extremal first eigenfunction of the sub-Laplacian

$$|\nabla^2 f|^2 \geq \frac{1}{4n}\left[\langle \nabla^2 f, g\rangle^2 + \sum_{s=1}^{3}\langle \nabla^2 f, \omega_s\rangle^2\right]. \tag{3.4.5}$$

3.5 Hyper-Kähler and 3-Sasakian Manifolds

Recall from Section 2.2.2 that a Hermitian manifold (M^{2n}, J, h) with a closed fundamental 2-form $\omega(X,Y) := h(JX, Y)$ is a Kähler manifold. Suppose there were three complex structures $\{I_1, I_2, I_3\} \subset \mathrm{End}(TM)$ satisfying the relations of the imaginary quaternions, ${I_s}^2 = -\mathrm{id}$ and $I_1 I_2 = -I_2 I_1 = I_3$, such that each (M, I_s, h) is a Hermitian manifold. Then, if each of the three fundamental 2-forms $\omega_s(X,Y) := h(I_s X, Y)$ is closed, i.e. each (M, ω_s) is Kähler, then (M, I_1, I_2, I_3, h) is said to be a *hyper-Kähler manifold*.

A manifold (M, J, g) meeting the hypotheses of the CR Obata-type Theorem 2.2.5 is CR-equivalent to the Sasakian sphere in $\mathbb{C}^n$, and this Sasakian structure on M can be characterized by the condition that the Riemannian cone $(\mathcal{C}(M), \hat{J}, \hat{g})$ is Kähler (cf. Chapter 2.2.2). We shall see that a manifold meeting the hypotheses of the QC Obata-type Theorem 3.7.1 will be QC-equivalent to the 3-*Sasakian* sphere, which is a specific type of QC-structure on the unit sphere in $\mathbb{R}^{4n+4}$. This 3-Sasakian structure also has a characterization in terms of the Riemannian cone of a manifold with an "almost-contact 3-structure" (cf. [8, 9]) being hyper-Kähler. However, more relevant to this dissertation is the fact that the 3-Sasakian manifolds are locally the only QC-Einstein (3.3.13) manifolds:

Theorem 3.5.1 (S. Ivanov, I. Minchev, & D. Vassilev, [25])**.** *Let $(M^{4n+3}, g, \mathbb{Q})$ be a QC-manifold with positive QC-Scalar curvature $Scal > 0$, assumed to be constant if $n = 1$. The next conditions are equivalent:*

1. *$(M^{4n+3}, g, \mathbb{Q})$ is a QC-Einstein manifold.*
2. *M is locally 3-Sasakian, i.e. locally there exists an $SO(3)$ matrix Ψ with smooth entries, such that, the local QC-structure $\left(\frac{16n(n+2)}{Scal}\Psi\eta, \mathbb{Q}\right)$ is 3-Sasakian.*
3. *The torsion of the Biquard connection is identically zero.*

The QC-Scalar curvature of a QC-Einstein manifold is a global constant ([25, Theorem 4.9] for $n > 1$ and [28, Theorem 1.1] for $n = 1$). From Lemma 3.3.6 we have that a manifold is QC-Einstein if and only if the torsion endomorphism T_ξ

vanishes identically. In particular, a QC-Einstein manifold of positive QC-Scalar curvature, assumed in addition to be constant if $n = 1$, is an Einstein manifold of positive Riemannian Scalar curvature.

3.6 The QC Lichernowicz-type Theorem

As in the CR case, the corresponding sub-Laplacian is a sub-elliptic operator, hence the compactness of M implies the spectrum of Δ is discrete. The QC Lichnerowicz-type theorem was found in [31] for the $n > 1$ case, and in [30] in the $n = 1$ case.

Theorem 3.6.1 (S. Ivanov, A. Petkov, & D. Vassilev, [30, 31]). *Let (M, η) be a compact QC-manifold of dimension $4n+3$. Suppose, for $\alpha_n = \frac{2(2n+3)}{2n+1}$, $\beta_n = \frac{4(2n-1)(n+2)}{(2n+1)(n-1)}$, $\beta_1 = 0$, and for any $X \in H$*

$$\mathcal{L}(X, X) \overset{def}{=} 2Sg(X, X) + \alpha_n T^0(X, X) + \beta_n U(X, X) \geq 4g(X, X). \tag{3.6.1}$$

If $n = 1$, assume in addition the positivity of the P-function of any eigenfunction. Then, any eigenvalue of the sub-Laplacian Δ satisfies the inequality $\lambda \geq 4n$.

The inequality (3.6.1) is the QC-analogue of the lower-Ricci bound (2.1.1). Similar to (2.1.3) and (2.2.6), we have the *QC-Bochner formula* [31, (4.1)]:

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + 2(n+2)S|\nabla f|^2 + 2(n+2)T^0(\nabla f, \nabla f) \\ + 2(2n+1)U(\nabla f, \nabla f) + 4\sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f). \tag{3.6.2}$$

Assuming that $\Delta f = \lambda f$, integrate (3.6.2) over the compact M to arrive at

$$0 \geq \int_M \left(|\nabla^2 f|^2 - \frac{1}{4n}\left[\langle \nabla^2 f, g\rangle^2 + \sum_{s=1}^{3} \langle \nabla^2 f, \omega_s \rangle^2 \right] \right) \mathrm{Vol}_\eta - \frac{3}{4n}\int_M P_f(\nabla f)\mathrm{Vol}_\eta \\ + \frac{2n+1}{2}\int_M \left(\mathcal{L}(\nabla f, \nabla f) - \frac{\lambda}{n}|\nabla f|^2 \right) \mathrm{Vol}_\eta$$

The Hessian inequality (3.4.5), Theorem 3.4.2, and the Lichnerowicz-type bound $\mathcal{L}(\nabla f, \nabla f) \geq 4|\nabla f|^2$ imply the above inequality can only hold when $\lambda \geq 4n$. In addition, when $\lambda = 4n$ those same assumptions now show that an extremal first eigenfunction f enjoys several more properties:

- it satisfies the *horizontal Hessian equation*:

$$\nabla^2 f(X,Y) = -f g(X,Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X,Y), \tag{3.6.3}$$

- its gradient ∇f achieves equality in the Lichnerowicz-type bound (3.6.1):

$$\mathcal{L}(\nabla f, \nabla f) = 4|\nabla f|^2,$$

- the integral of its P-function vanishes:

$$\int_M P_f(\nabla f)\mathrm{Vol}_\eta = 0.$$

3.7 The QC Obata-type Theorem

By *QC-conformal transformation* we mean a diffeomorphism between QC-manifolds $F : (M,\eta) \to (\bar{M},\bar{\eta})$ that pulls $\bar{\eta}$ back to a form conformal to η: $F^*(\bar{\eta}) = \mu\Psi\eta \in [\eta]$, where $0 < \mu \in C^\infty(M)$ and $\Psi \in \mathrm{SO}(3)$ with smooth functions as entries. Then, (M^{4n+3},η) is QC-homothetic to the unit 3-Sasakian sphere $(\bar{M},\bar{\eta}) := (\mathbb{S}^{4n+3},\mathring{\eta})$ (c.f. [25, Section 8.3]) if there is such a diffeomorphism with μ a positive constant.

Theorem 3.7.1 (S. Ivanov, A. Petkov, & D. Vassilev, [32])**.** *Let (M,η) be a compact QC-manifold of dimension $4n+3$ which satisfies a Lichnerowicz-type bound $\mathcal{L}(X,X) \geq 4g(X,X)$. Then, there is a function f with $\Delta f = 4nf$ if and only if*

1. *when $n > 1$, M is QC-homothetic to the 3-Sasakian sphere;*

2. *when $n = 1$, and M is QC-Einstein, i.e. $T^0 = 0$, then M is QC-homothetic to the 3-Sasakian sphere.*

Under these hypotheses it was shown that

$$\Delta f = 4nf \qquad \Longleftrightarrow \qquad \nabla^2 f = -fg - \sum_{s=1}^{3} df(\xi_s)\omega_s$$

since the reverse implication holds by definition $\Delta f = -\mathrm{tr}^g(\nabla^2 f)$. Similar to the Riemannian and CR-cases, when $n > 1$ the proof of this theorem relies on a result analogous to Obata's Theorem 2.1.2 in the QC-case concerning complete manifolds that admit functions whose Hessians are as in (3.6.3).

Theorem 3.7.2 (S. Ivanov, A. Petkov, & D. Vassilev, [32])**.** *Let (M, η) be a quaternionic contact manifold of dimension $4n + 3 > 7$ which is complete with respect to the associated Riemannian metric $h = g+(\eta_1)^2+(\eta_2)^2+(\eta_3)^2$. There exists a smooth $f \not\equiv const$, such that,*

$$\nabla df(X, Y) = -fg(X, Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X, Y)$$

if and only if the QC-manifold $(M, \eta, g, \mathbb{Q})$ is QC-homothetic to the unit 3-Sasakian sphere.

When $n > 1$, Theorem 3.7.2 implies Theorem 3.7.1 since a compact Riemannian manifold is necessarily complete. Also when $n > 1$, [32, Lemma 3.10] states that a QC-manifold which satisfies the conditions of Theorem 3.7.2 is necessarily QC-Einstein. Let ∇^h denote the Levi-Civita connection of the Riemannian metric h of (3.3.6). For any $A, B \in \Gamma(TM)$ the relationship between the Hessians of the Levi-Civita and Biquard connections is [32, (3.75)]:

$$(\nabla^h)^2 f(A, B) = \nabla^2 f(A, B) + \frac{1}{2}\left[h(T(A, B), df) - h(T(B, df), A) + h(T(df, A), B)\right].$$

Then, if (M^{4n+3}, h) is complete and $(\nabla^h)^2 f = -fh$ we can invoke Obata's theorem 2.1.2 to conclude that (M^{4n+3}, h) is isometric to the round unit sphere $(\mathbb{S}^{4n+3}, \mathring{g})$.

Lemma 3.7.3 (S. Ivanov, A. Petkov, & D. Vassilev, [32])**.** *Let $(M, \eta, g, \mathbb{Q})$ be a QC-Einstein manifold, $T^0 = U = 0$, of dimension $4n+3 > 7$. Let h be the associated Riemannian metric (3.3.6). If f is a smooth function whose horizontal Hessian satisfies (3.6.3), then the Riemannian Hessian of f with respect to the metric h satisfies (2.1.5).*

A Riemannian manifold (M, g) is locally conformally-flat (i.e. locally conformally diffeomorphic to $\mathbb{R}^n$ with its flat metric δ) if and only if its Weyl tensor vanishes. The flat model of a QC-manifold is the quaternionic Heisenberg group $G(\mathbb{H}) \cong \mathbb{H}^n \times \mathrm{Im}(\mathbb{H})$ with its standard contact form Θ (cf. [25, Section 5.2]), and the obstruction for a QC-manifold (M, η) to be locally QC conformally-flat (i.e. locally conformally diffeomorphic to $(G(\mathbb{H}), \Theta)$) is the conformally invariant *QC-conformal curvature tensor* W^{qc}.

Theorem 3.7.4 (S. Ivanov & D. Vassilev, [33])**.** *A QC-structure on a $(4n+3)$-dimensional smooth manifold is locally quaternionic contact conformal to the standard flat QC-structure on the quaternionic Heisenberg group $G(\mathbb{H})$ if and only if the QC-conformal curvature vanishes, $W^{qc} = 0$.*

The Cayley transform provides a QC-conformal diffeomorphism from $(G(\mathbb{H}), \Theta)$ to the 3-Sasakian sphere without a point $(\mathbb{S} - \{q\}, \mathring{\eta})$ (cf. [25, Section 8.3]). Therefore, as a corollary to the above theorem we also have

Theorem 3.7.5 (S. Ivanov & D. Vassilev, [33])**.** *A QC-manifold is locally quaternionic contact conformal to the quaternionic sphere $\mathbb{S}^{4n+3}$ if and only if the QC-conformal curvature tensor vanishes, $W^{qc} = 0$.*

In [32, Section 3.8] it is shown that a QC-manifold satisfying the hypotheses of Obata-type Theorem 3.7.1 indeed has its QC-conformal curvature tensor vanish.

In order to conclude from here that $(M, \eta, g, \mathbb{Q})$ is QC-homothetic to the unit 3-Sasakian sphere the authors invoke the following QC Liouville-type Theorem (see also [38, Section 8.3]):

Theorem 3.7.6 (A. Căp, J. Slovák,[10]). *Every QC-conformal transformation between open subsets of the* 3*-Sasakian unit sphere is the restriction of a global QC-conformal transformation.*

Hence, there is a diffeomorphism $F : (M^{4n+3}, \eta) \to (\mathbb{S}^{4n+3}, \mathring{\eta})$ such that $F^*(\mathring{\eta}) = \Psi\eta \in [\eta]$, i.e. M is QC-homothetic to the unit 3-Sasakian sphere.

3.7.1 The Open Problem

In the proof of the QC Obata-type theorem 3.7.1 the authors show that a manifold satisfying its hypotheses is QC-Einstein, and this is equivalent to the vanishing of the tensors T^0 and U from (3.3.8). When $n > 1$, they arrive at the following formulas [32, Lemma 3.8]:

$$
\begin{aligned}
|\nabla f|^4 T^0(X,Y) &= -\frac{2n}{n-1} U(\nabla f, \nabla f) \left[3df(X)df(Y) - \sum_{s=1}^{3} df(I_sX)df(I_sY) \right] \\
|\nabla f|^4 U(X,Y) &= -\frac{U(\nabla f, \nabla f)}{n-1} \left[|\nabla f|^2 g(X,Y) - n df(X)df(Y) \right] \\
&\qquad + n\frac{U(\nabla f, \nabla f)}{n-1} \sum_{s=1}^{3} df(I_sX)df(I_sY).
\end{aligned}
$$

Thus, to reach the conclusion that such an M is QC-Einstein when $n > 1$ they need only show that $U(\nabla f, \nabla f)$ vanishes. However, when $n = 1$ the tensor U vanishes identically, the formulas above become trivial, and the QC-Einstein condition was taken as an assumption. The general QC Obata-type 3.7.1 result in dimension 7 (without the additional assumption that M is QC-Einstein) remained open, which motivated the investigations that lead to the Main Theorem 1.0.1 in this dissertation.

Chapter 4

Proof of the Main Theorem 1.0.1

4.1 7-Dimensional QC-Structures

For ease of reference we collect the relevant identities from Chapter 3.3.3 working only in the 7-dimensional setting. In particular, the tensor u of Theorem 3.3.5 vanishes and therefore the torsion endomorphism $T_\xi = T^0_\xi$ is symmetric. This implies that $U(X, Y)$ from (3.3.8) also vanishes and the identites in (3.3.12) simplify when $n = 1$:

$$\begin{aligned}
&\mathrm{Ric}(X, Y) = 4T^0(X, Y) + 6Sg(X, Y),\\
&\zeta_s(X, I_sY) = \frac{3}{4}T^0(X, Y) + \frac{1}{4}T^0(I_sX, I_sY) + \frac{S}{2}g(X, Y),\\
&T(\xi_i, \xi_j) = -S\xi_k - [\xi_i, \xi_j]|_H, \qquad S = -h(T(\xi_1, \xi_2), \xi_3),\\
&g(T(\xi_i, \xi_j), X) = -\rho_k(I_iX, \xi_i) = -\rho_k(I_jX, \xi_j) = -h([\xi_i, \xi_j], X).
\end{aligned} \tag{4.1.1}$$

Let $\{e_\gamma\}_{\gamma=1}^{4n}$ be a local orthonormal frame for H, $f \in C^\infty(M)$, and the summation convention (0.0.2) hold. The sub-Laplacian Δf, and the norm of the horizontal gradient ∇f, are defined by

$$\Delta f = -\mathrm{tr}^g_H(\nabla^2 f) = -\nabla^2 f(e_\gamma, e_\gamma), \qquad |\nabla f|^2 = df(e_\gamma)df(e_\gamma).$$

Following the notation set in (3.3.2), we see that

$$\langle \nabla^2 f, g\rangle = \nabla^2 f(e_\gamma, e_\gamma) = -\Delta f.$$

On the other hand, the Ricci identity $\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = -2\omega_s(X,Y)df(\xi_s)$ and the fact that $\nabla^2 f(e_\gamma, I_s e_\gamma) = -\nabla^2 f(I_s e_\gamma, e_\gamma)$ implies

$$\langle \nabla^2 f, \omega_s\rangle = \nabla^2 f(e_\gamma, I_s e_\gamma) = -4df(\xi_s) = -4f_s, \tag{4.1.2}$$

where we set

$$f_s \overset{def}{=} df(\xi_s). \tag{4.1.3}$$

4.2 First Equations

We shall assume throughout all of the hypotheses of Theorem 1.0.1. The function f will denote an eigenfunction of the sub-Laplacian, $\Delta f = \lambda f$, achieving the lowest possible eigenvalue $\lambda = 4$. It was shown in [30, Remark 4.1] that, with the made assumptions, the horizontal Hessian of f is given by the *Hessian equation*:

$$\nabla^2 f(Y,X) = -fg(Y,X) - \sum_{s=1}^{3} f_s\,\omega_s(Y,X) \tag{4.2.1}$$

for $Y, X \in \Gamma(H)$, recalling the notation $f_s = df(\xi_s)$ set in (4.1.3). In addition,

$$\mathcal{L}(\nabla f, \nabla f) - 4|\nabla f|^2 = 2\left[(S-2)|\nabla f|^2 + \frac{5}{3}T^0(\nabla f, \nabla f)\right] = 0, \tag{4.2.2}$$

where $\mathcal{L}$ is as in Theorem 1.0.1, and we have that

$$\int_M P_f(\nabla f)\,\mathrm{Vol}_\eta = 0.$$

We note that the compactness of M was essential in order to obtain the above identities by integrating the qc-Bochner formula. Furthermore, if f satisfies (4.2.1) then differentiating the Hessian equation we obtain the identity, [32, Lemma 3.1],

$$\nabla^3 f(A,Y,X) = -df(A)g(Y,X) - \sum_{s=1}^{3} \nabla^2 f(A,\xi_s)\omega_s(Y,X) \tag{4.2.3}$$

for $A \in \Gamma(TM)$ and $Y, X \in \Gamma(H)$. We note that [32] assumes $n > 1$, but the cited lemma and its proof do not make use of this assumption. In addition, the argument leading to [32, (3.8)] is valid in the case $n = 1$ as well, i.e., we have the following identity

$$\sum_{s=1}^{3} \nabla^2 f(I_s X, \xi_s) = (1 - 2S)df(X) - \frac{2}{3}T^0(X, \nabla f), \tag{4.2.4}$$

which follows from the Ricci identity $\nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) = T(\xi_s, X, \nabla f)$ applied to the left-hand side of [32, (3.8)], noting that the Ricci idenity and (3.3.10) give

$$\nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) = -\frac{1}{4}[T^0(I_s X, \nabla f) + T^0(X, I_s \nabla f)]. \tag{4.2.5}$$

It will be convenient to define the quadratic symmetric (0,2)-tensor $\mathcal{P}$ by

$$\mathcal{P}(X, Y) \overset{def}{=} 2\left[\mathcal{L}(X, Y) - 4g(X, Y)\right] = 4\left[(S - 2)g(X, Y) + \frac{5}{3}T^0(X, Y)\right] \tag{4.2.6}$$

The Lichnerowicz-type bound (1.0.1) implies that $\mathcal{P}$ is non-negative $\mathcal{P}(X, X) \geq 0$. Hence, taking into account that T^0 is a traceless tensor, we have $S \geq 2$; while by (4.2.2) it follows that

$$\mathcal{P}(\nabla f, \nabla f) = 0.$$

Lemma 4.2.1. *The P-form of f is $\mathcal{P}(X, \nabla f)$, i.e.,*

$$P_f(X) = \mathcal{P}(X, \nabla f) = 4(S - 2)df(X) + \frac{20}{3}T^0(X, \nabla f). \tag{4.2.7}$$

Furthermore, $\mathcal{P}(X, \nabla f) = 0$, hence

$$T^0(X, \nabla f) = -\frac{3}{5}(S - 2)df(X). \tag{4.2.8}$$

Proof. Taking the indicated traces in (4.2.3) we obtain

$$\begin{aligned}
\nabla^3 f(X, e_\gamma, e_\gamma) &= -4df(X),\\
\sum_{s=1}^{3} \nabla^3 f(I_s X, e_\gamma, I_s e_\gamma) &= -4\sum_{s=1}^{3} \nabla^2 f(I_s X, \xi_s)\\
&= -4(1 - 2S)df(X) + \frac{8}{3}T^0(X, \nabla f).
\end{aligned}$$

A substitution of the above two identities in the definition (3.4.1) of the P-form of f gives (4.2.7). The non-negativity of $\mathcal{P}$, Cauchy-Schwarz' inequality and $\mathcal{P}(\nabla f, \nabla f) = 0$ imply $\mathcal{P}(X, \nabla f) = 0$. $\square$

An immediate consequence of the Hermitian compatibility of the horizontal metric g and (4.2.8) are the following identities,

$$T^0(I_s \nabla f, \nabla f) = 0, \qquad s = 1, 2, 3. \tag{4.2.9}$$

In addition, the covariant derivative of (4.2.8) along a horizontal vector Y and the Hessian equation (4.2.1) yield the following

$$\begin{aligned} \nabla T^0(Y, X, \nabla f) = -\frac{3}{5} dS(Y) df(X) + f \left(T^0(Y, X) + \frac{3}{5}(S-2) g(Y, X) \right) \\ + \sum_{s=1}^{3} f_s \left(T^0(I_s Y, X) + \frac{3}{5}(S-2) \omega_s(Y, X) \right). \end{aligned} \tag{4.2.10}$$

Next, we will obtain formulas for the individual terms in the sum (4.2.4) and their derivatives. This will be achieved by computing the horizontal QC-Ricci tensor ζ_s in two different ways.

Lemma 4.2.2. *The following identities hold true*

$$\nabla^2 f(X, \xi_i) = \frac{1}{5}(1 + 2S) df(I_i X) - \frac{2}{3} T^0(X, I_i \nabla f) \tag{4.2.11}$$

and

$$\begin{aligned} \nabla^3 f(Y, X, \xi_i) = \frac{1}{5}(1 + 2S) \left[f\omega_i(Y, X) - f_i g(Y, X) + f_j \omega_k(Y, X) - f_k \omega_j(Y, X) \right] \\ + \frac{2}{3} \left[f T^0(X, I_i Y) - f_i T^0(X, Y) + f_j T^0(X, I_k Y) - f_k T^0(X, I_j Y) \right] \\ + \frac{2}{5} dS(Y) df(I_i X) - \frac{2}{3} \nabla T^0(Y, X, I_i \nabla f). \end{aligned} \tag{4.2.12}$$

Proof. First, we compute $\zeta_i(I_iX,\nabla f)$ using its definition (3.3.11) and then apply the following third order Ricci identity

$$R(X,Y,\nabla f,Z)=\nabla^3 f(X,Y,Z)-\nabla^3 f(Y,X,Z)+2\sum_{s=1}^{3}\nabla^2 f(\xi_s,Z)\omega_s(X,Y),$$

which give for a fixed i the formula

$$\begin{aligned}4\zeta_i(I_iX,\nabla f)&\overset{def}{=}R(e_\gamma,I_iX,\nabla f,I_ie_\gamma)\\&=\nabla^3 f(e_\gamma,I_iX,I_ie_\gamma)-\nabla^3 f(I_iX,e_\gamma,I_ie_\gamma)+2\sum_{s=1}^{3}\nabla^2 f(\xi_s,I_ie_\gamma)\omega_s(e_\gamma,I_iX).\end{aligned}$$

An application of (4.2.3) and (4.2.4) to the above equation brings us to

$$\begin{aligned}4\zeta_i(I_iX,\nabla f)&=-df(X)-\sum_{s=1}^{3}\nabla^2 f(e_\gamma,\xi_s)\omega_s(I_iX,I_ie_\gamma)\\&\quad+\sum_{s=1}^{3}\nabla^2 f(I_iX,\xi_s)\omega_s(e_\gamma,I_ie_\gamma)+2\sum_{s=1}^{3}\nabla^2 f(\xi_s,I_ie_\gamma)\omega_s(e_\gamma,I_iX)\\&=-df(X)-\big[\nabla^2 f(I_iX,\xi_i)-\nabla^2 f(I_jX,\xi_j)-\nabla^2 f(I_kX,\xi_k)\big]\\&\quad+4\nabla^2 f(I_iX,\xi_i)+2[\nabla^2 f(\xi_i,I_iX)-\nabla^2 f(\xi_j,I_jX)-\nabla^2 f(\xi_k,I_kX)].\end{aligned}\tag{4.2.13}$$

Invoking (4.2.5) to re-write the last bracket in (4.2.13) we come to

$$\begin{aligned}4\zeta_i(I_iX,\nabla f)&=-df(X)-\big[\nabla^2 f(I_iX,\xi_i)-\nabla^2 f(I_jX,\xi_j)-\nabla^2 f(I_kX,\xi_k)\big]\\&\quad+2[\nabla^2 f(I_iX,\xi_i)-\nabla^2 f(I_jX,\xi_j)-\nabla^2 f(I_kX,\xi_k)]-\frac{1}{2}[T^0(X,\nabla f)-T^0(I_iX,I_i\nabla f)]\\&\quad+4\nabla^2 f(I_iX,\xi_i)+\frac{1}{2}[T^0(X,\nabla f)-T^0(I_jX,I_j\nabla f)]+\frac{1}{2}[T^0(X,\nabla f)-T^0(I_kX,I_k\nabla f)]\\&=-df(X)+6\nabla^2 f(I_iX,\xi_i)-\sum_{s=1}^{3}\nabla^2 f(I_sX,\xi_s)+T^0(X,\nabla f)+T^0(I_iX,I_i\nabla f)\\&\quad-\frac{1}{2}\left[T^0(X,\nabla f)+\sum_{s=1}^{3}T^0(I_sX,I_s\nabla f)\right].\end{aligned}$$

Finally, using (3.3.9) and (4.2.4) the last equation takes the form

$$\begin{aligned}4\zeta_i(I_iX,\nabla f)&=2(S-1)df(X)+6\nabla^2 f(I_iX,\xi_i)\\&\quad+\frac{5}{3}T^0(X,\nabla f)+T^0(I_iX,I_i\nabla f).\end{aligned}\tag{4.2.14}$$

On the other hand, from (4.1.1) we have the following formula for $\zeta_i(I_iX, \nabla f)$

$$4\zeta_i(I_iX, \nabla f) = -3T^0(I_iX, I_i\nabla f) - T^0(X, \nabla f) - 2Sdf(X). \tag{4.2.15}$$

From (4.2.15) and (4.2.14) we obtain

$$3\nabla^2 f(X, \xi_i) = (2S-1)df(I_iX) + \frac{4}{3}T^0(I_iX, \nabla f) - 2T^0(X, I_i\nabla f). \tag{4.2.16}$$

A substitution of (4.2.8) into (4.2.16) gives (4.2.11).

The second identity in the lemma is obtained by taking the covariant derivative of (4.2.11) along $Y \in \Gamma(H)$, noting that from (3.3.5) the terms involving the connection 1-forms coming from the covariant derivatives of I_i and ξ_i cancel, which gives

$$\begin{aligned}\nabla^3 f(Y, X, \xi_i) = &\frac{2}{5}dS(Y)df(I_iX) - \frac{2}{3}\nabla T^0(Y, X, I_i\nabla f)\\ &+ \frac{1}{5}(1+2S)\nabla^2 f(Y, I_iX) + \frac{2}{3}fT^0(X, I_iY) + \frac{2}{3}\sum_{s=1}^{3} f_sT^0(X, I_iI_sY).\end{aligned}$$

Finally, using the Hessian equation (4.2.1) in the above formula gives (4.2.12). □

Some of the above identities can be viewed as versions of formulas found in [32] that hold when $n = 1$. Other than (4.2.1) and (4.2.3) coming directly from [32] as stated above, (4.2.4) can be obtained from [32, (3.8)] by setting $U = 0$, $n = 1$, and then applying a Ricci identity.

On the other hand, identity (4.2.8) can be *formally* obtained from [32, (3.5)] by setting $U = 0$ and $n = 1$. When $n > 1$, the proof of [32, Lemma 3.2] shows that [32, (3.5)] is found by subtracting [32, (3.7)] from [32, (3.8)]. However, if $n = 1$ then [32, (3.7)] is identical to [32, (3.8)] and therefore we cannot obtain [32, (3.5)] when $n = 1$ following [32]. In our case, in order to prove (4.2.8) we used the compactness of M to show that the P-form of f vanishes instead. However, once (4.2.8) is known, substituting it into (4.2.4), and then using a Ricci identity, will yield [32, (3.9)]. In addition, (4.2.9) implies that [32, (3.6)] continues to hold when $n = 1$. Finally, (4.2.11) and (4.2.12) correspond to [32, Lemma 3.3] and [32, (3.18)], respectively, but now they hold in the case $n = 1$.

4.2.1 A Key Identity

Since the canonical connection preserves the type of a tensor and T^0 is symmetric, we can compute terms of the form $\nabla T^0(Y, \nabla f, I_i\nabla f)$ by finding $\nabla T^0(Y, I_i\nabla f, \nabla f)$ from (4.2.10); we cannot obtain in this way explicit formulas for the $\nabla T^0(Y, I_j\nabla f, I_i\nabla f)$. However, with the help of the previous Lemmas, we will find a certain relation between the torsion T^0, the normalized QC-scalar curvature S, and their derivatives, which will lead to a system that can be solved for the "unknown" components. To formulate it, we need the following covariant tensors that will also play a prominent role for the remainder of the chapter,

$$\begin{aligned}\Gamma^i(Y,X) \overset{def}{=} \omega_j(Y,X)\rho_k(I_i\nabla f,\xi_i) - \omega_k(Y,X)\rho_j(I_i\nabla f,\xi_i) + df(I_kY)\rho_j(I_iX,\xi_i)\\ + df(I_kX)\rho_j(I_iY,\xi_i) - df(I_jY)\rho_k(I_iX,\xi_i) - df(I_jX)\rho_k(I_iY,\xi_i),\end{aligned} \tag{4.2.17}$$

where the last four terms constitute its symmetric part:

$$\begin{aligned}\Gamma^i_{\mathrm{sym}}(Y,X) \overset{def}{=} df(I_kY)\rho_j(I_iX,\xi_i) + df(I_kX)\rho_j(I_iY,\xi_i)\\ - df(I_jY)\rho_k(I_iX,\xi_i) - df(I_jX)\rho_k(I_iY,\xi_i)\end{aligned} \tag{4.2.18}$$

Lemma 4.2.3. *The following identity holds true for any cyclic permutation of the* *indices* (i,j,k),

$$5\nabla T^0(Y,X,I_i\nabla f)-3\nabla T^0(X,Y,I_i\nabla f)=-3[\nabla T^0(\nabla f,Y,I_iX)+\nabla T^0(\nabla f,I_iY,X)]$$
$$+3dS(Y)df(I_iX)-\frac{9}{5}dS(X)df(I_iY)+\frac{6}{5}(4+3S)f_ig(Y,X)+12\sum_{s=1}^{3}\nabla^2 f(\xi_i,\xi_s)\omega_s(Y,X)$$
$$-\frac{12}{5}(1+2S)\left[f\omega_i(X,Y)+\sum_{s=1}^{3}f_s\omega_s(Y,I_iX)\right]+f[5T^0(X,I_iY)-3T^0(I_iX,Y)]$$
$$+f_i[6T^0(I_iX,I_iY)-8T^0(X,Y)]+f_j[5T^0(X,I_kY)+6T^0(I_jX,I_iY)+3T^0(I_kX,Y)]$$
$$+f_k[6T^0(I_kX,I_iY)-5T^0(X,I_jY)-3T^0(I_jX,Y)]-12\Gamma^i(Y,X). \quad (4.2.19)$$

Proof. We begin by finding another formula for $\nabla^3 f(Y,X,\xi_i)$, see (4.2.22) below, besides the already known identity (4.2.12). We begin by using the third order Ricci identity

$$\nabla^3 f(Y,X,\xi_i)=\nabla^3 f(\xi_i,Y,X)+\nabla^2 f(T(\xi_i,Y),X)+\nabla^2 f(Y,T(\xi_i,X))$$
$$+df((\nabla_Y T)(\xi_i,X))+R(\xi_i,Y,X,\nabla f). \quad (4.2.20)$$

Next, we compute each of the terms in the right-hand side of (4.2.20) separately. The first term can be simplified with the help of (4.2.3), which gives

$$\nabla^3 f(\xi_i,Y,X)=-f_ig(Y,X)-\sum_{s=1}^{3}\nabla^2 f(\xi_i,\xi_s)\,\omega_s(Y,X).$$

The Hessian equation (4.2.1) and (4.2.5) show that

$$\nabla^2 f(T(\xi_i,Y),X)=\frac{1}{4}f[T^0(I_iY,X)+T^0(Y,I_iX)]$$
$$-\frac{1}{4}\sum_{t=1}^{3}df(\xi_t)[T^0(I_iY,I_tX)+T^0(Y,I_iI_tX)].$$

The third term in the right-hand side of (4.2.20) is handled similarly. Next, use (3.3.9) to simplify the sum of the above formulas for the second and third terms, which give

$$\nabla^2 f(T(\xi_i, Y), X) + \nabla^2 f(Y, T(\xi_i, X)) = \frac{1}{2} f[T^0(I_i X, Y) + T^0(X, I_i Y)] \\ + \frac{1}{2} f_j [T^0(X, I_k Y) - T^0(I_k X, Y)] + \frac{1}{2} f_k [T^0(I_j X, Y) - T^0(X, I_j Y)].$$

To simplify the fourth term, we differentiate (3.3.10), using (3.3.5), which gives

$$\begin{aligned} df((\nabla_Y T)(\xi_i, X)) &= (\nabla_Y T)(\xi_i, X, \nabla f) \\ &= -\frac{1}{4}[\nabla T^0(Y, I_i X, \nabla f) + \nabla T^0(Y, X, I_i \nabla f)]. \end{aligned}$$

At this point we need the following general formula for the curvature, cf. [33, 35],

$$\begin{aligned} R(\xi_i, Y, X, Z) &= -\frac{1}{4}[\nabla T^0(X, I_i Z, Y) + \nabla T^0(X, Z, I_i Y)] \\ &+ \frac{1}{4}[\nabla T^0(Z, I_i X, Y) + \nabla T^0(Z, X, I_i Y)] + \omega_j(Y, X)\rho_k(I_i Z, \xi_i) \\ &- \omega_k(Y, X)\rho_j(I_i Z, \xi_i) + \omega_k(Y, Z)\rho_j(I_i X, \xi_i) + \omega_k(X, Z)\rho_j(I_i Y, \xi_i) \\ &- \omega_j(Y, Z)\rho_k(I_i X, \xi_i) - \omega_j(X, Z)\rho_k(I_i Y, \xi_i). \end{aligned} \tag{4.2.21}$$

Letting $Z = \nabla f$ in (4.2.21) gives the following formula for the fifth term in the right-hand side of (4.2.20),

$$\begin{aligned} R(\xi_i, Y, X, \nabla f) &= -\frac{1}{4}[\nabla T^0(X, I_i \nabla f, Y) + \nabla T^0(X, \nabla f, I_i Y)] \\ &+ \frac{1}{4}[\nabla T^0(\nabla f, I_i X, Y) + \nabla T^0(\nabla f, X, I_i Y)] + \Gamma^i(Y, X), \end{aligned}$$

recalling the tensor Γ^i defined in (4.2.17). A substitution of the above identities into (4.2.20) yields the sought formula for $\nabla^3 f(Y, X, \xi_i)$, i.e.,

$$\begin{aligned} \nabla^3 f(Y, X, \xi_i) &= -f_i g(Y, X) - \sum_{s=1}^{3} \nabla^2 f(\xi_i, \xi_s)\omega_s(Y, X) + \frac{1}{2} f[T^0(I_i X, Y) + T^0(X, I_i Y)] \\ &+ \frac{1}{2} f_j [T^0(X, I_k Y) - T^0(I_k X, Y)] + \frac{1}{2} f_k [T^0(I_j X, Y) - T^0(X, I_j Y)] \\ &- \frac{1}{4}[\nabla T^0(Y, I_i X, \nabla f) + \nabla T^0(Y, X, I_i \nabla f)] - \frac{1}{4}[\nabla T^0(X, I_i \nabla f, Y) + \nabla T^0(X, \nabla f, I_i Y)] \\ &+ \frac{1}{4}[\nabla T^0(\nabla f, I_i X, Y) + \nabla T^0(\nabla f, X, I_i Y)] + \Gamma^i(Y, X). \end{aligned} \tag{4.2.22}$$

After this initial calculation, we use (4.2.10) and the symmetry in the last two indices of ∇T^0 to rewrite and expand the terms $\nabla T^0(Y, I_iX, \nabla f)$ and $\nabla T^0(X, \nabla f, I_iY)$ in (4.2.22). After a small simplification we obtain

$$\begin{aligned}\nabla^3 f(Y,X,\xi_i) = &-\frac{1}{10}(4+3S)f_i g(Y,X) - \sum_{s=1}^{3}\nabla^2 f(\xi_i,\xi_s)\omega_s(Y,X)\\ &+\frac{1}{4}f[T^0(I_iX,Y)+T^0(X,I_iY)] + \frac{3}{20}[dS(Y)df(I_iX)+dS(X)df(I_iY)]\\ &-\frac{1}{4}f_j[T^0(I_jY,I_iX)+T^0(I_jX,I_iY)-2T^0(X,I_kY)+2T^0(I_kX,Y)]\\ &-\frac{1}{4}f_k[T^0(I_kY,I_iX)+T^0(I_kX,I_iY)-2T^0(I_jX,Y)+2T^0(X,I_jY)]\\ &-\frac{1}{2}f_iT^0(I_iX,I_iY) - \frac{1}{4}[\nabla T^0(Y,X,I_i\nabla f)+\nabla T^0(X,Y,I_i\nabla f)]\\ &+\frac{1}{4}[\nabla T^0(\nabla f,I_iX,Y)+\nabla T^0(\nabla f,X,I_iY)] + \Gamma^i(Y,X).\end{aligned} \quad (4.2.23)$$

Next, we subtract (4.2.12) from (4.2.23) and collect the terms containing $\nabla T^0(\,\cdot\,,\,\cdot\,,I_i\nabla f)$ on one side leaving the terms containing the "unknown" components of ∇T^0 and the vertical Hessian of f on the other side. With the help of (3.3.9) we simplify the bracketed terms multiplying the vertical derivatives of f, which gives the claimed formula (4.2.19). □

For several calculations we will need the symmetric part of (4.2.19), which is given by

$$\begin{aligned}\nabla T^0(Y,X,I_i\nabla f)+\nabla T^0(X,Y,I_i\nabla f) = &-3[\nabla T^0(\nabla f,I_iY,X)+\nabla T^0(\nabla f,Y,I_iX)]\\ &+\frac{3}{5}[dS(Y)df(I_iX)+dS(X)df(I_iY)] - \frac{6}{5}(S-2)g(Y,X)f_i - 12\Gamma^i_{\mathrm{sym}}(Y,X)\\ &+f[T^0(Y,I_iX)+T^0(I_iY,X)] + f_i[6T^0(I_iY,I_iX)-8T^0(Y,X)]\\ &+f_j[4T^0(I_kY,X)+4T^0(Y,I_kX)+3T^0(I_iY,I_jX)+3T^0(I_jY,I_iX)]\\ &+f_k[3T^0(I_kY,I_iX)+3T^0(I_iY,I_kX)-4T^0(Y,I_jX)-4T^0(I_jY,X)]\end{aligned} \quad (4.2.24)$$

recalling (4.2.18) and also taking into account that

$$\sum_{s=1}^{3} f_s[\omega_s(Y, I_iX) + \omega_s(X, I_iY)] = f_i[g(I_iY, I_iX) + g(I_iX, I_iY)]$$
$$+ f_j[g(I_jY, I_iX) + g(I_jX, I_iY)] + f_k[g(I_kY, I_iX) + g(I_kX, I_iY)]$$
$$= f_i[g(Y, X) + g(X, Y)] + f_j[g(Y, I_kX) - g(I_kX, I_iY)]$$
$$+ f_k[g(I_jY, X) - g(X, I_jY)] = 2f_i g(X, Y).$$

4.2.2 Unique Continuation and a Special Frame

Let h be the Riemannian metric (3.3.6) and Δ^h be the associated elliptic Laplacian. In the following lemma we will give a version of [32, Lemma 3.6] for the case $n = 1$. In particular, this will allow the construction at almost every point of M of a global orthonormal frame of the horizontal space using the horizontal gradient of f.

Lemma 4.2.4. *The eigenfunction f obeys the following identity*

$$\Delta^h f = \left(\frac{19 + 8S}{5}\right) f - \frac{2}{5} dS(\nabla f). \tag{4.2.25}$$

In particular, f and its horizontal gradient ∇f do not vanish on any open set. Thus, if we let

$$I_0 \overset{def}{=} \mathrm{id}_H \qquad \text{and} \qquad \sigma_\alpha \overset{def}{=} |\nabla f|^{-1} I_\alpha \nabla f$$

then $\{\sigma_\alpha\}_{\alpha=0}^{3}$ is an orthonormal frame for the horizontal space H at almost every point of M.

Proof. Following [31, Lemma 5.1] the Riemannian Laplacian Δ^h and sub-Laplacian Δ are related by

$$\Delta^h f = \Delta f - \sum_{s=1}^{3} \nabla^2 f(\xi_s, \xi_s). \tag{4.2.26}$$

Taking the trace $X = e_\gamma$, $Y = I_i e_\gamma$ of (4.2.19) using that T^0 is completely trace-free gives

$$5\nabla T^0(I_i e_\gamma, e_\gamma, I_i \nabla f) - 3\nabla T^0(e_\gamma, I_i e_\gamma, I_i \nabla f) = 3dS(I_i e_\gamma)df(I_i e_\gamma) \\ + \frac{9}{5}dS(e_\gamma)df(e_\gamma) - \frac{48}{5}(1+2S)f + 48\nabla^2 f(\xi_i, \xi_i) - 12\Gamma^i(I_i e_\gamma, e_\gamma). \quad (4.2.27)$$

From $\nabla T^0(e_\gamma, I_i e_\gamma, X) = -\nabla T^0(I_i e_\gamma, e_\gamma, X)$ and $\Gamma^i(I_i e_\gamma, e_\gamma) = 0$ by (4.2.17) we can solve for the component of the vertical Hessian of f in (4.2.27) which gives

$$\nabla^2 f(\xi_i, \xi_i) = \frac{1}{6}\nabla T^0(e_\gamma, I_i e_\gamma, I_i \nabla f) + \frac{1}{10}dS(\nabla f) - \frac{1}{5}(1+2S)f. \quad (4.2.28)$$

Using (3.3.9) in which we take $X = I_i e_\gamma$, $Y = I_i \nabla f$, and (3.3.5), gives the following trace formula

$$\nabla T^0(e_\gamma, e_\gamma, \nabla f) + \sum_{s=1}^{3} \nabla T^0(e_\gamma, I_s e_\gamma, I_s \nabla f) = 0. \quad (4.2.29)$$

Then, (4.2.10) with $X = Y = e_\gamma$ shows that the divergence of T^0 satisfies

$$\nabla T^0(e_\gamma, e_\gamma, \nabla f) = -\frac{3}{5}\left[dS(\nabla f) - 4(S-2)f\right]. \quad (4.2.30)$$

Therefore, (4.2.28) with (4.2.29) and (4.2.30) implies

$$\sum_{s=1}^{3} \nabla^2 f(\xi_s, \xi_s) = \frac{2}{5}dS(\nabla f) + \frac{1}{5}(1-8S)f. \quad (4.2.31)$$

A substitution of (4.2.31) into (4.2.26), taking into account that $\Delta f = 4f$, shows (4.2.25). The final part of the Lemma follows from Aronszajn's unique continuation result [1]. □

4.3 The Components of T^0 and their Derivatives

Since in lemma 4.2.4 we found a global orthonormal frame $\{\sigma_\alpha, \xi_s\}$ valid a.e., it will be convenient to use the index notation for the components of the involved tensors constructed from the torsion as follows

$$T_{\alpha\beta} \overset{def}{=} T^0(I_\alpha \nabla f, I_\beta \nabla f), \qquad \nabla T^0(I_\gamma \nabla f, I_\alpha \nabla f, I_\beta \nabla f) = T_{\alpha\beta;\gamma}, \tag{4.3.1}$$

where $I_0 \overset{def}{=} \mathrm{id}_H$. In particular, the fact that T^0 is a symmetric tensor can be written as $T_{\alpha\beta} = T_{\beta\alpha}$ and (3.3.9) becomes

$$T_{00} + T_{11} + T_{22} + T_{33} = 0. \tag{4.3.2}$$

Furthermore, from the properties of the connection we have $T_{\alpha\beta;\gamma} = T_{\beta\alpha;\gamma}$.

Next, we will show that $T_{i0;0}$ vanish. This will yield a relation between the vertical derivatives f_s and the torsion components $T_{\alpha\beta}$.

Lemma 4.3.1. *The following identities between the components $T_{\alpha\beta}$ of the torsion tensor and the vertical derivatives f_i of the eigenfunction f hold true*

$$f_s T_{00} = f_1 T_{s1} + f_2 T_{s2} + f_3 T_{s3}, \qquad s = 1, 2, 3, \tag{4.3.3}$$

and

$$T_{i0;0} = 0. \tag{4.3.4}$$

Proof. From (4.2.11) and (4.2.8) we have

$$\nabla^2 f(X, \xi_i) = df(I_i X) - \frac{2}{3} T^0(I_i X, \nabla f) - \frac{2}{3} T^0(X, I_i \nabla f). \tag{4.3.5}$$

By (4.2.9) we have $T_{i0} = 0$, hence the above identity shows $\nabla^2 f(\nabla f, \xi_i) = 0$. Therefore, for any $\xi \in \Gamma(V)$ and $A \in \Gamma(TM)$, we have

$$\nabla^2 f(\nabla f, \xi) = \nabla^2 f(\nabla f, \nabla_A \xi) = 0 \tag{4.3.6}$$

taking into account (3.3.5). The covariant derivative along ∇f of the identity

$\nabla^2 f(\nabla f, \xi_i) = 0$ gives

$$0 = \nabla^3 f(\nabla f, \nabla f, \xi_i) - f\,\nabla^2 f(\nabla f, \xi_i) - \sum_{s=1}^{3} f_s \nabla^2 f(I_s \nabla f, \xi_i) - \nabla^2 f(\nabla f, \nabla_{\nabla f} \xi_i)$$
$$= \nabla^3 f(\nabla f, \nabla f, \xi_i) - \sum_{s=1}^{3} f_s \left[df(I_i I_s \nabla f) - \frac{2}{3} T^0(I_i I_s \nabla f, \nabla f) - \frac{2}{3} T^0(I_s \nabla f, I_i \nabla f) \right]$$
$$= \nabla^3 f(\nabla f, \nabla f, \xi_i) + |\nabla f|^2 f_i + \frac{2}{3}[-f_i T_{00} + f_i T_{ii} + f_j T_{ji} + f_k T_{ki}]$$

using the Hessian equation (4.2.1), (4.3.6), $T_{i0} = 0$ by (4.2.9), and (4.3.5). However, it follows from (4.2.10) that

$$T_{i0;0} = -f_i T_{00} + f_i T_{ii} + f_j T_{ji} + f_k T_{ki}, \tag{4.3.7}$$

hence,

$$\nabla^3 f(\nabla f, \nabla f, \xi_i) + |\nabla f|^2 f_i + \frac{2}{3} T_{i0;0} = 0. \tag{4.3.8}$$

On the other hand, from (4.2.23) we have

$$\nabla^3 f(\nabla f, \nabla f, \xi_i) = -\frac{1}{5}(1 + 2S)|\nabla f|^2 f_i + \frac{2}{3}\left[f T^0(\nabla f, I_i \nabla f) - f_i T^0(\nabla f, \nabla f) \right]$$
$$+ \frac{2}{3}\left[f_j T^0(\nabla f, I_k \nabla f) - f_k T^0(\nabla f, I_j \nabla f) \right] + \frac{2}{5} dS(\nabla f) df(I_i \nabla f)$$
$$- \frac{2}{3} \nabla T^0(\nabla f, \nabla f, I_i \nabla f) = -\frac{1}{5}(1 + 2S)|\nabla f|^2 f_i - \frac{2}{3} f_i T_{00} - \frac{2}{3} T_{i0;0} \tag{4.3.9}$$

using $T_{i0} = 0$ by (4.2.9) to obtain the last equality. Now, (4.3.8) and (4.3.9) give

$$\nabla^3 f(\nabla f, \nabla f, \xi_i) = -\frac{1}{2} T_{i0;0} - |\nabla f|^2 f_i. \tag{4.3.10}$$

A substitution of (4.3.10) into (4.3.8) shows (4.3.4), which together with (4.3.7) give (4.3.3). □

4.3.1 The Components $T_{ij;0}$ and the QC-Ricci 2-forms

At this stage, from (4.2.10) we can compute the components $T_{\alpha\beta;\gamma}$ only when either $\alpha = 0$ or $\beta = 0$. However, by evaluating (4.2.19) on the $\{\sigma_\alpha\}_{\alpha=0}^{3}$ frame, cf. Lemma

4.2.4, we will be able to use the components $T_{0i;j}$ to determine not only $T_{ij;0}$, but also the qc-Ricci 2-forms ρ_s defined in (3.3.11). We will use the following identities for the QC-Ricci 2-forms, cf. [33, Theorem 3.1] or [35, Theorem 4.3.11],

$$\begin{aligned} 18\rho_s(\xi_s, X) &= 3dS(X) + \frac{1}{2}\nabla T^0(e_\gamma, e_\gamma, X) - \frac{3}{2}\nabla T^0(e_\gamma, I_s e_\gamma, I_s X), \\ 18\rho_i(\xi_j, I_k X) &= -18\rho_i(\xi_k, I_j X) \\ &= 3dS(X) - \frac{5}{2}\nabla T^0(e_\gamma, e_\gamma, X) - \frac{3}{2}\nabla T^0(e_\gamma, I_i e_\gamma, I_i X). \end{aligned} \tag{4.3.11}$$

We begin by using (4.2.24) and the symmetry of the $T_{\alpha\beta;\gamma}$ in the first two indices to prove the following Lemma.

Lemma 4.3.2. *We have*

$$\rho_k(I_j\nabla f, \xi_j) = -\frac{3}{5}(S-2)f_k, \tag{4.3.12}$$

$$T_{ij;0} = \frac{1}{4}[f_k(T_{ii} - T_{jj}) + f_j T_{kj} - f_i T_{ki}]. \tag{4.3.13}$$

Proof. Letting $Y = I_j\nabla f$ and $X = \nabla f$ in (4.2.24) we have, taking into account $T_{i0} = 0$ and $T_{i0;0} = 0$ by (4.2.9) and (4.3.4), respectively, the following identity

$$\begin{aligned} T_{i0;j} + T_{j0;i} = &-3T_{ji;0} + fTji + 6f_i T_{ki} + 7f_j T_{jk} \\ &+ f_k(-3T_{ii} + 3T_{kk} - 4T_{jj} + 4T_{00}) - 12|\nabla f|^2\rho_k(I_i\nabla f, \xi_i). \end{aligned}$$

From (4.2.10) we can find another formula for $T_{i0;j} + T_{j0;i}$, which together with the above identity and (4.3.3) gives

$$T_{ji;0} = -3|\nabla f|^2\rho_k(I_i\nabla f, \xi_i) + \frac{3}{2}f_k T_{00} - [f_k T_{jj} - f_j T_{kj}] + \frac{1}{2}[f_i T_{ki} - f_k T_{ii}]. \tag{4.3.14}$$

On the other hand, by first taking (4.2.24) for j, and then working as above but using $Y = I_i\nabla f$ and $X = \nabla f$ we obtain the identity

$$T_{ij;0} = 3|\nabla f|^2\rho_k(I_j\nabla f, \xi_j) - \frac{3}{2}f_k T_{00} + [f_k T_{ii} - f_i T_{ki}] - \frac{1}{2}[f_j T_{kj} - f_k T_{jj}]. \tag{4.3.15}$$

Therefore, the symmetry of $T_{ij;0}$ in i and j, together with the last line in (4.1.1), (4.3.2) and (4.3.3) give

$$
\begin{aligned}
0 = T_{ij;0} - T_{ji;0} = 3|\nabla f|^2[\rho_k(I_j\nabla f, \xi_j) + \rho_k(I_i\nabla f, \xi_i)] - 3f_kT_{00} \\
+ \frac{3}{2}\left[f_k(T_{ii} + T_{jj}) - f_iT_{ki} - f_jT_{kj}\right] = 6|\nabla f|^2\rho_k(I_j\nabla f, \xi_j) - 6f_kT_{00},
\end{aligned}
$$

which, by (4.2.8), implies (4.3.12). Similarly, (4.3.14) and (4.3.15) yield

$$
\begin{aligned}
2T_{ij;0} &= T_{ij;0} + T_{ji;0} \\
&= 3|\nabla f|^2[\rho_k(I_j\nabla f, \xi_j) - \rho_k(I_i\nabla f, \xi_i)] + \frac{1}{2}[f_k(T_{ii} - T_{jj}) + f_jT_{kj} - f_iT_{ki}].
\end{aligned}
$$

By (4.1.1) the term in the first brackets is zero, hence we conclude (4.3.13). □

4.3.2 The Components $T_{ii;0}$ and the vertical Hessian of f

With the results of the previous section, we can begin to determine the components of dS. In particular, we can now show that one of the components of $dS|_H$ vanishes.

Lemma 4.3.3. *The normalized QC-scalar curvature S satisfies the following relations at almost every point of M*

$$dS(\nabla f) = 0, \qquad T_{00;0} = 0, \quad \textit{and} \qquad T_{ii;0} = \frac{1}{2}[f_jT_{ki} - f_kT_{ji}]. \tag{4.3.16}$$

Proof. Letting $X = I_i\nabla f$ and $Y = \nabla f$ in (4.2.24) and taking (4.2.9) into account, we have the identity

$$
\begin{aligned}
&T_{ii;0} + T_{0i;i} = \\
&\quad - 3[T_{ii;0} - T_{00;0}] - \frac{3}{5}|\nabla f|^2 dS(\nabla f) - f[T_{00} - T_{ii}] + f_jT_{ki} - f_kT_{ji}.
\end{aligned} \tag{4.3.17}
$$

From the formula for $\nabla T^0(Y, X, \nabla f)$ in (4.2.10) we can compute that

$$T_{i0;i} = f[T_{ii} - T_{00}] + f_kT_{ij} - f_jT_{ki} \tag{4.3.18}$$

and $5T_{00;0} = -3|\nabla f|^2 dS(\nabla f)$; using this and (4.3.18) in (4.3.17) we see that

$$T_{ii;0} = -\frac{3}{5}|\nabla f|^2 dS(\nabla f) + \frac{1}{2}[f_j T_{ki} - f_k T_{ji}]. \tag{4.3.19}$$

On the other hand, from the $Sp(1)Sp(1)$-invariance of (4.3.2) it follows that

$$T_{00;0} + T_{11;0} + T_{22;0} + T_{33;0} = 0$$

which together with (4.3.19) and (4.3.18) yield $|\nabla f|^2 dS(\nabla f) = 0$, hence (4.3.16). $\square$

Now we can determine the components of the vertical Hessian of f, which will then be used in the proofs that the remaining components of dS vanish and, eventually, in the final section that the torsion vanishes.

Lemma 4.3.4. *With the assumptions of Theorem 1.0.1, if f satisfies (4.2.1) then we have the following identities for the vertical Hessian of f,*

$$|\nabla f|^2 \nabla^2 f(\xi_i, \xi_i) = -\frac{1}{5}(1+2S)|\nabla f|^2 f - \frac{2}{3}[fT_{ii} + f_k T_{ij} - f_j T_{ki}], \tag{4.3.20}$$

$$|\nabla f|^2 \nabla^2 f(\xi_i, \xi_j) = -\frac{2}{3} f T_{ij} + \frac{1}{2}(4-S)|\nabla f|^2 f_k - \frac{11}{12}[f_i T_{ki} - f_k T_{ii}] - \frac{1}{4}[f_j T_{kj} - f_k T_{jj}], \tag{4.3.21}$$

$$\nabla^2 f(\xi_i, \xi_j) - \nabla^2 f(\xi_j, \xi_i) = \frac{2}{5}(3+S) f_k. \tag{4.3.22}$$

Proof. First, letting $X = I_i \nabla f$ and $Y = \nabla f$ in (4.2.19) and taking into account $\Gamma^i(\nabla f, I_i \nabla f) = 0$ from (4.2.17) we have

$$5T_{ii;0} - 3T_{i0;i} = 3T_{00;0} - 3T_{ii;0} - 3|\nabla f|^2 dS(\nabla f) + 5fT_{ii} + 3fT_{00} + 12|\nabla f|^2 \nabla^2 f(\xi_i, \xi_i) + \frac{12}{5}(1+2S)|\nabla f|^2 f + f_k T_{ij} - f_j T_{ki}.$$

Using the formulas in (4.3.16) and (4.3.18), we can expand the above

$$\frac{5}{2}[f_jT_{ki} - f_kT_{ji}] - 3(f[T_{ii} - T_{00}] + f_kT_{ij} - f_jT_{ki}) = -\frac{3}{2}[f_jT_{ki} - f_kT_{ji}] + 5fT_{ii} + 3fT_{00} + 12|\nabla f|^2\nabla^2 f(\xi_i, \xi_i) + \frac{12}{5}(1+2S)|\nabla f|^2 f + f_kT_{ij} - f_jT_{ki}$$

and then solve this for the vertical Hessian of f to obtain (4.3.20). Next, let $X = I_j\nabla f$ and $Y = \nabla f$ in (4.2.19); then use that $\Gamma^i(\nabla f, I_j\nabla f) = 2|\nabla f|^2\rho_k(I_i\nabla f, \xi_i)$ from (4.2.17) to see

$$5T_{ji;0} - 3T_{0i;j} = -3T_{0k;0} - 3T_{ij;0} + 5fT_{ji} + 12|\nabla f|^2\nabla^2 f(\xi_i, \xi_j) - \frac{12}{5}(1+2S)|\nabla f|^2 f_k - 24|\nabla f|^2\rho_k(I_i\nabla f, \xi_i) + 6f_iT_{ki} + 5f_jT_{jk} - 6f_kT_{ii} - 5f_kT_{jj} + 3f_kT_{00}.$$

From (4.2.10) with $Y = I_j\nabla f$, $X = I_i\nabla f$ we can compute that $T_{0i;j} = fT_{ij} + f_iT_{ki} + f_k[T_{00} - T_{ii}]$. Then this, along with $T_{i0;0} = 0$ from (4.3.4), the formula for $\rho_k(I_j\nabla f, \xi_j)$ in (4.3.12), and for $T_{ij;0}$ in (4.3.13), applied to the above gives

$$\begin{aligned}\frac{5}{4}[f_k(T_{ii} - T_{jj}) + f_jT_{kj} - f_iT_{ki}] - 3[fT_{ij} + f_iT_{ki} + f_k(T_{00} - T_{ii})]\\ = -\frac{3}{4}[f_k(T_{ii} - T_{jj}) + f_jT_{kj} - f_iT_{ki}] + 5fT_{ji} + 12|\nabla f|^2\nabla^2 f(\xi_i, \xi_j)\\ - \frac{12}{5}(1+2S)|\nabla f|^2 f_k + \frac{72}{5}(S-2)|\nabla f|^2 f_k + 6f_iT_{ki} + 5f_jT_{jk}\\ - 6f_kT_{ii} - 5f_kT_{jj} + 3f_kT_{00}.\end{aligned}$$

Using that $5T_{00} = -3(S-2)|\nabla f|^2$ from (4.2.8) and solving the above for the term $|\nabla f|^2\nabla^2 f(\xi_i, \xi_j)$ yields (4.3.21).

Finally, recall that $\{\xi_s\}_{s=1}^3$ is an orthornormal frame for V with respect to the Riemannian metric (3.3.6). With this, and the orthonormal frame $\{\sigma_\alpha\}_{\alpha=0}^3$ for H, we can expand

$$T(\xi_i, \xi_j) = |\nabla f|^{-2}\sum_{\alpha=0}^{3} h(T(\xi_i, \xi_j), I_\alpha\nabla f)I_\alpha\nabla f + \sum_{s=1}^{3} h(T(\xi_i, \xi_j), \xi_s)\xi_s.$$

By the last two lines of (4.1.1) we have $h(T(\xi_i,\xi_j),I_\alpha\nabla f) = -\rho_k(I_iI_\alpha\nabla f,\xi_i)$ and $h(T(\xi_i,\xi_j),\xi_s) = -S\delta_{ks}$. Thus, (4.3.12) and the Ricci identity

$$\nabla^2 f(\xi_i,\xi_j) - \nabla^2 f(\xi_j,\xi_i) = -df(T(\xi_i,\xi_j))$$

shows (4.3.22). □

4.4 The QC-Scalar Curvature is Constant

Here we obtain first a formula for the horizontal part $dS|_H$ of the differential of S and therefore one for the horizontal Hessian $\nabla^2 S(X,Y)$ as well. The latter will then be used to show that $dS|_V = dS|_H = 0$ and allow us to conclude that S is constant.

Several divergences of the torsion tensor T^0 will appear in the next calculations, so we remind the notation set in (0.0.3). In particular, we will use that if $\alpha \neq 0$ then $\nabla^*_\alpha T^0(X) = -\nabla T^0(I_\alpha e_\gamma, e_\gamma, X)$.

Lemma 4.4.1. *The next identity holds at almost every point of M,*

$$dS(I_t\nabla f) = -2(S-2)f_t, \qquad t = 1,2,3. \tag{4.4.1}$$

Proof. With (4.3.12), the last line of (4.1.1), and second line of (4.3.11), we arrive at the identity

$$\begin{aligned} -\frac{3}{5}(S-2)f_i &= \rho_i(I_k\nabla f,\xi_k) \\ &= -\frac{1}{6}dS(I_i\nabla f) + \frac{5}{36}\nabla^* T^0(I_i\nabla f) - \frac{1}{12}\nabla^*_i T^0(\nabla f). \end{aligned} \tag{4.4.2}$$

By (4.2.10) and the fact that T^0 is completely trace-free we have the identity

$$\nabla^*_i T^0(\nabla f) = \frac{3}{5}[dS(I_i\nabla f) + 4(S-2)f_i]. \tag{4.4.3}$$

Therefore, we need only determine $\nabla^*T^0(I_i\nabla f)$. For this, take the trace $X = e_\gamma$, $Y = I_\alpha e_\gamma$ in (4.2.19):

$$\begin{aligned}5\nabla T^0(I_\alpha e_\gamma, e_\gamma, I_i\nabla f) - 3\nabla T^0(e_\gamma, I_\alpha e_\gamma, I_i\nabla f) = 3dS(I_\alpha e_\gamma)df(I_i e_\gamma)\\ -\frac{9}{5}dS(e_\gamma)df(I_iI_\alpha e_\gamma) + \frac{6}{5}(4+3S)g(I_\alpha e_\gamma, e_\gamma)f_i + 12\sum_{s=1}^{3}\nabla^2 f(\xi_i,\xi_s)\omega_s(I_\alpha e_\gamma, e_\gamma)\\ -\frac{12}{5}(1+2S)\left(f\omega_i(e_\gamma, I_\alpha e_\gamma) + \sum_{s=1}^{3} f_s\,\omega_s(I_\alpha e_\gamma, I_i e_\gamma)\right) - 12\Gamma^i(I_\alpha e_\gamma, e_\gamma).\end{aligned} \tag{4.4.4}$$

For $\alpha = 0$ equation (4.4.4) becomes

$$\nabla^*T^0(I_i\nabla f) = -\frac{3}{5}[dS(I_i\nabla f) + 4(S-2)f_i] - 12\Gamma^i(e_\gamma, e_\gamma). \tag{4.4.5}$$

Using (4.2.17) and (4.3.11) we see that

$$\begin{aligned}\Gamma^i(e_\gamma, e_\gamma) &= 2[\rho_j(I_j\nabla f, \xi_i) + \rho_k(I_k\nabla f, \xi_i)]\\ &= \frac{2}{3}dS(I_i\nabla f) - \frac{5}{9}\nabla^*T^0(I_i\nabla f) + \frac{1}{6}[\nabla^*_jT^0(I_k\nabla f) - \nabla^*_kT^0(I_j\nabla f)],\end{aligned} \tag{4.4.6}$$

thus a substitution of (4.4.6) into (4.4.5) gives

$$\begin{aligned}\nabla^*T^0(I_i\nabla f) = \frac{3}{7}\left(\frac{23}{5}dS(I_i\nabla f) + \frac{12}{5}(S-2)f_i\right)\\ + \frac{3}{7}\left[\nabla^*_jT^0(I_k\nabla f) - \nabla^*_kT^0(I_j\nabla f)\right].\end{aligned} \tag{4.4.7}$$

Now we write (4.4.4) for j instead of i and then let $\alpha = k$ in the result. Then we use (4.2.17) to see that $\Gamma^j(I_k e_\gamma, e_\gamma) = -4\rho_i(I_j\nabla f, \xi_j)$ and thus by (4.3.12):

$$\nabla^*_kT^0(I_j\nabla f) = -\frac{3}{5}dS(I_i\nabla f) - \frac{6}{5}(7-S)f_i + 6\nabla^2 f(\xi_j, \xi_k). \tag{4.4.8}$$

Next, we do one more permutation of the indices and consider (4.4.4) for k instead of i and then let $\alpha = j$, which taking into account $\Gamma^k(I_j e_\gamma, e_\gamma) = 4\rho_i(I_k\nabla f, \xi_k)$ and (4.3.12) gives an identity for the remaining divergence

$$\nabla^*_jT^0(I_k\nabla f) = \frac{3}{5}dS(I_i\nabla f) + \frac{6}{5}(7-S)f_i + 6\nabla^2 f(\xi_k, \xi_j). \tag{4.4.9}$$

Therefore, subtracting (4.4.8) from (4.4.9) and applying (4.3.22) we come to

$$\nabla_j^* T^0(I_k\nabla f) - \nabla_k^* T^0(I_j\nabla f) = \frac{6}{5}[dS(I_i\nabla f) - 4(S-2)f_i]. \tag{4.4.10}$$

Lastly, a substitution of (4.4.10) into (4.4.7) gives

$$\nabla^* T^0(I_i\nabla f) = \frac{3}{7}\left(\frac{29}{5}dS(I_i\nabla f) - \frac{12}{5}(S-2)f_i\right) \tag{4.4.11}$$

which after using it together with (4.4.3) in (4.4.2) shows (4.4.1). □

Lemma 4.4.2. *The normalized QC-scalar curvature S is constant, in fact $S=2$. In particular,*

$$T_{00} = 0, \qquad T_{11}+T_{22}+T_{33} = 0, \qquad f_1T_{s1}+f_2T_{s2}+f_3T_{s3} = 0 \tag{4.4.12}$$

for $s = 1,2,3$, and for any cyclic permutation (i,j,k) of $(1,2,3)$ we have

$$f_kT_{ik} - f_iT_{kk} = f_iT_{jj} - f_jT_{ij}. \tag{4.4.13}$$

Proof. First we will show that the differential of S vanishes on the vertical space, $dS|_V = 0$. With (4.3.16) and (4.4.1) we can write the horizontal gradient of S in the $\{\sigma_\alpha\}_{\alpha=0}^3$ frame in the form

$$|\nabla f|^2\nabla S = -2(S-2)\sum_{t=1}^3 f_t\, I_t\nabla f. \tag{4.4.14}$$

The covariant derivative of (4.4.14) along a horizontal vector Y, using (3.3.5), the horizontal Hessian equation (4.2.1), and (4.2.11) for the term $\nabla^2 f(Y,\xi_i)$, gives the equation

$$\begin{aligned}\frac{1}{2}|\nabla f|^2\nabla^2 S(Y,X) &= f\,df(Y)dS(X) + \sum_{t=1}^3 f_t[df(I_tY)dS(X) + df(I_tX)dS(Y)]\\ &+ (S-2)\sum_{t=1}^3\left[f_t\nabla^2 f(Y,I_tX) + \left(\frac{1}{5}(1+2S)df(I_tY) - \frac{2}{3}T^0(Y,I_t\nabla f)\right)df(I_tX)\right].\end{aligned} \tag{4.4.15}$$

Using (4.2.1) again, the identities in (4.3.16) and (4.4.1) we find $\nabla^2 S(I_i\nabla f, \nabla f) = \nabla^2 S(\nabla f, I_i \nabla f)$. Hence, by the Ricci identity

$$\nabla^2 S(X,Y) - \nabla^2 S(Y,X) = -2\sum_{t=1}^{3} dS(\xi_t)\omega_t(X,Y)$$

we have

$$-2\sum_{t=1}^{3} dS(\xi_t)\omega_t(I_i\nabla f, \nabla f) = \nabla^2 S(I_i\nabla f, \nabla f) - \nabla S(\nabla f, I_i\nabla f) = 0,$$

which implies

$$dS(\xi_t) = 0, \qquad t = 1, 2, 3. \tag{4.4.16}$$

Now we can show that the differential of S vanishes on all horizontal vectors as well, $dS|_H = 0$. From (4.1.2) and (4.4.16) we find $\nabla^2 S(e_\gamma, I_i e_\gamma) = 0$. On the other hand, using (4.4.15) with (4.4.1) we also have

$$|\nabla f|^2 \nabla^2 S(e_\gamma, I_i e_\gamma) = -2f dS(I_i\nabla f).$$

Thus, since $f \neq 0$ a.e., see Lemma 4.2.4, we conclude

$$dS(I_t\nabla f) = 0 \qquad t = 1, 2, 3.$$

Hence, since in addition we have $dS(\nabla f) = 0$ by (4.3.16), it follows that $dS|_H = 0$. Therefore, taking into account that dS vanishes on the Reeb vector fields as proven above, it follows that $dS = 0$ and hence S is constant.

In order to determine the constant we note that from (4.4.1), either $S = 2$ or $f_1 = f_2 = f_3 = 0$ on some open set. Arguing by contradiction, suppose the latter, then for any horizontal vector X we would have, by (3.3.5) and the assumption $f_s = 0$, the idenity

$$0 = Xf_i = \nabla^2 f(X, \xi_i) - \alpha_j(X) f_k + \alpha_k(X) f_j = \nabla^2 f(X, \xi_i).$$

Then it would follow from (4.2.11) that $10\,T_{ii} = -3(1+2S)|\nabla f|^2$. On the other hand, the component T_{00} can be computed from (4.2.8), which gives $T_{00} = -\frac{3}{5}(S-2)|\nabla f|^2$. Therefore, by (4.3.2) we have

$$0 = \sum_{\alpha=0}^{3} T_{\alpha\alpha} = \frac{3}{10}(1-8S)|\nabla f|^2,$$

hence, $S = 1/8$. This is a contradiction since the Lichnerowicz-type bound (1.0.1) implies, due to T^0 being a trace-free tensor, that $S \geq 2$. Thus we must have $S = 2$, and consequently (4.2.8) now implies $T_{00} = 0$. With this, (4.4.12) follows from (4.3.2) and (4.3.3).

Finally, a substitution of the second identity in (4.4.12) into the third one written for $s = i$ shows

$$0 = f_i T_{ii} + f_j T_{ij} + f_k T_{ik} = f_i(-T_{jj} - T_{kk}) + f_j T_{ij} + f_k T_{ik}$$

from which (4.4.13) follows. $\square$

4.5 Vanishing of the Torsion

The last application of (4.2.19) is to finally show that $T^0 = 0$. We begin with a simple lemma describing the consequences of $S = 2$ on the components of the divergences $\nabla^*_i T^0(X) = \nabla T^0(e_\gamma, I_i e_\gamma, X)$ defined in (0.0.3).

Lemma 4.5.1. *The divergences of the torsion satisfy the following identities,*

$$|\nabla f|^2 \nabla^*_i T^0(I_i \nabla f) = -4[f T_{ii} + f_k T_{ij} - f_j T_{ki}] \tag{4.5.1}$$

$$|\nabla f|^2 \nabla^*_j T^0(I_k \nabla f) = |\nabla f|^2 \nabla^*_k T^0(I_j \nabla f) = -4[f T_{jk} + f_j T_{ij} - f_i T_{jj}]. \tag{4.5.2}$$

Proof. Since $S = 2$ by Lemma 4.4.2, equation (4.4.10) implies $\nabla_k T^0(I_j \nabla f) = \nabla_j T^0(I_k \nabla f)$, which gives the first equality in (4.5.2). Furthermore, (4.3.21) now takes the simpler form

$$12|\nabla f|^2\nabla^2 f(\xi_j,\xi_k) = -8fT_{jk} + 12|\nabla f|^2 f_i - 11f_jT_{ij} + 11f_iT_{jj} - 3f_kT_{ik} + 3f_iT_{kk}. \quad (4.5.3)$$

Applying (4.4.13) to (4.5.3) we find

$$12|\nabla f|^2\nabla^2 f(\xi_j,\xi_k) = -8fT_{jk} + 12|\nabla f|^2 f_i - 8f_jT_{ij} + 8f_iT_{jj}. \quad (4.5.4)$$

Substituting (4.5.4) into (4.4.8) yields the second equality of (4.5.2).

Finally, let $\alpha = i$ in (4.4.4), which due to $\nabla^*_\alpha T^0(X) = -\nabla T^0(I_\alpha e_\gamma, e_\gamma, X)$ takes the form (for i fixed)

$$\begin{aligned} -8\nabla^*_i T^0(I_i\nabla f) &= 5\nabla T^0(I_ie_\gamma, e_\gamma, I_i\nabla f) - 3\nabla T^0(e_\gamma, I_ie_\gamma, I_i\nabla f) \\ &= -48f - 48\nabla^2 f(\xi_i,\xi_i) - 12\Gamma^i(I_ie_\gamma, e_\gamma) = -48\left[f + \nabla^2 f(\xi_i,\xi_i)\right] \end{aligned}$$

using $\Gamma^i(I_ie_\gamma, e_\gamma) = 0$ from (4.2.17). An application of (4.3.20) to the last equation gives (4.5.1). □

In the last lemma needed for the proof of the main theorem we derive the key relation between the components of the torsion tensor. We continue the use of the notation T_{ij} for the components of the torsion set in (4.3.1).

Lemma 4.5.2. *The next identities hold at almost every point,*

$$fT_{jk} = \frac{1}{4}[f_iT_{kk} - f_kT_{ki}] = \frac{1}{4}[f_jT_{ij} - f_iT_{jj}] \quad (4.5.5)$$

$$fT_{ii} = \frac{1}{4}[f_kT_{ij} - f_jT_{ki}]. \quad (4.5.6)$$

Proof. First, let us dispose with the trivial case, by noting that the second identity in (4.5.5) follows directly from (4.4.13).

We turn to the proof of the first equality in (4.5.5). Let $A(Y,X)$ denote the tensor in the left-hand side of (4.2.19). Therefore,

$$16\nabla T^0(Y,X,I_i\nabla f)=5A(Y,X)+3A(X,Y).$$

Taking into account that the scalar curvature is constant and $T_{00}=0$ the above equation takes the following explicit form

$$\begin{aligned}8\nabla T^0(Y,X,I_i\nabla f)&=-12\nabla T^0(\nabla f,I_iX,Y)-12\nabla T^0(\nabla f,X,I_iY)+48g(X,Y)f_i\\&-\sum_{s=1}^3 f_s[30g(I_iX,I_sY)+18g(I_sX,I_iY)]+12\sum_{s=1}^3\nabla^2 f(\xi_i,\xi_s)g(X,I_sY)\\&+8fT^0(X,I_iY)+12fg(X,I_iY)+f_i[24T^0(I_iX,I_iY)-32T^0(X,Y)]\\&+f_j[17T^0(X,I_kY)+15T^0(I_jX,I_iY)+9T^0(I_iX,I_jY)+15T^0(I_kX,Y)]\\&+f_k[15T^0(I_kX,I_iY)+9T^0(I_iX,I_kY)-17T^0(X,I_jY)-15T^0(I_jX,Y)]-48\Gamma^i(X,Y).\end{aligned}\tag{4.5.7}$$

Now, let $X=Y=I_j\nabla f$ in (4.5.7) and use (4.2.9) to obtain

$$\begin{aligned}8T_{ji:j}&=-24T_{jk;0}+8fT_{jk}+f_i[24T_{kk}-32T_{jj}]\\&\qquad-32f_jT_{ij}-24f_kT_{ik}-48\Gamma^i(I_j\nabla f,I_j\nabla f).\end{aligned}$$

Next, consider (4.5.7) written for j, and then let $X=I_i\nabla f$, $Y=I_j\nabla f$. Using (4.4.12) and $T_{i0;0}=0$ by (4.3.4) it follows

$$\begin{aligned}8T_{ij;j}&=12T_{kj;0}+12|\nabla f|^2f_i-12|\nabla f|^2\nabla^2 f(\xi_j,\xi_k)\\&\qquad-32f_jT_{ij}+26f_kT_{ik}+f_i[-9T_{kk}+17T_{ii}-15T_{jj}]-48\Gamma^j(I_i\nabla f,I_j\nabla f).\end{aligned}$$

By the symmetry of $T_{\alpha\beta;\gamma}$ in its first two indices and the above identities for $T_{ji;j}$ and $T_{ij;j}$ we have

$$\begin{aligned}0=8T_{ji;j}-8T_{ij;j}&=-36T_{jk;0}+8fT_{jk}+33f_iT_{kk}-17f_iT_{jj}-50f_kT_{ki}-17f_iT_{ii}\\&-12|\nabla f|^2f_i+12|\nabla f|^2\nabla^2 f(\xi_j,\xi_k)-48[\Gamma^i(I_j\nabla f,I_j\nabla f)-\Gamma^j(I_i\nabla f,I_j\nabla f)].\end{aligned}\tag{4.5.8}$$

Since $S = 2$, (4.2.30) and (4.4.11) imply $\nabla^* T^0 = 0$. Therefore by the last lines in (4.3.11), (4.1.1) we have

$$\rho_k(I_j X, \xi_i) = -\rho_k(I_i X, \xi_j) = \frac{1}{12}\nabla^*_k T^0(I_k X). \tag{4.5.9}$$

The definition of $\Gamma^i(Y,X)$ in (4.2.17), together with (4.5.9) and (4.5.2) show that

$$\Gamma^i(I_j\nabla f, I_j\nabla f) - \Gamma^j(I_i\nabla f, I_j\nabla f) = -\frac{1}{4}|\nabla f|^2\nabla^*_k T^0(I_j\nabla f) = fT_{jk} + f_jT_{ij} - f_iT_{jj},$$

which gives a formula for the last term in (4.5.8). The latter, together with the identities (4.3.13) for the term $T_{jk;0}$ and (4.5.3) for the term $|\nabla f|^2\nabla^2 f(\xi_j,\xi_k)$, allows to rewrite (4.5.8) as follows

$$\begin{aligned} 0 = &-9[f_i(T_{jj}-T_{kk}) + f_kT_{ik} - f_jT_{ij}] + 8fT_{jk} + 33f_iT_{kk} - 17f_iT_{jj} - 50f_kT_{ki} - 17f_iT_{ii} \\ &- 12|\nabla f|^2 f_i - 8fT_{jk} + 12|\nabla f|^2 f_i - 8f_jT_{ij} + 8f_iT_{jj} - 48[fT_{jk} + f_jT_{ij} - f_iT_{jj}] \\ &= 30f_iT_{jj} + 42f_iT_{kk} - 17f_iT_{ii} - 47f_jT_{ij} - 59f_kT_{ik} - 48fT_{jk}. \end{aligned}$$

From (4.4.12) we have that $-17f_iT_{ii} = 17f_jT_{ij} + 17f_kT_{ik}$, therefore the above reads

$$0 = 30f_iT_{jj} + 42f_iT_{kk} - 30f_jT_{ij} - 42f_kT_{ik} - 48fT_{jk}. \tag{4.5.10}$$

In addition, (4.4.12) also gives

$$f_jT_{ij} - f_iT_{jj} = f_jT_{ij} + f_iT_{ii} + f_iT_{kk} = f_iT_{kk} - f_iT_{ik}.$$

Applying this to (4.5.10) shows

$$0 = 30f_iT_{jj} + 42f_iT_{kk} - 30f_jT_{ij} - 42f_kT_{ik} - 48fT_{jk} = 12f_iT_{kk} - 12f_kT_{ik} - 48fT_{jk}$$

from which the first identity in (4.5.5) follows.

We turn to the proof of (4.5.6). Choosing X and Y in the obvious ways, equation (4.5.7) written for j and k, respectively, implies the following identities

$$\begin{aligned} 8T_{kj;i} = 12T_{kk;0} - 12T_{ii;0} - 12|\nabla f|^2\nabla^2 f(\xi_j,\xi_j) - 8fT_{kk} - 12|\nabla f|^2 f \\ - 56f_jT_{ki} - 6f_kT_{ij} - 2f_iT_{jk} - 48\Gamma^j(I_k\nabla f, I_i\nabla f) \end{aligned}$$

and

$$8T_{jk;i} = 12T_{ii;0} - 12T_{jj;0} + 12|\nabla f|^2\nabla^2 f(\xi_k, \xi_k) + 8fT_{jj} + 12|\nabla f|^2 f$$
$$- 56f_kT_{ij} - 2f_iT_{jk} - 6f_jT_{ki} - 48\Gamma^k(I_j\nabla f, I_i\nabla f).$$

Therefore, we have

$$\begin{aligned} 0 = 8T_{kj;i} - 8T_{jk;i} &= 12[T_{kk;0} - 2T_{ii;0} + T_{jj:0}] \\ &- 12|\nabla f|^2[\nabla^2 f(\xi_j, \xi_j) + \nabla^2 f(\xi_k, \xi_k)] - 24|\nabla f|^2 f - 8f[T_{kk} + T_{jj}] \\ &- 50[f_jT_{ki} - f_kT_{ij}] - 48[\Gamma^j(I_k\nabla f, I_i\nabla f) - \Gamma^k(I_j\nabla f, I_i\nabla f)]. \end{aligned} \tag{4.5.11}$$

By the definition of Γ^i in (4.2.17), followed by the identity (4.5.9) for ρ_s, and (4.5.1), we find

$$\begin{aligned} \Gamma^j(I_k\nabla f, I_i\nabla f) - \Gamma^k(I_j\nabla f, I_i\nabla f) &= \frac{1}{12}|\nabla f|^2\nabla^*_k T^0(I_k\nabla f) \\ - \frac{1}{6}|\nabla f|^2\nabla^*_i T^0(I_i\nabla f) &+ \frac{1}{12}|\nabla f|^2\nabla^*_j T^0(I_j\nabla f) = fT_{ii} - f_jT_{ki} + f_kT_{ij}. \end{aligned}$$

Then, using the above along with (4.3.19) and (4.3.20) in (4.5.11) gives

$$\begin{aligned} 0 = 12&\left[\frac{1}{2}(f_iT_{jk} - f_jT_{ij}) - (f_jT_{ki} - f_kT_{ji}) + \frac{1}{2}(f_kT_{ij} - f_iT_{kj})\right] \\ - 12&\left[-|\nabla f|^2 f - \frac{2}{3}(fT_{jj} + f_iT_{jk} - f_kT_{ij}) - |\nabla f|^2 f - \frac{2}{3}(fT_{kk} + f_jT_{ki} - f_iT_{jk})\right] \\ &- 24|\nabla f|^2 f - 8f[T_{kk} + T_{jj}] - 50[f_jT_{ki} - f_kT_{ij}] - 48[fT_{ii} - f_jT_{ki} + f_kT_{ij}] \\ &= -12f_jT_{ki} + 12f_kT_{ij} - 48fT_{ii} \end{aligned}$$

from which (4.5.6) follows. □

4.6 Proof of Theorem 1.0.1

We can now finally prove that M is QC-Einstein and hence conclude the Main Theorem. With the notation set in (4.3.1) we have $T_{00} = T_{0i} = 0$, see (4.4.12) and

(4.2.9), hence

$$|\nabla f|^4|T^0|^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}^2 + 2T_{23}^2 + 2T_{31}^2$$
$$= \sum_{(i\,j\,k)}[T_{ii}^2 + 2T_{ij}^2] = \sum_{(i\,j\,k)}[T_{ii}^2 + 2T_{jk}^2], \quad (4.6.1)$$

recalling that $\sum_{(i\,j\,k)}$ indicates a cyclic sum. Using the identities

$$4fT_{jk} = f_iT_{kk} - f_kT_{ki} = f_jT_{ij} - f_iT_{jj} \qquad \text{and} \qquad 4fT_{ii} = f_kT_{ij} - f_jT_{ki}$$

by (4.5.5) and (4.5.6), we obtain

$$4f|\nabla f|^4|T^0|^2 = \sum_{(i\,j\,k)}[T_{ii}\,(f_kT_{ij} - f_jT_{ki}) + T_{jk}\,(f_iT_{kk} - f_kT_{ki}) + T_{jk}\,(f_jT_{ij} - f_iT_{jj})]$$
$$= \sum_{(i\,j\,k)}[f_kT_{ii}T_{ij} - f_jT_{ii}T_{ki} + f_iT_{jk}T_{kk} - f_kT_{jk}T_{ki} + f_jT_{jk}T_{ij} - f_iT_{jk}T_{jj}]$$
$$= \sum_{(i\,j\,k)}[f_kT_{ii}T_{ij} - f_kT_{jj}T_{ij} + f_kT_{ij}T_{jj} - f_kT_{jk}T_{ki} + f_kT_{ki}T_{jk} - f_kT_{ij}T_{ii}] = 0.$$

By Lemma 4.2.4 it follows $T^0 \equiv 0$. Thus, M is a QC-Einstein manifold. Finally, since f is an eigenfunction of the sub-Laplacian of eigenvalue 4 and M is QC-Einstein of constant normalized QC-Scalar curvature $S = 2$, it follows from part 2 of Theorem 3.7.1 that (M, η) is QC-equivalent to the standard seven-dimensional 3-Sasakian sphere. The main steps in the proof of this QC-equivalence are given in Section 3.7.

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