6-21-1998

Robust non-fragile LQ controllers: the static state feedback case

Chaouki T. Abdallah

Domenico Famularo

Ali Jadbabaie

Peter Dorato

Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

Robust Non-fragile LQ Controllers: The Static State Feedback Case

Domenico Famularo
DEIS
Università della Calabria
Rende (CS), 87036, ITALY
domenico.famularo@unical.it

Chaouki T. Abdallah, Ali Jadbabaie and Peter Dorato
EECE Dept.
University of New Mexico
Albuquerque, NM 87131, USA
{chaouki,alij,peter}@eece.unm.edu

Wassim M. Haddad
School of Aerospace Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0150, USA
wassim.haddad@aerospace.gatech.edu

Abstract

This paper describes the synthesis of non-fragile or *resilient* regulators for linear systems. The general framework for fragility is described using state-space methodologies, and the LQ/$\mathcal{H}_2$ static state-feedback case is examined in detail. We discuss the multiplicative structured uncertainties case, and propose remedies of the fragility problem using a convex programming framework (LMIs) as a possible solution scheme. The benchmark problem is taken as an example to show how controller gain variations can affect the performance of the closed-loop system.

Keywords: Fragility, Linear quadratic regulator, Multiplicative structured uncertainties, Linear Matrix Inequalities (LMIs).

1. Introduction

The purpose of this paper is to address and understand the effects of controller uncertainties in the implementation of robust regulators which optimize a given performance index in linear systems. In the literature, there are different algorithms that give an answer to the classical problem shown in Figure 1:

Given a linear plant P with additive uncertainties ΔP find a feedback controller K which internally stabilizes the family $P + \Delta P$ and satisfies a given performance measure. In

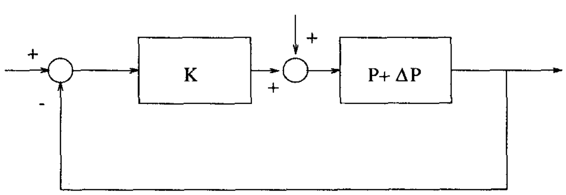

Figure 1: Robust Control Scheme

this paper we will consider structured uncertainties in the plant, to represent the effect of (generally) time-varying parameters whose exact values are unknown but which are known to belong to a given set [1]. Virtually all control algorithms proposed in the literature do not consider the problems introduced by implementing uncertain controllers. We first remark that it is reasonable to consider only structured uncertainties in the controller since by design, one can choose its exact structure even though the designer may not be able to exactly implement that nominal configuration. The controllers obtained using most robust design approaches are thus *optimal* if implemented *exactly*. There are however many reasons to believe that one can never exactly implement a compensator which theoretically meets all objectives (see [2] for an example of a compensator that cannot be implemented). Moreover, it is easy to argue that even when exact implementation is possible, some tuning by the control engineer is required on the actual controller in order to achieve a "safety" margin with respect to sampling procedures, roundoff errors etc. We thus consider the more realistic problem depicted in Figure 2 wherein both the plant and controller are uncertain. In a recent paper, Keel and Bhattacharyya [3]

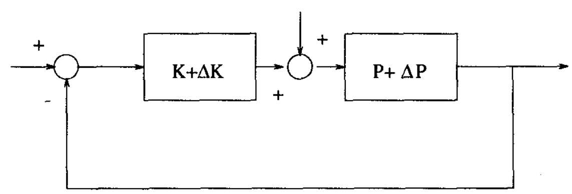

Figure 2: Robust Fragility Control Scheme

have shown that, in the case of unstructured uncertainties in the plant, and using weighted $\mathcal{H}_\infty$, μ or l_1 synthesis techniques, the resulting controllers exhibit a poor stability margin if not implemented exactly! This so-called "fragility" is displayed regardless of whether these controllers are optimal when implemented using their nominal parameters. Reference [3] gives the following suggestions to overcome the fragility problem:

1. Develop synthesis algorithms which take into account some structured uncertainties in the controllers and search for the "best" solution that guarantees a compromise between optimality and fragility;

2. Examine the structure of the controller in order to

0-7803-4530-4/98 $10.00 © 1998 AACC

parameterize it in a useful way (lower-order or fixed-structure controllers).

Reference [4] addresses and solves a special case of the fragility problem by considering a *structured uncertain* dynamic compensator for a noise-driven linear plant. The authors in [4] obtain sufficient conditions by bounding the uncertainties in the controller using classical quadratic Lyapunov bounds [5]. The resulting controllers are proven to be "resilient" in the sense that even when they are not exactly implemented, stability and some measure of performance are maintained.

It is true that other authors have hinted at the problem of fragility, see for example page 75 of Ackermann [6], and that many critics have dismissed the issue, since robust controllers are not designed to be resilient. On the other hand, the problem is reminiscent of the Linear Quadratic Gaussian (LQG) optimal controllers which were only useful when implemented on the exact plant, and had no guaranteed robustness margins if the plant was uncertain. This lack of robustness was corrected using Linear Quadratic Gaussian synthesis with Loop Transfer Recovery (LQG/LTR) [7]. In addition, even robust controllers will eventually have to be implemented on an actual system using digital hardware and should be resilient both to implementation errors and to tuning [6].

The aim of this paper is to extend the ideas in [3, 4] and to analyze the robust fragility problem by considering the combined effect of structured uncertainties in the plant and in the compensator. The basic idea is that, instead of computing the controller as a single point in the parameters space, we look for a set of controllers allowing the parameters to lie in a region of uncertainty. This is reminiscent of the design of Ackermann [6] and Barmish *et al.* [8].

This paper is organized as follows. In Section 2, we present the synthesis of static state-feedback controllers for linear systems while allowing structured uncertainties in the feedback gain matrix. We then further restrict our study to multiplicative structured uncertainties in the plant. In Section 3, a numerical example using Linear Matrix Inequalities as a computational tool is given. Our conclusions and directions for future research are finally given in Section 4.

2. Outline of the problem

Consider the following time-varying linear system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + Bu(t)\,,\ t \geq 0\,, \\ y(t) & = & Cx(t)\,, \end{array} \right. \tag{1}$$

where

- $x(t) \in \mathbb{R}^n$, is the state vector,
- $u(t) \in \mathbb{R}^m$, is the control input,
- $y(t) \in \mathbb{R}^p$, is the output measurements vector,
- $A(t)$, $t \geq 0$, contains affine uncertainties (see [9]) of the form
$$A(t) = A_0 + \sum_{i=1}^{q} \alpha_i(t) A_i,$$
where the scalar coefficients $\alpha_i(t)$, $t \geq 0$, are Lebesgue measurable functions on $[0, \infty)$ representing unknown and time -varying coefficients whose values belong to an uncertainty interval
$$\underline{\alpha}_i \leq \alpha_i(t) \leq \overline{\alpha}_i,\ 1 \leq i \leq q,\ t \geq 0\,. \tag{2}$$

The system (1) can then be written in the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \left(A_0 + \sum_{i=1}^{q} \alpha_i A_i\right) x(t) + Bu(t) \ = \\ & = & (A_0 + \delta A)x(t) + Bu(t)\,, \\ y(t) & = & Cx(t)\,. \end{array} \right. \tag{3}$$

Now, we assume that the initial condition $x(0)$ is a random variable with mean $x(0)$ and covariance matrix equal to I_n and proceed to find a state-feedback compensator $u(t) = Kx(t)$ which minimizes the Linear Quadratic (LQ) performance index, given by

$$J = \mathcal{E}\left[\int_0^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right)\, dt\right]\,, \tag{4}$$

where $Q = C^T C$, R is a symmetric positive-definite matrix and $\mathcal{E}$ denotes the expectation with respect to the initial state $x(0)$.

2.1. Non-fragile controller synthesis

Although one finds the controller $u = Kx$, the actual controller implemented is

$$u = (K + \delta K)x = \tilde{K}x\,, \tag{5}$$

where K is the nominal controller gain, and the term δK represents controller gain variations. In this case, the performance index (4) becomes a function of K, the uncertain term δK, and the uncertainties α_i in (3) as shown below so that

$$J = J(K, \delta K, \alpha_i).$$

A possible solution to the fragility problem may be stated as follows:

1. Letting $\delta K = 0$, design a "nominal" controller and find a bound $\bar{J}$ on the performance index (4) so that
$$J(K, 0, \alpha_i) \leq \bar{J}(K).$$
Then, solve a standard guaranteed-cost problem [5] for the controller gain such that $\bar{K}$ minimizes $\bar{J}(K)$;

2. Fix the uncertainty range δK and find a new bound $\hat{J}$ to (4), that is,

$$J(K, \delta K, \alpha_i) \leq \hat{J}(K).$$

Note that $\bar{J}(K) \leq \hat{J}(K)$. Now, solving the new guaranteed-cost problem we seek controller gains $\hat{K}$ that minimize $\hat{J}(K)$ and satisfy

$$\left|\hat{J}\left(\hat{K}\right) - \bar{J}\left(\bar{K}\right)\right| < M, \qquad (6)$$

where M is a level that can be fixed *a priori*. If (6) does not hold, we reduce the uncertainty level δK.

With this scheme in mind, we now study the multiplicative uncertainty case of equation (5) in greater detail.

2.2. Multiplicative structured uncertainties

Let the nominal state-feedback matrix K be an $m \times n$ ($m < n$) matrix. If we allow relative percentage drift from the nominal entries of the matrices K and represent each entry of the perturbed matrix as a multiplicative scalar uncertainty, we have

$$\begin{aligned}(K+\delta K) &= \begin{bmatrix} k_{11}(1+\delta_{11}) & \cdots & k_{1n}(1+\delta_{1n}) \\ \vdots & \ddots & \vdots \\ k_{m1}(1+\delta_{m1}) & \cdots & k_{mn}(1+\delta_{mn}) \end{bmatrix} \\ &= \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{bmatrix} + \begin{bmatrix} k_{11}\delta_{11} & \cdots & k_{1n}\delta_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1}\delta_{m1} & \cdots & k_{mn}\delta_{mn} \end{bmatrix}, \\ & -1 < \underline{\delta}_{ij} \leq \delta_{ij} \leq \bar{\delta}_{ij} < 1. \end{aligned} \qquad (7)$$

Equation (7) then leads to the uncertain controller structure:

$$u(t) = \left(K + \sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij}\Psi_i^{(m)} K \Psi_j^{(n)}\right) x(t) = K_\Delta x(t), \qquad (8)$$

where $\Psi_i^{(m)}$, $\Psi_j^{(n)}$ are $m \times m$ and $n \times n$ rank-one matrices with a "1" entry located at the i-th and j-th position of the main diagonal, respectively. In this case, the closed loop system is given by

$$\dot{x}(t) = \left(A_0 + \sum_{i=1}^{q} \alpha_i A_i + B\left(K + \sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij}\Psi_i^{(m)} K \Psi_j^{(n)}\right)\right) x(t). \qquad (9)$$

Note that the closed-loop system matrix has structured uncertainty of the form:

$$\tilde{A} = A_0 + \sum_{i=1}^{q} \alpha_i A_i + BK + \sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij} B_i K \Psi_j^{(n)}, \qquad (10)$$

where $B_i = B\Psi_i^{(m)}$. Now, substituting (8) into (4) yields

$$J = \mathcal{E}\left[\int_0^\infty \left(x^T Q x + x^T K_\Delta^T R K_\Delta x\right) dt\right]. \qquad (11)$$

Note that, using (7) it follows that

$$x^T K_\Delta^T R K_\Delta x \leq \alpha(K) x^T K^T R K x, \qquad (12)$$

where

$$\alpha(K) = \frac{\sup_{\delta_{ij}} \lambda_{\max}\left(K_\Delta^T R K_\Delta\right)}{\lambda_{\min}\left(K^T R K\right)},$$

and where the supremum operation is performed over the uncertainty set and $\lambda_{\max}$ and $\lambda_{\min}$ indicate, respectively, the maximum and the minimum eigenvalues of a Hermitian matrix. It is easy to see that the bound (12) is verifiable only when K is known in advance. The performance index (11) is then bounded by

$$\mathcal{J} = \mathcal{E}\left[\int_0^\infty \left(x^T Q x + \alpha(K) x^T K^T R K x\right) dt\right], \qquad (13)$$

which gives rise to a *non-convex* dynamic optimization problem which is in general difficult to solve but, in particular cases, it reduces to a convex optimization problem [10]. In the following we analyze some of these special cases.

2.2.1 Special Cases

In the single input case (i.e., $m = 1$) (7) reduces to

$$K + \delta K = \left[\ k_1(1+\delta_1) \quad \ldots \quad k_n(1+\delta_n)\ \right], \qquad (14)$$

where δ_j, $j = 1, \ldots, n$, are scalar coefficients such that $-1 < \underline{\delta}_j \leq \delta_j \leq \bar{\delta}_j < 1$. We can then write the controller (8) as

$$u = K(I_n + \sum_{j=1}^{n} \delta_j \Psi_j^{(n)})x = K\Lambda(\delta)x, \qquad (15)$$

where the term $\Lambda(\delta)$ denotes a diagonal matrix whose entries are $1+\delta_j$, $j = 1, \ldots, n$. In this case, the performance index J can be easily bounded by noting that

$$\Lambda(\delta) R \Lambda(\delta) \leq (1+\theta)^2 R,$$

where $\theta = \max_j |\delta_j|$, $j = 1, \ldots, n$, so that

$$J \leq \mathcal{J} = \mathcal{E}\left[\int_0^\infty \left(x^T Q x + (1+\theta)^2 x^T K^T R K x\right) dt\right]. \qquad (16)$$

Now, the closed-loop system is given by

$$\dot{x}(t) = \left(A + \sum_{i=1}^{q} \alpha_i A_i + BK\Lambda(\delta)\right) x(t) \qquad (17)$$

and the closed-loop dynamic matrix $\tilde{A}$ can be rewritten as

$$\tilde{A} = A + BK + \sum_{i=1}^{q} \alpha_i A_i + \sum_{j=1}^{n} \delta_j BK\Psi_j^{(n)}.$$

In this case the problem is equivalent to a static output feedback problem [11] and cannot be reduced to a full-state feedback problem because in the state-space transformation

$$\xi(t) = \Lambda(\delta)x(t)\,,$$

$\Lambda(\delta)$ is a matrix whose coefficients are uncertain. The only case that can be reduced to a full-state feedback problem is when all entries of the state-feedback gain matrix K are perturbed by the same amount (i.e., $\delta_1 = \delta_2 = \ldots = \delta_n = \delta$), as was done in [4]. In this case the performance index (16) is bounded by the trace of a symmetric positive definite matrix $\mathcal{P}$, that is,

$$\mathcal{J} \leq \operatorname{tr} \mathcal{P}\,, \tag{18}$$

where $\mathcal{P}$ satisfies a modified regulator Riccati equation given by (20). This leads to the following guaranteed-cost optimization problem:

Find K such that

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \left(A + \sum_{i=1}^{q} \alpha_i A_i\right) x(t) + (1+\delta)\, B\hat{u}(t)\,, \\ \hat{u}(t) & = & Kx(t)\,, \end{array} \right. \tag{19}$$

is asymptotically stable and $\operatorname{tr}\mathcal{P}$ *is minimized.*

The solution of this problem results in the following convex optimization problem

$$\begin{array}{ll} \text{Min} & \operatorname{tr}\mathcal{P} \\ \text{subject to} & \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} + Q + (1+\theta)^2\, K^T R K < 0\,, \end{array} \tag{20}$$

where

$$\mathcal{A} = \mathcal{A}\left(\tilde{\alpha}_i, \tilde{\delta}, K\right) = A_0 + \sum_{i=1}^{q} \tilde{\alpha}_i A_i + (1+\tilde{\delta})BK\,,$$

and

$$\tilde{\alpha}_i \in \Omega = \{\{\underline{\alpha}_i, \overline{\alpha}_i\}\,,\ i = 1, \ldots, q\}\ ,\ \tilde{\delta} \in \Delta = \{\underline{\delta}, \overline{\delta}\}\ . \tag{21}$$

Note that, in this case, the sets (21) are sets and not intervals and the number of Linear Matrix Inequalities in (20) is equal to 2^{q+1} because the affine linear system (19) has $q+1$ parameters.

The proposed guaranteed-cost scheme, formulated as a convex optimization problem, can then be numerically used to provide a quantitative study of non-fragile synthesis controllers over the closed-loop performance of the system.

3. A numerical example using LQ/$\mathcal{H}_2$ non-fragile design

Consider the mechanical system shown in Figure 3, known as the "Benchmark Problem" [9], where

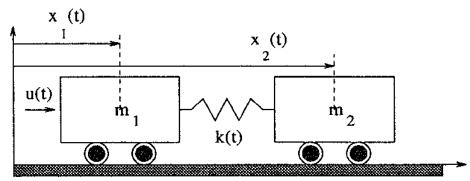

Figure 3: Benchmark Problem

1. $u(t)$ is the control input;
2. x_1, x_2 are the positions, with respect to a reference system, of the masses m_1, m_2, respectively;
3. the masses m_1, m_2 are equal to 1 in the appropriate units;
4. the stiffness $k(t)$, $t \geq 0$, is a time-varying parameter in the interval [0.5, 2].

The linear time-varying model which describes the behavior of the system is given by

$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} & = & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k(t) & k(t) & 0 & 0 \\ k(t) & -k(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t)\,, \\ y(t) & = & \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + u(t)\,. \end{array} \right. \tag{22}$$

It is easy to see that we can represent (22) as an affine uncertain model where the matrix $A(t)$, $t \geq 0$, is given by

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + k(t) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = A_0 + k(t)A_1\,,$$

and the matrices B, C, D are constant.

Using the MATLAB$^{\text{TM}}$ LMI toolbox and the function `msfsyn`, a nominal LQ/$\mathcal{H}_2$ static state-feedback controller was designed. The guaranteed LQ/$\mathcal{H}_2$ performance was found to be 1.54 and the controller gain vector is given by

$$K = \begin{bmatrix} -2.7917 & 1.7912 & -2.3651 & -0.1045 \end{bmatrix}. \tag{23}$$

An affine family of uncertain controllers given by

$$\tilde{K} = (1+\delta)K\,, \tag{24}$$

were generated, where δ is a parameter which corresponds to a drift in the nominal values k_i, $i = 1, \ldots, 4$. In this case each component of K was considered to have the same relative uncertainty range [4]. The fragility of the controller was tested by varying δ and, using MATLAB$^{\text{TM}}$

LMI Toolbox standard routines quadstab and pdlstab, the values of δ for which the closed-loop system is no longer quadratically stable [10, 12, 9] or, less conservatively, does not admit a parameter-dependent Lyapunov function [10, 12, 9] were checked. For this particular system the nominal controller (23) was implemented to obtain the closed-loop system

$$\dot{x}(t) = (A_0 + kA_1 + (1+\delta)BK)\, x(t),$$

It was observed that if δ is greater than 0.1 quadratic stability is lost, and if δ is greater than 0.78 the system does not admit a parameter-dependent Lyapunov function. Now, letting δ lie in the interval $-0.1 \leq \delta \leq 0.1$ a new design was performed using the convex optimization problem (20) where

1. $\Omega = \{0.5, 2\}$, $\tilde{\delta} \in \{-0.1, 0.1\}$ and $\theta = 0.1$,
2. $Q = C^T C$ and $R = 1$.

Four matrix inequalities were obtained because two parameters (k, δ) are involved in the inequalities in problem (20). Once again, using the function msfsyn a new "center" value for the K vector was obtained

$$K = [\ -3.0930 \quad 2.0916 \quad -2.6365 \quad -0.0396\ ]$$

and the guaranteed LQ/$\mathcal{H}_2$ performance in this case was equal to 1.7. The difference between the two guaranteed costs is equal to

$$|1.7 - 1.54| = 0.16\,,$$

which corresponds to a 10.36% degradation of the LQ/$\mathcal{H}_2$ cost as a price paid to guarantee non-fragility. The effect of this type of synthesis is now clear: A trade-off exists between performance and fragility of the compensator.

4. Conclusions

In this paper, the effect of the LQ robust synthesis of uncertain static state feedback controllers for linear systems with structured uncertainties in the dynamic matrix was considered. Simple theoretical results and upper bounds on the performance index were obtained when multiplicative structured uncertainties are allowed in the controller. A guaranteed-cost approach, using Linear Matrix Inequalities, was formulated and used in a numerical experiment involving the benchmark control problem. The results show that there exists a trade-off between controller resiliency and system performance. Future research directions include the synthesis of dynamic non-fragile controllers, and the relation between controller order and its fragility characteristics.

References

[1] S. Bhattacharyya, H. Chapellat, and L. Keel, *Robust Control: The Parametric Approach.* Upper Saddle River, NJ: Prentice-Hall PTR, 1st ed., 1995.

[2] J. Rosenthal and X. Wang, "Output feedback pole placement with dynamic compensators," *IEEE Trans. Autom. Control*, vol. AC-41, pp. 830–843, 1996.

[3] L. Keel and S. Bhattacharyya, "Robust, fragile, or optimal?," *IEEE Trans. Autom. Control*, vol. AC-42, pp. 1098–1105, 1997.

[4] W. Haddad and J. Corrado, "Non-fragile controller design via quadratic Lyapunov bounds," in *Proceedings of the 35th IEEE Conf. on Dec. and Control*, (San Diego, CA), pp. 2678–283, 1997.

[5] D. Bernstein and W. Haddad, "Robust stability and performance via fixed-order dynamic compensation with guaranteed cost bounds," *Math. Control Signals Systems*, vol. 3, pp. 139–163, 1990.

[6] J. Ackermann, *Robust Control - System with Uncertain Physical Parameters.* Berlin: Springer-Verlag, 1st ed., 1993.

[7] P. Dorato, C. Abdallah, and V. Cerone, *Linear-Quadratic Control, An Introduction.* Englewood Cliffs, N.J.: Prentice Hall, 1995.

[8] B. Barmish, C. Hollot, F. Kraus, and R. Tempo, "Extreme point results for robust stabilization of interval plants with first order compensators," *IEEE Trans. Autom. Control*, vol. AC-37, pp. 707–714, 1992.

[9] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox.* The Mathworks, 1995.

[10] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control theory.* Philadelphia, PA: SIAM, 1994.

[11] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback–a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[12] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions for real parametric uncertainty," in *Proceedings of the 32th IEEE Conf. on Dec. and Contr.*, (Lake Buena Vista, FL), pp. 2026–2031, 1994.