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Eigenfunction Restriction Estimates for Curves with Nonvanishing Geodesic Curvatures in Compact Riemannian Surfaces with Nonpositive Sectional Curvatures

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DISSERTATION

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DEDICATION

To my family

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ABSTRACT

For $2 \leq p < 4$, we study the L^p norms of restrictions of eigenfunctions of the Laplace-Beltrami operator on smooth compact 2-dimensional Riemannian manifolds. Burq, Gérard, and Tzvetkov [12], and Hu [21] found eigenfunction restriction estimates for a curve with nonvanishing geodesic curvatures. We will explain how the proof of the known estimates helps us to consider the case where the given smooth compact Riemannian manifold has nonpositive sectional curvatures. For $p = 4$, we will also obtain a logarithmic analogous estimate, by using arguments in Xi and Zhang [37], Sogge [33], and Bourgain [10].

At the end of this dissertation, we will talk about a future work, which is a follow up study for higher dimensional analogues of the above curve cases.

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Chapter 1

Introduction

Let (M, g) be a smooth compact n-dimensional Riemannian manifold without boundary and Σ a k-dimensional embedded submanifold. We denote by Δ_g the associated negative Laplace-Beltrami operator on M. By the compactness of M, the spectrum of $-\Delta_g$ is discrete. If e_λ is any L^2 normalized eigenfunction, then we write

$$\Delta_g e_\lambda = -\lambda^2 e_\lambda, \quad \|e_\lambda\|_{L^2(M)} = 1, \quad \lambda \geq 0.$$

Here $L^p(M)$ is the space of L^p functions with respect to the Riemannian measure. There have been many ways of measuring possible concentrations of the eigenfunctions of the Laplace-Beltrami operator on a manifold so far. One of the ways of measuring the possible concentrations of e_λ on a manifold is to study the possible growth of the L^p norm of the restrictions of e_λ to submanifolds of M. This dissertation deals with the concentrations of the restrictions of e_λ to a curve with nonvanishing geodesic curvatures of 2-dimensional manifold M.

We first review the previous results. We consider the operator $\mathbb{1}_{[\lambda,\lambda+h(\lambda)]}(\sqrt{-\Delta_g})$, which projects a function onto all eigenspaces of $\sqrt{-\Delta_g}$ whose corresponding eigenvalue lies in $[\lambda, \lambda + h(\lambda)]$, which are approximations to eigenfunctions, or quasimodes in the sense that

$$\|(\Delta_g + \lambda^2)\mathbb{1}_{[\lambda,\lambda+h(\lambda)]}(\sqrt{-\Delta_g})f\|_{L^2(M)} \leq C\lambda h(\lambda)\|f\|_{L^2(M)}, \quad 0 \leq h(\lambda) \leq C',$$

for some uniform constant $C, C' > 0$. We will review the construction of $\sqrt{-\Delta_g}$ in Chapter 2. Recall that the exact eigenfunctions can also be considered as quasimodes in that

$$\mathbb{1}_{[\lambda,\lambda+h(\lambda)]}(\sqrt{-\Delta_g})e_\lambda = e_\lambda.$$

For $h(\lambda) \equiv 1$ case, there are well-known estimates of Sogge [29] which state that, for a uniform constant $C > 0$ depending only on M,

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(M)} \leq C\lambda^{\delta(p,n)}, \quad \lambda \geq 1, \tag{1.1}$$

where

$$\delta(p,n)=\begin{cases}\frac{n-1}{2}-\frac{n}{p}, & \text{if} \quad p_c\le p\le\infty,\\ \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right), & \text{if} \quad 2\le p\le p_c,\end{cases} \qquad p_c=\frac{2(n+1)}{n-1}.$$

It follows immediately that

$$\|e_\lambda\|_{L^p(M)}\le C\lambda^{\delta(p,n)}. \tag{1.2}$$

The exponent p_c is a so-called "critical" exponent. The work of Sogge [29] (see also [30, pp.142-145]) also showed that the estimates (1.1) are sharp in that, for all $\lambda\ge 1$, there exist a function f_λ, or a quasimode, such that

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})f_\lambda\|_{L^p(M)}\ge c\lambda^{\delta(p,n)}\|f_\lambda\|_{L^2(M)}, \quad \text{for some uniform } c>0.$$

Sogge [28] showed that the estimates (1.2) are sharp for an infinite family of exact eigenfunctions e_λ in that

$$\|e_\lambda\|_{L^p(\mathbb{S}^n)}\ge c\lambda^{\delta(p,n)}, \quad \text{for some uniform } c>0,$$

where M is the round sphere. Specifically, the $p_c\le p\le\infty$ case is saturated by a sequence of the zonal harmonics on the sphere, whereas $2\le p\le p_c$ case is sharp due to the highest weight spherical harmonics on the sphere. The estimates (1.1) or (1.2) are sometimes called "universal estimates" since they are satisfied on any smooth compact Riemannian manifold. If one assumes nonpositive curvatures or no conjugate points on M, the phenomenas are a bit different. For example, the geodesic flow in negatively curved manifolds behave chaotically, and so, there may be smaller concentration of the restrictions of eigenfunctions of the Laplace-Beltrami operator to geodesics in the negatively curved manifolds.

If (M,g) has nonpositive sectional curvatures, we have some estimates of the case $h(\lambda)=(\log\lambda)^{-1}$

$$\|\mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(M)}\le C_p\frac{\lambda^{\delta(p,n)}}{(\log\lambda)^{\sigma(p,n)}}, \tag{1.3}$$

for some constant $\sigma(p,n)>0$. By using methods of Bérard [3], Hassell and Tacy showed in [19] that the estimates (1.3) hold for $\sigma(p,n)=\frac{1}{2}$ with $p_c<p\le\infty$. This case was also recently investigated by Canzani and Galkowski [13] under more general hypotheses. The case $2<p\le p_c$ was investigated by Blair and Sogge [7–9], Sogge [31], and Sogge and Zelditch [34].

There are analogues of (1.1) and (1.3) when we replace $\mathbb{1}_{[\lambda,\lambda+h(\lambda)]}$ by $\mathcal{R}_\Sigma\circ\mathbb{1}_{[\lambda,\lambda+h(\lambda)]}$, where $\mathcal{R}_\Sigma$ denotes the restriction map as $\mathcal{R}_\Sigma f=f|_\Sigma$. The metric g endows Σ with induced measures, and thus, we can also consider the Lebesgue spaces $L^p(\Sigma)$. Works of Burq, Gérard, and Tzvetkov [12], and Hu [21] studied estimates of the form

$$\|\mathcal{R}_\Sigma\circ\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\Sigma)}\le C\lambda^{\rho_k(p,n)}, \quad \lambda\ge 1, \tag{1.4}$$

where

$$\rho_k(p,n)=\begin{cases}\frac{n-1}{4}-\frac{n-2}{2p}, & \text{if } k=n-1 \text{ and } 2\le p\le\frac{2n}{n-1},\\ \frac{n-1}{2}-\frac{n-1}{p}, & \text{if } k=n-1, \text{ and } \frac{2n}{n-1}\le p\le\infty,\end{cases}$$

which in turn implies that

$$\|e_\lambda\|_{L^p(\Sigma)} \le C\lambda^{\rho_k(p,n)}. \tag{1.5}$$

These estimates are also called universal estimates since they hold on any smooth compact Riemannian manifold. The exponent $\frac{2n}{n-1}$ is the critial exponent in this case. They also considered other cases $k \le n-2$, but we focus on $k = n-1$ here and below, since we will talk about $(n,k) = (2,1)$ mainly in this dissertation. The work of Tacy [35] considers generalizations of (1.4) in semiclassical setting. In [12], Burq, Gérard, and Tzvetkov also showed the estimates (1.4) are sharp by showing that, for all $\lambda \ge 1$, there exists a function f_λ such that

$$\|\mathcal{R}_\Sigma \circ \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g}) f_\lambda\|_{L^p(\Sigma)} \ge c\lambda^{\rho_k(p,n)}\|f_\lambda\|_{L^2(M)}, \quad \text{for some uniform } c > 0,$$

on any compact Riemannian manifold, and the estimates (1.5) are sharp by showing that

$$\|e_\lambda\|_{L^p(\Sigma)} \ge c\lambda^{\rho_k(p,n)}, \quad \text{for some } c > 0,$$

if the e_λ are the zonal harmonics or the highest weight spherical harmonics on the round sphere $M = \mathbb{S}^n$.

Focusing on the case $(n,k,p) = (2,1,2)$ in (1.4), they showed that, for an arbitrary curve,

$$\|\mathcal{R}_\Sigma \circ \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^2(\Sigma)} \le C\lambda^{\frac{1}{4}}, \quad \lambda \ge 1.$$

Burq, Gérard, and Tzvetkov [12], and Hu [21] showed that if Σ is a curve γ with nonvanishing geodesic curvatures, then $\lambda^{1/4}$ can be replaced by $\lambda^{1/6}$.

Theorem 1.1 (Theorem 2 in [12], Theorem 1.2 in [21])**.** Suppose $\dim M = 2$ and the curve γ is a unit-length curve having nonvanishing geodesic curvatures, that is,

$$g(D_t\gamma', D_t\gamma') \neq 0, \tag{1.6}$$

where D_t is the covariant derivatives along the curve γ. We then have that, for a uniform constant C,

$$\|\mathcal{R}_\gamma \circ \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^2(\gamma)} \le C\lambda^{\frac{1}{6}}, \quad \lambda \gg 1. \tag{1.7}$$

If follows immediately from this that

$$\|e_\lambda\|_{L^2(\gamma)} \le C\lambda^{\frac{1}{6}}, \quad \lambda \gg 1. \tag{1.8}$$

This estimate was generalized to a higher dimensional analogue in [21, Theorem 1.4]. Again, the work of Hassell and Tacy [18] obtains generalizations of (1.7) in semiclassical settings.

Burq, Gérard, and Tzvetkov [12, Section 5.2 and Remark 5.4] also showed that the estimate (1.7) is sharp by finding a function $f = f_\lambda$ as above, and the estimate

(1.8) is also sharp when M is the standard sphere $\mathbb{S}^2$, and γ is any curve with non-vanishing geodesic curvatures. See also Tacy [36] for constructing sharp examples for exact eigenfunctions on $\mathbb{S}^n$ or quasimodes. We will prove Theorem 1.1 again in this dissertation in a different point of view, since we need estimates in our proof to prove Theorem 1.2, which will be illustrated below.

Similarly, when (M, g) has nonpositive curvatures, it has been studied that

$$\|\mathcal{R}_\Sigma \circ \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\Sigma)} \le C\frac{\lambda^{\rho_k(p,n)}}{(\log\lambda)^{\sigma_k(p,n)}}, \quad \lambda \ge 1, \tag{1.9}$$

for some constant $\sigma_k(p, n) > 0$ with the same constant $\rho_k(p, n)$ as in (1.4). In [14], Chen obtained $\sigma_k(p, n) = \frac{1}{2}$ in (1.9) for the cases $k = n - 1$ with $p > \frac{2n}{n-1}$.

For $k = 1$, there also have been studies of critical or subcritical exponent. For subcritical cases, Sogge and Zelditch [34] showed that for any $\epsilon > 0$ there exists a $\lambda(\epsilon) < \infty$ such that

$$\sup_{\gamma\in\Pi}\left(\int_\gamma |e_\lambda|^2\, ds\right)^{1/2} \le \epsilon\lambda^{\frac{1}{4}}, \quad \lambda > \lambda(\epsilon),\ \dim M = 2, \tag{1.10}$$

where Π is the space of all unit-length geodesics in M, and ds is the arc-length measure on γ. By using the methods in [34] with Toponogov's comparison theorem, Blair and Sogge [8] obtained $\sigma_1(2, 2) = \frac{1}{4}$, which is an improvement of ϵ in (1.10). The works of Blair [5], and Xi and Zhang [37] obtain $\sigma_1(4, 2) = \frac{1}{4}$ for (unit-length) geodesics, which is a critical exponent in that $p = \frac{2n}{n-1}$.

As in the universal estimates, for the case $(n, k, p) = (2, 1, 2)$ in (1.9), we can expect that $\lambda^{1/4}$ may be replaced by $\lambda^{1/6}$ if γ has nonvanishing geodesic curvatures, analogous to (1.7). Moreover, by [12, Theorem 2] and [21, Theorem 1.2], we know that

$$\|\mathcal{R}_\gamma \circ \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\gamma)} \le C\lambda^{\frac{1}{3}-\frac{1}{3p}}, \quad \lambda \ge 1, \quad 2 \le p \le 4.$$

We want to show the analogue of this for $2 \le p < 4$ in the presence of nonpositive sectional curvatures.

Theorem 1.2. Let (M, g) be a compact 2-dimensional smooth Riemannian manifold (without boundary) with nonpositive sectional curvatures pinched between -1 and 0. Also suppose that γ is a fixed unit-length curve with (1.6), i.e., $g(D_t\gamma', D_t\gamma') \ne 0$. Then, for a uniform constant $C_p > 0$ and $\lambda \ge 1$,

$$\|\mathcal{R}_\gamma \circ \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\gamma)} \le C_p\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{(\log\lambda)^{\frac{1}{2}}}, \quad 2 \le p < 4, \tag{1.11}$$

where $C_p \to \infty$ as $p \to 4$.

It follows from this that

$$\|e_\lambda\|_{L^p(\gamma)} \le C_p\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{(\log\lambda)^{\frac{1}{2}}}, \quad \lambda \ge 1, \quad 2 \le p < 4.$$

We are assuming that the curvatures of M are pinched between -1 and 0, just for convenience. We remark that, by scaling the metric, the bound (1.11) applies to any compact Riemannian manifold with nonpositive sectional curvatures. Using Theorem 1.2, we can show the following estimate at the critical exponent $p = 4$.

Corollary 1.3. Let (M, g) be a compact 2-dimensional smooth Riemannian manifold (without boundary) with nonpositive sectional curvatures pinched between -1 and 0. Also suppose that γ is a fixed unit-length curve with $g(D_t\gamma', D_t\gamma') \neq 0$. Then, for a uniform constant $C > 0$ and $\lambda \gg 1$,

$$\|\mathcal{R}_\gamma \circ \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^4(\gamma)} \leq C\frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{8}}}.$$

It then follows that

$$\|e_\lambda\|_{L^4(\gamma)} \leq C\frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{8}}}, \quad \lambda \gg 1.$$

This corollary is a curved curve analogue to Blair [5, Theorem 1.1], and Xi and Zhang [37, Theorem 1, Theorem 2].

1.1 Outline of the work

Even though Theorem 1.1 is already proved in [12, Theorem 2] and [21, Theorem 2], we go through a variation of the proof of Theorem 1.1 in Chapter 3, since we need some results from the proof to show Theorem 1.2.

In Chapter 2, we introduce some tools to prove Theorem 1.1. We will use pseudo-differential cutoffs Q_j as in [8] to reduce our problem to Proposition 2.3, 2.4, and 2.5. The support properties of the Q_j in ξ are similar to a partition of unity in [12, Section 6].

We will prove Theorem 1.1 by showing Proposition 2.3, 2.4, and 2.5 in Chapter 3. Stationary phase arguments, Young's inequality, and Egorov's theorem (cf. [32], [38]) will be the key points in the section.

By using Proposition 2.3 and 2.5, we reduce Theorem 1.2 to a simpler version in Chapter 4. To show the reduced estimates, we lift the remaining problem to the universal cover of the given manifold by the Cartan-Hadamard theorem. We will use the Hadamard parametrix there to compute the remaining part. We will need Proposition 4.8 to convert our problem to oscillatory integral operator problems. To finish the proof of Theorem 1.2, we may need support properties of the oscillatory integral operators. We will use the Hessian comparison there (cf. [23, Theorem 11.7]) to figure out the support properties.

Using Theorem 1.2, and the strategies in Xi and Zhang [37], Sogge [33], and Bourgain [10], we will show Corollary 1.3 in Chapter 5.

1.2 Notation

1. For nonnegative numbers A and B, $A \lesssim B$ means $A \leq CB$ for some uniform constant $C > 0$ which depends only on the manifold under consideration.

2. $A \approx B$ means $cB \leq A \leq CB$ for some uniform constants $c > 0$ and $C > 0$, or $|A - B| \leq \epsilon$ for a sufficiently small $0 \leq \epsilon \leq 1$.

3. $A \gg B$ means $A \geq CB$ for a sufficiently large $C > 0$.

4. The constant $C > 0$ may be assumed to be a uniform constant, if there is no further notice. The uniform constant $C > 0$ can also be different from each other at any different lines.

5. For geometric terminologies, the notation draws largely from Lee [23].

6. For terminologies of the pseudodifferential operator theory and Egorov's theorem, the notation draws largely from Sogge [30], [32], and Zworski [38].

7. We use $\rho(x, y)$ for the Riemannian distance between x and y.

8. Certain variables may be redefined in different places when the arguments there are independent of each other. For example,

    - ∇ may represent the gradient of functions in some places, and may be the Levi-Civita connection in other places.
    - α may represent a multi-index in some places, and may be deck transformations in other places, defined in the context of the universal cover of the base manifold M.
    - ∂ represents partial derivative, but $\partial^2\phi$ represents the Hessian of ϕ.
    - N may represent a unit normal vector to a given curve γ in some places, but may represent integers $N = 1, 2, 3, \cdots$ in other places.
    - Tildes over letters usually denote the corresponding letters in the universal cover of the base manifold, but sometimes, we also use letters with tildes (or bars) when changing variables if needed.
    - $\epsilon > 0$ appears in many places, and the meanings of $\epsilon > 0$ there may be slightly different, but all of them are sufficiently small but fixed at the end of the computations in each computation,

    and so on. However, the context in which we are using the notations will be clear.

Chapter 2

Some Tools and Reductions for Theorem 1.1

Let $P = \sqrt{-\Delta_g}$. We first review the construction of P. We review basic concepts of pseudodifferential operators from [30] and [32]. We say that a function $P(x,\xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a *symbol of order* m, denoted by $P(x,\xi) \in S^m$, if, for all multi-indices α and β,

$$\left| \left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta P(x,\xi) \right| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}.$$

To a given symbol $P(x,\xi)$ we associate the operator

$$\begin{aligned} P(x,D)u(x) &= (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} P(x,\xi) u(y)\, d\xi\, dy \\ &= (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} P(x,\xi) \widehat{u}(\xi)\, d\xi. \end{aligned}$$

We then say that an operator $P : C_0^\infty \to C^\infty$ is a *pseudodifferential operator of order* m if it is equal to $P(x,D)$, for some $P(x,\xi) \in S^m$ as above. It is known that the pseudodifferential operator can also be extended to an operator from $\mathcal{S}'$ to $\mathcal{S}'$, where $\mathcal{S}'$ is the set of tempered distributions (see also [25, Definition 3.4]).

If we have symbols $P_j(x,\xi) \in S^{m_j}$, where $m_0 \geq m_1 \geq \cdots$ and $m_j \to -\infty$, then we write

$$P \sim \sum_{j=0}^{\infty} P_j, \quad \text{or} \quad P \sim \sum P_j,$$

when we have that

$$P(x,\xi) - \sum_{j=0}^{N-1} P_j(x,\xi) \in S^{m_N}, \quad \text{for any } N.$$

We then can define classical pseudodifferential operators on a compact Riemannian manifold M. Suppose that Ω_ν is a local coordinate patch and ψ_0 and ψ_1 are in $C_0^\infty(\Omega_\nu)$. Also suppose that $\psi_1 = 1$ in a neighborhood of $\mathrm{supp}(\psi_0)$. If the operator

$$\tilde{P}_\nu u(y) = \psi_0(x) \cdot P(\psi_1 \cdot u \circ \kappa_\nu)(x), \quad u = \kappa_\nu(x), \quad u \in C^\infty(\tilde{\Omega}_\nu)$$

is a pseudodifferential operator of order m in $\mathbb{R}^n$, then a map $P : C^\infty(M) \to C^\infty(M)$ is called a *pseudodifferential operator of order* m on a compact manifold M. The operator P is said to be *classical* if, in every local coordinate system, we have

$$P(x,\xi) \sim \sum P_{m-j}(x,\xi),$$

where P_{m-j} is homogeneous of degree $m-j$. In this case, we write $P \in \Psi^m_{cl}(M)$.

We are now ready to review the construction of $\sqrt{-\Delta_g}$.

Theorem 2.1 (Theorem 3.3.1 in [30])**.** Let Q be self-adjoint and positive with $m > 0$. Then the operator $Q^{1/m}$ defined by the spectral theorem is in Ψ^1_{cl}. Its principal symbol is $(q(x,\xi))^{1/m}$, if $q(x,\xi)$ is the principal symbol of Q.

In our case, we consider $Q = -\Delta_g$ and $m = 2$ in Theorem 2.1.

Theorem 2.2 (Theorem 4.2.15 in [32])**.** Let Δ_g be the Laplace-Beltrami operator on a compact Riemannian manifold (M,g). Then $P = \sqrt{-\Delta_g}$ is a first order self-adjoint classical pseudodifferential operator with principal symbol given by

$$p(x,\xi) = \sqrt{\sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k}.$$

Here, the matrix (g^{jk}) is the inverse of the matrix (g_{jk}), and $g_{jk}(q) = g_q\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)$.

We consider $n = 2$ throughout this dissertation except the last chapter for future work. For some $\epsilon_0 > 0$ sufficiently small, let $\chi \in \mathcal{S}(\mathbb{R})$ be an even function such that

$$\begin{aligned} &\chi(0) = 1, \quad \chi(t) > 0 \text{ for } |t| \le 1, \\ &\mathrm{supp}(\widehat{\chi}) \subset \{t : \epsilon_0/2 \le |t| \le \epsilon_0\}, \quad \mathrm{supp}(\widehat{\chi^2}) \subset [-2\epsilon_0, 2\epsilon_0], \end{aligned} \tag{2.1}$$

so that

$$\chi(\lambda - P)e_\lambda = e_\lambda.$$

Assume that γ has a unit-speed (parametrized by arc-length). With this in mind, to prove (1.7), we now want to show

$$\|\chi(\lambda - P)f\|_{L^2(\gamma)} \lesssim \lambda^{\frac{1}{6}}\|f\|_{L^2(M)}, \tag{2.2}$$

that is, we can replace the spectral projector $\mathbb{1}_{[\lambda,\lambda+1]}(P)$ by $\chi(\lambda - P)$. Indeed, the operator $\chi(\lambda - P)$ is invertible on the range of the spectral projector $\mathbb{1}_{[\lambda,\lambda+1]}(P)$ and

$$\|\chi(\lambda - P)^{-1} \circ \mathbb{1}_{[\lambda,\lambda+1]}(P)\|_{L^2(M)\to L^2(M)} \lesssim 1,$$

and so, it suffices to show (2.2).

Fix $\chi_0 \in C_0^\infty(\mathbb{R})$ satisfying $\chi_0(t) = 1$ for $|t| \le 1$ and $\chi_0(t) = 0$ for $|t| \ge 2$. We also fix $\tilde{\chi}_0 \in C_0^\infty(\mathbb{R})$ that satisfies $\tilde{\chi}_0(t) = 1$ for $|t| \le 3$ and $\tilde{\chi}_0(t) = 0$ for $|t| \ge 4$. Choose a Littlewood-Paley bump function $\chi_1 \in C_0^\infty(\mathbb{R})$ that satisfies $\chi_1(t) = 0$ if $t \notin (1/2, 2)$ so that we write

$$\sum_{j=-\infty}^{\infty} \chi_1(2^j t) = 1, \quad \text{for } t \neq 0.$$

We will use Fermi coordinates frequently in the rest of this dissertation. We recall basic properties of Fermi coordinates briefly here. Let γ and M be as above, let N be an element of the normal bundle $N\gamma$, let $\mathcal{E} \subset TM$ be the domain of the exponential map of M, let $\mathcal{E}_p = \mathcal{E} \cap N\gamma$, let $E : \mathcal{E}_p \to M$ be the restriction of exp (the exponential map of M) to $\mathcal{E}_p$, and let $U \subset M$ be a normal neighborhood of γ with $U = E(V)$ for an appropriate open subset $V \subset N\gamma$. If (W_0, ψ) is a smooth coordinate chart for γ, we define $B : \psi(W_0) \times \mathbb{R} \to N\gamma|_{W_0}$ by

$$B(x_1, v_1) = (q, v_1 N|_q), \quad \text{where } q = \psi^{-1}(x_1),$$

by shrinking W_0 if necessary. Setting $V_0 = V \cap N\gamma|_{W_0} \subset N\gamma$ and $U_0 = E(V_0) \subset M$, we define a smooth coordinate map $\varphi : U_0 \to \mathbb{R}^2$ by $\varphi = B^{-1} \circ (E|_{V_0})^{-1}$,

$$\varphi : E(q, v_1 N_q) \mapsto (x_1(q), v_1).$$

Coordinates of this form are called Fermi coordinates. We list here properties of Fermi coordinates from [23, Proposition 5.26].

1. $\gamma \cap U_0$ is the set of points where $v_1 = 0$. Setting $x_2 = v_1$, the curve γ is $\{x_2 = 0\} = \{(x_1, 0) : |x_1| \le \epsilon\}$ in Fermi coordinates. We can take a small $0 < \epsilon \ll 1$ by a partition of unity if necessary.

2. At each point $q \in \gamma \cap U_0$, the metric components satisfy that

$$g_{ij} = g_{ji} = \begin{cases} 0, & i = 1 \quad \text{and} \quad j = 2, \\ 1, & i = j = 2. \end{cases}$$

3. For every $q \in \gamma \cap U_0$ and $v = v_1 E_1|_q \in N_q\gamma$, the geodesic γ_v starting at q with initial velocity v is the curve with coordinate expression $\gamma_v(t) = (x_1(q), tv_1)$.

For detail, see [17, Chapter 2], [23, Chapter 5], etc. If we identify a covector ξ with a vector, then, in Fermi coordinates, we have

$$|\xi|_{g(x)} = g^{11}(x)\xi_1^2 + \xi_2^2, \quad \text{for } x \in \gamma,$$

where $(g^{ij}) = (g_{ij})^{-1}$. Also, we observe that $g^{11}(x_1, 0) = 1$ for $x = (x_1, 0) \in \gamma$ in Fermi coordinates, by the arc-length parametrization.

Suppose ξ is a covector and N is a unit vector field normal to γ. Here, $\xi(N)$ means $\langle \xi^{\#}, N\rangle_g$, where $\xi^{\#}$ is the sharp of ξ as a musical isomorphism. In Fermi coordinates, $N = \frac{\partial}{\partial x_2}$. Set $J = \lfloor \log_2 \lambda^{\frac{1}{3}} \rfloor$. We write

$$1 = \sum_{j=-\infty}^{J-1} \chi_1\left(2^j \frac{|\xi(N)|}{|\xi|_g}\right) + \tilde{\chi}_J\left(\lambda^{\frac{1}{3}} \frac{|\xi(N)|}{|\xi|_g}\right),$$

where

$$\tilde{\chi}_J(t) = 1 - \sum_{j=-\infty}^{J-1} \chi_1(t), \quad \tilde{\chi}_J \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}(\tilde{\chi}_J) \subset \{t : |t| \lesssim 1\}.$$

Here, if $j \ll -1$, the term $\chi_1(2^j|\xi(N)|/|\xi_g|)$ is zero, and thus, the sum is a finite sum, since $|\xi(N)| \lesssim |\xi|_g$.

We will consider decomposition using pseudodifferential cutoffs in Smith and Sogge [27], and Blair [4]. In Fermi coordinates, if $j \le J-1$, we define the compound symbols

$$q_j(x, y, \xi) = \chi_0(\epsilon_0^{-1}\rho(x,\gamma))\tilde{\chi}_0(\epsilon_0^{-1}\rho(y,\gamma))\chi_1(2^j|\xi_2|/|\xi|_g)\Upsilon(|\xi|_g/\lambda), \tag{2.3}$$

where $d_g = \rho$, and $\Upsilon \in C_0^\infty(\mathbb{R})$ satisfies

$$\Upsilon(t) = 1, \text{ for } t \in [c_1, c_1^{-1}], \quad \Upsilon(t) = 0, \text{ for } t \notin \left[\frac{c_1}{2}, 2c_1^{-1}\right],$$

with a small fixed number $c_1 > 0$. Invariantly, we can also define the compound symbols by

$$q_j(x, y, \xi) = \chi_0(\epsilon_0^{-1}\rho(x,\gamma))\tilde{\chi}_0(\epsilon_0^{-1}\rho(y,\gamma))\chi_1\left(2^j \frac{|\xi(N)|}{|\xi|_g}\right)\Upsilon(|\xi|_g/\lambda). \tag{2.4}$$

If $j = J$, we define, in Fermi coordinates,

$$q_J(x, y, \xi) = \chi_0(\epsilon_0^{-1}\rho(x,\gamma))\tilde{\chi}_0(\epsilon_0^{-1}\rho(y,\gamma))\tilde{\chi}_J(\lambda^{\frac{1}{3}}|\xi_2|/|\xi|_g)\Upsilon(|\xi|_g/\lambda), \quad 0 < \epsilon_0 \ll 1,$$

or invariantly,

$$q_J(x, y, \xi) = \chi_0(\epsilon_0^{-1}\rho(x,\gamma))\tilde{\chi}_0(\epsilon_0^{-1}\rho(y,\gamma))\tilde{\chi}_J\left(\lambda^{\frac{1}{3}} \frac{|\xi(N)|}{|\xi|_g}\right)\Upsilon(|\xi|_g/\lambda), \quad 0 < \epsilon_0 \ll 1.$$

Let Q_j be the pseudodifferential operator with compound symbol q_j whose kernel $Q_j(x, w)$ is defined by

$$Q_j(x, w) = \frac{1}{(2\pi)^2} \int e^{i(x-w)\cdot\eta} q_j(x, w, \eta)\, d\eta. \tag{2.5}$$

As in [8], in Fermi coordinates, we know from the homogeneity in ξ and $|\xi| \approx \lambda$ that

$$\begin{aligned}
&|D_{x,w}^\beta D_{\xi_1}^{\alpha_1} D_{\xi_2}^{\alpha_2} q_j(x, w, \xi)| \le C_{\alpha_1,\alpha_2,\beta} 2^{j|\alpha_2|}\lambda^{-|\alpha_1|-|\alpha_2|}, \quad \text{for all } \alpha_1, \alpha_2, \\
&|\partial_{x,w}^\beta Q_j(x, w)| \le C_N 2^{-j}\lambda^{2+|\beta|}(1 + \lambda|x_1 - y_1| + \lambda 2^{-j}|x_2 - y_2|)^{-N}, \text{ for } N = 1, 2, 3, \cdots, \\
&\sup_x \int |Q_j(x, w)|\, dw, \quad \sup_w \int |Q_j(x, w)|\, dx \lesssim 1.
\end{aligned} \tag{2.6}$$

Now, for (2.2), we are reduced to showing that

$$\| \sum_{j \le J} Q_j \circ \chi(\lambda - P) f \|_{L^2(\gamma)} \lesssim \lambda^{\frac{1}{6}} \|f\|_{L^2(M)}, \tag{2.7}$$

and

$$\|(I - \sum_{j \le J} Q_j) \circ \chi(\lambda - P) f \|_{L^2(\gamma)} \le C_N \lambda^{-N} \|f\|_{L^2(M)}, \quad N = 1, 2, 3, \cdots. \tag{2.8}$$

The estimate (2.8) follows from Young's inequality and the analysis of its kernel.

Proposition 2.3. The kernel $(I - \sum_{j \le J} Q_j) \circ \chi(\lambda - P)(x, y)$ satisfies

$$(I - \sum_{j \le J} Q_j) \circ \chi(\lambda - P)(x, y) = O(\lambda^{-N}),$$

for any $N \ge 1$.

We will talk about this later in Section 3.3. To see (2.7), we consider $j = J$ separately.

Proposition 2.4. We have

$$\|Q_J \circ \chi(\lambda - P) f\|_{L^2(\gamma)} \lesssim \lambda^{\frac{1}{6}} \|f\|_{L^2(M)}.$$

We will talk about this proposition in the next chapter. Assuming this proposition is true, we would have (2.7) if we could show that

$$\sum_{j \le J-1} \|Q_j \circ \chi(\lambda - P) f\|_{L^2(\gamma)} \lesssim \lambda^{\frac{1}{6}} \|f\|_{L^2(M)},$$

which follows from

$$\|Q_j \circ \chi(\lambda - P) f\|_{L^2(\gamma)} \lesssim 2^{\frac{j}{2}} \|f\|_{L^2(M)}, \quad j \le J - 1.$$

To see this, we further split Q_j into two operators $Q_{j,\pm}$

$$Q_j = Q_{j,+} + Q_{j,-},$$

where the compound symbols $q_{j,\pm}$ of the $Q_{j,\pm}$ are

$$q_{j,\pm}(x, y, \xi) = \chi_0(x_2) \tilde{\chi}_0(y_2) \chi_1(\pm 2^j \xi_2 / |\xi|_g) \Upsilon(|\xi|_g / \lambda),$$

in Fermi coordinates. We would have (2.7) if we could show the following.

Proposition 2.5. If $j \le J - 1$, we have

$$\|Q_{j,\pm} \circ \chi(\lambda - P) f\|_{L^2(\gamma)} \lesssim 2^{\frac{j}{2}} \|f\|_{L^2(M)}.$$

We will also prove this proposition in the next chapter.

2.1 Notation for symbols of pseudodifferential operators

The pseudodifferential operator Q_j above is defined by using the compound symbols $q_j(x, y, \xi)$, but sometimes we will identify the compound symbol $q_j(x, y, \xi)$ with the usual symbol $q_j(x, \xi)$, modulo smoothing errors, especially when we apply Egorov's theorem and the theorem is stated with usual symbols of pseudodifferential operators. Indeed, recall from the pseudodifferential operator theory (cf. [30, p.97] or [32, pp.92-pp.93]) that there exists a symbol $\tilde{q}_j(x, \xi)$ such that

$$\int e^{i(x-y)\cdot\xi} q_j(x, y, \xi)\, d\xi - \int e^{i(x-y)\cdot\xi} \tilde{q}_j(x, \xi)\, d\xi$$

is smoothing of any order.

In fact, our case is simpler then the case in [30] or [32]. By construction, if $j \leq J - 1$, then

$$q_j(x, y, \xi) = \chi_0(\epsilon_0^{-1} d_g(x, \gamma))\tilde{\chi}_0(\epsilon_0^{-1} d_g(y, \gamma))\chi_1\left(2^j \frac{|\xi(N)|}{|\xi|_g}\right)\Upsilon(|\xi|_g/\lambda) = \tilde{q}_j(x, \xi)\psi(y),$$

where

$$\tilde{q}_j(x, \xi) = \chi_0(\epsilon_0^{-1} d_g(x, \gamma))\chi_1\left(2^j \frac{|\xi(N)|}{|\xi|_g}\right)\Upsilon(|\xi|_g/\lambda), \quad \psi(y) = \tilde{\chi}_0(\epsilon_0^{-1} d_g(y, \gamma)).$$

Note that

$$\tilde{q}_j(x, \xi) = q_j(x, y, \xi) + \tilde{q}_j(x, \xi)(1 - \psi)(y).$$

We want to show that the contribution related with $\tilde{q}_j(x, \xi)(1 - \psi)(y)$ is negligible, that is,

$$A(x, y)(1 - \psi)(y) := \int e^{i(x-y)\cdot\xi} \tilde{q}_j(x, \xi)(1 - \psi)(y)\, d\xi$$

is smoothing. By the construction of χ_0 and $\tilde{\chi}_0$, if $y \in \operatorname{supp}(A(x, \cdot)(1 - \psi)(\cdot))$, then

$$|x - y| \gtrsim 1, \quad \text{for } x \in \operatorname{supp}(\tilde{q}_j(\cdot, \xi)).$$

Since $\nabla_\xi((x - y) \cdot \xi) = x - y$, we have, for $k = 0, 1, 2, \cdot$,

$$\begin{aligned}\int e^{i(x-y)\cdot\xi} \tilde{q}_j(x, \xi)(1 - \psi)(y)\, d\xi &= \int (-\Delta_\xi)^k (e^{i(x-y)\cdot\xi}) \tilde{q}_j(x, \xi)(1 - \psi)(y) \frac{1}{|x - y|^{2k}}\, d\xi \\ &= \int e^{i(x-y)\cdot\xi} (\Delta_\xi)^k (\tilde{q}_j(x, \xi))(1 - \psi)(y) \frac{1}{|x - y|^{2k}}\, d\xi.\end{aligned}$$

Since we know $|x - y| \gtrsim 1$ here, it follows that the last expression is smoothing. We thus can replace $q_j(x, y, \xi)$ by $\tilde{q}_j(x, \xi)$ in the following analysis without any loss whenever we want. Same arguments will work for $j = J$. For simplicity, we may write $\tilde{q}_j(x, \xi)$ as $q_j(x, \xi)$.

The same principle is applied to any other symbols of pseudodifferential operators unless otherwise specified.

Chapter 3

Proof of Theorem 1.1

Note that

$$\chi^2(\lambda - P)f(x) = \frac{1}{2\pi}\int e^{i(\lambda-P)t}\widehat{\chi^2}(t)f(x)\,dt = \frac{1}{2\pi}\int e^{it\lambda}\widehat{\chi^2}(t)e^{-itP}f(x)\,dt.$$

We first recall from [30] or [38] that, by the Lax parametrix, there exist φ and a such that, up to smoothing errors,

$$e^{-itP}(x,w) = \int e^{i\varphi(t,x,\xi)-iw\cdot\xi}a(t,x,\xi)\,d\xi.$$

Here, the phase $\varphi = \varphi(t,x,\xi)$ satisfies, for small enough t,

$$\begin{aligned} &\kappa_t(d_\xi\varphi(t,x,\xi),\xi) = (x, d_x\varphi(t,x,\xi)), \quad (\text{or, } \kappa_t(y,\xi(0)) = (x,\xi(t))) \\ &\partial_t\varphi + p(x, d_x\varphi) = 0, \quad \varphi(0,x,\xi) = \langle x,\xi\rangle, \end{aligned} \tag{3.1}$$

where, for Hamiltonian $p(x,\xi) = |\xi|_{g(x)}$, we have Hamilton's equation

$$\dot{x} = d_\xi p, \quad \dot{\xi} = -d_x p,$$

that is, with $|\xi|_g^2 = g^{11}(x)\xi_1^2 + \xi_2^2$ in Fermi coordinates, we have that

$$\begin{aligned} \dot{x}_1(t) &= \frac{g^{11}(x)\xi_1}{|\xi|_{g(x)}}, \quad \dot{\xi}_1(t) = -\frac{\partial_{x_1}g^{11}(x)\xi_1^2}{|\xi|_{g(x)}}, \\ \dot{x}_2(t) &= \frac{\xi_2}{|\xi|_{g(x)}}, \quad \dot{\xi}_2(t) = -\frac{\partial_{x_2}g^{11}(x)\xi_1^2}{|\xi|_{g(x)}}, \end{aligned} \tag{3.2}$$

and, $\kappa_t : \mathbb{R}^4 \to \mathbb{R}^4$ is the Hamiltonian flow of $p(x,\xi) = |\xi|_{g(x)}$ and homogeneous in ξ. Also, the amplitude a satisfies

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)| \le C_{j,\alpha,\beta}(1+|\xi|)^{-|\beta|},$$

and so,

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\lambda\xi)| \le C_{j,\alpha,\beta}\lambda^{|\beta|}(1+\lambda|\xi|)^{-|\beta|}. \tag{3.3}$$

Note that the right hand side is dominated by $C_{j,\alpha,\beta}\lambda^{|\beta|}(1+\lambda|\xi|)^{-|\beta|} \lesssim C_{j,\alpha,\beta}$ if $|\xi| \approx 1$.

In this chapter, we prove Proposition 2.5, Proposition 2.4, and (2.8) in order.

3.1 Proof of Proposition 2.5

By a TT^* argument, Proposition 2.5 follows from

$$\|Q_{j,\pm} \circ \chi^2(\lambda - P) \circ Q_{j,\pm}^* f\|_{L^2(\gamma)} \lesssim 2^j \|f\|_{L^2(\gamma)}, \quad j \le J-1. \tag{3.4}$$

We focus on the operator $Q_{j,+}\circ\chi^2(\lambda-P)\circ Q_{j,+}^*$. The argument for $Q_{j,-}\circ\chi^2(\lambda-P)\circ Q_{j,-}^*$ is similar. We write

$$\begin{aligned} Q_{j,+} \circ \chi^2(\lambda - P) \circ Q_{j,+}^* f(x) &= \left[Q_{j,+} \circ \left(\frac{1}{2\pi} \int e^{it\lambda} \widehat{\chi^2}(t) e^{-itP} \, dt \right) \circ Q_{j,+}^* \right] f(x) \\ &= \frac{1}{2\pi} \int K_{j,+}(x,y) f(y) \, dy, \end{aligned}$$

where

$$K_{j,+}(x,y) = \int e^{it\lambda} \widehat{\chi^2}(t) \left(Q_{j,+} \circ e^{-itP} \circ Q_{j,+}^* \right)(x,y) \, dt. \tag{3.5}$$

By Egorov's theorem (cf. [38, Theorem 11.1], and/or [32, Chapter 4]), we have

$$Q_{j,+} \circ e^{-itP} = e^{-itP} \circ B_{t,j,+},$$

where $B_{t,j,+}$ has a symbol

$$b_{t,j,+} = \kappa_t^* q_{j,+} + b' = q_{j,+} \circ \kappa_t + b'.$$

Here, $\kappa_t^* q_{j,+}$ is homogeneous of degree 1 in ξ, and $|\partial^\alpha b'| \le C'_\alpha \lambda^{-1} 2^{2j} 2^{j|\alpha|} \le C_\alpha \lambda^{-\frac{1}{3}} 2^{j|\alpha|}$ for all α. We will ignore the remainder b'. Indeed, let B' be the operator whose symbol is b' such that

$$B'(x,y) = \frac{1}{(2\pi)^2} \int e^{i(x-y)\cdot\xi} b'(x,y,\xi) \, d\xi.$$

The size estimates $|\partial^\alpha b'| \le C_\alpha \lambda^{-\frac{1}{3}} 2^{j|\alpha|}$ are better than the size estimates of $\kappa_t^* q_{j,+}$ and $q_{j,+}$. Also, the symbol b' is compactly supported with $\mathrm{supp}(b') \subset \mathrm{supp}(\kappa_{-t}(\mathrm{supp}(q_j)))$. Thus, the arguments below will work when we replace $\kappa_t^* q_{j,+}$ by b' but with better estimates. Hence, for simplicity, we write $b_{t,j,+} = \kappa_t^* q_{j,+}$.

Without loss of generality, we can assume that $a(t,x,\lambda\xi)$ is compactly supported in ξ, independent of λ. Indeed, first let $h_t(z,y,\eta)$ be the symbol of $B_{t,j,+} \circ Q_j^*$. By construction, $h_t(z,y,\eta)$ is supported near $|\eta| \approx \lambda^{-1}$. Let $\beta \in C_0^\infty$ be a bump function with $\beta \equiv 1$ in a neighborhood of $\mathrm{supp}(h_t(z,y,\lambda(\cdot)))$. We then have that

$$\begin{aligned} &(e^{-itP} \circ B_{t,j,+} \circ Q_j^*)(x,y) \\ &= \frac{\lambda^4}{(2\pi)^2} \iiint e^{i\lambda[\varphi(t,x,\xi) - z\cdot\xi + (z-y)\cdot\eta]} a(t,x,\lambda\xi) h_t(z,y,\lambda\eta) \, dz \, d\xi \, d\eta. \end{aligned}$$

Integrating by parts in z, we have, for $N = 1, 2, 3, \cdots$,

$$\begin{aligned}&\left|\iiint e^{i\lambda[\varphi(t,x,\xi)-z\cdot\xi+(z-y)\cdot\eta]}(1-\beta(\xi))a(t,x,\lambda\xi)h_t(z,y,\lambda\eta)\,dz\,d\xi\,d\eta\right|\\ &\lesssim \iiint_{|\xi|\not\approx 1,\ |\eta|\approx 1}(1+\lambda|\xi-\eta|)^{-N}|(1-\beta(\xi))a(t,x,\lambda\xi)h_t(z,y,\lambda\eta)|\,dz\,d\xi\,d\eta\\ &\lesssim \iint_{|\xi-\eta|\gtrsim 1,\ |\eta|\approx 1}(1+\lambda|\xi-\eta|)^{-N}\,d\xi\,d\eta \lesssim \int_{|\xi|\gtrsim 1}(\lambda|\xi|)^{-N}\,d\xi \lesssim \lambda^{-N}.\end{aligned}$$

Since we can ignore this contribution, we can assume that $a(t,x,\lambda\xi)$ is compactly supported in ξ, by replacing $a(t,x,\lambda\xi)$ with $\beta(\xi)a(t,x,\lambda\xi)$ if needed.

We write

$$(e^{-itP}\circ B_{t,j,+}\circ Q^*_{j,+})(x,y) = \frac{\lambda^6}{(2\pi)^4}\int e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)}V_{j,+}(t,x,y,\xi)\,d\xi,$$

where

$$\begin{aligned}&V_{j,+}(t,x,y,\xi) = \iiiint e^{i\lambda\Phi(w,\eta,z,\zeta)}v_{j,+}(t,w,\eta,z,\zeta)\,dw\,d\eta\,dz\,d\zeta,\\ &\Phi(w,\eta,z,\zeta) = (y-w)\cdot\xi + (w-z)\cdot\eta + (z-y)\cdot\zeta,\\ &v_{j,+}(t,w,\eta,z,\zeta) = v_{j,+}(x,y;t,w,\eta,z,\zeta) = a(t,x,\lambda\xi)b_{t,j,+}(w,z,\lambda\eta)q_{j,+}(y,z,\lambda\zeta).\end{aligned} \tag{3.6}$$

We first consider the kernel $e^{-itP}\circ B_{t,j,+}\circ Q^*_{j,+}$.

Lemma 3.1. Let Φ, v be as in (3.6). We have

$$\begin{aligned}(e^{-itP}\circ B_{t,j,+}\circ Q^*_{j,+})(x,y) = \lambda^2\int & e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)}\tilde{a}_j(t,x,y,\xi)\,d\xi\\ &+ \frac{\lambda^6}{(2\pi)^4}R_N(t,y)\int e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)}a(t,x,\lambda\xi)\,d\xi,\end{aligned} \tag{3.7}$$

where

$$\tilde{a}_j(t,x,y,\xi) = \sum_{l=0}^{N-1}\lambda^{-l}L_l\Big(v_{j,+}(x,y;t,y,\xi,y,\xi)\Big), \quad \text{and} \quad |R_N| \le C_N\lambda^{-\frac{N}{3}}, \tag{3.8}$$

and the L_l are the differential operators of order at most $2l$ with respect to the variables w, η, z, and ζ, acting on $v_{j,+}$ at the critical point of $\Phi(w,\eta,z,\zeta)$.

Proof. Since

$$\nabla_{w,\eta,z,\zeta}\Phi = (\Phi'_w, \Phi'_\eta, \Phi'_z, \Phi'_\zeta) = (-\xi+\eta, w-z, \zeta-\eta, z-y),$$

the critical point is $(w,\eta,z,\zeta)=(y,\xi,y,\xi)$. We consider the stationary phase argument. The Hessian of Φ, denoted by $\partial^2\Phi$, is

$$\partial^2\Phi = \begin{pmatrix} \Phi''_{ww} & \Phi''_{w\eta} & \Phi''_{wz} & \Phi''_{w\zeta} \\ \Phi''_{\eta w} & \Phi''_{\eta\eta} & \Phi''_{\eta z} & \Phi''_{\eta\zeta} \\ \Phi''_{zw} & \Phi''_{z\eta} & \Phi''_{zz} & \Phi''_{z\zeta} \\ \Phi''_{\zeta w} & \Phi''_{\zeta\eta} & \Phi''_{\zeta z} & \Phi''_{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} O & I & O & O \\ I & O & -I & O \\ O & -I & O & I \\ O & O & I & O \end{pmatrix}.$$

By standard properties of the determinant, we have $|\det(\partial^2\Phi)|=1$. We begin by computing the signum of Φ. Let e be an eigenvalue of $\partial^2\Phi$, that is, $\det(\partial^2\Phi - eI)=0$. If $e=0$, then $|\det(\partial^2\Phi-eI)|=1\neq 0$, which is a contradiction, and so, $e\neq 0$. With this in mind, using the properties of block matrices (cf. [24], [26], etc.), we have

$$\begin{aligned}
&\det(\partial^2\Phi - eI) \\
&= e^4\left(\left(e-\frac{1}{e}\right)^2 - 1\right)^2 = \left((e^2-1)^2-e^2\right)^2 = (e^2-e-1)^2(e^2+e-1)^2 \\
&= \left(e-\frac{1+\sqrt{5}}{2}\right)^2\left(e-\frac{1-\sqrt{5}}{2}\right)^2\left(e-\frac{-1+\sqrt{5}}{2}\right)^2\left(e-\frac{-1-\sqrt{5}}{2}\right)^2.
\end{aligned}$$

This gives us $\mathrm{sgn}(\partial^2\Phi)=0$.

By construction and homogeneity, we have the size estimates for radial and generic derivatives

$$\left|\partial^\alpha_{w,z}(\eta\cdot\nabla_\eta)^{l_1}(\zeta\cdot\nabla_\zeta)^{l_2}\partial^\beta_{\eta,\zeta}\Big(b_{t,j,+}(w,z,\lambda\eta)q_j(y,z,\lambda\zeta)\Big)\right| \le C_{\alpha,k,l_1,l_2,\beta}2^{j|\beta|},$$

which in turn implies

$$|\partial^\alpha_{w,\eta,z,\zeta}v_{j,+}(t,w,\eta,z,\zeta)| \le C_\alpha 2^{j|\alpha|}.$$

Here, we used the homogeneity of $b_{t,j,+}=\kappa_t^* q_j$, and the fact that the size estimates of $b_{t,j,+}=\kappa_t^*q_j$ are comparable to those of q_j by [38, Lemma 11.11] with small t.

By the method of stationary phase (cf. Theorem 7.7.5 and (3.4.6) in [20]), we have that

$$\begin{aligned}
V_{j,+}(t,x,y,\xi) &= \iiiint e^{i\lambda\Phi(w,\eta,z,\zeta)}v_{j,+}(t,w,\eta,z,\zeta)\,dw\,d\eta\,dz\,d\zeta \\
&= e^{i\lambda\Phi(y,\xi,y,\xi)}\left(\frac{\lambda}{2\pi}\right)^{-\frac{8}{2}}|\det\partial^2\Phi|^{-\frac{1}{2}}e^{\frac{\pi i}{4}\mathrm{sgn}(\partial^2\Phi)}\sum_{l<N}\lambda^{-l}L_l v_{j,+}(t,y,\xi,y,\xi) \\
&\qquad + R_N(t,y)a(t,x,\lambda\xi),
\end{aligned}$$

for $N=1,2,3,\cdots$, where, at the critical point (y,ξ,y,ξ),

$$|R_N| \le C_N\lambda^{-N}\sup_{|\alpha|\le 2N}|\partial^\alpha v_{j,+}| \le \tilde{C}_N\lambda^{-N}(2^j)^{2N} \lesssim \tilde{C}_N\lambda^{-\frac{N}{3}}.$$

Here, the L_l are differential operators of order at most $2l$ acting on $v_{j,+}$ at the critical point (y, ξ, y, ξ), and $2^{-j} \gtrsim \lambda^{-\frac{1}{3}}$. It follows that

$$V_{j,+}(t,x,y,\xi) = (2\pi)^4 \lambda^{-4} \left(\sum_{k=0}^{N-1} \lambda^{-k} L_k v_{j,+}(t,y,\xi,y,\xi) \right) + R_N(t,y) a(t,x,\lambda\xi),$$

which in turn implies that

$$(e^{-itP} \circ B_{t,j,+} \circ Q^*_{j,+})(x,y) = \lambda^2 \int e^{i\lambda(\varphi(t,x,\xi) - y\cdot\xi)} \tilde{a}_j(t,x,y,\xi)\, d\xi \\ + \frac{\lambda^6}{(2\pi)^4} R_N(t,y) \int e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)} a(t,x,\lambda\xi)\, d\xi,$$

where

$$\tilde{a}_j(t,x,y,\xi) = \sum_{l=0}^{N-1} \lambda^{-l} L_l v_{j,+}(x,y;t,y,\xi,y,\xi).$$

This completes the proof of Lemma 3.1. □

Remark 3.2. Note that the proof of this lemma also works for $j = J$, since we only used the fact that $2^{-j} \gtrsim \lambda^{-\frac{1}{3}}$ for j, which is also satisfied for $j = J$. We will use this later to prove Proposition 2.4.

We first want to show that we can ignore the contribution of the second term in the right hand side of (3.7). If we replace $(Q_j \circ e^{-itP} \circ Q^*_j)$ by the second term modulo smoothing errors, by (3.5), the contribution of the second term in $K_{j,+}$ is

$$\frac{\lambda^6}{(2\pi)^4} \iint e^{i\lambda[t+\varphi(t,x,\xi)-y\cdot\xi]} \widehat{\chi^2}(t) R_N(t,y) a(t,x,\lambda\xi)\, d\xi\, dt.$$

We can ignore this contribution.

Lemma 3.3. If $N \geq 18$, then

$$\left| \frac{\lambda^6}{(2\pi)^4} \iint e^{i\lambda[t+\varphi(t,x,\xi)-y\cdot\xi]} \widehat{\chi^2}(t) R_N(t,y) a(t,x,\lambda\xi)\, d\xi\, dt \right| \leq C_N \lambda^{6-\frac{N}{3}} = O(1).$$

Proof. Recall that we may assume that $a(t,x,\xi)$ is compactly supported in ξ. The function $\widehat{\chi^2}(t)$ is also compactly supported in t. It then follows that

$$\left| \frac{\lambda^6}{(2\pi)^4} \iint e^{i\lambda[t+\varphi(t,x,\xi)-y\cdot\xi]} \widehat{\chi^2}(t) R_N(t,y) a(t,x,\lambda\xi)\, d\xi\, dt \right| \\ \lesssim \lambda^6 \sup_{t,y} |R_N| \lesssim \lambda^{6-\frac{N}{3}} = O(1),$$

when $N \geq 18$. □

By Lemma 3.3, the contribution of the second term of the right hand side of (3.7) is $O(1)$ by the generalized Young's inequality, which is better than what we need to show.

We thus focus on the first term in the right hand side of (3.7), that is, modulo $O(1)$ errors,

$$K_{j,+}(x,y) = \lambda^2 \iint e^{i\lambda(t+\varphi(t,x,\xi)-y\cdot\xi)}\widehat{\chi^2}(t)\tilde{a}_j(t,x,y,\xi)\,d\xi\,dt.$$

Recall that

$$\tilde{a}_j(t,x,y,\xi) = \sum_{l=0}^{N-1} \lambda^{-l} L_l\Big(v_{j,+}(x,y;t,y,\xi,y,\xi)\Big),$$

and, by the discussion in Section 2.1, we can write

$$\begin{aligned}
&v_{j,+}(t,y,\xi,y,\xi)\\
&= a(t,x,\lambda\xi)b_{t,j,+}(y,y,\lambda\xi)q_{j,+}(y,y,\lambda\xi)\\
&= a(t,x,\lambda\xi)q_{j,+}(\kappa_t(y,\lambda\xi))q_{j,+}(y,\lambda\xi)\\
&= a(t,x,\lambda\xi)q_{j,+}(x,\lambda\xi(t))q_{j,+}(y,\lambda\xi)\\
&= a(t,x,\lambda\xi)\chi_0(\epsilon_0^{-1}d_g(x,\gamma))\chi_0(\epsilon_0^{-1}d_g(y,\gamma))\\
&\qquad\qquad \times \chi_1\left(2^j\frac{\langle\xi(t),N\rangle_g}{|\xi(t)|_g}\right)\chi_1\left(2^j\frac{\langle\xi,N\rangle_g}{|\xi|_g}\right)\Upsilon(|\xi(t)|_g)\Upsilon(|\xi|_g),
\end{aligned}$$

and thus, by (3.1), in Fermi coordinates,

$$\begin{aligned}
v_{j,+}(t,y,\xi,y,\xi) &= a(t,x,\lambda\xi)\chi_0(\epsilon_0^{-1}d_g(x,\gamma))\chi_0(\epsilon_0^{-1}d_g(y,\gamma))\\
&\quad \times \chi_1\left(2^j\frac{\partial_{x_2}\varphi(t,x,\xi)}{|\partial_x\varphi(t,x,\xi)|_g}\right)\chi_1\left(2^j\frac{\xi_2}{|\xi|_g}\right)\Upsilon(|\partial_x\varphi(t,x,\xi)|_g)\Upsilon(|\xi|_g).
\end{aligned}\tag{3.9}$$

We will consider the contribution of the first term of (3.7) in Fermi coordinates. Recall that we focus only on small t by (2.1). We show that $|\partial_{\xi_2}\varphi(t,x,\xi)| = |t|$, if $\xi_1(s) = 0$ for some small s.

Lemma 3.4. If $\xi_1(s) = 0$ for some $|s| \ll 1$, then we have that $|\partial_{\xi_2}\varphi(t,x,\xi)| = |t|$.

Proof. Suppose $\xi_1(s_0) = 0$ for some small s_0. Let $(x(s),\xi(s))$ be the curve as in (3.1) with

$$x(t) = x,\ x(0) = y = d_\xi\varphi(t,x,\xi),\ \xi(0) = \xi,\ x_2(t) = 0,$$

in Fermi coordinates. By (3.2), the curve $(x(s),\xi(s))$ satisfies

$$\dot{x}_1(s) = \frac{g^{11}(x)\xi_1}{|\xi|_{g(x)}},\quad \dot{x}_2(s) = \frac{\xi_2}{|\xi|_{g(x)}},\quad \dot{\xi}_1(s) = -\frac{\partial_{x_1}g^{11}(x)\xi_1^2}{|\xi|_{g(x)}},\quad \dot{\xi}_2(s) = -\frac{\partial_{x_2}g^{11}(x)\xi_1^2}{|\xi|_{g(x)}}.$$

Note that if $\xi_1 \equiv 0$, then $\xi_1(s_0) = 0$ automatically, and, in Fermi coordinates,

$$|\xi|_{g(x)} = \sqrt{g^{11}(x)\xi_1^2 + \xi_2^2} = |\xi_2| = \pm\xi_2.$$

Thus, if $\xi_1 \equiv 0$, then the curve $(x(s), \xi(s))$ satisfies

$$\dot{x}_1(s) = 0, \quad \dot{x}_2(s) = \frac{\xi_2}{|\xi|_{g(x)}} = \pm 1, \quad \dot{\xi}_1(s) = 0, \quad \dot{\xi}_2(s) = 0.$$

We then observe that

$$x_1(s) \equiv x_1(0), \quad x_2(s) = x_2(0) \pm s, \quad \xi_1(s) \equiv \xi_1(0), \quad \xi_2(s) \equiv \xi_2(0),$$

is the solution to the ODE above. Since we focus on small s from the support properties of χ (2.1), this is a unique solution of this ODE by the uniqueness of the solutions to the ODEs. Since we know

$$(x_1(0), x_2(0)) = (\partial_{\xi_1}\varphi(t,x,\xi), \partial_{\xi_2}\varphi(t,x,\xi)),$$

we have that

$$|\partial_{\xi_2}\varphi(t,x,\xi)| = |x_2(0)| = |x_2(t) \pm t| = |\pm t| = |t|,$$

as required. □

We next consider the case of $\xi_1(s) \neq 0$ for any small s.

Lemma 3.5. For $|s| \ll 1$, suppose $\xi(s) \in \operatorname{supp}(\tilde{a}_j(s,x,y,\cdot))$. Let γ be as above. If $\xi_1(s) \neq 0$ for any small s, then, for $x, y \in \gamma$, in Fermi coordinates, we have either $\dot{\xi}_2(s) > 0$ or $\dot{\xi}_2(s) < 0$.

Proof. We know from the curvature assumption of γ, (1.6), that

$$|\nabla_{\partial_1}\partial_1|_g \neq 0, \quad \text{where } \partial_1 = \frac{\partial}{\partial x_1} \text{ and } \partial_2 = \frac{\partial}{\partial x_2},$$

where ∇ denotes the Levi-Civita connection. Indeed, we know that

$$|\nabla_{\partial_1}\partial_1|_g^2 = |\nabla_{\dot{\gamma}}\dot{\gamma}|_g^2 = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}) = g(D_t\dot{\gamma}, D_t\dot{\gamma}) \neq 0.$$

Note that

$$0 = \frac{\partial}{\partial x_1}\langle \partial_1, \partial_2\rangle_g = \langle \nabla_{\partial_1}\partial_1, \partial_2\rangle_g + \langle \partial_1, \nabla_{\partial_1}\partial_2\rangle_g.$$

But since $[\partial_1, \partial_2] = 0$ and the Levi-Civita connection is symmetric, we have that

$$\langle \partial_1, \nabla_{\partial_1}\partial_2\rangle_g = \langle \partial_1, \nabla_{\partial_2}\partial_1\rangle_g + \langle \partial_1, [\partial_1, \partial_2]\rangle_g = \frac{1}{2}\frac{\partial}{\partial x_2}|\partial_1|_g^2 = \frac{1}{2}\frac{\partial}{\partial x_2}g_{11}(x_1, 0).$$

Combining these two, we have that

$$\frac{\partial}{\partial x_2} g_{11}(x_1, 0) = -2\langle \nabla_{\partial_1}\partial_1, \partial_2\rangle_g. \tag{3.10}$$

Since $|\partial_1|_g = 1$ along $x_2 = 0$ by the arc-length parametrization, we have that

$$\langle \nabla_{\partial_1}\partial_1, \partial_1\rangle_g = \frac{1}{2}\frac{\partial}{\partial x_1}|\partial_1|_g^2 = 0,$$

and thus,

$$\nabla_{\partial_1}\partial_1 = \langle \nabla_{\partial_1}\partial_1, \partial_1\rangle_g \partial_1 + \langle \nabla_{\partial_1}\partial_1, \partial_2\rangle_g \partial_2 = c\partial_2,$$

for some $c \neq 0$ due to the assumption $|\nabla_{\partial_1}\partial_1|_g \neq 0$. By (3.10) with this, we have that

$$-\frac{\partial}{\partial x_2} g^{11}(x) \neq 0, \quad \text{on } x_2 = 0,$$

since $g^{11} = g_{11}^{-1}$ (cf. [23, Proposition 5.26]), and this also holds on a neighborhood of $x_2 = 0$. Since we are assuming $\xi_1 \neq 0$, by the above Hamilton's equation, we have that

$$\dot{\xi}_2(s) = -\partial_2 g^{11}(x)\xi_1^2 \neq 0, \quad \text{along } x_2 = 0.$$

This completes the proof. ∎

We next consider the ξ_2 derivative of φ. By (3.8) and (3.9), we have $\tilde{a}_j(t, x, y, \xi) = 0$ unless

$$\chi_1\left(2^j \frac{\partial_{x_2}\varphi(t, x, \xi)}{|d_x\varphi(t, x, \xi)|_g}\right) \neq 0, \quad \chi_1\left(2^j \frac{\xi_2}{|\xi|_g}\right) \neq 0, \quad \text{and} \quad |\xi|_g \in [\frac{c_1}{2}, 2c_1^{-1}],$$

and so, we may assume that

$$\xi \in \operatorname{supp}(q_{j,+}(x, y, \lambda(\cdot))), \text{ and } d_x\varphi(t, x, \xi) \in \operatorname{supp}(q_{j,+}(x, y, \lambda(\cdot))).$$

Lemma 3.6. Suppose

$$\xi \in \operatorname{supp}(q_{j,+}(x, y, \lambda(\cdot))), \ \xi_1 \neq 0, \text{ and } d_x\varphi(t, x, \xi) \in \operatorname{supp}(q_{j,+}(x, y, \lambda(\cdot))),$$

for some $x \in \gamma$, i.e., $x_2 = 0$ in Fermi coordinates. Then there exists a uniform constant $C > 0$ such that $|\partial_{\xi_2}\varphi(t, x, \xi)| \geq C2^{-j}|t|$.

Proof. We use Fermi coordinates to prove this lemma. Suppose $(x(s), \xi(s))$ is the curve such that $z(t) = x$ and $\zeta(0) = \xi$, as in (3.1). Without loss of generality, by homogeneity we assume $|\xi|_g = 1$, since $|\xi|_g \approx 1$ for $\xi \in \operatorname{supp}(q_{j,+}(x, y, \lambda(\cdot)))$ and φ is also homogeneous. It follows from (3.1) that $d_x\varphi(t, x, \xi) = \xi(t)$ and $d_\xi\varphi(t, x, \xi) = x(0)$, and thus,

$$\chi_1(2^j\xi_2(0)) \neq 0 \quad \text{and} \quad \chi_1(2^j\xi_2(t)) \neq 0, \quad \text{i.e.,} \quad \xi_2(0) \approx 2^{-j} \quad \text{and} \quad \xi_2(t) \approx 2^{-j}.$$

Thus, if $0 \le s \le t$, we have $\xi_2(s) \approx 2^{-j}$ since the map $s \mapsto \xi_2(s)$ is monotonic in s, due to the fact that either $\dot{\xi}_2(s) > 0$ or $\dot{\xi}_2(s) < 0$ by Lemma 3.5. Similarly, the map $s \mapsto \xi_2(s)$ is monotonic in s when $t \le s \le 0$. In any case, we have $|\xi_2(s)| \ge C2^{-j}$ when s is between 0 and t.

Now recall that $\dot{x}_2(s) = \xi_2(s)$ since, by (3.2),

$$\dot{x}_2(s) = \frac{\xi_2}{|\xi|_{g(x)}} = \xi_2(s), \quad \text{in Fermi coordinates.}$$

Thus, since $x(t) = x \in \gamma$, we have $x_2(t) = 0$, and so, the mean value theorem gives

$$0 = x_2(0) + t\dot{x}_2(\tilde{c}),$$

for some $\tilde{c}$ between 0 and t. This gives

$$|\partial_{\xi_2}\varphi(t,x,\xi)| = |x_2(0)| = |t\dot{x}_2(\tilde{c})| = |t\xi_2(\tilde{c})| \ge C|t|2^{-j},$$

for some uniform constant $C > 0$. □

We now return to the kernels $K_{j,+}(x,y)$. By Lemma 3.1, (3.8), and (3.9), we write

$$K_{j,+}(x,y) = \lambda^2 \iint e^{i\lambda(t+\varphi(t,x,\xi)-y\cdot\xi)}\tilde{a}_j(t,x,y,\xi)\widehat{\chi^2}(t)\, d\xi\, dt,$$

where $\tilde{a}_j(t,x,y,\xi) = 0$ unless

$$\chi_1\left(2^j \frac{\partial_{x_2}\varphi(t,x,\xi)}{|d_x\varphi(t,x,\xi)|_g}\right) \ne 0, \quad \chi_1\left(2^j \frac{\xi_2}{|\xi|_g}\right) \ne 0, \quad \text{and} \quad |\xi|_g \in [\frac{c_1}{2}, 2c_1^{-1}],$$

for some small constant $c_1 > 0$. Moreover, we have

$$|\partial_t^k \partial_{\xi_1}^l \partial_{\xi_2}^m \tilde{a}_j| \le C_{k,l,m} 2^{jm}. \tag{3.11}$$

Here, we used (3.3) and size estimates of q_j and $\kappa_t^* q_j$, since $|\xi|_g \approx 1$ by the support properties of Υ. Also, note that $y_2 = 0$ in Fermi coordinates if $y = (y_1, y_2) \in \gamma$. If we set

$$L_\xi = \frac{1 - i\lambda(\partial_{\xi_1}\varphi(t,x,\xi) - y_1)\partial_{\xi_1} - i\lambda 2^{-2j}\partial_{\xi_2}\varphi(t,x,\xi)\partial_{\xi_2}}{1 + \lambda^2|\partial_{\xi_1}\varphi(t,x,\xi) - y_1|^2 + \lambda^2 2^{-2j}|\partial_{\xi_2}\varphi(t,x,\xi)|^2}$$

then we have

$$L_\xi(e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}) = e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}.$$

Integration by parts gives us that

$$\left| \int e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}\tilde{a}_j(t,x,y,\xi)\, d\xi \right| = \left| \int e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}(L_\xi^T)^N(\tilde{a}_j(t,x,y,\xi))\, d\xi \right|,$$

where L_ξ^T is the transpose of L_ξ. For simplicity, we set

$$w_1 = \lambda(\partial_{\xi_1}\varphi - y_1), \quad w_2 = \lambda 2^{-j}\partial_{\xi_2}\varphi,$$
$$L = L_\xi = \frac{1 - iw_1\partial_{\xi_1} - iw_2 2^{-j}\partial_{\xi_2}}{1 + w_1^2 + w_2^2}.$$

Here, w_1 and w_2 are functions of λ, t, x, y, ξ and we suppress the arguments for convenience if necessary. We then write (up to signs)

$$L^T \tilde{a}_j = A_0 + A_1 + A_2 + A_3 + A_4,$$

where

$$A_0 = \frac{1}{1 + w_1^2 + w_2^2}\tilde{a}_j, \quad A_1 = \frac{iw_1}{1 + w_1^2 + w_2^2}\partial_{\xi_1}\tilde{a}_j, \quad A_2 = \frac{iw_2}{1 + w_1^2 + w_2^2}2^{-j}\partial_{\xi_2}\tilde{a}_j,$$
$$A_3 = \tilde{a}_j\partial_{\xi_1}\left(\frac{iw_1}{1 + w_1^2 + w_2^2}\right), \quad A_4 = \tilde{a}_j 2^{-j}\partial_{\xi_2}\left(\frac{iw_2}{1 + w_1^2 + w_2^2}\right).$$

By (3.11), we have

$$|A_0|,\ |A_1|,\ |A_2| \leq \frac{1}{(1 + w_1^2 + w_2^2)^{\frac{1}{2}}}. \tag{3.12}$$

We note that A_3 is

$$\tilde{a}_j\partial_{\xi_1}\left(\frac{iw_1}{1 + w_1^2 + w_2^2}\right) = \tilde{a}_j\partial_{w_1}\left(\frac{iw_1}{1 + w_1^2 + w_2^2}\right)\partial_{\xi_1}w_1 + \tilde{a}_j\partial_{w_2}\left(\frac{iw_1}{1 + w_1^2 + w_2^2}\right)\partial_{\xi_1}w_2,$$

where $\partial_{\xi_1}w_1 = \lambda\partial_{\xi_1}^2\varphi$ and $\partial_{\xi_1}w_2 = \lambda 2^{-j}\partial_{\xi_1\xi_2}^2\varphi$. Similarly, A_4 is

$$\tilde{a}_j 2^{-j}\partial_{w_1}\left(\frac{iw_2}{1 + w_1^2 + w_2^2}\right)\partial_{\xi_2}w_1 + \tilde{a}_j 2^{-j}\partial_{w_2}\left(\frac{iw_2}{1 + w_1^2 + w_2^2}\right)\partial_{\xi_2}w_2,$$

where $\partial_{\xi_2}w_1 = \lambda\partial_{\xi_1\xi_2}^2\varphi$ and $\partial_{\xi_2}w_2 = \lambda\partial_{\xi_2}^2\varphi$. Both A_3 and A_4 contain terms of the form $\partial^\alpha\varphi$ for $|\alpha| \geq 2$, and we want to approximate these first. Recall that we are assuming $|t| \lesssim 1$, by the support properties of χ.

Lemma 3.7. If $|\alpha| \geq 2$ for $\alpha = (\alpha_1, \alpha_2)$, then

$$|\partial_\xi^\alpha\varphi| \lesssim \begin{cases} |\xi_2|^2|t|, & \text{if } \alpha_2 = 0, \\ |\xi_2||t|, & \text{if } \alpha_2 = 1, \\ |t|, & \text{for any } |\alpha| \geq 2. \end{cases}$$

Proof. It follows from (3.1) that

$$\varphi(t, x, \xi) = x \cdot \xi - \int_0^t p(x, \nabla_x\varphi(s, x, \xi))\, ds,$$

where $p(x,\xi) = |\xi|_{g(x)}$. For any $|\alpha| \geq 2$, we obtain

$$|\partial_\xi^\alpha \varphi| = \left| -\int_0^t \partial_\xi^\alpha (p(x, \nabla_x \varphi(s,x,\xi)))\, ds \right| \leq |t| \sup_\xi \left[\partial_\xi^\alpha (p(x, \nabla_x \varphi(s,x,\xi))) \right] \leq C_\alpha |t|. \tag{3.13}$$

We now focus on $\alpha_2 = 0$ or $\alpha_2 = 1$.

On the other hand, if $\Phi(\xi)$ is homogeneous of degree $-k$, then, by Euler's homogeneous theorem, we have

$$\xi_1 \partial_{\xi_1} \Phi + \xi_2 \partial_{\xi_2} \Phi = -k\Phi. \tag{3.14}$$

Since $|\xi| \approx 1$, we have either $|\xi_1| \approx 1$ or $|\xi_2| \approx 1$. The case of $|\xi_1| \leq |\xi_2|$ is simpler. Indeed, if $|\xi_1| \leq |\xi_2|$, then $|\xi_2| \approx 1$, and so, by (3.13), we have $|\partial_\xi^\alpha \varphi| \leq C_\alpha |\xi_2|^l |t|$ for any nonnegative integer l.

Thus, we may assume that $|\xi_2| \leq |\xi_1|$, and so, $|\xi_1| \approx 1$. Taking $\Phi = \partial_{\xi_2}\varphi$ with $k = 0$ in (3.14), it follows from (3.13) that

$$|\partial^2_{\xi_1\xi_2}\varphi| = \left| \frac{\xi_2}{\xi_1} \partial^2_{\xi_2}\varphi \right| \lesssim |\xi_2||t|. \tag{3.15}$$

Using this, if we take $\Phi = \partial_{\xi_1}\varphi$ with $k = 0$ in (3.14), then we have that

$$|\partial^2_{\xi_1}\varphi| = \left| \frac{\xi_2}{\xi_1} \partial^2_{\xi_2\xi_1}\varphi \right| \lesssim |\xi_2|(|\xi_2||t|) = |\xi_2|^2|t|.$$

We can also compute $\partial^3_{\xi_1\xi_1\xi_2}\varphi$ taking $\Phi = \partial^2_{\xi_1\xi_2}\varphi$ with $k = -1$

$$\partial^3_{\xi_1\xi_1\xi_2}\varphi = -\frac{1}{\xi_1}\partial^2_{\xi_1\xi_2}\varphi - \frac{\xi_2}{\xi_1}\partial^3_{\xi_1\xi_2\xi_2}\varphi.$$

By (3.13) and (3.15), we have $|\partial^3_{\xi_1\xi_1\xi_2}\varphi| \lesssim |\xi_2||t|$. Similarly, we can find the estimate for $\partial^3_{\xi_1\xi_1\xi_1}\varphi$. The higher order derivatives of φ are bounded by induction and repeated use of (3.15). □

By Lemma 3.6, we have $|\partial_{\xi_2}\varphi| \gtrsim |\xi_2||t|$. By this and Lemma 3.7, we have that

$$\begin{aligned}
|\partial_{\xi_1} w_1| &= |\lambda \partial^2_{\xi_1}\varphi| \lesssim \lambda |\xi_2|^2 |t| \lesssim \lambda |\partial_{\xi_2}\varphi||\xi_2| \lesssim \lambda |\partial_{\xi_2}\varphi| 2^{-j} \lesssim |w_2|, \\
|2^{-j}\partial_{\xi_2} w_1| &= |\lambda 2^{-j} \partial^2_{\xi_1\xi_2}\varphi| \lesssim \lambda 2^{-j}|\xi_2||t| \lesssim \lambda 2^{-j}|\partial_{\xi_2}\varphi| \lesssim |w_2|, \\
|\partial_{\xi_1} w_2| &= |\lambda 2^{-j} \partial^2_{\xi_1\xi_2}\varphi| \lesssim \lambda 2^{-j}|\xi_2||t| \lesssim \lambda 2^{-j}|\partial_{\xi_2}\varphi| = |w_2|, \\
|2^{-j}\partial_{\xi_2} w_2| &= |\lambda (2^{-j})^2 \partial^2_{\xi_2}\varphi| \lesssim \lambda 2^{-j}|\xi_2||t| \lesssim \lambda 2^{-j}|\partial_{\xi_2}\varphi| = |w_2|.
\end{aligned}$$

We also have that

$$\partial_{w_l}\left(\frac{w_k}{1 + w_1^2 + w_2^2} \right) \lesssim \frac{1}{1 + w_1^2 + w_2^2}, \quad l, k \in \{1, 2\}.$$

Combining these together, we have that

$$|A_3|,\ |A_4| \lesssim \frac{|w_2|}{1+w_1^2+w_2^2} \le \frac{(1+w_1^2+w_2^2)^{\frac{1}{2}}}{1+w_1^2+w_2^2} = \frac{1}{(1+w_1^2+w_2^2)^{\frac{1}{2}}}.$$

By this and (3.12), we have

$$|L^T\tilde{a}_j| \lesssim \frac{1}{(1+w_1^2+w_2^2)^{\frac{1}{2}}}. \tag{3.16}$$

Inductively, we can obtain

$$|(L^T)^N\tilde{a}_j| \lesssim (1+w_1^2+w_2^2)^{-\frac{N}{2}} \lesssim (1+|w_1|+|w_2|)^{-N}.$$

Hence, integration by parts gives, for $x, y \in \gamma$,

$$\begin{aligned}
&\left|\int e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}\tilde{a}_j(t,x,y,\xi)\,d\xi\right| \\
&= \left|\int (L_\xi)^N(e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)})\tilde{a}_j(t,x,y,\xi)\,d\xi\right| \\
&= \left|\int e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}(L_\xi^T)^N(\tilde{a}_j(t,x,y,\xi))\,d\xi\right| \\
&\lesssim \int (1+\lambda|\partial_{\xi_1}\varphi(t,x,\xi)-y_1|+\lambda 2^{-j}|\partial_{\xi_2}\varphi(t,x,\xi)|)^{-N}\,d\xi,
\end{aligned}$$

and thus, we have that

$$\begin{aligned}
&|K_{j,+}(x_1,0,y_1,0)| \\
&\le C_N\lambda^2 \iint_{\mathrm{supp}(q_j)} |\widehat{\chi^2}(t)| \left(1+\lambda|\partial_{\xi_1}\varphi(t,x,\xi)-y_1|+\lambda 2^{-j}|\partial_{\xi_2}\varphi(t,x,\xi)|\right)^{-N}\,d\xi\,dt.
\end{aligned}$$

In Fermi coordinates, we can write $\gamma = \{(x_1,0) : |x_1| \le \epsilon\}$ for some small $\epsilon > 0$, and so, we may write $x = (x_1, 0)$ and $y = (y_1, 0)$. To show (3.4), we now want to show that

$$\int |K_{j,+}(x_1,0,y_1,0)|\,dx_1 \lesssim 2^j, \quad \text{and} \quad \int |K_j(x_1,0,y_1,0)|\,dy_1 \lesssim 2^j.$$

To see these, first note that, by Lemma 3.4 and Lemma 3.6, we have $|\partial_{\xi_2}\varphi(t,x,\xi)| \gtrsim 2^{-j}|t|$ in both cases $\xi_1 \neq 0$ and $\xi_1 = 0$, and so, we have that

$$\begin{aligned}
&|K_{j,+}(x_1,0,y_1,0)| \\
&\le C_N\lambda^2 \iint_{\xi_2\approx 2^{-j},|\xi|\approx 1} |\widehat{\chi^2}(t)|(1+\lambda|\partial_{\xi_1}\varphi(t,(x_1,0),\xi)-y_1|+\lambda 2^{-2j}|t|)^{-N}\,dt\,d\xi,
\end{aligned}$$

and thus, the second inequality follows from

$$\begin{aligned}
&\int |K_{j,+}(x_1,0,y_1,0)|\,dy_1\\
&\le C_N\lambda^2\\
&\quad\times\int_{\xi_2\approx 2^{-j},|\xi|\approx 1}\left(\iint |\widehat{\chi^2}(t)|(1+\lambda|\partial_{\xi_1}\varphi(t,(x_1,0),\xi)-y_1|+\lambda 2^{-2j}|t|)^{-N}\,dt\,dy_1\right)d\xi\\
&\lesssim \lambda^2(\lambda 2^{-2j})^{-1}\lambda^{-1}\mathrm{Vol}(\{\xi_2\approx 2^{-j},\ |\xi|\approx 1\})\\
&\lesssim 2^{2j}2^{-j}=2^j.
\end{aligned}$$

Here, we gained λ^{-1} from y_1 integration, $\lambda 2^{-2j}$ from t integration, and $\mathrm{Vol}(\{|\xi_2|\approx 2^{-j},|\xi|\approx 1\})$ from ξ_2 integration.

The proof that

$$\int |K_{j,+}(x_1,0,y_1,0)|\,dx_1\lesssim 2^j$$

is similar, but it uses that $|\partial^2_{x_1\xi_1}\varphi(t,x,\xi)|\ge c>0$ for some small $c>0$, for $|\xi|_g\approx 1$ and $\xi_2\approx 2^{-j}$, i.e., $|\xi_1|\approx 1$.

To see $|\partial^2_{x_1\xi_1}\varphi(t,x,\xi)|\gtrsim 1$, we recall that φ satisfies $\varphi(0,x,\xi)=\langle x,\xi\rangle$ (cf. [38, Lemma 10.5 (ii)]). By this, we have $|\partial^2_{x_1\xi_1}\varphi(t,x,\xi)|=1$ at $t=0$, and so, $|\partial^2_{x_1\xi_1}\varphi(t,x,\xi)|\gtrsim 1$ for small t by continuity, but we can focus only on small t by taking $\epsilon_0>0$ to be sufficiently small in (2.1), and hence $|\partial^2_{x_1\xi_1}\varphi(t,x,\xi)|\gtrsim 1$ in the support of $K_{j,+}$.

This completes the proof of Proposition 2.5.

3.2 Proof of Proposition 2.4

In this section, by the TT^* argument, we want to show that

$$\|Q_J\circ\chi^2(\lambda-P)\circ Q_J^*f\|_{L^2(\gamma)}\lesssim\lambda^{\frac13}\|f\|_{L^2(\gamma)},\quad J=\lfloor\log_2\lambda^{\frac13}\rfloor. \tag{3.17}$$

We obtain K_J, $\tilde a_J$, v_J, etc., by replacing j by J in the settings of the previous section. We also ignore the contribution of the remainder after using Egorov's theorem.

Using the proof of Lemma 3.1, we have the following lemma.

Lemma 3.8. We have

$$\begin{aligned}
(e^{-itP}\circ B_{t,J}\circ Q_J^*)(x,y)&=\lambda^2\int e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)}\tilde a_J(t,x,y,\xi)\,d\xi\\
&\quad+\frac{\lambda^6}{(2\pi)^4}\int e^{i\lambda(\varphi(t,x,\xi)-y\cdot\xi)}R_N(t,y)a(t,x,\lambda\xi)\,d\xi,
\end{aligned}\tag{3.18}$$

where

$$\tilde a_J(t,x,y,\xi)=\sum_{l=0}^{N-1}\lambda^{-l}L_lv_J(x,y;t,y,\xi,y,\xi),\quad |\partial_t^\alpha R_N|\le C_{N,\alpha}\lambda^{-\frac N3},$$

and the L_l are the differential operators with respect to (w, η, z, ζ) of order at most $2l$ acting on v_J at the point $(w, \eta, z, \zeta) = (y, \xi, y, \xi)$.

By Lemma 3.3 and the generalized Young's inequality again, the contribution of the second term of the right hand side of (3.18) is $O(1)$ with N large, and so, we focus on the first term in (3.18).

Using the proof of Lemma 3.5, we can also show that $\dot{\xi}_2$ is nonvanishing.

Lemma 3.9. For $|s| \ll 1$, suppose $\xi(s) \in \operatorname{supp}(\tilde{a}_J(s, x, y, \cdot))$. Let γ be as above. If $\xi_1(s) \neq 0$ for any small s, then, for $x, y \in \gamma$, in Fermi coordinates, we have either $\dot{\xi}_2(s) > 0$ or $\dot{\xi}_2(s) < 0$.

With this in mind, we figure out the support properties of $\tilde{a}_J$.

Lemma 3.10. Suppose $\xi \in \operatorname{supp}(q_J(x, y, \lambda(\cdot)))$, and $d_x\varphi(t, x, \xi) \in \operatorname{supp}(q_J(x, y, \lambda(\cdot)))$ for some $x \in \gamma$, i.e., $x_2 = 0$ in Fermi coordinates. If $|t| \gg \lambda^{-\frac{1}{3}}$, then $\tilde{a}_J(t, x, y, \xi) = 0$, and thus, $\tilde{a}_J$ is supported where $|t| \lesssim \lambda^{-\frac{1}{3}}$.

Proof. Suppose $(z(s), \xi(s))$ is the curve such that $z(t) = x, \xi(0) = \xi$. It follows that

$$d_x\varphi(t, x, \xi) = \xi(t) = (\xi_1(t), \xi_2(t)), \quad d_\xi\varphi(t, x, \xi) = z(0) = (z_1(0), z_2(0)).$$

By construction, we have $\tilde{a}_J(t, x, y, \xi) = 0$ in Fermi coordinates unless

$$\tilde{\chi}_J\left(\lambda^{\frac{1}{3}} \frac{|\xi_2(t)|}{|\xi(t)|_g}\right) \neq 0, \quad \text{and} \quad \tilde{\chi}_J\left(\lambda^{\frac{1}{3}} \frac{|\xi_2|}{|\xi|_g}\right) \neq 0.$$

By the support properties of Υ, we have $|\xi|_g \approx 1$ and $|\xi(t)|_g \approx 1$, and so, we have $\tilde{a}_J(t, x, y, \xi) = 0$ unless

$$|\xi_2(t)| \lesssim \lambda^{-\frac{1}{3}}, \quad |\xi_2(0)| \lesssim \lambda^{-\frac{1}{3}}.$$

We want to show that we cannot have $|\xi_2(t)| \lesssim \lambda^{-\frac{1}{3}}$ when $|t| \gg \lambda^{-\frac{1}{3}}$. We note that $\xi_1(s) \neq 0$ for any small s. Indeed, if $|\xi_2| \lesssim \lambda^{-\frac{1}{3}}$ and $|\xi| \approx 1$, then $|\xi_1| \gtrsim 1$.

By the mean value theorem, we have

$$\xi_2(t) = \xi_2(0) + \dot{\xi}_2(c_t)t, \tag{3.19}$$

where c_t is between 0 and t. Since $\widehat{\chi^2}$ is compactly supported in $[-2\epsilon_0, 2\epsilon_0]$ for small $\epsilon_0 > 0$ by (2.1), by the proof of Lemma 3.9, there exists a $\tilde{c} > 0$ such that $|\dot{\xi}_2(s)| \geq \tilde{c}$. If $|\xi_2(0)| \gg \lambda^{-\frac{1}{3}}$, then we have $\tilde{a}_J$ vanishes automatically. If $|\xi_2(0)| \lesssim \lambda^{-\frac{1}{3}}$ and $|t| \gg \lambda^{-\frac{1}{3}}$, then, by (3.19) and $|\dot{\xi}_2(s)| \geq \tilde{c}$, we have

$$|\xi_2(t)| \geq |\dot{\xi}_2(c_t)||t| - |\xi_2(0)| \gg \lambda^{-\frac{1}{3}}.$$

Hence, the amplitude $\tilde{a}_J$ is supported where $|t| \lesssim \lambda^{-\frac{1}{3}}$. $\square$

In Fermi coordinates, by Lemma 3.8, modulo $O(1)$ errors, we write

$$K_J(x,y) = \lambda^2 \iint e^{i\lambda[t+\varphi(t,x,\xi)-y\cdot\xi]}\widehat{\chi^2}(t)\tilde{a}_J(t,x,y,\xi)\,d\xi\,dt,$$

where, by Lemma 3.10, $\tilde{a}_J(t,x,y,\xi)$ is supported where $|t| \lesssim \lambda^{-\frac{1}{3}}$. Moreover, we have

$$|\partial_t^k \partial_{\xi_1}^l \partial_{\xi_2}^m \tilde{a}_J| \le C_{k,l,m}(\lambda^{\frac{1}{3}})^m. \tag{3.20}$$

As before, here we used (3.3) and size estimates of q_j and $\kappa_t^* q_j$, since $|\xi|_g \approx 1$ by the support properties of Υ. Note that $y_2 = 0$ in Fermi coordinates for $y = (y_1, y_2) \in \gamma$. As before, if we set

$$L_\xi = \frac{1 - i w_1 \partial_{\xi_1}}{1 + |w_1|^2}, \quad w_1 = \lambda(\partial_{\xi_1}\varphi(t,x,\xi) - y_1),$$

then we have

$$L_\xi(e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}) = e^{i\lambda(t+\varphi(t,x,\xi)-y_1\xi_1)}.$$

By Lemma 3.7, we have

$$|\partial_{\xi_1}^k \varphi| \lesssim |\xi_2|^2|t|, \quad \text{for } k \ge 2,$$

which in turn implies that

$$|\partial_{\xi_1}^k w_1| \lesssim \lambda|\xi_2|^2|t| \lesssim \lambda(\lambda^{-\frac{1}{3}})^2\lambda^{-\frac{1}{3}} \lesssim 1, \quad k \ge 1. \tag{3.21}$$

Integration by parts, as before, gives, for $x, y, \in \gamma$,

$$\left| \int e^{i\lambda[t+\varphi(t,x,\xi)-y_1\xi_1]}\tilde{a}_J(t,x,y,\xi)\,d\xi \right| = \left| \int e^{i\lambda[t+\varphi(t,x,\xi)-y_1\xi_1]}(L_\xi^T)^N(\tilde{a}_J(t,x,y,\xi))\,d\xi \right|,$$

where L_ξ^T is the transpose of L_ξ. As above, we write (up to signs)

$$L_\xi^T \tilde{a}_J = B_0 + B_1 + B_2,$$

where

$$B_0 = \frac{1}{1+w_1^2}\tilde{a}_J, \quad B_1 = \frac{i w_1}{1+w_1^2}\partial_{\xi_1}\tilde{a}_J, \quad B_2 = \tilde{a}_J \partial_{\xi_1}\left(\frac{i w_1}{1+w_1^2}\right).$$

By (3.20), we have

$$|B_0|,\ |B_1| \lesssim \frac{1}{(1+w_1^2)^{\frac{1}{2}}} \lesssim \frac{1}{1+|w_1|}. \tag{3.22}$$

Since we have

$$B_2 = \tilde{a}_J \partial_{w_1}\left(\frac{i w_1}{1+w_1^2}\right)\partial_{\xi_1}w_1 = \tilde{a}_J \frac{i(1-w_1^2)}{(1+w_1^2)^2}\partial_{\xi_1}w_1,$$

it follows from (3.21) that

$$|B_2| \le |\tilde{a}_J| \frac{1+w_1^2}{(1+w_1^2)^2} |\partial_{\xi_1} w_1| \lesssim \frac{1}{1+w_1^2} \lesssim \frac{1}{(1+|w_1|)^2}.$$

By this and (3.22), we have

$$|B_0|,\ |B_1|,\ |B_2| \lesssim \frac{1}{1+|w_1|}.$$

Hence, integration by parts gives, for $x, y, \in \gamma$,

$$|K_J(x,y)| \le C_N \lambda^2 \iint_{|t| \lesssim \lambda^{-\frac{1}{3}}, |\xi_2| \lesssim \lambda^{-\frac{1}{3}}, |\xi|_g \approx 1} |\widehat{\chi^2}(t)| (1 + \lambda |\partial_{\xi_1} \varphi(t,x,\xi) - y_1|)^{-N} \, d\xi \, dt.$$

In Fermi coordinates, we write $\gamma = \{(x_1, 0) : |x_1| \le \epsilon\}$ for $\epsilon > 0$ small, and so, $x = (x_1, 0)$ and $y = (y_1, 0)$. We thus want to show that

$$\int |K_J(x_1,0,y_1,0)| \, dx_1 \lesssim \lambda^{\frac{1}{3}}, \quad \int |K_J(x_1,0,y_1,0)| \, dy_1 \lesssim \lambda^{\frac{1}{3}}. \tag{3.23}$$

Indeed, this and Young's inequality imply (3.17) immediately.

We first focus on $\int |K_J(x_1,0,y_1,0)| \, dy_1$. We take $\tilde{C} > 0$ sufficiently large, and bound

$$\begin{aligned}
&\int |K_J(x_1,0,y_1,0)| \, dy_1 \\
&\le C_N \lambda^2 \iiint_{|t| \lesssim \lambda^{-\frac{1}{3}}, |\xi_2| \lesssim \lambda^{-\frac{1}{3}}, |\xi|_g \approx 1} |\widehat{\chi^2}(t)| (1 + \lambda |\partial_{\xi_1} \varphi(t,x,\xi) - y_1|)^{-N} \, dy_1 \, d\xi \, dt \\
&\lesssim \lambda^2 \lambda^{-1} \lambda^{-\frac{1}{3}} \mathrm{Vol}(\{|\xi_2| \lesssim \lambda^{-\frac{1}{3}}, |\xi| \approx 1\}) \lesssim \lambda^{\frac{1}{3}}.
\end{aligned}$$

Here, we gained λ^{-1} from y_1 integration, $\lambda^{-\frac{1}{3}}$ from t integration due to $|t| \lesssim \lambda^{-\frac{1}{3}}$, and $\mathrm{Vol}(\{|\xi_2| \lesssim \lambda^{-\frac{1}{3}}, |\xi| \approx 1\})$ from ξ_2 integration.

The proof of the second inequality in (3.23) is similar, but uses that $|\partial^2_{x_1 \xi_1} \varphi(t,x,\xi)| \gtrsim 1$ for small t as in the case $j \le J-1$.

This completes the proof of Proposition 2.4.

3.3 Proof of Proposition 2.3

As we promised before, we talk about Proposition 2.3 here. Let $\tilde{Q} = I - \sum_{j \le J} Q_j$. By the Fourier inversion formula, we write

$$\tilde{Q} f(x) = \int \tilde{Q}(x,y) f(y) \, dy,$$

where

$$\tilde{Q}(x,y) = \frac{1}{(2\pi)^2} \int e^{i(x-y)\cdot\xi} \Big(1 - \sum_{j \le J} q_j(x,y,\xi)\Big) \, d\xi.$$

Setting

$$\tilde{q}(x,y,\xi) = 1 - \sum_{j\le J} q_j(x,y,\xi),$$

we write

$$\begin{aligned}\tilde{q}(x,y,\xi) &= 1 - \chi_0(\rho(x,\gamma))\tilde{\chi}_0(\rho(y,\gamma)) \sum_{j\le J} \tilde{\chi}_j \left(2^j \frac{|\xi(N)|}{|\xi|_g}\right) \Upsilon(|\xi|_g/\lambda) \\ &= 1 - \chi_0(\rho(x,\gamma))\tilde{\chi}_0(\rho(y,\gamma))\Upsilon(|\xi|_g/\lambda).\end{aligned}$$

Let $\tilde{Q}$ be a pseudodifferential operator whose kernel is $\tilde{Q}(x,y)$. Since $\chi_0, \tilde{\chi}_0$, and Υ are compactly supported bump functions, we have

$$|\partial^\alpha_{x,y,\xi}\tilde{q}(x,y,\lambda\xi)| \le C_\alpha,$$

and so, we can consider integration by parts below easily.

We write the kernel of $\tilde{Q}\circ\chi(\lambda - P)$ as

$$(\tilde{Q}\circ\chi(\lambda-P))(x,y) = \frac{\lambda^4}{(2\pi)^3}\iiiint e^{i\lambda\Psi(t,z,\eta,\xi)}\widehat{\chi}(t)\tilde{q}(x,z,\lambda\eta)a(t,z,\lambda\xi)\,dt\,dz\,d\eta\,d\xi,$$

where

$$\Psi(t,z,\eta,\xi) = (x-z)\cdot\eta + \varphi(t,z,\xi) - y\cdot\xi.$$

We note that, on the support of $\tilde{q}(x,z,\lambda\eta)$ in η,

$$\begin{aligned}|\nabla_{t,z}\Psi(t,z,\eta,\xi)| = |(\Psi'_t,\Psi'_z)| &= \sqrt{|1-|\nabla_z\varphi(t,z,\xi)|_{g(z)}|^2 + |\nabla_z\varphi(t,z,\xi)-\eta|^2} \\ &\gtrsim |1-|\nabla_z\varphi(t,z,\xi)|_{g(z)}| + ||\nabla_z\varphi(t,z,\xi)|_{g(z)} - |\eta|| \\ &\ge |1-|\eta|| \gtrsim 1+|\eta|.\end{aligned}$$

With this in mind, we first consider the integral

$$\frac{\lambda^4}{(2\pi)^3}\iiiint e^{i\lambda\Psi(t,z,\eta,\xi)}\widehat{\chi}(t)\tilde{q}(x,z,\lambda\eta)a(t,z,\lambda\xi)(1-\Upsilon(|\xi|))\,dt\,dz\,d\eta\,d\xi. \tag{3.24}$$

On the support of $1-\Upsilon(\xi)$ in ξ, we have that

$$|\nabla_{t,z}\Psi(t,z,\eta,\xi)| \gtrsim |1-|\nabla_z\varphi(t,z,\xi)|_{g(z)}| = |1-|\xi|_{g(\nabla_\xi\varphi(t,z,\xi))}| \approx |1-|\xi|| \approx 1+|\xi|,$$

when we choose $c_1>0$ small enough in (2.3). Integration by parts in t and z then gives us that the integral (3.24) is dominated by

$$C\lambda^4\lambda^{-N}\iiiint_{t\in\text{supp}(\widehat{\chi}),|z|\lesssim 1}(1+|\eta|)^{-N'}(1+|\xi|)^{-N'}\,dt\,dz\,d\eta\,d\xi \lesssim \lambda^{4-N},$$

when we take N, N' large enough. Using the generalized Young's inequality, this satisfies the estimates (2.8), and thus, we focus on the integral

$$\frac{\lambda^4}{(2\pi)^3} \iiiint e^{i\lambda\Psi(t,z,\eta,\xi)} \widehat{\chi}(t) \tilde{q}(x, z, \lambda\eta) a(t, z, \lambda\xi) \Upsilon(|\xi|) \, dt \, dz \, d\eta \, d\xi.$$

In this case, the amplitude of the integral is compactly supported in ξ, and so, we do not need to consider $|\Psi_t'|$ separately. Thus, integration by parts in t and z, the integral is dominated by

$$C\lambda^4 \lambda^{-N} \iiiint_{t \in \operatorname{supp}(\widehat{\chi}), |z| \lesssim 1, |\xi| \approx 1} (1 + |\eta|)^{-N} \, dt \, dz \, d\eta \, d\xi \lesssim \lambda^{4-N},$$

when we take N large enough, which proves Proposition 2.3.

This shows (2.8) by using the generalized Young's inequality, and thus, completes the proof of Theorem 1.1.

Chapter 4

Proof of Theorem 1.2

In this chapter, assuming nonpositive sectional curvatures on M, we want to prove Theorem 1.2. Let

$$T = c_0 \log \lambda, \tag{4.1}$$

where $c_0 > 0$ is small but fixed, which will be specified later. Let $P = \sqrt{-\Delta_g}$ as before. As in Theorem 1.1, we would have Theorem 1.2, if we could show that

$$\|\chi(T(\lambda - P))f\|_{L^p(\gamma)} \le C_p \frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T^{\frac{1}{2}}} \|f\|_{L^2(M)}, \quad 2 \le p < 4, \tag{4.2}$$

where $C_p \to \infty$ as $p \to 4$. Indeed, it is enough to show (4.2), since the operator $\chi(T(\lambda - P))$ is invertible on the range of the spectral projector $\mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)$ and

$$\|\chi(T(\lambda - P))^{-1} \circ \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)\|_{L^2(M)\to L^2(M)} \lesssim 1.$$

By the spectral theorem, we write

$$P = \sum_{j=1}^{\infty} \lambda_j E_j, \quad I = \sum_{j=1}^{\infty} E_j,$$

where the $E_j : L^2 \to L^2$ are the projection operators that project onto the eigenspaces with eigenvalues λ_j. Recall the definitions of the spectral projection (approximate) operators.

$$\chi(\lambda - P)f = \sum_{j=0}^{\infty} \chi(\lambda - \lambda_j)E_j f, \quad \chi(T(\lambda - P))f = \sum_{j=0}^{\infty} \chi(T(\lambda - \lambda_j))E_j f,$$
$$\mathbb{1}_{[\lambda,\lambda+1)}(P)f = \sum_{\lambda_j\in[\lambda,\lambda+1)} E_j f.$$

Let $\mathcal{R}_\gamma$ be the restriction to γ, defined by $\mathcal{R}_\gamma f = f|_\gamma$. Note the following estimate by a well-known argument using the estimates for the unit window projections $\mathbb{1}_{[\lambda,\lambda+1]}$ and size estimates of χ (cf. [33], [37], [9], etc.).

Lemma 4.1. We have

$$\|(I-\chi(\lambda-P))\circ\chi(T(\lambda-P))f\|_{L^2(\gamma)}\lesssim\frac{\lambda^{\frac{1}{6}}}{T}\|f\|_{L^2(M)}.$$

Proof. By duality and Theorem 1.1, we have

$$\|\mathbb{1}_{[k,k+1)}(P)\circ\mathcal{R}_\gamma^* g\|_{L^2(M)}\lesssim k^{\frac{1}{6}}\|g\|_{L^2(\gamma)}. \tag{4.3}$$

If we set

$$\psi_T(\zeta)=(1-\chi(\zeta))\chi(T\zeta),\quad \psi_T(\lambda-P)=(I-\chi(\lambda-P))\chi(T(\lambda-P)),$$

then Lemma 4.1 follows from

$$\|\psi_T(\lambda-P)f\|_{L^2(\gamma)}\lesssim\frac{\lambda^{\frac{1}{6}}}{T}\|f\|_{L^2(M)}. \tag{4.4}$$

We note that

$$|\psi_T(\zeta)|\le C_N T^{-1}(1+T|\xi|)^{-N},\quad N=1,2,3,\cdots. \tag{4.5}$$

Indeed, since $\chi(0)=1$ and $\chi\in\mathcal{S}(\mathbb{R})$, by the mean value theorem, there exists a $\tilde{c}$ between 0 and ζ such that

$$\begin{aligned}|\psi_T(\zeta)|&=|(\chi(0)-\chi(\zeta))\chi(T\zeta)|\\&=|-\chi'(\tilde{c})\zeta\chi(T\zeta)|\le\|\chi'\|_\infty|\zeta||\chi(T\zeta)|\le C_N T^{-1}(1+T|\zeta|)^{-N}.\end{aligned}$$

In the last inequality, we used the fact that $\chi\in\mathcal{S}(\mathbb{R})$ and that

$$|\zeta||\chi(T\zeta)|\le C_N T^{-1}(1+T|\zeta|)^{-N}.$$

Indeed, if $|\zeta|\le T^{-1}$, then

$$|\zeta||\chi(T\zeta)|\le C_N T^{-1}(1+T|\zeta|)^{-N},$$

and, if $|\zeta|\ge T^{-1}$, then

$$\begin{aligned}|\zeta||\chi(T\zeta)|&\le C_{N+1}|\zeta|(1+T|\zeta|)^{-(N+1)}\\&\le C_{N+1}|\zeta|(T|\zeta|)^{-N-1}=C_{N+1}T^{-1}(T|\zeta|)^{-N}\le C_N' T^{-1}(1+T|\zeta|)^{-N}.\end{aligned}$$

By (4.5), we have, for $\lambda\gg 1$,

$$\begin{aligned}&|\psi_T(\lambda-\tau)\sum_{k=0}^{\infty}\mathbb{1}_{[k,k+1)}(\tau)|\\&\le C_N\frac{1}{T}\sum_{k=0}^{\infty}(1+T|\lambda-\tau|)^{-N}\mathbb{1}_{[k,k+1)}(\tau)\\&\le C_N\frac{1}{T}\sum_{k<\lfloor\lambda\rfloor}(1+T(\lfloor\lambda\rfloor-k-1))^{-N}\mathbb{1}_{[k,k+1)}(\tau)+C_N\frac{1}{T}\mathbb{1}_{[\lfloor\lambda\rfloor,\lfloor\lambda\rfloor+1)}(\tau)\\&\qquad+C_N\frac{1}{T}\sum_{k\ge\lfloor\lambda\rfloor+1}(1+T(k-\lfloor\lambda\rfloor))^{-N}\mathbb{1}_{[k,k+1)}(\tau).\end{aligned} \tag{4.6}$$

Recall that, for a function ϕ, we have $\|\phi(P)\|_{L^2(M)\to L^2(M)} \le \sup_\tau |\phi(\tau)|$. Indeed, we have

$$\|\phi(P)f\|^2_{L^2(M)} = \sum_{j=0}^{\infty} \phi(\lambda_j)^2 \|E_j f\|^2_{L^2(M)} \le (\sup_j |\phi(\lambda_j)|)^2 \sum_{j=0}^{\infty} \|E_j f\|^2_{L^2(M)}$$
$$\le (\sup_\tau |\phi(\tau)|)^2 \|f\|^2_{L^2(M)}.$$

With this in mind, since the operator $\mathbb{1}_{[k,k+1)}(P)$ is a projection operator, we know $\mathbb{1}^2_{[k,k+1)}(P) = \mathbb{1}_{[k,k+1)}(P)$, which in turn implies that, by (4.3) and (4.6),

$$\begin{aligned}
&\|\psi_T(\lambda - P) \circ \mathcal{R}^*_\gamma g\|^2_{L^2(M)} \\
&= \sum_{k=0}^{\infty} \|\psi_T(\lambda - P) \circ \mathbb{1}^2_{[k,k+1)}(P) \circ \mathcal{R}^*_\gamma g\|^2_{L^2(M)} \\
&= \sum_{k=0}^{\infty} \|[\psi_T(\lambda - P) \circ \mathbb{1}_{[k,k+1)}(P)] \circ [\mathbb{1}_{[k,k+1)}(P) \circ \mathcal{R}^*_\gamma] g\|^2_{L^2(M)} \\
&\le \frac{C_N}{T^2} \sum_{0\le k\le \lfloor\lambda\rfloor - 1} \Big((1 + T(\lfloor\lambda\rfloor - k - 1))^{-N} \|\mathbb{1}_{[k,k+1)} \circ \mathcal{R}^*_\gamma g\|_{L^2(M)} \Big)^2 \\
&\qquad + \frac{C_N}{T^2} \|\mathbb{1}_{[\lfloor\lambda\rfloor, \lfloor\lambda\rfloor+1)}(P) \circ \mathcal{R}^*_\gamma g\|^2_{L^2(M)} \\
&\qquad + C_N \frac{1}{T^2} \sum_{k \ge \lfloor\lambda\rfloor + 1} \Big((1 + T(k - \lfloor\lambda\rfloor))^{-N} \|\mathbb{1}_{[k,k+1)} \circ \mathcal{R}^*_\gamma g\|_{L^2(M)} \Big)^2 \\
&\le \frac{C_N}{T^2} \sum_{0\le k\le \lfloor\lambda\rfloor - 1} (1 + T(\lfloor\lambda\rfloor - k - 1))^{-2N} (k^{\frac16})^2 \|g\|^2_{L^2(\gamma)} + \frac{C_N}{T^2} (\lfloor\lambda\rfloor^{\frac16})^2 \|g\|^2_{L^2(\gamma)} \\
&\qquad + \frac{C_N}{T^2} \sum_{k \ge \lfloor\lambda\rfloor + 1} (1 + T(k - \lfloor\lambda\rfloor))^{-2N} (k^{\frac16})^2 \|g\|^2_{L^2(\gamma)} \\
&\le \frac{C_N}{T^2} \sum_{0\le k\le \lfloor\lambda\rfloor - 1} (1 + T(\lfloor\lambda\rfloor - k - 1))^{-2N} k^{\frac13} \|g\|^2_{L^2(\gamma)} + \frac{C_N}{T^2} \lambda^{\frac13} \|g\|^2_{L^2(\gamma)} \\
&\qquad + \frac{C_N}{T^2} \sum_{k \ge \lfloor\lambda\rfloor + 1} (1 + T(k - \lfloor\lambda\rfloor))^{-2N} k^{\frac13} \|g\|^2_{L^2(\gamma)}.
\end{aligned}$$

We next want to show that

$$\sum_{0\le k\le \lfloor\lambda\rfloor - 1} (1 + T(\lfloor\lambda\rfloor - k - 1))^{-2N} k^{\frac13} \lesssim \lambda^{\frac13}, \quad \sum_{k \ge \lfloor\lambda\rfloor + 1} (1 + T(k - \lfloor\lambda\rfloor))^{-2N} k^{\frac13} \lesssim \lambda^{\frac13}.$$

We prove the first inequality here. Similar arguments will work for the second one. Recall that $(a+b)^{\frac13} \le a^{\frac13} + b^{\frac13}$ for $a, b \ge 0$. With this in mind, changing indices, we have

$$\begin{aligned}\sum_{0\le k<\lfloor\lambda\rfloor}(1+T(\lfloor\lambda\rfloor-k-1))^{-2N}k^{\frac{1}{3}} &= \sum_{k=0}^{\lfloor\lambda\rfloor-1}\frac{(\lfloor\lambda\rfloor-1-k)^{\frac{1}{3}}}{(1+Tk)^{2N}} \le \sum_{k=0}^{\lfloor\lambda\rfloor-1}\frac{(\lambda+k)^{\frac{1}{3}}}{(1+Tk)^{2N}}\\
&\le \sum_{k=0}^{\lfloor\lambda\rfloor-1}\frac{\lambda^{\frac{1}{3}}+k^{\frac{1}{3}}}{(1+Tk)^{2N}}\\
&\le \lambda^{\frac{1}{3}}+\sum_{k=1}^{\lfloor\lambda\rfloor-1}\left(\frac{\lambda^{\frac{1}{3}}}{T^{2N}k^{2N}}+\frac{1}{T^{2N}k^{2N-\frac{1}{3}}}\right)\\
&\le \lambda^{\frac{1}{3}}+\frac{\lambda^{\frac{1}{3}}}{T^{2N}}\sum_{k=1}^{\infty}\frac{1}{k^{2N}}+\frac{1}{T^{2N}}\sum_{k=1}^{\infty}\frac{1}{k^{2N-\frac{1}{3}}}\\
&\lesssim \lambda^{\frac{1}{3}}+\frac{\lambda^{\frac{1}{3}}}{T^{2N}}+\frac{1}{T^{2N}}\\
&\lesssim \lambda^{\frac{1}{3}},\end{aligned}$$

by taking $N \gg 1$.

Putting these together, we have

$$\|\psi_T(\lambda-P)\circ\mathcal{R}_\gamma^* g\|_{L^2(M)}^2 \lesssim \frac{\lambda^{\frac{1}{3}}}{T^2}\|g\|_{L^2(\gamma)}^2,$$

and thus,

$$\|\psi_T(\lambda-P)\circ\mathcal{R}_\gamma^* g\|_{L^2(M)} \lesssim \frac{\lambda^{\frac{1}{6}}}{T}\|g\|_{L^2(\gamma)}.$$

By this and duality, we have (4.4), which completes the proof of Lemma 4.1. □

Remark 4.2. In fact, compared to the above lemma, we can have an easier proof of

$$\|(I-\chi(\lambda-P))\circ\chi(T(\lambda-P))f\|_{L^2(\gamma)} \lesssim \frac{\lambda^{\frac{1}{6}}}{T^{\frac{1}{2}}}\|f\|_{L^2(M)}, \tag{4.7}$$

which is still enough in our case. Considering the proof of Lemma 4.1, (4.7) follows from

$$\frac{1}{T}\sum_{k=0}^{\infty}(1+T|\lambda-k|)k^{\frac{1}{3}} \lesssim \frac{\lambda^{\frac{1}{3}}}{T}.$$

We first want to replace the sum in the left hand side by the integral

$$\frac{1}{T}\int_0^\infty (1+|\lambda-s|)^{-N}s^{\frac{1}{3}}\,ds.$$

Indeed, first note that

$$\frac{1}{T}\sum_{k=0}^{\infty}(1+T|\lambda-k|)^{-N}k^{\frac{1}{3}} = \frac{1}{T}\sum_{k=0}^{\infty}\int_k^{k+1}(1+T|\lambda-k|)^{-N}k^{\frac{1}{3}}\,ds.$$

For $k \le s \le k+1$ and $T = c_0 \log \lambda \gg 1$, we have

$$\begin{aligned}(1+T|\lambda-k|)^{-N} &\le (1+\frac{1}{2}|\lambda-k|)^{-N} \\ &\le (1+\frac{1}{2}|\lambda-s|-\frac{1}{2}|k-s|)^{-N} \\ &\le (1-\frac{1}{2}+\frac{1}{2}|\lambda-s|)^{-N} = 2^N(1+|\lambda-s|)^{-N},\end{aligned}$$

which in turn implies that

$$\begin{aligned}\frac{1}{T}\sum_{k=0}^{\infty}\int_k^{k+1}(1+T|\lambda-k|)^{-N}\,ds &\le \frac{2^N}{T}\sum_{k=0}^{\infty}\int_k^{k+1}(1+|\lambda-s|)^{-N}k^{\frac{1}{3}}\,ds \\ &\le \frac{C_N}{T}\sum_{k=0}^{\infty}\int_k^{k+1}(1+|\lambda-s|)^{-N}s^{\frac{1}{3}}\,ds \\ &= \frac{C_N}{T}\int_0^{\infty}(1+|\lambda-s|)^{-N}s^{\frac{1}{3}}\,ds.\end{aligned}$$

It then suffices to show that

$$\frac{1}{T}\int_0^{\infty}(1+|\lambda-s|)^{-N}s^{\frac{1}{3}}\,ds \lesssim \frac{\lambda^{\frac{1}{3}}}{T}.$$

But we have that

$$\begin{aligned}\frac{1}{T}\int_0^{\infty}(1+|\lambda-s|)^{-N}s^{\frac{1}{3}}\,ds &= \frac{1}{T}\int_{-\infty}^{\lambda}\frac{(\lambda-s)^{\frac{1}{3}}}{(1+|s|)^N}\,ds \\ &\le \frac{1}{T}\int_{-\infty}^{\infty}\frac{(\lambda+|s|)^{\frac{1}{3}}}{(1+|s|)^N}\,ds \\ &\le \frac{1}{T}\int_{-\infty}^{\infty}\frac{\lambda^{\frac{1}{3}}}{(1+|s|)^N}\,ds + \frac{1}{T}\int_{-\infty}^{\infty}\frac{|s|^{\frac{1}{3}}}{(1+|s|)^N}\,ds \\ &\lesssim \frac{\lambda^{\frac{1}{3}}}{T}+\frac{1}{T} \lesssim \frac{\lambda^{\frac{1}{3}}}{T},\end{aligned}$$

when we take N large enough, which completes the proof.

We have shown that

$$\|(I-\chi(\lambda-P))\circ\chi(T(\lambda-P))f\|_{L^2(\gamma)} \lesssim \frac{\lambda^{\frac{1}{6}}}{T}\|f\|_{L^2(M)}.$$

Similarly, using [12, Theorem 1] (see also [21, Theorem 1.1]) instead of Theorem 1.1, we have that

$$\|(I-\chi(\lambda-P))\circ\chi(T(\lambda-P))f\|_{L^4(\gamma)} \lesssim \frac{\lambda^{\frac{1}{4}}}{T}\|f\|_{L^2(M)}.$$

By interpolation, we have that

$$\|(I-\chi(\lambda-P))\circ\chi(T(\lambda-P))f\|_{L^p(\gamma)}\lesssim\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T}\|f\|_{L^2(M)},\quad 2\le p\le 4.$$

We would therefore have (4.2) if we could show

$$\|\chi(\lambda-P)\circ\chi(T(\lambda-P))f\|_{L^p(\gamma)}\le C_p\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T^{\frac{1}{2}}}\|f\|_{L^2(M)},\quad 2\le p<4.$$

This follows from

$$\sum_{j\le J}\|Q_j\circ\chi(\lambda-P)\circ\chi(T(\lambda-P))f\|_{L^p(\gamma)}\le C_p\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T^{\frac{1}{2}}}\|f\|_{L^2(M)},\quad 2\le p<4,\tag{4.8}$$

and, for $N=1,2,3,\cdots$,

$$\|(I-\sum_{j\le J}Q_j)\circ\chi(\lambda-P)\circ\chi(T(\lambda-P))f\|_{L^p(\gamma)}\lesssim\lambda^{-N}\|f\|_{L^2(M)},\quad 2\le p<4.\tag{4.9}$$

We first show (4.9). Recall that

$$\|\chi(T(\lambda-P))f\|_{L^2(M)}\lesssim\|f\|_{L^2(M)}.\tag{4.10}$$

By Proposition 2.3 and (4.10), we have, for $2\le p<4$ and $N=1,2,3,\cdots$,

$$\begin{aligned}\|(I-\sum_{j\le J}Q_j)\circ\chi(\lambda-P)\circ\chi(T(\lambda-P))f\|_{L^p(\gamma)}&\le C_N\lambda^{-N}\|\chi(T(\lambda-P))f\|_{L^2(M)}\\&\lesssim\lambda^{-N}\|f\|_{L^2(M)},\end{aligned}$$

which is better than (4.9), and so, we are left to show (4.8).

Before we proceed further, let us look at the $L^2(M)\to L^4(\gamma)$ estimate of $Q_j\circ\chi(\lambda-P)$.

Lemma 4.3. For $j\le J$, we have

$$\|Q_j\circ\chi(\lambda-P)f\|_{L^4(\gamma)}\le C\lambda^{\frac{1}{4}}\|f\|_{L^2(M)}.$$

It follows from (4.10) that

$$\|Q_j\circ\chi(\lambda-P)\circ\chi(T(\lambda-P))f\|_{L^4(\gamma)}\lesssim\lambda^{\frac{1}{4}}\|\chi(T(\lambda-P))f\|_{L^2(M)}\lesssim\lambda^{\frac{1}{4}}\|f\|_{L^2(M)}.\tag{4.11}$$

Proof. In Fermi coordinates as above, we write, for $\epsilon>0$ small,

$$\gamma=\{(r,0):|r|\le\epsilon\},\quad \gamma_c=\{(x_1,x_2):|x_1|\le\epsilon,\ x_2=c\}.$$

We first show that

$$\|\mathcal{R}_\gamma\circ Q_jg\|_{L^4(\gamma)}\lesssim\sup_{|c|\le\epsilon}\|\mathcal{R}_{\gamma_c}g\|_{L^4(\gamma_c)},\tag{4.12}$$

where $\mathcal{R}_\gamma g$ and $\mathcal{R}_{\gamma_c} g$ are the restrictions of g onto γ and γ_c, respectively.

We can write

$$(\mathcal{R}_\gamma \circ Q_j)(r, y) = \frac{1}{(2\pi)^2} \int e^{i[(r-y_1)\xi_1 - y_2\xi_2]} q_j(r, 0, \xi)\, d\xi.$$

We may assume $|y_1|, |y_2| \le \epsilon$ by a partition of unity if necessary. By (2.6), integration by parts then gives

$$\begin{aligned}|(\mathcal{R}_\gamma \circ Q_j)(r, y)| &\le C_N \lambda^2 2^{-j} (1 + \lambda|r - y_1| + \lambda 2^{-j}|y_2|)^{-2N} \\ &\le C_N \lambda^2 2^{-j} (1 + \lambda|r - y_1|)^{-N} (1 + \lambda 2^{-j}|y_2|)^{-N}, \quad N = 1, 2, 3, \cdots .\end{aligned}$$

This implies that

$$\int |(\mathcal{R}_\gamma \circ Q_j)(r, y_1, y_2)|\, dr, \quad \int |(\mathcal{R}_\gamma \circ Q_j)(r, y_1, y_2)|\, dy_1 \lesssim C_N \lambda 2^{-j} (1 + \lambda 2^{-j}|y_2|)^{-N}.$$

By Young's inequality, we then have that

$$\|\mathcal{R}_\gamma \circ Q_j g(\cdot, y_2)\|_{L^4_r([-\epsilon,\epsilon])} \lesssim \lambda 2^{-j} (1 + \lambda 2^{-j}|y_2|)^{-N} \|g(\cdot, y_2)\|_{L^4_{y_1}([-\epsilon,\epsilon])}.$$

By this, (2.4), and Minkowski's inequality for integrals, we have that, for $0 < \epsilon \ll 1$,

$$\begin{aligned}\|\mathcal{R}_\gamma \circ Q_j g\|_{L^4(\gamma)} &= \left(\int \left| \int \left[\int (\mathcal{R}_\gamma \circ Q_j)(r, y_1, y_2) g(y_1, y_2)\, dy_1 \right] dy_2 \right|^4 dr \right)^{\frac{1}{4}} \\ &\le \int \|(\mathcal{R}_\gamma \circ Q_j) g(\cdot, y_2)\|_{L^4([-\epsilon,\epsilon])}\, dy_2 \\ &\lesssim \lambda 2^{-j} \int (1 + \lambda 2^{-j}|y_2|)^{-N} \|g(\cdot, y_2)\|_{L^4([-\epsilon,\epsilon])}\, dy_2 \\ &\lesssim \sup_{|y_2| \le \epsilon} \|g(\cdot, y_2)\|_{L^4([-\epsilon,\epsilon])} = \sup_{|c| \le \epsilon} \|g(\cdot, c)\|_{L^4([-\epsilon,\epsilon])} = \sup_{|c| \le \epsilon} \|\mathcal{R}_{\gamma_c} g\|_{L^4(\gamma_c)},\end{aligned}$$

which proves (4.12).

By (the proof of) [12, Theorem 1] and [21, Theorem 1.1], we know that

$$\sup_{|c| \le \epsilon} \|\mathcal{R}_{\gamma_c} \circ \chi(\lambda - P) f\|_{L^4(\gamma_c)} \lesssim \lambda^{\frac{1}{4}} \|f\|_{L^2(M)}.$$

Here, the implicit constants are uniform, which are stable under C^∞ perturbation of γ. Combining this and (4.12) with $g = \chi(\lambda - P) f$, we obtain that

$$\|\mathcal{R}_\gamma \circ Q_j \circ \chi(\lambda - P) f\|_{L^4(\gamma)} \lesssim \sup_{|c| \le \epsilon} \|\mathcal{R}_{\gamma_c} \circ \chi(\lambda - P) f\|_{L^4(\gamma_c)} \lesssim \lambda^{\frac{1}{4}} \|f\|_{L^2(M)}.$$

This completes the proof. $\square$

By Proposition 2.5 and Lemma 4.3, we have that, for $j \le J$,

$$\|Q_j \circ \chi(\lambda - P) f\|_{L^2(\gamma)} \le C 2^{\frac{j}{2}} \|f\|_{L^2(M)},$$
$$\|Q_j \circ \chi(\lambda - P) f\|_{L^4(\gamma)} \le C \lambda^{\frac{1}{4}} \|f\|_{L^2(M)}.$$

By interpolation, we have

$$\|Q_j \circ \chi(\lambda - P) f\|_{L^p(\gamma)} \le C 2^{\frac{j}{2}(\frac{4}{p}-1)} \lambda^{\frac{1}{4}(2-\frac{4}{p})} \|f\|_{L^2(M)}, \quad 2 \le p < 4. \tag{4.13}$$

Let $\epsilon > 0$ be a fixed but small number, which will be specified later. By (4.13) and (4.10), if $2 \le p < 4$, then

$$\begin{aligned}
&\sum_{j \le \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor} \|Q_j \circ \chi(\lambda - P) \circ \chi(T(\lambda - P)) f\|_{L^p(\gamma)} \\
&\qquad \le C \sum_{j \le \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor} 2^{\frac{j}{2}(\frac{4}{p}-1)} \lambda^{\frac{1}{4}(2-\frac{4}{p})} \|\chi(T(\lambda - P)) f\|_{L^2(M)} \\
&\qquad \le \frac{2C}{1 - 2^{-\frac{1}{2}(\frac{4}{p}-1)}} \lambda^{\frac{1}{3} - \frac{1}{3p} - \frac{\epsilon}{2}(\frac{4}{p}-1)} \|f\|_{L^2(M)} \\
&\qquad \le \frac{2C}{1 - 2^{-\frac{1}{2}(\frac{4}{p}-1)}} \frac{\lambda^{\frac{1}{3} - \frac{1}{3p}}}{T^{\frac{1}{2}}} \|f\|_{L^2(M)},
\end{aligned}$$

which satisfies (4.8).

Remark 4.4. We note that we cannot relax the condition $C_p \to \infty$ as $p \to 4$ in our argument. Indeed, note that

$$\lim_{\lambda \to \infty} \lim_{\epsilon \to 0} \frac{\lambda^{\frac{1}{3} - \frac{1}{3p} - \frac{\epsilon}{2}(\frac{4}{p}-1)}}{\lambda^{\frac{1}{3} - \frac{1}{3p}} / T^{\frac{1}{2}}} = \lim_{\lambda \to \infty} T^{\frac{1}{2}} = \infty.$$

Also, if we set

$$C_p = \frac{2C}{1 - 2^{-\frac{1}{2}(\frac{4}{p}-1)}},$$

then our argument gives

$$\sum_{j \le \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor} \|Q_j \circ \chi(\lambda - P) \circ \chi(T(\lambda - P)) f\|_{L^p(\gamma)} \le C_p \frac{\lambda^{\frac{1}{3} - \frac{1}{3p}}}{T^{\frac{1}{2}}} \|f\|_{L^2(M)},$$

but we have that $\lim\limits_{p \to 4} C_p = \infty$.

If we set

$$\chi_T(\zeta) = \chi(\zeta / T), \quad \mu_T(\zeta) = \chi_T(\zeta) \chi(\zeta), \tag{4.14}$$

we have $\chi(\lambda - P)\chi(T(\lambda - P)) = \mu_T(T(\lambda - P))$. Also, since $\widehat{\chi}_T(\zeta) = T\widehat{\chi}(T\zeta)$ and $\widehat{\mu}_T(t) = (2\pi)^{-1}\widehat{\chi}_T * \widehat{\chi}(t)$, we have, by (2.1),

$$\mathrm{supp}(\widehat{\mu}_T) \subset \mathrm{supp}(\widehat{\chi}_T) + \mathrm{supp}(\widehat{\chi}) \subset [-\frac{\epsilon_0}{T}, \frac{\epsilon_0}{T}] + [-\epsilon_0, \epsilon_0] \subset [-2\epsilon_0, 2\epsilon_0],$$

and so,

$$\mathrm{supp}(\widehat{\mu_T^2}) \subset \mathrm{supp}(\widehat{\mu_T}) + \mathrm{supp}(\widehat{\mu_T}) \subset [-4\epsilon_0, 4\epsilon_0], \tag{4.15}$$

since $T = c_0 \log \lambda \gg 1$. We have shown that

$$\sum_{j \le \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor} \|Q_j \circ \mu_T(T(\lambda - P))f\|_{L^p(\gamma)} \le \frac{2C}{1 - 2^{-\frac{1}{2}(\frac{4}{p}-1)}} \frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T^{\frac{1}{2}}} \|f\|_{L^2(M)},$$

For the rest of (4.8), we want to show that, for $2 \le p < 4$,

$$\|Q_j \circ \mu_T(T(\lambda - P))f\|_{L^p(\gamma)} \lesssim \frac{2^{j\left(\frac{2}{p}-\frac{1}{2}\right)}\lambda^{\frac{1}{2}-\frac{1}{p}}}{T^{\frac{1}{2}}} \|f\|_{L^2(M)}, \quad \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor \le j \le J.$$

Indeed, we have

$$\sum_{\lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor \le j \le J} \frac{2^{j\left(\frac{2}{p}-\frac{1}{2}\right)}\lambda^{\frac{1}{2}-\frac{1}{p}}}{T^{\frac{1}{2}}} \lesssim \frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{T^{\frac{1}{2}}}.$$

Here, we take $\epsilon > 0$ to be sufficiently small and choose a small $c_0 > 0$ in (4.1).

By a TT^* argument, we would have (4.8) if we could show either, for $2 \le p < 4$,

$$\|Q_j \circ \mu_T^2(T(\lambda - P)) \circ Q_j^* f\|_{L^p(\gamma)} \lesssim \frac{2^{j\left(\frac{4}{p}-1\right)}\lambda^{1-\frac{2}{p}}}{T} \|f\|_{L^{p'}(\gamma)}, \quad 2 \le p < 4. \tag{4.16}$$

We want to lift this problem to the universal cover of M. Let $\tilde{M}$ be the universal cover of M with the pullback metric $\tilde{g}$ under the covering map $p : \tilde{M} \to M$. By the Cartan-Hadamard theorem, $\tilde{M}$ is diffeomorphic to $\mathbb{R}^2$ with the diffeomorphism $T_{x_0}M \cong \mathbb{R}^2 \to \tilde{M}$ for any $x_0 \in M$, so that the map $p = \exp_{x_0} : T_{x_0}M \to M$ is a smooth covering map. Without loss of generality, we write $p : \mathbb{R}^2 \cong \tilde{M} \to M$.

Let $D \subset \mathbb{R}^2$ be a fundamental domain of the universal covering p so that every point in $\mathbb{R}^2$ is the translate of exactly one point in D. Without loss of generality, we may assume that γ and other amplitudes like q_j are supported in D°, where D° is the interior of D, i.e., $\gamma \subset D^\circ$, and $\mathrm{supp}(q_j) \subset D^\circ$, etc. We write tildes over letters to express that those letters are defined in $\mathbb{R}^2 \cong \tilde{M}$. For example, for any $x \in M$, let $\tilde{x} \in D$ be the unique point so that $p(\tilde{x}) = x$, $p(\tilde{\gamma}) = \gamma$, and the metric $\tilde{g}$ on $\mathbb{R}^2 \cong \tilde{M}$ is the pullback metric of g, $\tilde{\rho}(\tilde{x}, \tilde{y})$ is the Riemannian distance $d_{\tilde{g}}(\tilde{x}, \tilde{y})$, and so on. Let Γ be the group of deck transformations α's, which are diffeomorphisms satisfying $p \circ \alpha = p$. With this in mind, if we have a function $\tilde{f}$ on D, we can extend this $\tilde{f}$ to $\mathbb{R}^2 \cong \tilde{M}$ by setting

$$\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x})) \quad \text{for } \tilde{x} \in D.$$

Here, since $p : \mathbb{R}^2 \to M$ is a local diffeomorphism, abusing notation we write

$$\tilde{Q}_j(\tilde{x}, \tilde{w}) = \frac{\lambda^2}{(2\pi)^2} \int e^{i\lambda(\tilde{x}-\tilde{w})\cdot\eta} \tilde{q}_j(\tilde{x}, \tilde{w}, \lambda\eta)\, d\eta,$$

$$\tilde{Q}_j^*(\tilde{z}, \tilde{y}) = \overline{\tilde{Q}_j(\tilde{y}, \tilde{z})} = \overline{\tilde{Q}_j(\alpha(\tilde{y}), \alpha(\tilde{z}))} = \frac{\lambda^2}{(2\pi)^2} \int e^{-i\lambda(\alpha(\tilde{y})-\alpha(\tilde{z}))\cdot\zeta} \overline{\tilde{q}_j}(\alpha(\tilde{y}), \alpha(\tilde{z}), \lambda\zeta)\, d\zeta.$$

Recall that we know from [34] that

$$(\cos tP)(x, y) = \sum_{\alpha\in\Gamma} (\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})), \quad \tilde{x}, \tilde{y} \in D.$$

Also recall that, by a counting argument and finite propagation speed as in [34], there are at most $O(e^{Ct})$ many nonzero terms in the sum.

Using Euler's formula, we have, modulo $O(\lambda^{-N})$ errors,

$$\begin{aligned} \chi^2(T(\lambda - P))(x, y) &= \frac{1}{\pi T} \int e^{it\lambda} \widehat{\chi^2}(t/T) \cos(tP)(x, y)\, dt - \chi^2(T(\lambda + P))(x, y) \\ &= \frac{1}{\pi T} \sum_{\alpha\in\Gamma} \int e^{it\lambda} \widehat{\chi^2}(t/T) \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y}))\, dt, \end{aligned}$$

since $\chi^2(T(\lambda + P))(x, y) = O(\lambda^{-N})$.

We want to show that the estimate for $\alpha = \mathrm{Id}$ satisfies (4.16).

Lemma 4.5. If $\alpha = \mathrm{Id}$ and $2 \le p \le 4$, then

$$\begin{aligned} &\left\| \frac{1}{\pi T} \iint e^{it\lambda} \widehat{\mu_T^2}(t/T) (\tilde{Q}_j \circ \cos(t\sqrt{-\Delta_{\tilde{g}}})(\cdot, \alpha(\cdot)) \circ \tilde{Q}_j^*)(\tilde{\gamma}(\cdot), \tilde{\gamma}(s)) f(s)\, dt\, ds \right\|_{L^p(\gamma)} \\ &\qquad\qquad \lesssim \frac{2^{j\left(\frac{4}{p}-1\right)} \lambda^{1-\frac{2}{p}}}{T} \|f\|_{L^{p'}(\gamma)}, \end{aligned}$$

which satisfies the estimate (4.16).

Proof. We choose $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(t) = 1 \text{ for } |t| \le c, \text{ and } \beta(t) = 0 \text{ for } |t| \ge 2c, \tag{4.17}$$

for a small $c > 0$, which will be specified later in this proof. Since $\beta(t)\widehat{\mu_T^2}(t/T)$ is compactly supported in t and

$$|\partial_t^k [\beta(t) \widehat{\mu_T^2}(t/T)]| \le C_k,$$

the term $\beta(t)\widehat{\mu_T^2}(t/T)$ plays the same role as $\widehat{\chi^2}(t)$ in Chapter 3. Thus, by the proof of Theorem 1.1, we have, for $\alpha = \mathrm{Id}$,

$$\begin{aligned} &\left\| \frac{1}{\pi T} \iint e^{it\lambda} \beta(t) \widehat{\mu_T^2}(t/T) (\tilde{Q}_j \circ \cos(t\sqrt{-\Delta_{\tilde{g}}})(\cdot, \cdot) \circ \tilde{Q}_j^*)(\tilde{\gamma}(\cdot), \tilde{\gamma}(s)) f(s)\, dt\, ds \right\|_{L^2(\gamma)} \\ &\qquad\qquad \lesssim \frac{2^j}{T} \|f\|_{L^2(\gamma)}. \end{aligned}$$

The difference between this and Theorem 1.1 is that we use the Hadamard parametrix about the cosine propagator $\cos(t\sqrt{-\Delta_{\tilde g}})$ here, and we used the Lax parametrix about e^{-itP} there.

Similarly, instead of using Theorem 1.1, by using the proof of (4.11) with a TT^* argument, we can obtain, for $\alpha = \mathrm{Id}$,

$$\left\| \frac{1}{\pi T} \iint e^{it\lambda} \beta(t) \widehat{\mu_T^2}(t/T) (\tilde Q_j \circ \cos(t\sqrt{-\Delta_{\tilde g}})(\cdot,\cdot) \circ \tilde Q_j^*)(\tilde\gamma(\cdot), \tilde\gamma(s)) f(s)\, dt\, ds \right\|_{L^4(\gamma)} \lesssim \frac{\lambda^{\frac12}}{T} \|f\|_{L^{\frac43}(\gamma)}.$$

The desired estimate then follows from interpolation.

It then suffices to show that, for $\alpha = \mathrm{Id}$ and $N = 1, 2, 3, \cdots$,

$$\left\| \frac{1}{\pi T} \iint e^{it\lambda} (1-\beta(t)) \widehat{\mu_T^2}(t/T) (\tilde Q_j \circ \cos(t\sqrt{-\Delta_{\tilde g}})(\cdot,\cdot) \circ \tilde Q_j^*)(\tilde\gamma(\cdot), \tilde\gamma(s)) f(s) dt ds \right\|_{L^p(\gamma)} \lesssim \lambda^{-N} \|f\|_{L^{p'}(\gamma)}.$$

We show this as in [15, Lemma 3.1]. We first consider the kernel of the integral operator inside the L^2 norm without Q_j and Q_j^* compositions

$$\frac{1}{\pi T} \int e^{it\lambda} (1-\beta(t)) \widehat{\mu_T^2}(t/T) \cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x, \tilde y)\, dt, \quad \tilde x, \tilde y \in D.$$

We recall properties of the cosine propagator (cf. [6, (5.14)], etc.)

$$\operatorname{sing\,supp}(\cos t\sqrt{-\Delta_{\tilde g}})(\cdot,\cdot) \subset \{(\tilde x, \tilde z) \in \mathbb{R}^2 \times \mathbb{R}^2 : \tilde\rho(\tilde x, \tilde z) = |t|\},$$

that is, $\cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x, \tilde z)$ is smooth if $\tilde\rho(\tilde x, \tilde z) \neq |t|$. Since $1 - \beta(t) = 0$ for $|t| \le c$ where $c > 0$ is as in (4.17), we may assume that $|t| \ge c > 0$. Here, we choose a sufficiently small $c > 0$, compared to the injectivitiy radius of M. For $\alpha = \mathrm{Id}$, by a partition of unity if necessary, we may assume that $\tilde\rho(\tilde x, \alpha(\tilde y)) \le c/2$ for $\tilde x, \tilde y \in D$, and thus,

$$|t| \ge c > \tilde\rho(\tilde x, \alpha(\tilde y)), \ \alpha = \mathrm{Id}, \ \tilde x, \tilde y \in D, \quad \text{that is, } \tilde\rho(\tilde x, \alpha(\tilde y)) \neq |t|.$$

This implies that $\cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x, \tilde z)$ is smooth for $\tilde x, \tilde z \in \mathbb{R}^2$, and thus, integration by parts in t implies that

$$\frac{1}{\pi T} \int e^{it\lambda} (1-\beta(t)) \widehat{\mu_T^2}(t/T) \cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x, \alpha(\tilde y))\, dt = O(\lambda^{-N}), \quad \alpha = \mathrm{Id}, \ \tilde x, \tilde y \in \tilde\gamma.$$

For the contribution after compositions of Q_j and Q_j^*, by (2.6), we note that

$$\begin{aligned}
&\left|\frac{1}{\pi T}\int e^{it\lambda}(1-\beta(t))\widehat{\mu_T^2}(t/T)(\tilde{Q}_j\circ\cos(t\sqrt{-\Delta_{\tilde{g}}})(\cdot,\alpha(\cdot))\circ\tilde{Q}_j^*)(\tilde{\gamma}(r),\tilde{\gamma}(s))\,dt\right|\\
&\lesssim\frac{1}{T}\left|\iint\tilde{Q}_j(\tilde{\gamma}(r),z)\right.\\
&\qquad\qquad\left.\times\left(\int e^{it\lambda}(1-\beta(t))\widehat{\mu_T^2}(t/T)\cos(t\sqrt{-\Delta_{\tilde{g}}})(z,w)\,dt\right)\tilde{Q}_j^*(w,\tilde{\gamma}(s))\,dz\,dw\right|\\
&\lesssim\sup_{z,w}\left(\frac{1}{T}\int e^{it\lambda}(1-\beta(t))\widehat{\mu_T^2}(t/T)\cos(t\sqrt{-\Delta_{\tilde{g}}})(z,w)\,dt\right)\\
&\qquad\qquad\times\iint|\tilde{Q}_j(\tilde{\gamma}(r),z)||Q_j^*(w,\tilde{\gamma}(s))|\,dz\,dw\\
&\lesssim\lambda^{-N}\sup_{\tilde{\gamma}(r)}\int|\tilde{Q}_j(\tilde{\gamma}(r),z)|\,dz\ \sup_{\tilde{\gamma}(s)}\int|\tilde{Q}_j^*(w,\gamma(s))|\,dw\lesssim\lambda^{-N}.
\end{aligned}$$

This completes the proof. □

By Lemma 4.5, we can ignore the contribution of $\alpha=\mathrm{Id}$. Using Euler's formula, we know

$$\mu_T^2(T(\lambda-P))(x,y)=\frac{1}{\pi T}\int e^{it\lambda}\widehat{\mu_T^2}(t/T)(\cos tP)(x,y)\,dt-\mu_T^2(T(\lambda+P))(x,y),$$

and also know that $\mu_T^2(T(\lambda+P))(x,y)=O(\lambda^{-N})$. As in Chapter 3, if we set

$$K_j(x,y)=\frac{1}{2\pi T}\int e^{it\lambda}\widehat{\mu_T^2}(t/T)(Q_j\circ e^{-itP}\circ Q_j^*)(x,y)\,dt,$$

then, by Euler's formula, modulo $O(\lambda^{-N})$ errors, we have

$$K_j(x,y)=\frac{1}{\pi T}\int e^{it\lambda}\widehat{\mu_T^2}(t/T)(Q_j\circ\cos tP\circ Q_j^*)(x,y)\,dt,$$

which is the kernel of $Q_j\circ\mu_T^2(T(\lambda-P))\circ Q_j^*$ modulo $O(\lambda^{-N})$ errors. From now on, we focus on $\lfloor\log_2\lambda^{\frac{1}{3}-\epsilon}\rfloor\le j<J$. Similar arguments will also work for $j=J$.

By a version of Egorov's theorem in [11] and the subsequent observation in [1, Theorem 4.2.4], we have

$$e^{-it\sqrt{-\Delta_{\tilde{g}}}}\circ Q_j^*=\tilde{B}_{t,j}\circ e^{-it\sqrt{-\Delta_{\tilde{g}}}},\tag{4.18}$$

where $\tilde{B}_{t,j}$ has a symbol

$$\tilde{b}_{t,j}=\kappa_{-t}^*\tilde{q}_j^*+b'=\tilde{q}_j\circ\kappa_{-t}+b',$$

with the Hamiltonian flow κ_t, and $|b'|=O(\lambda^{-1+\frac{2}{3}+2\Lambda c_0})=O(\lambda^{-\frac{1}{3}+2\Lambda c_0})$ for some fixed $\Lambda>0$. Since M is compact, we have $|b'|=O(\lambda^{-\frac{1}{3}+2\Lambda c_0})=O(\lambda^{-\frac{1}{3}+\epsilon'})$ for some small $\epsilon'>0$ when taking $c_0>0$ to be sufficiently small in (4.1) for a uniform constant Λ.

As before, we will ignore the contribution of the remainder b', and write $b_{t,j} = \kappa_t^* q_j$. Using Euler's formula again, we can replace $e^{-it\sqrt{-\Delta_{\tilde g}}}$ by $\cos t\sqrt{-\Delta_{\tilde g}}$ modulo $O(\lambda^{-N})$ errors in (4.18).

With this in mind, modulo $O(\lambda^{-N})$ errors, we have that

$$K_j(x,y) = \frac{1}{\pi T}\int e^{it\lambda}\widehat{\mu_T^2}(t/T)(Q_j\circ B_{t,j}\circ\cos tP)(x,y)\,dt, \tag{4.19}$$

where $\tilde B_{t,j}$ is the lift of $B_{t,j}$. Since p is a local isometry, we may assume $|\det p| = 1$ in Riemannian measure. Using $w = p(\tilde w)$ and $z = p(\tilde z)$, we write

$$\begin{aligned}
&(Q_j\circ B_{t,j}\circ\cos t\sqrt{-\Delta_g})(x,y)\\
&=\sum_\alpha\iint_{D^2}\tilde Q_j(\tilde x,\tilde w)\tilde B_{t,j}(\tilde w,\tilde z)(\cos t\sqrt{-\Delta_{\tilde g}})(\tilde z,\alpha(\tilde y))|\det p|^2\,d\tilde w\,d\tilde z\\
&=\sum_\alpha\iint_{D^2}\tilde Q_j(\tilde x,\tilde w)\tilde B_{t,j}(\tilde w,\tilde z)(\cos t\sqrt{-\Delta_{\tilde g}})(\tilde z,\alpha(\tilde y))\,d\tilde w\,d\tilde z.
\end{aligned}$$

By the Hadamard parametrix (cf. [3], [32], [34], etc.), we write, for $\tilde x\in D$ and $\tilde w\in\alpha(D)$,

$$(\cos t\sqrt{-\Delta_{\tilde g}})(\tilde x,\tilde w) = \tilde K_N(t,\tilde x;\tilde w) + \tilde R_N(t,\tilde x;\tilde w),$$

where

$$\tilde K_N(t,\tilde x;\tilde w) = \begin{cases}\sum_{\nu=0}^N u_\nu(\tilde x,\tilde w)\partial_t E_\nu(t,\tilde\rho(\tilde x,\tilde w)), & t\ge 0,\\ -\sum_{\nu=0}^N u_\nu(\tilde x,\tilde w)\partial_t E_\nu(-t,\tilde\rho(\tilde x,\tilde w)), & t<0.\end{cases}$$

We explain the u_ν and E_ν below.

For simplicity, from now on, we focus on $t\ge 0$. Similar arguments work for $t\le 0$. Here, the C^∞ functions u_ν are as in [3, §2 in B and (10)] and [32, p.35]:

$$\begin{aligned}
&u_0(\tilde x,\tilde w) = \Theta^{-\frac12}(\tilde x,\tilde w),\\
&u_\nu(\tilde x,\tilde w) = \Theta^{-\frac12}(\tilde x,\tilde w)\int_0^1 s^{\nu-1}\Theta^{1/2}(\tilde x,\tilde x_s)\Delta_{\tilde g,\tilde w}u_{\nu-1}(\tilde x,\tilde x_s)\,dx, \quad \nu=1,2,3,\cdots,\\
&\Theta(\tilde x,\tilde w) = |\det D_{\exp_{\tilde x}^{-1}(\tilde w)}\exp_{\tilde x}|,
\end{aligned}$$

where $\tilde x_s$ is the minimizing geodesic from $\tilde x$ to $\alpha(\tilde w)$ parametrized by arc length and

$$\Theta = (\det(\tilde g_{jk}))^{\frac12}.$$

As in [32, Chapter 1], the distributions E_ν are, in $\mathbb{R}^n$, for $\nu = 0,1,2,\cdots$,

$$E_\nu(t,x) = \lim_{\epsilon\to 0+}\nu!(2\pi)^{-n-1}\iint_{\mathbb{R}^{1+n}} e^{ix\cdot\xi+it\tau}(|\xi|^2-(\tau-i\epsilon)^2)^{-\nu-1}\,d\xi\,d\tau,$$

and

$$\begin{aligned}
&E_0(t,x) = H(t)\times(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin t|\xi|}{|\xi|}\,d\xi,\\
&\Box E_\nu = \nu E_{\nu-1}, \quad -2\frac{\partial E_\nu}{\partial x} = xE_{\nu-1}, \quad 2\frac{\partial E_\nu}{\partial t} = tE_{\nu-1}, \quad \nu = 1,2,3,\cdots,
\end{aligned}$$

where $H(t)$ is the Heaviside function

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

We have $n = 2$ in our work. Here, $E_0(t,x)$ is interpreted as

$$E_0(t,x) = \begin{cases} (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} \, d\xi, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and

$$\langle E_0(t,\cdot), f\rangle = (2\pi)^{-2} H(t) \int_{\mathbb{R}^2} \frac{\sin t|\xi|}{|\xi|} \hat{f}(\xi) \, d\xi,$$

that is, the Fourier transform of $E_0(t,\cdot)$ is $\frac{\sin t|\xi|}{|\xi|}$. Also, since the E_ν are radial in x, we may abuse notation, for example, $E_\nu(t,x) = E_\nu(t,|x|)$.

We will ignore the contribution of $\tilde{R}_N$. We first recall a result in [3], [32, Theorem 3.1.5], and [22, Proposition 3.1], adapted to our settings.

Lemma 4.6 ([3], [32], [22])**.** For $|t| \leq T$, we have $\tilde{R}_N \in C^{N-5}([-T,T] \times D \times D)$ and

$$|\partial^\beta_{t,x,y} \tilde{R}_N(t,\tilde{x};\tilde{w})| \lesssim e^{C_\beta T}, \quad \text{if } |\beta| \ll N.$$

Consider the operator whose kernel is

$$\frac{1}{\pi T} \int e^{it\lambda} \widehat{\mu^2_T}(t/T) \tilde{R}_N(t,\tilde{x};\tilde{w}) \, dt. \tag{4.20}$$

By Lemma 4.6, integration by parts in t gives, for $N = 1,2,3,\cdots$,

$$\frac{1}{\pi T} \int e^{it\lambda} \widehat{\mu^2_T}(t/T) \tilde{R}_N(t,\tilde{x};\tilde{w}) \, dt = O(T^{-1}(2T)\lambda^{-N'} e^{C_{N'}T}) = O(\lambda^{-N}).$$

By (2.6) again as in the proof of Lemma 4.5, we can obtain, for $N = 1,2,3,\cdots$,

$$\left(\tilde{Q}_j \circ \left(\frac{1}{\pi T} \int e^{it\lambda} \widehat{\mu^2_T}(t/T) \tilde{R}_N(t,\cdot;\cdot) \, dt \right) \circ \tilde{Q}^*_j \right) (\tilde{\gamma}(r), \tilde{\gamma}(s)) = O(\lambda^{-N}),$$

and thus, by Young's inequality, we ignore the contribution of (4.20), when we take $N \gg 1$.

We can also ignore the contribution of E_ν for $\nu \geq 1$.

Lemma 4.7 (Theorem 3.4 in [14])**.** We have, for $\tilde{x} \in D$ and $\tilde{w} \in \alpha(D)$,

$$\int e^{it\lambda} \widehat{\mu^2_T}(t/T) \partial_t E_\nu(t, \tilde{\rho}(\tilde{x},\tilde{w})) \, dt = O(\lambda^{1-2\nu}), \quad \nu = 0,1,2,\cdots.$$

By the same arguments as in $\tilde{R}_\lambda$, Lemma 4.7 gives, for $\nu = 0, 1, 2, \cdots$,

$$\left(\tilde{Q}_j \circ \left(\frac{1}{\pi T}\int e^{it\lambda}\widehat{\mu_T^2}(t/T)\partial_t E_\nu(t, \tilde{\rho}(\cdot,\cdot))\,dt\right)\circ \tilde{Q}_j^*\right)(\tilde{\gamma}(r), \tilde{\gamma}(s)) = O(\lambda^{1-2\nu}).$$

By Young's inequality, the contribution of this is better than (4.16) when $\nu \geq 1$, and so, we only need to consider $\nu = 0$. With this in mind, we may write, modulo $O(\lambda^{-1})$ errors,

$$\begin{aligned}(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{z}, \alpha(\tilde{y})) &= u_0(\tilde{z}, \alpha(\tilde{y}))\partial_t E_0(t, \tilde{\rho}(\tilde{z}, \alpha(\tilde{y}))) \\ &= \frac{1}{(2\pi)^2}u_0(\tilde{z}, \alpha(\tilde{y}))\int e^{i\Phi(\tilde{z},\alpha(\tilde{y}))\cdot\xi}\cos(t|\xi|)\,d\xi,\end{aligned} \tag{4.21}$$

where $|\Phi(\tilde{z}, \alpha(\tilde{y}))| = \tilde{\rho}(\tilde{z}, \alpha(\tilde{y}))$ (cf. [14, p.4026], [32], etc.). Using (orthogonal) coordinate changes if necessary, we may assume that

$$\Phi(\tilde{z}, \alpha(\tilde{y}))\cdot\xi = \tilde{z}\cdot\xi, \quad \text{in normal coordinates at } \alpha(\tilde{y}).$$

Modulo $O(\lambda^{-1})$ errors, it follows from (4.21) that

$$\begin{aligned}&(Q_j \circ B_{t,j}\circ\cos(t\sqrt{-\Delta_g}))(x, y) \\ &= (2\pi)^{-2}\sum_\alpha \iint_{D^2}\int \tilde{Q}_j(\tilde{x}, \tilde{w})\tilde{B}_{t,j}(\tilde{w}, \tilde{z})u_0(\tilde{z}, \alpha(\tilde{y}))e^{i\Phi(\tilde{z},\alpha(\tilde{y}))\cdot\xi}\cos t|\xi|\,d\xi\,d\tilde{w}\,d\tilde{z} \\ &= \frac{1}{2(2\pi)^2}\sum_\alpha\sum_\pm \iiint \tilde{Q}_j(\tilde{x}, \tilde{w})\tilde{B}_{t,j}(\tilde{w}, \tilde{z})u_0(\tilde{z}, \alpha(\tilde{y}))e^{i\Phi(\tilde{z},\alpha(\tilde{y}))\cdot\xi\pm it|\xi|}\,d\xi\,d\tilde{w}\,d\tilde{z}.\end{aligned}$$

We now write

$$\begin{aligned}&(Q_j \circ B_{t,j}\circ\cos(t\sqrt{-\Delta_g}))(x, y) \\ &= \frac{\lambda^6}{2(2\pi)^6}\sum_{\alpha,\pm}\int e^{i\lambda[(\tilde{x}-\tilde{w})\cdot\eta+(\tilde{w}-\tilde{z})\cdot\zeta+\Phi(\tilde{z},\alpha(\tilde{y}))\cdot\xi\pm t|\xi|]}\tilde{q}_j(\tilde{x}, \tilde{w}, \lambda\eta)\tilde{b}_{t,j}(\tilde{w}, \tilde{z}, \lambda\zeta) \\ &\qquad\qquad \times u_0(\tilde{z}, \alpha(\tilde{y}))\,d\tilde{w}\,d\eta\,d\tilde{z}\,d\zeta\,d\xi.\end{aligned}$$

By (4.19), modulo $O(\lambda^{-1})$ errors, we write

$$K_j(x, y) = \sum_{\alpha,\pm} U_{\alpha,j,\pm}(\tilde{x}, \tilde{y}),$$

where

$$\begin{aligned}&U_{\alpha,j,\pm}(\tilde{x}, \tilde{y}) = \frac{\lambda^6}{(2\pi)^7 T}\int e^{i\lambda\tilde{\psi}_{\alpha,\pm}(t,\xi,\tilde{w},\eta,\tilde{z},\zeta)}a_j(t, \tilde{w}, \eta, \tilde{z}, \zeta)\,d\tilde{w}\,d\eta\,d\tilde{z}\,d\zeta\,d\xi\,dt, \\ &a_j(t, \tilde{w}, \eta, \tilde{z}, \zeta) = a_j(\tilde{x}, \tilde{y}; t, \tilde{w}, \eta, \tilde{z}, \zeta) = \widehat{\mu_T^2}(t/T)\tilde{q}_j(\tilde{x}, \tilde{w}, \lambda\eta)\tilde{b}_{t,j}(\tilde{w}, \tilde{z}, \lambda\zeta)u_0(\tilde{z}, \alpha(\tilde{y})), \\ &\tilde{\psi}_{\alpha,\pm}(t, \xi, \tilde{w}, \eta, \tilde{z}, \zeta) = t + (\tilde{x}-\tilde{w})\cdot\eta + (\tilde{w}-\tilde{z})\cdot\zeta + \Phi(\tilde{z}, \alpha(\tilde{y}))\cdot\xi \pm t|\xi|.\end{aligned}$$

In geodesic normal coordinates centered at $\alpha(\tilde{y})$, using suitable orthogonal coordinate changes, we have

$$\tilde{\psi}_{\alpha,\pm}(t,\xi,\tilde{w},\eta,\tilde{z},\zeta) = t + (\tilde{x}-\tilde{w})\cdot\eta + (\tilde{w}-\tilde{z})\cdot\zeta + \tilde{z}\cdot\xi \pm t|\xi|.$$

By Lemma 4.5, we can focus only on $\alpha \neq \mathrm{Id}$. We would then have (4.16), if we could show that, for $2 \le p < 4$,

$$\left\| \int U_{\alpha,j,\pm}(\tilde{\gamma}(\cdot),\tilde{\gamma}(s)) f(s)\, ds \right\|_p \lesssim \frac{\lambda^{\frac12}}{T} e^{CT} (2^{-j})^{\frac{2}{p}} \|f\|_{p'}, \quad \alpha \neq \mathrm{Id}, \quad \lfloor \log_2 \lambda^{\frac13-\epsilon} \rfloor \le j \le J. \tag{4.22}$$

We have the following analysis for $U_{\alpha,j,\pm}$.

Proposition 4.8. For $\alpha \neq \mathrm{Id}$ fixed, we have, modulo $O(\lambda^{-1})$ errors, that

$$\begin{aligned} &U_{\alpha,j,\pm}(\tilde{\gamma}(r),\tilde{\gamma}(s)) \\ &= \begin{cases} \frac{\lambda^{\frac12}}{T} e^{i\lambda\tilde{\rho}(\tilde{\gamma}(r),\alpha(\tilde{\gamma}(s)))} \tilde{a}_{\alpha,j}(r,s), & \text{if } |d_{\tilde{x}}\tilde{\rho}(\tilde{\gamma}(r),\alpha(\tilde{\gamma}(s)))(\tilde{N})| \approx 2^{-j}, \\ & \text{and } |d_{\tilde{y}}\tilde{\rho}(\tilde{\gamma}(r),\alpha(\tilde{\gamma}(s)))(\alpha_*(\tilde{N}))| \approx 2^{-j}, \\ O(\lambda^{-N}), & \text{otherwise,} \end{cases} \end{aligned}$$

where $\lfloor \log_2 \lambda^{\frac13-\epsilon} \rfloor \le j \le J$, $\tilde{\rho} = d_{\tilde{g}}$, and $|\tilde{a}_{\alpha,j}(r,s)| \lesssim e^{CT}$.

Proof. In normal coordinates at $\alpha(\tilde{y})$, we write $U_{\alpha,j,\pm}(\tilde{x},\tilde{y})$ as $U_{\alpha,j,\pm}(\tilde{x})$, where

$$U_{\alpha,j,\pm}(\tilde{x}) = \frac{\lambda^6}{(2\pi)^7 T} \int e^{i\lambda\tilde{\psi}_{\alpha,\pm}(t,\xi,\tilde{w},\eta,\tilde{z},\zeta)} \tilde{a}_j(t,\tilde{w},\eta,\tilde{z},\zeta)\, d\tilde{w}\, d\eta\, d\tilde{z}\, d\zeta\, d\xi\, dt,$$

where $\tilde{a}_j(t,\tilde{w},\eta,\tilde{z},\zeta)$ is the coordinate expression of $a_j(t,\tilde{w},\eta,\tilde{z},\zeta)$ in normal coordinates at $\alpha(\tilde{y})$. Let $\tilde{\beta} \in C_0^\infty(\mathbb{R}^2)$ be such that $\mathrm{supp}(\tilde{\beta}) \subset \{\xi : c_2 \le |\xi| \le c_2^{-1}\}$ for a small fixed $c_2 > 0$. We write

$$U_{\alpha,j,\pm}(\tilde{x}) = U^1_{\alpha,j,\pm}(\tilde{x}) + U^2_{\alpha,j,\pm}(\tilde{x}),$$

where

$$\begin{aligned} U^1_{\alpha,j,\pm}(\tilde{x}) &= \frac{\lambda^6}{(2\pi)^7 T} \int e^{i\lambda\tilde{\psi}_{\alpha,\pm}(t,\xi,\tilde{w},\eta,\tilde{z},\zeta)} \tilde{\beta}(\xi) \tilde{a}_j(t,\tilde{w},\eta,\tilde{z},\zeta)\, d\tilde{w}\, d\eta\, d\tilde{z}\, d\zeta\, d\xi\, dt, \\ U^2_{\alpha,j,\pm}(\tilde{x}) &= \frac{\lambda^6}{(2\pi)^7 T} \int e^{i\lambda\tilde{\psi}_{\alpha,\pm}(t,\xi,\tilde{w},\eta,\tilde{z},\zeta)} (1-\tilde{\beta}(\xi)) \tilde{a}_j(t,\tilde{w},\eta,\tilde{z},\zeta)\, d\tilde{w}\, d\eta\, d\tilde{z}\, d\zeta\, d\xi\, dt. \end{aligned}$$

We note that, choosing $c_2 > 0$ small in the support of $\tilde{\beta}$, we have $|\partial_t \tilde{\psi}_\alpha| = |1 \pm |\xi|| \gtrsim 1 + |\xi|$. Thus, integrating by parts in t as in the proof of Lemma 4.5, we can write $U_{\alpha,j,\pm}(\tilde{x},\tilde{y})$ as $U^1_{\alpha,j,\pm}(\tilde{x})$ in normal coordinates at $\alpha(\tilde{y})$, and we focus on $U^1_{\alpha,j,\pm}(\tilde{x})$.

We will focus on the minus sign in the phase function. Indeed, if we choose the plus sign, then we have $\partial_t \tilde{\psi}_{\alpha,+} = 1 + |\xi| > 0$, and thus, there is no critical point of the phase function. Hence, integration by parts in t again, we have $U^1_{\alpha,j,\pm}(\tilde{x}) = O(\lambda^{-N})$.

Set $\rho_0 = \tilde{\rho}(\tilde{x}, \alpha(\tilde{y}))$. Since $\alpha \neq \mathrm{Id}$, we know $\rho_0 > 0$, and thus, can consider the following change of variables.

$$\bar{t} = \frac{t}{\sqrt{\rho_0}}, \quad \bar{w} = \frac{\tilde{w}}{\sqrt{\rho_0}}, \quad \bar{\xi} = \sqrt{\rho_0}\xi, \quad \bar{\eta} = \sqrt{\rho_0}\eta, \quad \bar{z} = \frac{\tilde{z}}{\sqrt{\rho_0}}, \quad \bar{x} = \frac{\tilde{x}}{\sqrt{\rho_0}}, \quad \bar{\zeta} = \sqrt{\rho_0}\zeta. \tag{4.23}$$

This implies that

$$d\tilde{w}\, d\eta\, d\tilde{z}\, d\zeta\, d\xi\, dt = \frac{1}{\sqrt{\rho_0}} d\bar{w}\, d\bar{\eta}\, d\bar{z}\, d\bar{\zeta}\, d\bar{\xi}\, d\bar{t}.$$

Since we choose the minus sign in the phase function, we set

$$\begin{aligned}\tilde{\psi}_{\alpha,-}(t, \xi, \tilde{w}, \eta, \tilde{z}, \zeta) &= \sqrt{\rho_0}\bar{t} - \bar{t}|\bar{\xi}| + (\bar{x} - \bar{w}) \cdot \bar{\eta} + (\bar{w} - \bar{z}) \cdot \bar{\zeta} + \bar{z} \cdot \bar{\xi} \\ &=: \bar{\psi}(\bar{t}, \bar{\xi}, \bar{w}, \bar{\eta}, \bar{z}, \bar{\zeta}).\end{aligned}$$

Note that

$$\begin{aligned}\nabla\bar{\psi} &= (\partial_{\bar{t}}\bar{\psi}, \partial_{\bar{\xi}}\bar{\psi}, \partial_{\bar{w}}\bar{\psi}, \partial_{\eta}\bar{\psi}, \partial_{\bar{z}}\bar{\psi}, \partial_{\bar{\zeta}}\bar{\psi}) \\ &= (\sqrt{\rho_0} - |\bar{\xi}|, \bar{z} - \bar{t}\frac{\bar{\xi}}{|\bar{\xi}|}, \bar{\zeta} - \bar{\eta}, \bar{x} - \bar{w}, -\bar{\zeta} + \bar{\xi}, \bar{w} - \bar{z}),\end{aligned}$$

and thus, the critical point satisfies

$$\sqrt{\rho_0} = |\bar{\xi}|, \quad \bar{z} = \bar{t}\frac{\bar{\xi}}{|\bar{\xi}|}, \quad \bar{\zeta} = \bar{\eta}, \quad \bar{x} = \bar{w}, \quad \bar{\zeta} = \bar{\xi}, \quad \bar{w} = \bar{z}. \tag{4.24}$$

The Hessian $\partial^2\bar{\psi}$ is

$$\begin{aligned}\partial^2\bar{\psi} &= \begin{pmatrix} O_{1\times1} & \left(-\frac{\bar{\xi}^T}{|\bar{\xi}|}\right)_{1\times2} & O_{1\times2} & O_{1\times2} & O_{1\times2} & O_{1\times2} \\ \left(-\frac{\bar{\xi}}{|\bar{\xi}|}\right)_{2\times1} & A_{2\times2} & O_{2\times2} & O_{2\times2} & I_{2\times2} & O_{2\times2} \\ O_{2\times1} & O_{2\times2} & O_{2\times2} & -I_{2\times2} & O_{2\times2} & I_{2\times2} \\ O_{2\times1} & O_{2\times2} & -I_{2\times2} & O_{2\times2} & O_{2\times2} & O_{2\times2} \\ O_{2\times1} & I_{2\times2} & O_{2\times2} & O_{2\times2} & O_{2\times2} & -I_{2\times2} \\ O_{2\times1} & O_{2\times2} & I_{2\times2} & O_{2\times2} & -I_{2\times2} & O_{2\times2} \end{pmatrix} \\ &=: \begin{pmatrix} B_{7\times7} & C_{7\times4} \\ (C^T)_{4\times7} & D_{4\times4} \end{pmatrix},\end{aligned}$$

where

$$A_{2\times2} = \bar{\psi}''_{\bar{\xi}\bar{\xi}}.$$

By properties of determinants for block matrices (cf. [24], [26], etc.), we have, at the critical point,

$$\det(\partial^2\bar{\psi}) = \det(B - CD^{-1}C^T)\det D,$$

provided the matrix D is invertible. Since $\det D = 1$, by properties of block matrix determinants again, we have, at the critical point,

$$\det(\partial^2\bar{\psi}) = \det(B - CD^{-1}C^T) = \det\begin{pmatrix} O_{1\times1} & \left(-\frac{\bar{\xi}^T}{\sqrt{\rho_0}}\right)_{1\times2} & O_{1\times2} & O_{1\times2} \\ \left(-\frac{\bar{\xi}}{\sqrt{\rho_0}}\right)_{2\times1} & A_{2\times2} & I_{2\times2} & O_{2\times2} \\ O_{2\times1} & I_{2\times2} & O_{2\times2} & -I_{2\times2} \\ O_{2\times1} & O_{2\times2} & -I_{2\times2} & O_{2\times2} \end{pmatrix}$$

$$= \det\begin{pmatrix} O_{1\times1} & \left(-\frac{\bar{\xi}^T}{\sqrt{\rho_0}}\right)_{1\times2} & O_{1\times2} & O_{1\times2} \\ \left(-\frac{\bar{\xi}}{\sqrt{\rho_0}}\right)_{2\times1} & A_{2\times2} & O_{2\times2} & O_{2\times2} \\ O_{2\times1} & O_{2\times2} & O_{2\times2} & I_{2\times2} \\ O_{2\times1} & O_{2\times2} & I_{2\times2} & O_{2\times2} \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & -\frac{\bar{\xi}_1}{\sqrt{\rho_0}} & -\frac{\bar{\xi}_2}{\sqrt{\rho_0}} \\ -\frac{\bar{\xi}_1}{\sqrt{\rho_0}} & -\frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_2^{\,2} & \frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_1\bar{\xi}_2 \\ -\frac{\bar{\xi}_2}{\sqrt{\rho_0}} & \frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_1\bar{\xi}_2 & -\frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_1^2 \end{pmatrix} = \frac{\bar{t}|\bar{\xi}|^4}{\rho_0^{\frac{5}{2}}} = 1.$$

In the last equality, we used $\bar{t}, |\bar{\xi}| = \sqrt{\rho_0}$ at the critical point, since, by (4.23) and (4.24), we have that, for $\bar{t} > 0$,

$$|\bar{\xi}| = \sqrt{\rho_0}, \quad \bar{t} = |\bar{t}| = |\bar{z}| = |\bar{w}| = |\bar{x}| = \frac{1}{\sqrt{\rho_0}}|\tilde{x}| = \sqrt{\rho_0}.$$

This gives us that $|\det \partial^2\bar{\psi}| = 1$ at the critical point.

Remark 4.9. Since we have shown $\det(\partial^2\bar{\psi}) = 1$, we have

$$(\partial^2\bar{\psi})^{-1} = \frac{1}{\det(\partial^2\bar{\psi})}\operatorname{adj}(\partial^2\bar{\psi}) = \operatorname{adj}(\partial^2\bar{\psi}).$$

Each entry of the adjugate $\operatorname{adj}(\partial^2\bar{\psi})$ is a finite linear combination of multiplications of terms of the form

$$1, \ \frac{\bar{\xi}_1}{\sqrt{\rho_0}}, \ \frac{\bar{\xi}_2}{\sqrt{\rho_0}}, \ \frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_1^2, \ \frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_2^2, \ \frac{\bar{t}}{\sqrt{\rho_0}^3}\bar{\xi}_1\bar{\xi}_2.$$

These are all $O(1)$ near the critical point, since we have $|\bar{t}|, |\bar{\xi}| \approx \sqrt{\rho_0}$. This implies that the matrix norm of $\partial^2\bar{\psi}$ is $O(1)$, and thus, we can use the method of stationary phase below easily.

Continuing with our proof, in the normal coordinates at $\alpha(\tilde{y})$, by the stationary phase argument, we have, modulo $O(\lambda^{-1})$ errors, at the critical point,

$$U^1_{\alpha,j,-}(\tilde{x}) = \frac{\lambda^6}{(2\pi)^7\sqrt{\rho_0}T}\Bigg[\left(\frac{\lambda}{2\pi}\right)^{-\frac{11}{2}} e^{i\lambda\sqrt{\rho_0}|\bar{x}|}e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\bar{\psi})}\sum_{l<l_0}\lambda^{-l}L_l a_0$$
$$+O\left(\lambda^{-l_0}\sum_{|\beta|\le 2l_0}\sup|D^\beta a_0|\right)\Bigg],$$

where a_0 is defined by

$$a_0(\bar{t},\bar{\xi},\bar{w},\bar{\eta},\bar{z},\bar{\zeta}) = \tilde{\beta}\left(\frac{\bar{\xi}}{\sqrt{\rho_0}}\right)\widehat{\mu_T^2}\left(\frac{\sqrt{\rho_0}\bar{t}}{T}\right)\tilde{q}_j\left(\sqrt{\rho_0}\bar{x},\sqrt{\rho_0}\bar{w},\lambda\frac{\bar{\eta}}{\sqrt{\rho_0}}\right)$$
$$\times\tilde{b}_{\sqrt{\rho_0}\bar{t},j}\left(\sqrt{\rho_0}\bar{w},\sqrt{\rho_0}\bar{x},\lambda\frac{\bar{\eta}}{\sqrt{\rho_0}}\right)u_0(\sqrt{\rho_0}\bar{z}),$$

$u_0(\sqrt{\rho_0}\bar{z})$ is the coordinate expression of $u_0(\sqrt{\rho_0}\bar{z},\alpha(\tilde{y}))$ in normal coordinates at $\alpha(\tilde{y})$, and the L_l are the differential operators of order at most $2l$ acting on a_0 at the critical point. Recall that we can easily control the size estimates of $\tilde{q}_j$ by e^{CT} by construction, and the size estimates of u_0 by e^{CT} due to [22, Lemma B.1]. Also, by [11] and/or [38, Lemma 11.11], the size estimtaes for $\kappa_t^*q_j^*$ are the same as those for q_j, up to e^{CT}. Thus, the remainder is

$$O\left(\lambda^{-l_0}\sum_{|\beta|\leq 2l_0}\sup|D^\beta a_0|\right) = O(\lambda^{-l_0}(\lambda^{\frac{1}{3}})^{2l_0}e^{CT}) = O(\lambda^{-\frac{l_0}{3}}e^{CT}).$$

Taking l_0 large enough, we can ignore the contribution of the remainder.

As before, by (4.23) and (4.24), at the critical point, we have that

$$\bar{t} = |\bar{z}| = |\bar{w}| = |\bar{x}| = \frac{1}{\sqrt{\rho_0}}|\tilde{x}|, \quad |\bar{\xi}| = \sqrt{\rho_0}, \quad \bar{\xi} = \frac{|\bar{\xi}|}{\bar{t}}\bar{z}, \quad \bar{z} = \bar{w} = \bar{x}.$$

This gives us that, in the geodesic normal coordinates,

$$\bar{\xi} = \frac{|\bar{\xi}|}{\bar{t}}\bar{z} = \frac{\sqrt{\rho_0}}{|\bar{x}|}\bar{x} = \sqrt{\rho_0}\frac{\tilde{x}}{|\tilde{x}|} = \sqrt{\rho_0}\frac{d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))}{|d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))|_{\tilde{g}}}.$$

We then have, modulo $O(\lambda^{-1})$ errors, that

$$\begin{aligned}
&U^1_{\alpha,j,-}(\tilde{x},\tilde{y})\\
&= \frac{\lambda^{\frac{1}{2}}}{2\pi\sqrt{2\pi}T}\frac{1}{\sqrt{\rho_0}}e^{i\lambda\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))}e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\bar{\psi})}\\
&\times\sum_{l<l_0}\lambda^{-l}L_la_0\left(\frac{|\tilde{x}|}{\sqrt{\rho_0}},\sqrt{\rho_0}d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})),\frac{\tilde{x}}{\sqrt{\rho_0}},\sqrt{\rho_0}d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})),\frac{\tilde{x}}{\sqrt{\rho_0}},\sqrt{\rho_0}d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))\right)\\
&= \frac{\lambda^{\frac{1}{2}}}{2\pi\sqrt{2\pi}T}\frac{1}{\sqrt{\rho_0}}e^{i\lambda\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))}e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\bar{\psi})}\\
&\times\sum_{l<l_0}\lambda^{-l}L_l\left(\widehat{\mu_T^2}\left(|\tilde{x}|/T\right)u_0(\tilde{x},\alpha(\tilde{y}))\tilde{b}_{|\tilde{x}|,j}(\tilde{x},\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})))\tilde{q}_j(\tilde{x},\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})))\right).
\end{aligned}$$

By the discussion in Section 2.1 and the properties of the geodesic flow, we can write

$$\begin{aligned}
\tilde{b}_{|\tilde{x}|,j}(\tilde{x},\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))) &= \tilde{q}_j(\kappa_{|\tilde{x}|}(\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))))\\
&= \tilde{q}_j(\alpha(\tilde{y}),\alpha(\tilde{y}),-\lambda d_{\tilde{y}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))).
\end{aligned}$$

Hence, modulo $O(\lambda^{-1})$ errors, for $\alpha \neq \mathrm{Id}$,

$$U^1_{\alpha,j,-}(\tilde{x},\tilde{y}) = \frac{\lambda^{\frac{1}{2}}}{2\pi\sqrt{2\pi}T} e^{i\lambda\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))} a_j(\tilde{x},\alpha(\tilde{y})),$$

where

$$a_j(\tilde{x},\alpha(\tilde{y})) = \sum_{l<l_0} \frac{1}{(\tilde{\rho}(\tilde{x},\alpha(\tilde{y})))^{\frac{1}{2}}} \lambda^{-l} L_l \Big(\widehat{\mu_T^2}(\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))/T) u_0(\tilde{x},\alpha(\tilde{y})) \\ \times \tilde{q}_j(\alpha(\tilde{y}),\alpha(\tilde{y}),-\lambda d_{\tilde{y}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})))\tilde{q}_j(\tilde{x},\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))) \Big).$$

Since we have $\tilde{\rho}(\tilde{x},\alpha(\tilde{y})) \gtrsim 1$ for $\alpha \neq \mathrm{Id}$, we have, by construction,

$$|a_j(\tilde{x},\alpha(\tilde{y}))| \leq e^{CT}.$$

Recall that the ξ-support of $q_j(x,y,\xi)$ is contained in $\{\xi : \frac{|\xi(N)|}{|\xi|_g} \approx 2^{-j}\}$ where $\xi(N) = \langle \xi^{\#}, N\rangle_{\tilde{g}}$. Note that, in geodesic normal coordinates centered at $\alpha(\tilde{y})$, we have, for $\tilde{\rho}(\tilde{x},\tilde{z})$,

$$|d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))|_{\tilde{g}} = 1 = |d_{\tilde{z}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))|_{\tilde{g}}.$$

By the support properties of $\tilde{q}_j(\tilde{x},\tilde{x},\lambda d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y})))$, $a_j(\tilde{x},\alpha(\tilde{y}))$ is supported where

$$|d_{\tilde{x}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))(\tilde{N})| \approx 2^{-j}, \quad \text{if } \tilde{x},\tilde{y} \in \tilde{\gamma}.$$

Here, $\tilde{N}$ is a unit normal vector to $\tilde{\gamma}$, since α is an isometry. We also observe that $\alpha_*(\tilde{N})$ is normal to $\alpha \circ \tilde{\gamma}$. By the support properties of

$$\tilde{q}_j(\alpha(\tilde{y}),\alpha(\tilde{y}),-\lambda d_{\tilde{y}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))),$$

for each $\alpha \neq \mathrm{Id}$, if $\tilde{y} = \alpha \circ \tilde{\gamma}(s)$ for $|s| \ll 1$, then $a_j(\tilde{x},\alpha(\tilde{y}))$ is supported where

$$|d_{\tilde{y}}\tilde{\rho}(\tilde{x},\alpha(\tilde{y}))(\alpha_*(\tilde{N}))| = |d_{\tilde{y}}\tilde{\rho}(\tilde{x},\alpha \circ \tilde{\gamma}(s))(\alpha_*(\tilde{N}))| \approx 2^{-j}.$$

This completes the proof. □

We next consider the support properties of the amplitude of $U_{\alpha,j,\pm}$. Let $\tilde{\rho}_\alpha(\tilde{x},\tilde{y}) = \tilde{\rho}(\tilde{x},\alpha(\tilde{y}))$ for $\alpha \neq \mathrm{Id}$. Fix $r_0, s_0 \in [0,1]$ so that

$$|d_{\tilde{x}}\tilde{\rho}_\alpha(\tilde{\gamma}(r_0),\tilde{\gamma}(s_0))(\tilde{N})| \approx 2^{-j} \quad \text{and} \quad |d_{\tilde{y}}\tilde{\rho}_\alpha(\tilde{\gamma}(r_0),\tilde{\gamma}(s_0))(\alpha_*(\tilde{N}))| \approx 2^{-j}.$$

We can assume such r_0 and s_0 exist, or otherwise, by the above proposition, we have $U_{\alpha,j,\pm} = O(\lambda^{-N})$ for any $N = 1,2,3,\cdots$. Using a partition of unity, we may assume that

$$|d_{\tilde{x}}\tilde{\rho}_\alpha(\tilde{\gamma}(r),\tilde{\gamma}(s))(\tilde{N})| \approx 2^{-j} \quad \text{and} \quad |d_{\tilde{y}}\tilde{\rho}_\alpha(\tilde{\gamma}(r),\tilde{\gamma}(s))(\alpha_*(\tilde{N}))| \approx 2^{-j}.$$

happens only near (r_0, s_0). By Proposition 4.8, we may assume that $U_{\alpha,j,\pm}(\tilde{\gamma}(r), \tilde{\gamma}(s))$ is supported where

$$|d_{\tilde{x}}\tilde{\rho}_\alpha(\tilde{\gamma}(r), \tilde{\gamma}(s))(\tilde{N})|, \quad |d_{\tilde{y}}\tilde{\rho}_\alpha(\tilde{\gamma}(r), \tilde{\gamma}(s))(\alpha_*(\tilde{N}))| \in [2^{-j-1}, 2^{-j+1}].$$

Suppose $r, s \in [0, \epsilon_1] = I$ for some small $\epsilon_1 > 0$, and write $I = \cup_k I_k$, where $\{I_k\}_k$ is a collection of almost disjoint intervals with $|I_k| \approx e^{-CT}$ for some large $C > 0$. Let r_0 and s_0 be fixed points in a subinterval I_k and $I_{k'}$, respectively. We want to show the following.

Lemma 4.10. Suppose $|d_{\tilde{x}}\tilde{\rho}_\alpha(\tilde{\gamma}(r_0), \tilde{\gamma}(s_0))(\tilde{N})| \in [C_1 2^{-j}, C_2 2^{-j}]$. Then, choosing $C > 0$ sufficiently large with $|I_k| \approx e^{-CT}$, there exists a uniform constant $\tilde{C} > 0$ such that, for r and r_0 in a same subinterval I_k,

$$|d_{\tilde{x}}\tilde{\rho}_\alpha(\tilde{\gamma}(r), \tilde{\gamma}(s_0))(\tilde{N})| \notin [C_1 2^{-j}, C_2 2^{-j}], \quad \text{whenever } |r - r_0| \geq \tilde{C} 2^{-j}.$$

Similarly, if $|d_{\tilde{y}}\tilde{\rho}_\alpha(\tilde{\gamma}(r_0), \tilde{\gamma}(s_0))(\alpha_*(\tilde{N}))| \in [C_1 2^{-j}, C_2 2^{-j}]$, then, choosing $C > 0$ sufficiently large with $|I_k| \approx e^{-CT}$, there exists a uniform constant $\tilde{C} > 0$ such that, for s and s_0 in a same subinterval I_k,

$$|d_{\tilde{y}}\tilde{\rho}_\alpha(\tilde{\gamma}(r_0), \tilde{\gamma}(s))(\alpha_*(\tilde{N}))| \notin [C_1 2^{-j}, C_2 2^{-j}], \quad \text{whenever } |s - s_0| \geq \tilde{C} 2^{-j}.$$

Before we prove this lemma, we review some basic properties of the Hessian operator $\mathcal{H}_r$ in [23]: Suppose (M, g) is an n-dimensional Riemannian manifold, U is a normal neighborhood of a point $p \in M$, and $r : U \to \mathbb{R}$ is the radial distance function from the point p defined by

$$r(x) = \sqrt{(x_1)^2 + \cdots + (x_n)^2}, \tag{4.25}$$

where (x_i) are normal coordinates on U centered at p. We also define the radial vector field on $U \setminus \{p\}$, denoted by ∂_r, as

$$\partial_r = \sum_{i=1}^{n} \frac{x_i}{r(x)} \frac{\partial}{\partial x_i} = \operatorname{grad} r$$

(cf. [23, Corollary 6.10]), where $\operatorname{grad} f = (d_{\tilde{x}} f)^\#$ is the Riemannian gradient of f and $\#$ is the musical isomorphism sharp. Note that the radial vector field ∂_r is a unit vector field. Then, the $(1,1)$-tensor field $\mathcal{H}_r = \nabla(\partial_r)$, defined by

$$\mathcal{H}_r(w) = \nabla_w \partial_r, \quad \text{for all } w \in TM|_{U \setminus \{p\}},$$

is called the Hessian operator of r, where ∇ is the Levi-Civita connenction. By [23, Lemma 11.1], $\mathcal{H}_r$ is self-adjoint, $\mathcal{H}_r(\partial_r) \equiv 0$, and the restriction of $\mathcal{H}_r$ to vectors tangents to a level set of r is equal to the shape operator of the level set associated with the normal vector field $-\partial_r$.

Proof of Lemma 4.10. We prove the second case in this lemma. Similar arguments will work on the first one. In this proof, ∇ denotes the Levi-Civita connection. We write $\tilde{\rho}_0(\tilde{y}) = \tilde{\rho}_{0,\alpha}(\tilde{y}) = \tilde{\rho}_0(\tilde{\gamma}(r_0), \alpha(\tilde{y}))$ so that $\tilde{\rho}_0$ is the distance function as in (4.25), since r_0 is fixed. We also write the radial vector field as $\partial_{\tilde{\rho}_0} = \frac{\partial}{\partial \tilde{\rho}_0}$. Set

$$h(s) = \langle \mathrm{grad}_{\tilde{y}} \tilde{\rho}_0(\tilde{\gamma}(s)), \alpha_*(\tilde{N}) \rangle_{\tilde{g}},$$

where $\mathrm{grad}_{\tilde{y}}$ is the gradient for $\tilde{y}$. By assumption, we have $|h(s_0)| \in [C_1 2^{-j}, C_2 2^{-j}]$. We will work in geodesic normal coordinates centered at $\tilde{\gamma}(r_0)$.

We want to show that $|h'(s_0)| \approx 1$. Let $\tilde{\eta} = \tilde{\eta}_\alpha = \alpha \circ \tilde{\gamma}$. We then have that

$$\frac{d}{ds}(h(s)) = \langle \nabla_{\dot{\tilde{\eta}}(s)} \mathrm{grad}_{\tilde{y}} \tilde{\rho}_0(\tilde{\gamma}(s)), \alpha_*(\tilde{N}) \rangle_{\tilde{g}} + \langle \mathrm{grad}_{\tilde{y}} \tilde{\rho}_0(\tilde{\gamma}(s)), \nabla_{\dot{\tilde{\eta}}(s)} \alpha_*(\tilde{N}) \rangle_{\tilde{g}}. \quad (4.26)$$

For the first term in the right hand side, note that

$$\langle \nabla_{\dot{\tilde{\eta}}(s_0)} \mathrm{grad}_{\tilde{y}} \tilde{\rho}_0(\tilde{\gamma}(s_0)), \alpha_*(\tilde{N}) \rangle_{\tilde{g}} = \langle \mathcal{H}_{\tilde{\rho}_0}(\dot{\tilde{\eta}}(s_0)), \alpha_*(\tilde{N}) \rangle_{\tilde{g}},$$

where $\mathcal{H}_{\tilde{\rho}_0}$ is the Hessian operator of $\tilde{\rho}_0$.

Before going further, we show that

$$\left| \dot{\tilde{\eta}}(s_0) - \frac{\partial}{\partial \tilde{\rho}_0} \right|_{\tilde{g}} \approx 2^{-j}. \quad (4.27)$$

Indeed, we may assume that $|\dot{\tilde{\eta}}|_{\tilde{g}} = 1$ by arc-length parametrization, if necessary. Recall that $\partial_{\tilde{\rho}_0}$ is a unit vector field. Let θ be the angle between $\partial_{\tilde{\rho}_0}$ and $\alpha_*(\tilde{N})$. Note that

$$|h(s_0)| = |\langle \partial_{\tilde{\rho}_0}, \alpha_*(\tilde{N}) \rangle_{\tilde{g}}| = |\cos\theta| \approx 2^{-j},$$

since $|\partial_{\tilde{\rho}_0}|_{\tilde{g}} = |\alpha_*(\tilde{N})|_{\tilde{g}} = 1$ by the fact that the radial vector field is a unit vector field and α is an isometry. The angle between $\dot{\tilde{\eta}}(s_0)$ and $\partial_{\tilde{\rho}_0}$ is then $\frac{\pi}{2} - \theta$ if $0 \le \theta \le \frac{\pi}{2}$ or $\theta - \frac{\pi}{2}$ if $\frac{\pi}{2} \le \theta \le \pi$. In any case, we have that

$$\begin{aligned} \left| \dot{\tilde{\eta}}(s_0) - \frac{\partial}{\partial \tilde{\rho}_0} \right|_{\tilde{g}}^2 &= 2 - 2\cos\left(\frac{\pi}{2} - \theta\right) = 2 - 2\sin\theta \\ &= 2 - 2\sqrt{1 - |\cos\theta|^2} \approx 2 - 2\left(1 - \frac{1}{2} \cdot 2^{-2j}\right) = 2^{-2j}, \end{aligned}$$

which proves (4.27).

Since $\mathcal{H}_{\tilde{\rho}_0}(\partial_{\tilde{\rho}_0}) \equiv 0$, by (4.27), we have

$$|\mathcal{H}_{\tilde{\rho}_0}(\dot{\tilde{\eta}}(s_0))|_{\tilde{g}} = \left| \mathcal{H}_{\tilde{\rho}_0}\left(\dot{\tilde{\eta}}(s_0) - \frac{\partial}{\partial \tilde{\rho}_0} \right) \right|_{\tilde{g}} \le \|\mathcal{H}_{\tilde{\rho}_0}\| \left| \dot{\tilde{\eta}}(s_0) - \frac{\partial}{\partial \tilde{\rho}_0} \right|_{\tilde{g}} \approx 2^{-j} \|\mathcal{H}_{\tilde{\rho}_0}\|, \quad (4.28)$$

where $\|\mathcal{H}_{\tilde{\rho}_0}\|$ denotes the operator norm of $\mathcal{H}_{\tilde{\rho}_0}$ and $\frac{\partial}{\partial \tilde{\rho}_0}$ is a radial vector at $\alpha \circ \tilde{\gamma}(s_0)$. For the first term in (4.26) on the right hand side, we continue to show that $\|\mathcal{H}_{\tilde{\rho}_0}\| \lesssim 1$. Let

$$s_c(t) = \begin{cases} t, & \text{if } c = 0, \\ R \sin \frac{t}{R}, & \text{if } c = \frac{1}{R^2} > 0, \\ R \sinh \frac{t}{R}, & \text{if } c = -\frac{1}{R^2}. \end{cases}$$

If we call the curvature of our manifold κ, then we can assume $-1 \le \kappa \le 0$ by the assumption in Theorem 1.2. By the Hessian comparison (cf. Theorem 11.7 in [23]), we have that

$$\frac{1}{\tilde{\rho}_0}\pi_{\tilde{\rho}_0} = \frac{s_0'(\rho)}{s_0(\rho)}\pi_{\tilde{\rho}_0} \le \mathcal{H}_{\tilde{\rho}_0} \le \frac{s_{-1}'(\rho)}{s_{-1}(\rho)}\pi_{\tilde{\rho}_0} = \coth(\tilde{\rho}_0)\pi_{\tilde{\rho}_0}, \tag{4.29}$$

where $\pi_{\tilde{\rho}_0}$ is the orthogonal projection onto the tangent space of the level set of $\tilde{\rho}_0$ as in [23]. Here, $A \le B$ means $\langle Av, v\rangle_{\tilde{g}} \le \langle Bv, v\rangle_{\tilde{g}}$ for all vectors v. From the second inequality in (4.29), we have

$$\langle \mathcal{H}_{\tilde{\rho}_0} v, v\rangle_{\tilde{g}} \le \coth(\tilde{\rho}_0)\langle \pi_{\tilde{\rho}_0} v, v\rangle_{\tilde{g}} \lesssim \langle \pi_{\tilde{\rho}_0} v, \pi_{\tilde{\rho}_0} v + (v - \pi_{\tilde{\rho}_0} v)\rangle_{\tilde{g}} = |\pi_{\tilde{\rho}_0} v|_{\tilde{g}}^2 \le |v|_{\tilde{g}}^2.$$

Here, we used the fact that $\coth(\rho) = \frac{e^\rho + e^{-\rho}}{e^\rho - e^{-\rho}}$ with $1 \lesssim \tilde{\rho}_0 \le T$. We can make the same argument for the first inequality, and so, in summary, we have

$$0 \le \frac{1}{\tilde{\rho}_0}|\pi_{\tilde{\rho}_0} v|_{\tilde{g}}^2 \le \langle \mathcal{H}_{\tilde{\rho}_0} v, v\rangle_{\tilde{g}} \lesssim |v|_{\tilde{g}}^2,$$

from which it follows that $0 \le |\langle \mathcal{H}_{\tilde{\rho}_0} v, v\rangle| \lesssim |v|_{\tilde{g}}^2$. Since $\mathcal{H}_{\tilde{\rho}_0}$ is self-adjoint (cf. Lemma 11.1 in [23]), what we have shown is

$$\|\mathcal{H}_{\tilde{\rho}_0}\| = \sup_{|v|_{\tilde{g}}=1} |\langle \mathcal{H}_{\tilde{\rho}_0} v, v\rangle_{\tilde{g}}| \lesssim \sup_{|v|_{\tilde{g}}=1} |v|_{\tilde{g}}^2 = 1.$$

Combining this with (4.28), (4.26) is translated into

$$\begin{aligned} \frac{d}{ds}(h(s))\Big|_{s=s_0} &= O(2^{-j}) + \langle \mathrm{grad}_{\tilde{g}}\tilde{\rho}_0(\tilde{\gamma}(s_0)), \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}} \\ &= O(2^{-j}) + \left\langle \frac{\partial}{\partial \tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\right\rangle_{\tilde{g}}. \end{aligned} \tag{4.30}$$

For the second term in (4.26), we first note that

$$\nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s) = \langle \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s), \alpha_*(\tilde{N})\rangle_{\tilde{g}}\alpha_*(\tilde{N}).$$

Indeed, since $\tilde{\eta}$ can be parametrized by arc length, we have

$$\langle \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s), \dot{\tilde{\eta}}(s)\rangle_{\tilde{g}} = \frac{1}{2}\nabla_{\dot{\tilde{\eta}}(s)}(\langle \dot{\tilde{\eta}}(s), \dot{\tilde{\eta}}(s)\rangle_{\tilde{g}}) = \frac{1}{2}\nabla_{\dot{\tilde{\eta}}(s)}1 = 0,$$

which in turn implies that

$$\begin{aligned} \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s) &= \langle \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s), \dot{\tilde{\eta}}(s)\rangle_{\tilde{g}}\dot{\tilde{\eta}}(s) + \langle \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s), \alpha_*(\tilde{N})\rangle_{\tilde{g}}\alpha_*(\tilde{N}) \\ &= \langle \nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s), \alpha_*(\tilde{N})\rangle_{\tilde{g}}\alpha_*(\tilde{N}). \end{aligned}$$

Since $\alpha_*(\tilde{N})$ and $\dot{\tilde{\eta}}(s)$ are orthogonal (at $\tilde{\eta}(s_0)$), we have

$$\begin{aligned} |\nabla_{\dot{\tilde{\eta}}(s_0)}\dot{\tilde{\eta}}(s_0)|_{\tilde{g}} &= |\langle \nabla_{\dot{\tilde{\eta}}(s_0)}\dot{\tilde{\eta}}(s_0), \alpha_*(\tilde{N})\rangle_{\tilde{g}}| \\ &= |\langle \mathrm{II}(\dot{\tilde{\eta}}(s_0), \dot{\tilde{\eta}}(s_0)), \alpha_*(\tilde{N})\rangle_{\tilde{g}}| \\ &= |\langle \dot{\tilde{\eta}}(s_0), W_{\alpha_*(\tilde{N})}(\dot{\tilde{\eta}}(s_0))\rangle_{\tilde{g}}| \\ &= |\langle \dot{\tilde{\eta}}(s_0), \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}}| \approx \left| \left\langle \frac{\partial}{\partial \tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N}) \right\rangle_{\tilde{g}} \right|, \end{aligned}$$

where the map W_N is the Weingarten map in the direction of N and II is the second fundamental form of $\alpha(\gamma)$ in the universal cover $(\mathbb{R}^2, \tilde{g})$. In the last approximation, we used (4.27). Indeed, we know that

$$\langle \dot{\tilde{\eta}}(s_0), \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}} = \langle \partial_{\tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}} + \langle \dot{\tilde{\eta}}(s_0) - \partial_{\tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}},$$

and

$$|\langle \dot{\tilde{\eta}}(s_0) - \partial_{\tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(\tilde{N})\rangle_{\tilde{g}}| \lesssim 2^{-j} \ll 1,$$

when $1 \ll \lfloor \log_2 \lambda^{\frac{1}{3}-\epsilon} \rfloor \le j \le J$.

Since we know $|\nabla_{\dot{\tilde{\eta}}(s)}\dot{\tilde{\eta}}(s)|_{\tilde{g}} \approx 1$ by the assumption on the curvature of the given curve γ, (1.6), we have

$$\left| \left\langle \frac{\partial}{\partial \tilde{\rho}_0}, \nabla_{\dot{\tilde{\eta}}(s_0)}\alpha_*(N) \right\rangle_{\tilde{g}} \right| \approx |\nabla_{\dot{\tilde{\eta}}(s_0)}\dot{\tilde{\eta}}(s_0)|_{\tilde{g}} \approx 1.$$

Combining this with (4.30), we have that $|h'(s_0)| \approx 1$.

By Taylor's formula,

$$h(s) = h(s_0) + h'(s_0)(s - s_0) + O(|h''|(s - s_0)^2).$$

As a consequence of [22, Lemma B.2], there exists $C' > 0$ such that

$$h(s) = h(s_0) + h'(s_0)(s - s_0) + O(e^{C'T}(s - s_0)^2).$$

Since we are assuming $|s - s_0| \approx e^{-CT}$, for a sufficiently large $C > 0$, we have

$$h(s) = h(s_0) + (h'(s_0) + O(e^{(C'-C)T}))(s - s_0) \approx h(s_0) \pm |h'(s_0)|(s - s_0).$$

Since we have shown $|h'(s_0)| \approx 1$, there exists a $\tilde{C} > 0$ such that if $|s - s_0| \ge \tilde{C}2^{-j}$, then we have $|h(s)| \notin [C_1 2^{-j}, C_2 2^{-j}]$, which proves the lemma. □

By Lemma 4.10, we have, modulo $O(\lambda^{-1})$ errors, for $r \in I_k$, $s \in I_{k'}$,

$$\begin{aligned} &U_{\alpha,j,\pm}(\tilde{\gamma}(r), \tilde{\gamma}(s)) \\ &= \begin{cases} \frac{\lambda^{\frac{1}{2}}}{T} e^{i\lambda\tilde{\rho}(\tilde{\gamma}(r), \alpha(\tilde{\gamma}(s)))} \tilde{a}_{\alpha,j}(r,s), & \text{if } |r - r_0| \lesssim 2^{-j} \text{ and } |s - s_0| \lesssim 2^{-j}, \\ O(\lambda^{-N}), & \text{otherwise,} \end{cases} \end{aligned} \tag{4.31}$$

where $|\tilde{a}_{\alpha,j}(r,s)| \le Ce^{CT}$. Here, there is at most one cube of sidelength $C2^{-j}$ in $(r,s) \in I_k \times I_{k'} \subset I \times I = [0, \epsilon_1]^2$ for small $\epsilon_1 > 0$ such that the amplitude $\tilde{a}_{\alpha,j}(r,s)$ is nonzero, and (r_0, s_0) is the center of the cube $I_k \times I_{k'}$.

Remark 4.11. We observe that the way to find support properties of $U_{\alpha,j}$ here is similar to that of $K_j,+$ or K_J in the previous chapter. We used the assumption of nonvanishing geodesic curvatures on γ in both cases. We also used the properties of the Hessian operator and the Taylor expansion here, whereas used the properties of the solution to the eikonal equation φ and the mean value theorem there.

It follows from (4.31) that

$$\int |U_{\alpha,j,\pm}(\tilde{\gamma}(r),\tilde{\gamma}(s))|\,dr = \sum_k \int_{I_k} |U_{\alpha,j,\pm}(\tilde{\gamma}(r),\tilde{\gamma}(s))|\,dr \lesssim e^{C'T}\frac{\lambda^{\frac{1}{2}}}{T}2^{-j}.$$

Here, $e^{C'T}$ comes from the fact $|\tilde{a}_{\alpha,j}(\tilde{\gamma}(r),\tilde{\gamma}(s))| \le e^{C''T}$ and the fact that the number of $\{I_k\}$ is e^{CT} up to some constant, and 2^{-j} comes from the support property $|r-r_0| \lesssim 2^{-j}$ in I_k for some k. Similarly, we also have $\int |U_{\alpha,j,\pm}(\tilde{\gamma}(r),\tilde{\gamma}(s))|\,ds \lesssim e^{C'T}\frac{\lambda^{\frac{1}{2}}}{T}2^{-j}$. By Young's inequality, we have

$$\left\| \int U_{\alpha,j,\pm}(\tilde{\gamma}(\cdot),\tilde{\gamma}(s))f(s)\,ds \right\|_2 \lesssim \frac{\lambda^{\frac{1}{2}}}{T}e^{CT}2^{-j}\|f\|_2.$$

By (4.31), we also have that

$$\left\| \int U_{\alpha,j,\pm}(\tilde{\gamma}(\cdot),\tilde{\gamma}(s))f(s)\,ds \right\|_\infty \lesssim \frac{\lambda^{\frac{1}{2}}}{T}e^{CT}\|f\|_1.$$

By interpolation, we obtain

$$\left\| \int U_{\alpha,j,\pm}(\tilde{\gamma}(\cdot),\tilde{\gamma}(s))f(s)\,ds \right\|_p \lesssim \frac{\lambda^{\frac{1}{2}}}{T}e^{CT}(2^{-j})^{\frac{2}{p}}\|f\|_{p'}, \quad 2 \le p \le \infty,$$

which proves (4.22). This completes the proof.

Chapter 5

Proof of Corollary 1.3

Let $P = \sqrt{-\Delta_g}$, $\chi \in \mathcal{S}(\mathbb{R})$, and γ be as above. In this chapter, we heavily borrow arguments from Xi and Zhang [37], which was also motivated by Bourgain [10] and Sogge [33]. We first have an analogue of [37, Lemma 1].

Lemma 5.1. [Lemma 1 in [37]] We set $\lambda^{-1} \leq l \leq 1$. Let γ_l be a fixed subsegment of γ with length l. We then have that

$$\|\chi(\lambda - P)f\|_{L^2(\gamma_l)} \lesssim \lambda^{\frac{1}{4}} l^{\frac{1}{4}} \|f\|_{L^2(M)}.$$

Remark 5.2.

1. In fact, [37, Lemma 1] focuses on the case where γ is a geodesic segment, but the argument there applies equally well to any curve segment, by using $\rho(\gamma(r), \gamma(s)) \approx |r - s|$, which comes from $|r - s| \ll 1$ by a partition of unity if necessary.

2. As observed in [37, Remark 1], a similar argument gives the same estimate for $\chi(T_0(\lambda - P))$ if $T_0 \geq 1$. Indeed, the proof of Lemma 5.1 follows from the analsis of the kernel

$$\left[\int \widehat{\chi^2}(t) e^{it\lambda} e^{-itP}\, dt\right] (\gamma(r), \gamma(s)).$$

For the operator $\chi(T_0(\lambda - P))$, we consider the kernel

$$\left[\frac{1}{T_0} \int \widehat{\chi^2}(t/T_0) e^{it\lambda} e^{-itP}\, dt\right] (\gamma(r), \gamma(s)).$$

Since $\widehat{\chi^2}(\cdot/T_0)$ is supported in $|t| \leq 2\epsilon_0 T_0$ by (2.1), we split the interval

$$[-2\epsilon_0 T_0, 2\epsilon_0 T_0]$$

into $O(T_0)$ many subintervals with sidelength 1. Each piece of the kernel over a subinterval of size 1 gives us the same bound as in Lemma 5.1, by the fact that $\|e^{-it_0 P} f\|_{L^2(M)} = \|f\|_{L^2(M)}$ for any fixed t_0. If we sum up the $O(T_0)$ pieces from the partition, we have the same bound as in the lemma for the operator $\chi(T_0(\lambda - P))$.

Let T be as in (4.1). We show a weak L^4 estimate.

Proposition 5.3. Suppose (M,g) is a 2-dimensional compact Riemannian manifold with nonpositive curvatures. Then, for $\lambda \gg 1$, we have

$$\|\chi(T(\lambda - P))\|_{L^2(M)\to L^{4,\infty}(\gamma)} \lesssim \frac{\lambda^{\frac{1}{4}}}{(\log \lambda)^{\frac{1}{4}}}.$$

To show this, we will need a result from Bérard [3].

Lemma 5.4 ([3]). Let (M,g) be as above. Then there exists a constant $C = C(M,g)$ so that, for $T_0 \geq 1$ and $\lambda \gg 1$, we have that

$$|\chi^2(T_0(\lambda - P))(x,y)| \leq C\left[T_0^{-1}\left(\frac{\lambda}{\rho(x,y)}\right)^{\frac{1}{2}} + \lambda^{\frac{1}{2}}e^{CT}\right].$$

We now show Proposition 5.3.

Proof of Proposition 5.3. Assuming $\|f\|_{L^2(M)} = 1$, it suffices to show that

$$|\{x\in\gamma : |\chi(T(\lambda-P))f(x)| > \alpha\}| \leq C\alpha^{-4}\lambda(\log\lambda)^{-1}.$$

By the Chebyshev inequality and Theorem 1.2, we have

$$|\{x\in\gamma : |\chi(T(\lambda-P))f(x)| > \alpha\}| \leq \alpha^{-2}\int_\gamma |\chi(T(\lambda-P))f|^2\,ds \leq \alpha^{-2}\lambda^{\frac{1}{3}}(\log\lambda)^{-1}.$$

Note that, for large λ,

$$\alpha^{-2}\lambda^{\frac{1}{3}}(\log\lambda)^{-1} \leq \alpha^{-4}\lambda(\log\lambda)^{-1}, \quad \text{if } \alpha^2 \leq \lambda^{\frac{2}{3}}, \quad \text{i.e., } \alpha \leq \lambda^{\frac{1}{3}}.$$

We are left to show that, for $\|f\|_{L^2(M)} = 1$,

$$|\{x\in\gamma : |\chi(T(\lambda-P))f(x)| > \alpha\}| \leq C\alpha^{-4}\lambda(\log\lambda)^{-1}, \quad \text{when } \alpha \geq \lambda^{\frac{1}{3}}. \tag{5.1}$$

We set

$$A = A_\alpha = \{x\in\gamma : |\chi(T(\lambda-P))f(x)| > \alpha\}, \quad \text{and} \quad r = \lambda\alpha^{-4}(\log\lambda)^{-2}.$$

We consider a disjoint union $A = \cup_j A_j$, where $|A_j| \approx r$. Replacing A by a set of proportional measure, we may assume that $\operatorname{dist}(A_j, A_k) > C_1 r$, when $j \neq k$ for some $C_1 > 0$, which will be specified later.

Let $T_\lambda = \chi(T(\lambda - P)) : L^2(M) \to L^2(\gamma)$, and, for $x \in \gamma$, let

$$\psi_\lambda(x) = \begin{cases} \frac{T_\lambda f(x)}{|T_\lambda f(x)|}, & \text{if } T_\lambda f(x) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

We also write

$$S_\lambda = T_\lambda T_\lambda^*, \quad \text{and } a_j = \psi_\lambda \mathbb{1}_{A_j}.$$

By the Chebyshev inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\alpha|A| \le \left|\int_\gamma T_\lambda f \overline{\psi_\lambda \mathbb{1}_A}\, ds\right| &\le \left|\int_\gamma \sum_j T_\lambda f \overline{a_j}\, ds\right| \\ &= \left|\int_M \sum_j \overline{T_\lambda^* a_j} f\, dV_g\right| \le \left(\int_M |\sum_j T_\lambda^* a_j|^2\, dV_g\right)^{\frac{1}{2}}.\end{aligned}$$

We can then write

$$\alpha^2|A|^2 \le I + II,$$

where

$$I = \sum_j \int_M |T_\lambda^* a_j|^2\, dV_g, \quad II = \sum_{j\ne k} \int_\gamma S_\lambda a_j \overline{a_k}\, ds.$$

By duality and Remark 5.2, we have that

$$I \le C r^{\frac{1}{2}} \lambda^{\frac{1}{2}} \sum_j \int_\gamma |a_j|^2\, ds = C r^{\frac{1}{2}} \lambda^{\frac{1}{2}} |A| = C\lambda\alpha^{-2}(\log\lambda)^{-1}|A|.$$

For II, by Lemma 5.4, we note that the kernel $K_\lambda(s,s')$ of S_λ satisfies

$$|K_\lambda(s,s')| \le C\left[\frac{1}{T}\left(\frac{\lambda}{|s-s'|}\right)^{\frac{1}{2}} + \lambda^{\frac{1}{2}} e^{CT}\right] = C\left[\frac{1}{c_0\log\lambda}\left(\frac{\lambda}{|s-s'|}\right)^{\frac{1}{2}} + \lambda^{\frac{1}{2}+Cc_0}\right],$$

which in turn implies that

$$II \le C\left[\frac{1}{c_0\log\lambda}\left(\frac{\lambda}{C_1 r}\right)^{\frac{1}{2}} + \lambda^{\frac{1}{2}+Cc_0}\right] \sum_{j\ne k} \|a_j\|_{L^1}\|a_k\|_{L^1} \le \left[\frac{C}{c_0 C_1^{\frac{1}{2}}}\alpha^2 + C\lambda^{\frac{1}{2}+Cc_0}\right]|A|^2.$$

We now take c_0 to be sufficiently small so that $C\lambda^{\frac{1}{2}+Cc_0} \le \frac{1}{4}\lambda^{\frac{2}{3}} \le \frac{1}{4}\alpha^2$, since $\lambda \gg 1$ and $\alpha \ge \lambda^{\frac{1}{3}}$. Given the small $c_0 > 0$, we take $C_1 \gg 1$ so that $\frac{C}{c_0 C_1^{\frac{1}{2}}} \le \frac{1}{4}$. It then follows that

$$II \le \frac{1}{2}\alpha^2|A|^2.$$

Putting these all together, we have that

$$\alpha^2|A|^2 \le I + II \le C\lambda\alpha^{-2}(\log\lambda)^{-1}|A| + \frac{1}{2}\alpha^2|A|^2,$$

and thus,

$$|A| \le C\lambda\alpha^{-4}(\log\lambda)^{-1}, \quad \text{if } \alpha \ge \lambda^{\frac{1}{3}},$$

which proves (5.1). This completes the proof of Proposition 5.3. ∎

We are now ready to prove Corollary 1.3. We first recall a special case of a result in Bak and Seeger [2].

Lemma 5.5 ([2])**.** Suppose (M, g) is any 2-dimensional Riemannian manifold. If $\gamma \subset M$ is a curve segment in M, then

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(P)f\|_{L^{4,2}(\gamma)} \lesssim \lambda^{\frac{1}{4}}\|f\|_{L^2(M)}, \quad \lambda \geq 1.$$

We recall some properties of the Lorentz space $L^{p,q}(\gamma)$ (see also Grafakos [16], etc.). First, for a function u on M, the corresponding distribution function $d_u(\alpha)$ with respect to γ is defined by

$$d_u(\alpha) = |\{x \in \gamma : |u(x)| > \alpha\}|, \quad \alpha > 0.$$

The function u^* is the nondecreasing rearrangement of u on γ, defined by

$$u^*(t) = \inf\{\alpha : d_u(\alpha) \leq t\}, \quad t \geq 0.$$

For $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}(\gamma)$ is then

$$L^{p,q}(\gamma) = \left\{ u : \|u\|_{L^{p,q}(\gamma)} := \left(\frac{q}{p} \int_0^\infty [t^{\frac{1}{p}} u^*(t)]^q \, \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is also known that

$$\|\cdot\|_{L^{p,p}(\gamma)} = \|\cdot\|_{L^p(\gamma)}, \quad \text{and } \sup_{t>0} t^{\frac{1}{p}} u^*(t) = \sup_{\alpha>0} \alpha[d_u(\alpha)]^{\frac{1}{p}}.$$

We now take $u = \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)f$ with $\|f\|_{L^2(M)} = 1$. By Proposition 5.3, we have that

$$\sup_{t>0} t^{\frac{1}{4}} u^*(t) \lesssim \|u\|_{L^{4,\infty}} \lesssim \frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{4}}}. \tag{5.2}$$

Since $\mathbb{1}_{[\lambda,\lambda+1]}(P)u = u$, by Lemma 5.5, we have that

$$\|u\|_{L^{4,2}(\gamma)} \lesssim \lambda^{\frac{1}{4}}\|u\|_{L^2(M)} \lesssim \lambda^{\frac{1}{4}}. \tag{5.3}$$

By (5.2) and (5.3), we have

$$\begin{aligned} \|u\|_{L^4(\gamma)} &= \left(\int_0^\infty [t^{\frac{1}{4}} u^*(t)]^4 \, \frac{dt}{t} \right)^{\frac{1}{4}} \\ &\lesssim (\sup_{t>0} t^{\frac{1}{4}} u^*(t))^{\frac{1}{2}} \|u\|_{L^{4,2}(\gamma)}^{\frac{1}{2}} \lesssim \left(\frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{4}}} \right)^{\frac{1}{2}} \lambda^{\frac{1}{8}} = \frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{8}}}. \end{aligned}$$

This completes the proof.

Chapter 6

Future Work

6.1 Higher-dimensional analogues of Theorem 1.2 and Corollary 1.3

We have talked about eigenfunction restriction estimates for curves with nonvanishing geodesic curvatures when $\dim M = 2$. In fact, there is a known universal estimate for higher dimensional analogues of Theorem 1.1.

Theorem 6.1 (Theorem 1.4 in [21])**.** Let (M, g) be a smooth compact Riemannian manifold of dimension $d \geq 2$ and Σ be a smooth submanifold of dimension $d-1$. Suppose that the second fundamental form of Σ is (positive or negative) definite. Then, we have

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\Sigma)} \leq C\lambda^{\frac{d-1}{3}-\frac{2d-3}{3p}}, \quad 2 \leq p \leq \frac{2d}{d-1}.$$

We note that, when $d = 2$, the estimate in Theorem 6.1 is the same as in the one in Theorem 1.1 with $p = 2$. We want to find a logarithmic improved estimate of this estimate.

Conjecture 6.2. Let (M, g) be a d-dimensional compact Riemannian manifold with nonpositive sectional curvatures, and Σ be a hypersurface in M, where $d \geq 2$. If the second fundamental form of Σ is (positive or negative) definite, then there exists a uniform constant $C_p > 0$ such that

$$\|\mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})f\|_{L^p(\Sigma)} \leq C_p \frac{\lambda^{\frac{d-1}{3}-\frac{2d-3}{3p}}}{(\log\lambda)^{\frac{1}{2}}}\|f\|_{L^2(M)}, \quad \lambda \gg 1, \quad 2 \leq p < \frac{2d}{d-1},$$

where $C_p \to \infty$ as $p \to 3$.

If $d \geq 2$, the term $\lambda^{\frac{1}{2}}$ in Proposition 4.8 is replaced by $\lambda^{\frac{d-1}{2}}$ by the stationary phase argument in the proposition. When $d \geq 4$, it is hard to control the term $\lambda^{\frac{d-1}{2}}$. The $d = 3$ case may be easier, but it would be still difficult to consider the case $p = 2$ by a reason similar to $d \geq 4$. Fortunately, when $d = 3$, the case $2 < p < \frac{2d}{d-1} = 3$ may be manageable by using the arguments above, which is still an ongoing project.

If this conjecture is true, then we can obtain a logarithmic improved estimate at the critical exponent $p = \frac{2d}{d-1}$. As in the curve case, we use the arguments in [37], which was motivated by [33], which was also motivated by [10].

Corollary 6.3. Let (M, g) be a d-dimensional compact Riemannian manifold with nonpositive sectional curvatures, and Σ be a hypersurface in M. If the second fundamental form of Σ is (positive or negative) definite, then there exists a uniform constant $C > 0$ such that

$$\|\mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(\sqrt{-\Delta_g})f\|_{L^{\frac{2d}{d-1}}(\Sigma)} \le C\frac{\lambda^{\frac{d-1}{2d}}}{(\log\lambda)^{\frac{d-1}{2d^2}}}\|f\|_{L^2(M)}, \quad \lambda \gg 1.$$

We first need a restriction estimate for cubes with sidelength l, which is an analogue of [37, Lemma 1]. The following lemma can be proved by using the arguments in [12, §6].

Lemma 6.4. We have, in local coordinates,

$$\|\chi(\lambda - P)f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \lesssim \lambda^{\frac{1}{4}} l^{\frac{d-1}{4}}\|f\|_{L^2(M)},$$

where $\lambda^{-1} \le l \ll 1$.

Proof. By a TT^* argument, it suffices to show that

$$\|\chi(\lambda - P)f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^2)} \lesssim \lambda^{\frac{1}{2}} l^{\frac{d-1}{2}}\|f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^2)}.$$

We note that

$$\chi^2(\lambda - P)(x, y) = \frac{1}{2\pi}\int e^{it\lambda}\widehat{\chi^2}(t)e^{-itP}(x, y)\, dt.$$

By the proof of [30, Lemma 5.1.3], modulo $O(\lambda^{-N})$ errors, we have

$$\chi^2(\lambda - P)(x, y) = \begin{cases} \sum_{\pm} \frac{\lambda^{\frac{d-1}{2}}}{\rho(x,y)^{\frac{d-1}{2}}} e^{\pm i\lambda\rho(x,y)}a(x, y), & \text{if } \rho(x, y) \ge \lambda^{-1}, \\ O(\lambda^{d-1}), & \text{if } \rho(x, y) \le \lambda^{-1}, \end{cases} \tag{6.1}$$

where $|\partial^\alpha_{x,y}a(x, y)| \le C_\alpha$. We set

$$K(r, s) = \chi^2(\lambda - P)(\sigma(r), \sigma(s)).$$

We consider a partition of unity on $\{r \in \mathbb{R}^{d-1} : |r| \le l\}$,

$$1 = \chi_0(\lambda r) + \sum_{j=1}^{\log_2\lambda} \tilde{\chi}(2^j r),$$

where $\chi_0 \in C_0^\infty(\mathbb{R})$, and $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{d-1})$ with $\text{supp}(\tilde{\chi}) \subset \{r \in \mathbb{R}^{d-1} : \frac{l}{2} < |r| < 2l\}$. We then write

$$\begin{aligned} K(r,s) &= K(r,s)\chi_0(\lambda(r-s)) + \sum_{j=1}^{\log_2 \lambda} K(r,s)\tilde{\chi}(2^j(r-s)) \\ &=: K_0(r,s) + \sum_{j=1}^{\log_2 \lambda} K_j(r,s). \end{aligned}$$

For simplicity, we identify an operator as its kernel, for example, K_0 is the operator whose kernel is $K_0(r,s)$ in that

$$K_0 f = \int K_0(r,s) f(s)\, ds.$$

Using (6.1) and Young's inequality, we have

$$\|K_0 f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \lesssim \|f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})},$$

and thus, we focus on K_j for $j \geq 1$. Without loss of generality, we consider the plus sign in (6.1), and write

$$K_j(r,s) = \lambda^{\frac{d-1}{2}} e^{i\lambda\tilde{\rho}(r,s)} \frac{\tilde{\chi}(2^j(r,s))}{\tilde{\rho}(r,s)^{\frac{d-1}{2}}} \tilde{a}(r,s),$$

where $\tilde{\rho}(r,s) = \rho(\sigma(r),\sigma(s))$ and $\tilde{a}(r,s) = a(\sigma(r),\sigma(s))$. Similar argument will work for the minus sign.

We consider another parition of unity

$$1 = \sum_{p \in \mathbb{Z}^{d-1}} \chi_1(2^j r - lp),$$

where $\chi_1 \in C_0^\infty$ satisfies $\text{supp}(\chi_1) \subset \{r \in \mathbb{R}^{d-1} : |r| \lesssim l\}$. We write

$$K_j(r,s) = \sum_{p,\tilde{p} \in \mathbb{Z}^{d-1}} \chi_1(2^j r - lp) K_j(r,s) \chi_1(2^j s - l\tilde{p})$$

We denote by $R_{j,p,\tilde{p}}$ the operator whose kernel is

$$\chi_1(2^j r - lp) K_j(r,s) \chi_1(2^j s - l\tilde{p}).$$

By the support properties of χ_1 and K_j, we have

$$|p - \tilde{p}| \leq C_1, \quad \text{for some } C_1 > 0,$$

since

$$|lp - l\tilde{p}| \leq |lp - 2^j r| + |2^j r - 2^j s| + |2^j s - l\tilde{p}| \lesssim l.$$

By this, only the $R_{j,p,\tilde{p}}$ do not vanish if $|p-\tilde{p}| \le C_1$. As a consequence of almost orthogonality, we have

$$\|K_j\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1}) \to L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \lesssim C_1 \sup_{p,\tilde{p}} \|R_{j,p,\tilde{p}}\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1}) \to L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})}.$$

Using a translation and orthogonal (linear) transformation if necessary, we can assume $p=0$ and $g(p)=\mathrm{Id}$ for the metric. We set

$$R = 2^j r, \quad S = 2^j s, \quad \tilde{\rho}_j(R,S) = 2^j \tilde{\rho}(2^{-j}R, 2^{-j}S),$$

and $\underline{R_{j,0,0}}$ is the operator whose kernel is

$$(\lambda 2^j)^{\frac{d-1}{2}} e^{i\lambda 2^{-j}\tilde{\rho}_j(R,S)} \chi_1(R)\chi_1(S) \frac{\tilde{\chi}(R-S)}{\tilde{\rho}_j(R,S)^{\frac{d-1}{2}}} \tilde{a}(2^{-j}R, 2^{-j}S).$$

Note that, by shrinking the support of $\tilde{a}$, we can focus only on $j \gg 1$, and $\tilde{\rho}_j(R,S) \to |R-S|$ as $j \to \infty$ in the C^∞-topology (see [12, §6]). With this in mind, for large j's, by the proof of [30, Theorem 2.1.1], we have

$$\begin{aligned}\|\underline{R_{j,0,0}} f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} &\lesssim (\lambda 2^j)^{\frac{d-1}{2}} (\lambda 2^{-j})^{-\frac{d-2}{2}} l^{\frac{d-1}{2}} \|f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \\ &= \lambda^{\frac{1}{2}} 2^{\frac{2d-3}{2}j} l^{\frac{d-1}{2}} \|f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})}.\end{aligned}$$

Putting these together, we have

$$\begin{aligned}\|R_{j,0,0}\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1}) \to L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} &= 2^{-(d-1)j} \|\underline{R_{j,0,0}}\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1}) \to L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \\ &\lesssim \lambda^{\frac{1}{2}} 2^{-\frac{j}{2}} l^{\frac{d-1}{2}},\end{aligned}$$

from which it follows that

$$\|K_j\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1}) \to L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \lesssim C_1 \lambda^{\frac{1}{2}} 2^{-\frac{j}{2}} l^{\frac{d-1}{2}},$$

and thus, we have

$$\sum_{j=1}^{\log_2 \lambda} \|K_j f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})} \lesssim \lambda^{\frac{1}{2}} l^{\frac{d-1}{2}} \|f\|_{L^2([-\frac{l}{2},\frac{l}{2}]^{d-1})},$$

which completes the proof. □

Remark 6.5. As observed in [37, Remark 1] or Remark 5.2, for $T \ge 1$, the bound for $\chi(T(\lambda - P))$ is the same as the bound in Lemma 6.4 up to some uniform constant.

Before we proceed further, let us recall a property of the kernel $\chi^2(T(\lambda - P))$.

Lemma 6.6 ([3])**.** Let (M,g) be a d-dimensional compact Riemannian manifold with nonpositive sectional curvatures. Then there exists a constant $C = C(M,g)$ such that, for $T_0 \ge 1$ and $\lambda \gg 1$, we have that

$$|\chi^2(T_0(\lambda - P))(x,y)| \le C \left[\frac{1}{T_0} \frac{\lambda^{\frac{d-1}{2}}}{\rho(x,y)^{\frac{d-1}{2}}} + \lambda^{\frac{d-1}{2}} e^{CT} \right].$$

We now find a weak $L^{\frac{2d}{d-1}}$ estimate for $\chi(T(\lambda - P))$.

Proposition 6.7. If (M, g) be as above, then

$$\|\chi(T(\lambda - P))\|_{L^2(M)\to L^{\frac{2d}{d-1},\infty}(\Sigma)} \lesssim \frac{\lambda^{\frac{d-1}{2d}}}{(\log \lambda)^{\frac{d-1}{2d}}}.$$

Proof. We want to show that

$$|\{x \in \Sigma : |\chi(T(\lambda - P))f(x)| > \alpha\}| \lesssim \alpha^{-\frac{2d}{d-1}}\lambda(\log \lambda)^{-1}, \quad \|f\|_{L^2(M)} = 1 \tag{6.2}$$

By the conjecture and the Chebyshev inequality, we have

$$\begin{aligned} |\{x \in \Sigma : |\chi(T(\lambda - P))f(x)| > \alpha\}| &\le \alpha^{-p} \int_\Sigma |\chi(T(\lambda - P))f(s)|^p \, ds \\ &\lesssim \alpha^{-p} \frac{\lambda^{\frac{(d-1)p}{3} - \frac{2d-3}{3}}}{(\log \lambda)^{\frac{p}{2}}}, \quad 2 < p < \frac{2d}{d-1}. \end{aligned}$$

By a computation, we note that if $\alpha \le \lambda^{\frac{d-1}{3}}(\log \lambda)^{\frac{(d-1)(p-2)}{(4-2p)d+2p}}$ for $2 < p < \frac{2d}{d-1}$, then we have the required estimate (6.2), and thus, we want to show that, if $\|f\|_{L^2(M)} = 1$, then

$$|\{x \in \Sigma : |\chi(T(\lambda - P))f(x)| > \alpha\}| \lesssim \alpha^{-\frac{2d}{d-1}}\lambda(\log \lambda)^{-1}, \quad \text{when } \alpha \ge \lambda^{\frac{d-1}{3}}.$$

We set

$$A = A_\alpha = \{x \in \Sigma : |\chi(T(\lambda - P))f(x)| > \alpha\}, \quad \text{and } l = \alpha^{-\frac{4}{(d-1)^2}}\lambda^{\frac{1}{d-1}}(\log \lambda)^{-\frac{2}{d-1}}.$$

If we denote by I_j a cube with side-length l, we consider a disjoint union $A = \cup_j A_j$ with $A_j = A \cap I_j$. Replacing A by a set of proportional measure, we may assume that

$$\operatorname{dist}(A_j, A_k) > C_1 l, \quad \text{when } j \neq k,$$

for some $C_1 > 0$, which will be specified later.

For the operator $T_\lambda = \chi(T(\lambda - P)) : L^2(M) \to L^2(\Sigma)$, we define ψ_λ, for $x \in \gamma$, by

$$\psi_\lambda(x) = \begin{cases} \frac{T_\lambda(x)}{|T_\lambda(x)|}, & \text{if } T_\lambda f(x) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

We also set

$$S_\lambda = T_\lambda T_\lambda^*, \quad \text{and } a_j = \psi_\lambda \mathbb{1}_{A_j}.$$

By the Chebyshev inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \alpha|A| \le \left| \int_\Sigma T_\lambda f \overline{\psi_\lambda \mathbb{1}_A} \, ds \right| &\le \left| \int_\Sigma \sum_j T_\lambda f \overline{a_j} \, ds \right| \\ &= \left| \int_M \sum_j \overline{T_\lambda^* a_j} f \, dV_g \right| \le \left(\int_M |\sum_j T_\lambda^* a_j|^2 \, dV_g \right)^{\frac{1}{2}}. \end{aligned}$$

We can then write

$$\alpha^2|A|^2 \le \sum_j \int_M |T_\lambda^* a_j|^2 \, dV_g = I + II,$$

where

$$I = \sum_j \int_M |T_\lambda^* a_j|^2 \, dV_g, \quad II = \sum_{j\neq k} \int_\Sigma S_\lambda a_j \overline{a_k} \, ds.$$

By duality and Remark 6.5, we have that

$$I \le Cl^{\frac{d-1}{2}} \lambda^{\frac{1}{2}} \sum_j \int_\Sigma |a_j|^2 \, ds = Cl^{\frac{d-1}{2}} \lambda^{\frac{1}{2}} |A| = C\alpha^{-\frac{2}{d-1}} \lambda (\log \lambda)^{-1}.$$

For II, by Lemma 6.6, if $K_\lambda(r,s)$ denotes the kernel of S_λ, then

$$|K_\lambda(r,s)| \le C\left[\frac{\lambda^{\frac{d-1}{2}}}{T|r-s|^{\frac{d-1}{2}}} + \lambda^{\frac{d-1}{2}} e^{CT}\right] = C\left[\frac{1}{c_0 \log \lambda} \frac{\lambda^{\frac{d-1}{2}}}{|r-s|^{\frac{d-1}{2}}} + \lambda^{\frac{d-1}{2}+Cc_0}\right],$$

which in turn implies that

$$\begin{aligned} II &\le C\left[\frac{\lambda^{\frac{d-1}{2}}}{c_0 \log \lambda |r-s|^{\frac{d-1}{2}}} + \lambda^{\frac{d-1}{2}+Cc_0}\right] \sum_{j\neq k} \|a_j\|_{L^1} \|a_k\|_{L^1} \\ &\le C\left[\frac{\lambda^{\frac{d-1}{2}}}{c_0 \log \lambda C_1^{\frac{d-1}{2}} l^{\frac{d-1}{2}}} + \lambda^{\frac{d-1}{2}+Cc_0}\right] |A|^2 \\ &\le \left[\frac{C}{c_0 C_1^{\frac{d-1}{2}}} \alpha^{\frac{3d-2}{2(d-1)}} + C\alpha^{\frac{3}{2}+\frac{3}{d-1}Cc_0}\right] |A|^2, \end{aligned}$$

since we are assuming $\alpha \ge \lambda^{\frac{d-1}{3}}$. For a large $C > 0$, we take a sufficiently small $c_0 > 0$ so that

$$\alpha^{\frac{3}{2}+\frac{3}{d-1}Cc_0} \ll \frac{1}{4}\alpha^2, \quad \text{when } \alpha \ge \lambda^{\frac{d-1}{3}} \gg 1.$$

Also, given $C > 0$ and $c_0 > 0$, we choose $C_1 > 0$ large such that

$$\frac{C}{c_0 C_1^{\frac{d-1}{2}}} \alpha^{\frac{3d-2}{2(d-1)}} < \frac{1}{4}\alpha^2, \quad \text{when } \alpha \ge \lambda^{\frac{d-1}{3}} \gg 1.$$

Combining these two yields

$$II \le \frac{1}{2}\alpha^2 |A|^2, \quad \text{when } \lambda \gg 1,$$

and thus, we have that

$$\alpha^2 |A|^2 \le I + II \le C\alpha^{-\frac{2}{d-1}} \lambda (\log \lambda)^{-1} |A| + \frac{1}{2}\alpha^2 |A|^2,$$

from which it follows that

$$|A| \lesssim \alpha^{-\frac{2d}{d-1}}\lambda(\log\lambda)^{-1}.$$

This completes the proof. $\square$

We now prove Corollary 6.3. We first recall a special case of a result in [2].

Lemma 6.8 ([2]). Suppose (M,g) is a d-dimensional compact Riemannian manifold. If $\Sigma \subset M$ is any hypersurface, then

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(P)f\|_{L^{\frac{2d}{d-1},2}(\Sigma)} \lesssim \lambda^{\frac{d-1}{2d}}\|f\|_{L^2(M)}.$$

We briefly review some properties of the Lorentz space $L^{p,q}$ (see also Grafakos [16], etc.). If u is a function on M, the corresponding distribution function $d_u(\alpha)$ with respect to Σ is defined by

$$d_u(\alpha) = |\{x \in \Sigma : |u(x)| > \alpha\}|, \quad \alpha > 0.$$

The function u^* is the nondecreasing rearrangement of u on Σ, defined by

$$u^*(t) = \int \{\alpha : d_u(\alpha) \le t\}, \quad t \ge 0.$$

For $1 \le p, q \le \infty$, the Lorentz space $L^{p,q}(\Sigma)$ is

$$L^{p,q}(\Sigma) = \left\{ u : \|u\|_{L^{p,q}(\Sigma)} := \left(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}}u^*(t)]^q\frac{dt}{t}\right)^{\frac{1}{q}} < \infty \right\}.$$

It is also known that

$$\|\cdot\|_{L^{p,p}(\Sigma)} = \|\cdot\|_{L^p(\Sigma)}, \quad \sup_{t>0} t^{\frac{1}{p}}u^*(t) = \sup_{\alpha>0}\alpha[d_u(\alpha)]^{\frac{1}{p}}.$$

We now take $u = \mathbb{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)f$, where $\|f\|_{L^2(M)} = 1$. By Proposition 6.7, we have that

$$\sup_{t>0} t^{\frac{d-1}{2d}}u^*(t) \lesssim \|u\|_{L^{\frac{2d}{d-1},\infty}} \lesssim \frac{\lambda^{\frac{d-1}{2d}}}{(\log\lambda)^{\frac{d-1}{2d}}}.$$

Since $\mathbb{1}_{[\lambda,\lambda+1]}(P)u = u$, by Lemma 6.8, we have that

$$\|u\|_{L^{\frac{2d}{d-1},2}(\Sigma)} \lesssim \lambda^{\frac{d-1}{2d}}\|u\|_{L^2(M)} \lesssim \lambda^{\frac{d-1}{2d}}.$$

Putting these all together, we have that

$$\begin{aligned}\|u\|_{L^{\frac{2d}{d-1}}(\Sigma)} &= \left(\int_0^\infty [t^{\frac{d-1}{2d}}u^*(t)]^{\frac{2d}{d-1}}\frac{dt}{t}\right)^{\frac{d-1}{2d}} \\ &\lesssim \sup_{t>0}[t^{\frac{d-1}{2d}}u^*(t)]^{\frac{1}{d}}\|u\|^{\frac{d-1}{d}}_{L^{\frac{d-1}{2d},2}(\Sigma)} \lesssim \frac{\lambda^{\frac{d-1}{2d}}}{(\log\lambda)^{\frac{d-1}{2d^2}}}.\end{aligned}$$

This completes the proof.

6.2 Concluding remarks

In this dissertation, we have focused on eigenfunction restriction estimates for curved hypersurfaces. If the hypersurfaces are curves, then we considered the curves with nonvanishing geodesic curvatures. If the dimension of the hypersurfaces is greater than 1, then in this chapter we considered the case where the second fundamental form of the hypersurfaces is (positive or negative) definite. As we said above, Conjecture 6.2, a higher-dimensional analogue of Theorem 1.2, is still an on-going project.

If we remove the curvature assumptions on hypersurfaces, problems become subtle. For $p_c = \frac{2n}{n-1}$ and $n = \dim M$, Burq, Gérard, and Tzvetkov [12], and Hu [21] showed that

$$\|e_\lambda\|_{L^p(\Sigma)} = \begin{cases} O(\lambda^{\frac{n-1}{4} - \frac{n-2}{2p}}), & \text{if } 2 \le p \le p_c, \\ O(\lambda^{\frac{n-1}{2} - \frac{n-1}{p}}), & \text{if } p_c < p \le \infty, \end{cases}$$

where $\|e_\lambda\|_{L^2(M)} = 1$ as above, and Σ is any hypersurface. Chen [14] already found a logarithmic analogue of the case $p_c < p \le \infty$:

$$\|e_\lambda\|_{L^p(\Sigma)} = O\left(\frac{\lambda^{\frac{n-1}{2} - \frac{n-1}{p}}}{(\log \lambda)^{\frac{1}{2}}} \right), \quad \text{if } p_c < p \le \infty.$$

Finding a logarithmic analogue of the case $2 \le p \le p_c$ is still an open problem.

Burq, Gérard, and Tzvetkov [12], and Hu [21] also found

$$\|e_\lambda\|_{L^p(\gamma)} = O(\lambda^{\frac{1}{4}}), \quad \text{if } 2 \le p \le 4,$$

where γ is any curve. Logarithmic improved analogues of this estimate already have been studied by Blair and Sogge [8], Blair [5], and Xi and Zhang [37], where γ is a unit-length geodesic. Replacing a geodesic by any curve for logarithmic improvements here would be another open problem.

We conclude this dissertation with the following table showing a brief summary of eigenfunction restriction estimates for hypersurfaces on the next page.

Universal estimates	Logarithmic improvements
[12], [21] ($n=2$) $\|e_\lambda\|_{L^p(\gamma)} = O(\lambda^{\frac{1}{4}})$, $2 \le p \le 4$ γ: any curve	[8], [5], [37] ($n=2$) $\|e_\lambda\|_{L^p(\gamma)} = O\Big(\frac{\lambda^{1/4}}{(\log\lambda)^{1/4}}\Big)$, $2 \le p \le 4$ γ: unit-length geodesic
[12], [21] ($n=2$) $\|e_\lambda\|_{L^p(\gamma)} = O(\lambda^{\frac{1}{3}-\frac{1}{3p}})$, $2 \le p \le 4$ γ: a curved curve	Theorem 1.2 and Corollary 1.3 ($n=2$) $\|e_\lambda\|_{L^p(\gamma)} = O\Big(\frac{\lambda^{\frac{1}{3}-\frac{1}{3p}}}{(\log\lambda)^{\sigma(p)}}\Big)$, $2 \le p \le 4$ γ: a curved curve, $\sigma(p) = 1/2$ for $p \neq 4$, $\sigma(4) = 1/8$
[21] ($n \ge 2$) $\|e_\lambda\|_{L^p(\Sigma)} = O\big(\lambda^{\frac{n-1}{3}-\frac{2n-3}{3p}}\big)$, $2 \le p \le p_c$ Σ: a curved hypersurface	Future work ($n \ge 2$) Conjecture 6.2 and Corollary 6.3 Σ: a curved hypersurface
[12], [21] ($n \ge 2$) $\|e_\lambda\|_{L^p(\Sigma)} = O(\lambda^{\frac{n-1}{2}-\frac{n-1}{p}})$, $p_c < p \le \infty$ Σ: any hypersurface	[14] ($n \ge 2$) $\|e_\lambda\|_{L^p(\Sigma)} = O\Big(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}}}{(\log\lambda)^{1/2}}\Big)$, $p_c < p \le \infty$ Σ: any hypersurface
[12], [21] ($n \ge 2$) $\|e_\lambda\|_{L^p(\Sigma)} = O(\lambda^{\frac{n-1}{4}-\frac{n-2}{2p}})$, $2 \le p \le p_c$ Σ: any hypersurface	Future work ($n \ge 2$) (Work in progress) Σ: a (totally geodesic) hypersurface

Table 6.1: Eigenfunction restriction estimates for hypersurfaces

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