Isopathic Graphs and Airport Graphs

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This dissertation, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of The University of New Mexico in partial fulfillment of the requirements for the degree of

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ISOPATHIC GRAPHS AND AIRPORT GRAPHS

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ISOPATHIC GRAPHS AND AIRPORT GRAPHS

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ABSTRACT OF DISSERTATION

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This paper explores two kinds of graphs, isopathic graphs and airport graphs. A distance property of graphs in general is also examined.

Isopathic graphs are graphs in which every maximal path has the same length. The major theorem of this section characterizes isopathic graphs as extended stars, bipartite or hamiltonian. There is then a discussion of the latter two classes of isopathic graphs.

At the end of Section I, there is an introduction to isopathic digraphs, a natural concern after an exposure to isopathic graphs.

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The concept of the "distance of a graph" is presented in Section III. Here the degree of a point is seen to be of greater importance than is the condition of being a special point. (Special points corre-

spond to the snobs of Section II.)

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I

ISOPATHIC GRAPHS

DEFINITION 1:

A connected graph G is isopathic if every maximal path has the maximum length, M, of all paths in G. A graph is k-isopathic if each of its k components is isopathic.

NOTATION:

$d_w(u,v)$ represents the distance between u and v along a path containing w.

We will begin the study of isopathic graphs by examining isopathic trees. Clearly, if a graph G is a path, it is isopathic. Hence, our first theorem characterizes isopathic trees which are not paths.

THEOREM I:

A graph G, not a path, which is a tree is isopathic if, and only if, it has exactly one branch point p and G-p is a set of more than two paths of equal length.

PROOF:

Assume that G meets the conditions of the theorem. We must show that G is isopathic. Since G is a tree it is connected and p can be adjacent to precisely one point from each path of G-p. If p fails to be adjacent to any path of G-p, then G is disconnected; whereas, if p were adjacent to two points of any path of G-p, then a cycle is formed, contradicting the supposed acyclic character of G. If p were adjacent in G to any points other than endpoints of the paths of G-p, another branch

point would be created, in contradiction to the hypothesis. Thus, p must be adjacent to precisely one endpoint of each path of G-p.

Let the equal path lengths for the paths of G-p be k, and let P be an arbitrary path in G. Extend P to a maximal path $\hat{P}$ in any way desired. Because G is a tree, $\hat{P}$ ends at endpoints of G, and must include p, the cutpoint of G. From the way in which p connects G at path endpoints of G-p, we see that $M = 2k + 2 = 2(k + 1)$. Since P was arbitrary, we must have that all maximal paths are of length $2(k + 1)$, so that G is isopathic.

Conversely, let G be an isopathic tree. Denote by u and v two of its endpoints. Clearly the u-v path is maximal, and therefore has length M because every maximal path has maximum length. Let w be any other endpoint, and follow this branch until a point p of the u-v path is reached. This must happen because G is connected. The paths u-p-v, u-p-w, and w-p-v are all maximal, so of length M. We must therefore have:

$$d(u,p) + d(p,v) = d(u,p) + d(p,w) = d(w,p) + d(p,v). \quad (1)$$

From (1) it follows that:

$$d(u,p) = d(v,p) = d(w,p). \quad (2)$$

If t were any other endpoint, an analogous argument would show that the path beginning with t must also intersect the u-v path at p, so there can be but one branch point in G. Equation (2) shows that G-p will be a set of paths of equal length. QED.

An example of an isopathic tree is shown in *Figure 1*. Such a tree will be called an extended star.

COROLLARY:

An isopathic forest of k trees has k isomorphic trees, each with a single branch point, and each branch is a path of the same length as the other branches.

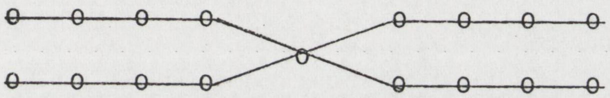


Figure 1.
An Isopathic Tree

THEOREM II.

If a line x lies on every maximal path of a graph G on n points none of which are isolated, then the graph is a path P_n.

PROOF:

The condition of the theorem implies that the graph is connected if there are no isolated points, because every line is on some maximal path. Let x = uv. The deg u < 3, and deg v < 3 because otherwise any two lines, except x, incident with u (or v) could be extended to a maximal path without including x. If deg u = deg v = 1, the theorem is trivial. We therefore assume that the degree of u is 2. Note further that x can not lie on a cycle for then the path from u to v, the long way around the cycle, could be extended to a maximal path which would exclude x. Denote by y the line wu incident, along with x, to u. Then y too must be on every maximal path, since it serves as a bridge to x and x is on no cycle. By the same reasoning as before, we conclude that deg w = 1 or deg w = 2. We now continue by induction to exhaust V. The induction must terminate since x is on no cycle. Of course, this pro-

cedure stops in precisely n-2 steps, when two 1-degree points have been reached. QED.

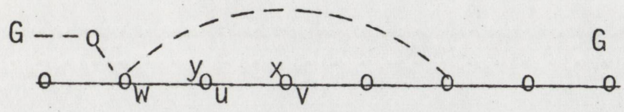


Figure 2.
Illustration of the Proof of Theorem II

THEOREM III:

If a (p,q) graph is isopathic, not a tree, and has $p > 2$, then it is hamiltonian or bipartite. (We note that trees are bipartite.)

PROOF:

Let G be the (p,q) graph. Since G is not a tree it has a cycle, and therefore a longest cycle C. Let c(G), the circumference of G, be the length of C. If $c(G) = M$, the maximum path length of G, there can be no points of G, except perhaps isolated points, which are not on C. Because G is connected, we see that G is hamiltonian.

Assume that $c(G) < M$. Trace around a longest cycle C from two consecutive points $u,v \in C$. Because $c(G) < M$, we can extend the u-v path from u or v or from both u and v. Note that these extensions must be disjoint. If they were not disjoint, a cycle of length $> c(G)$ would result. Denote by a,b the lengths of the extensions from u and v respectively. Let $w \in c$ and w adj v. As before, trace a v-w path, using the previous extension from v. In order to maintain isopathicity, the extension from w must have length a. If there is another point $t \in C$ adjacent on C to w, repeat this procedure using w, the previous w extension, and t. Continue this process until all the points of C have been used. The points of the cycle C thus alternate having extensions of length a and of length b.

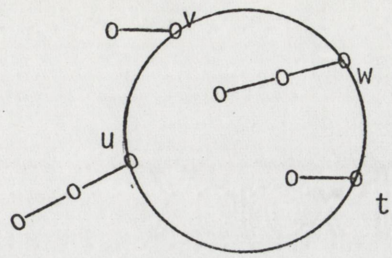


Figure 3.
Alternating a and b Extensions

Hence, if $a \neq b$, $c(G)$ must be an even number.

We may assume without loss of generality that $a \geq b$. There are two cases:

case 1 $\quad a = b > 0$

case 2 $\quad a > b \geq 0$

There are two subcases to case 2:

case 2.1 $\quad b = 0$

case 2.2 $\quad b > 0$

First let us examine case 2.1. $c(G) \geq 4$, and is even. Suppose that $c(G) = 4$. Denote the two points from which there is an extension a, a_1 and a_2. Call the last points in the two extensions e_1 and e_2 respectively. If the extensions do not intersect at any point, the path e_1-a_1-b_1-a_2-e_2 is clearly maximal. (b_i is the point of the cycle C adjacent to a_i in the counterclockwise direction.) Also, the path e_1-a_1-b_1-a_2-b_2 is maximal. Hence we have

$$2a + 2 = a + 3$$

$$\Longleftrightarrow a = 1$$

If $\deg e_i \neq 1$, a bigraph like *Figure* 5 with $e_1 = e_2$ results. We therefore assume $\deg e_1 = \deg e_2 = 1$.

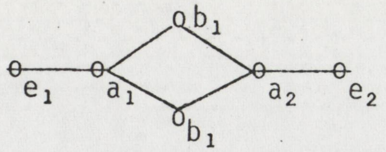


Figure 4.
A Diamond With Two Tails

If the degree of say $a_1 = k > 3$, then the degree of a_2 is also k, and the resulting graph has k-1 disjoint (except for endpoints) paths of length 2 between a_1 and a_2. Suppose a point s is adjacent to a_1. The path e_1-a_1-s has length 2 which implies it can be extended from s because G is isopathic and deg $e_2 = 1$. s cannot be adjacent to b_1 or b_2 for then a longer than maximal cycle would result; s cannot be adjacent to e_2 because deg $e_2 = 1$. Furthermore, s cannot be adjacent to still another point t because the path e_2-a_2-b_2-a_1-s-t would be too long. The only way out is for s to be adjacent to a_2. This isopathic graph is bipartite and is pictured in *Figure* 5.

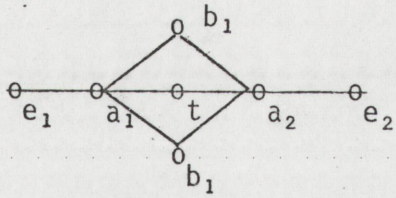


Figure 5.
Several Diamonds With Tails

This completes the discussion when c(G) = 4.

NOTATION:

$A = \{a_i \in C : a_i$ has an extension of length $a\}$.

$B = \{a_i \in C : a_i$ has an extension of length $b\}$.

Consider now c(G) = 6, b = 0. There are two possibilities; all extensions from the A points are mutually disjoint, or some pair of ex-

tensions intersect. It is not possible for all extensions to be disjoint because the path e_1-a_1-b_1-a_2-e_2 would be both maximal and shorter by two than the path e_1-a_1-a_2-a_3-e_3.

Assume therefore that some pair of extensions intersect at some point u. Since c(G) = 6, the A points from which the extensions arise must be separated on C by a single B point. In order to keep c(G) = 6, we must have again a = 1.

We use the fact that the A points are mutually nonadjacent, as are the B points (see Lemma III-A) to complete the discussion of c(G) = 6. Suppose the extensions from a_2 and a_3 intersect at u. If deg e_1 = 1, the isopathic graph of *Figure 6* results.

The degree of each of the A points must be 3 because any additional extensions intersecting or otherwise, would destroy the isopathicity as we shall see. Suppose there is another extension from, say a_3. This point, v, cannot be of degree 1 for obvious reasons. v cannot be adjacent to any of the A points because a maximal path of length 4 < M = 6 would be created, and v cannot be adjacent to any B point because c(G) = 6 < 7, i.e. a longer than maximal cycle would be produced.

Finally, $e_1 \neq u$ because b = 0, and this would cause a maximal path of length 5, whereas M = 6. The path b_1-a_1-u-a_3-b_2-a_2 would be a maximal path of length 5.

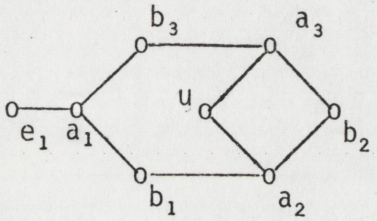


Figure 6
Isopathic Graph With c(G) = 6, p = 8

LEMMA III-A

Under the conditions of case 2.1 above, the A points are mutually nonadjacent.

PROOF:

Suppose some pair of A points is adjacent, say a_i is adjacent to a_j and $c(G) \geq 4$. The path b_j-a_i-a_j-b_i is maximal and too short, its length being $c(G) - 1 < M = c(G)$. The case $c(G) = 4$ is obvious.

If $c(G) = 4$, the B points of the diamond with two tails cannot be adjacent because M would then be 5 which is not possible, and, if the endpoints are identified, the B point still cannot be adjacent because a cycle on 5 points is then formed. If $c(G) = 6$, the isopathic graph of *Figure 6* can have no B points adjacent without destroying the isopathicity or causing a longer cycle.

Examine now the case where $c(G) = 8$. Assume that the degree of the B points is two. Since $c(G) + a = c(G) - 1 + 2a$, we still have $a = 1$. From this we get $M = 8$. None of the extension endpoints can then have degree 1 because a maximal path of length ≤ 6 would exist. It is also not possible for $e_i = e_{i+1} = e_{i+2}$, because of the resulting 6 point path b_{i+1}-a_{i+2}-e_i-a_i-b_i-a_{i+1}. This path is maximal because of the previous Lemma together with the assumption on the degree of the B points.

If $\deg b > 2$ or $\deg a_i > 3$, we still have G bipartite provided no a_i-a_j adjacency or b_k-b_ℓ adjacency is involved. The B points cannot be adjacent without changing a maximal path length or getting a longer cycle, and Lemma III-A disallows intra-A adjacencies.

Figure 7 illustrates that three consecutive e_i cannot be identical and *Figure 8* shows the two isopathic graphs with 8 point maximum cycles.

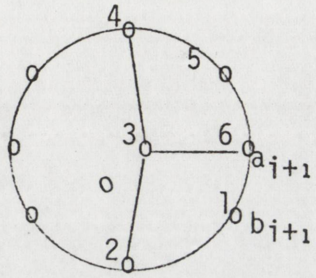


Figure 7.
Nonisopathic Graph

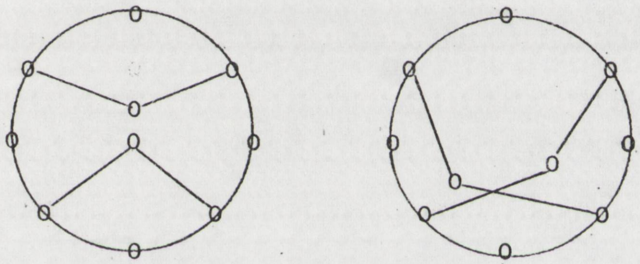


Figure 8.
Isopathic c(G) = 8 Graphs of M = 8

DEFINITION 2:

We will refer to C_4 subgraph of G, which includes 3 points of the maximum cycle C as a diamond. The left diagram of *Figure 8* has two diamonds. Two adjacent diamonds on a maximal cycle are two diamonds which have exactly one point of the cycle in common.

LEMMA III-B:

There are no isopathic graphs for $b > 0$.

PROOF:

Assume $a \geq b > 0$. The sets of the following discussion are A and B. The meaning of consecutive extensions and consecutive extensions from the same set are clear. We consider the three possibilities:

case i   There exist four consecutive disjoint extensions.

case ii   There is a pair of consecutive disjoint extensions from the same set.

case iii   All consecutive extensions from the same set intersect.

Proof of case i: Using first the cycle and two A extensions, and then the cycle and two B extensions, we are able to get the equations:

$$1 + M = c(G) - 1 + 2a \text{ and } 1 + M = c(G) - 1 + 2b \qquad (3)$$

But $1 + M = c(G) + a + b$, hence

$$a + b = 2a - 1 = 2b - 1 \qquad (4)$$

which is impossible.

Proof of case ii: We may assume that two consecutive A extensions are disjoint. As in case i, we must have $M = c(G) + 2a - 2$. In order not to exceed the maximum cycle C, the B extensions intersect immediately so that $b = 1$. Now the path e_1-a_1-b_2-f_2-b_1-a_2-e_2 uses $c(G) + 2a + 1$ points. Here the e_i are the endpoints of the a_i extensions and the f_i are the endpoints of the b_i extensions. From these equations we see that $2a + 1 = 2a - 1$ which is ridiculous.

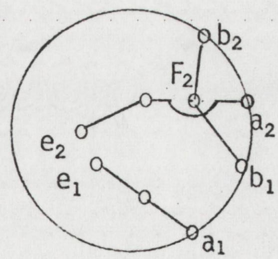


Figure 9.
Diagram for case ii

Proof of case iii: Let u be the first point of intersection of extensions a_1 and a_2, and denote the corresponding b_1 and b_2 point by v. If $u = v$, a cycle of length $c(G) + 1$ is created. If $u \neq v$, the cycle a_2-u-a_1-b_2-v-b_1-a_2 has length $c(G) + 2$. Case iii is therefore impossible too.

Hence, there are no isopathic graphs for $b > 0$.

We note that the condition $a > 0$ is equivalent to the statement that G is not hamiltonian. We have just proved that there are no isopathic graphs when $b > 0$, and that when $b = 0$, the A points are mutually nonadjacent (Lemma III-A). In order to complete the proof of Theorem III, we need only show that if $a > 0$ and $b = 0$, the B points are also mutually nonadjacent.

LEMMA III-C:

If $a > b = 0$, B points of G are mutually nonadjacent.

PROOF:

We recall that if $b = 0$, we must have $a = 1$ since $c(G) + a = c(G) - 1 + 2a$.

Suppose $b_i b_j \varepsilon G$, then either $e_{i+1} = e_{j+1}$ or $e_{i+1} \neq e_{j+1}$. If $e_{i+1} \neq e_{j+1}$, the path e_{i+1}-a_{i+1}-b_j-b_i-a_{j+1}-e_{j+1} uses $c(G) + 2$ points in contradiction to the fact that all isopathic graphs when $b = 0$ use $c(G) + 1$ points in maximal paths. If $e_{i+1} = e_{j+1}$, the path e_{i+1}-a_{i+1}-b_j-b_i-a_{j+1}-e_{j+1} is a cycle with $c(G) + 1$ points, which is a contradiction of the definition of $c(G)$.

Therefore, no B points can be adjacent so the proof of Lemma III-C is complete.

The Lemmas show us that is $a > b = 0$, G is bipartite, completing the proof of Theorem III.

Complete graphs, nearly complete graphs (defined below), $K_{m,m}$ graphs, $K_{m,m+1}$ graphs and C_{2m} graphs are all hamiltonian so that the hamiltonian class of isopathic graphs is neither disjoint from nor identical to the class of bipartite isopathic graphs.

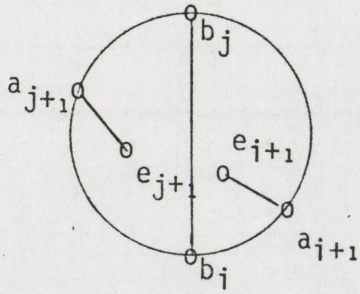


Figure 10.
Diagram for Proof of Lemma III-C

DEFINITION 3:

To deg B = 2, means deg b_i = 2, for all $b_i \varepsilon$ B.

We now turn our attention to isopathic bigraphs. If we require not only that b = 0, but also that deg B = 2, there are no isopathic graphs with c(G) > 8. This we show via the next two Lemmas and Theorem IV.

LEMMA IV-A:

For c(G) > 8 and deg B = 2, the existance of two diamonds with a common A-point and common extension point implies G is not isopathic.

PROOF:

Assume G is isopathic and let P = b_i-a_i-e_i-a_{i+2}-b_{i+1}-a_{i+1}. P must be extended from a_{i+1} to some point u not on C. Deg u is obviously not equal to 1, so u must be adjacent to some A-point, a_k, of C. But then the above path together with the path a_k-b_{i-1} or a_k-b_{i+2}, depending on which is shorter along C, is a maximal path with $8 + \left[\frac{c(G)-6}{2}\right]$ points. Our maximal paths have c(G) + 1 points, so we seek a solution to the inequality:

$$8 + \left[\frac{c(G)-6}{2}\right] < c(G) + 1 \qquad (5)$$

Thus, isopathicity is contradicted for $c(G) > 8$, completing the proof of the Lemma IV-A

LEMMA IV-B:

For $c(G) > 8$ and deg $B = 2$, a diamond on a maximal cycle C implies G is not isopathic.

PROOF:

Assume that G is isopathic and without loss of generality, we may assume the supposed diamond at a_1 and a_2. Lemma IV-A tells us that $e_1 \neq e_3$. Now the path b_1-a_1-e_1-a_2-a_3-e_3 must return to C, say at a_k because deg $B = 2$. The continuation along C to either $b_{c(G)}$ or to b_3, depending again on which is the shorter route, gives us the same inequality (5) as before. This contradiction completes the proof of Lemma IV-B

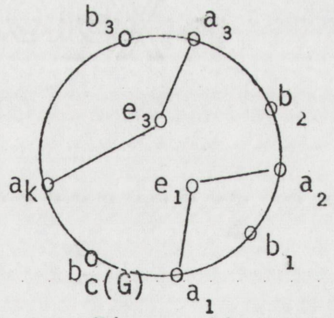


Figure 11.
The Paths of Lemma IV-B

THEOREM IV:

$c(G) > 8$ and every B point of degree 2 implies G is not isopathic.

PROOF:

Suppose G is isopathic. Either $e_1 \neq e_3$ or $e_1 = e_3$. If $e_1 \neq e_3$, the path e_1-a_1-$b_{c(G)}$-a_3-e_3 uses only $c(G) - 1$ points instead of the requisite $c(G) + 1$ points. Hence, e_1 or e_3 must be adjacent to a_2 because

$a = 1$ and a_2 is the only unused point of C. However, this cannot happen by Lemma IV-B, therefore, $e_1 = e_3$.

Assume that $e_1 = e_3$. The path $P = b_2\text{-}a_3\text{-}e_3\text{-}a_1\text{-}a_2\text{-}e_2$ must be extended from e_2 to some $a_k \in C$. P together with $a_k\text{-}b_{c(G)}$ or $a_k\text{-}b_3$ uses $8 + \frac{c(G)-6}{2}$ points and is maximal because $b = 0$. Again, the maximal path is too short if $c(G) > 8$ as the inequality (5) shows.

Therefore, there are no isopathic graphs if $c(G) > 8$ and deg B = 0. QED.

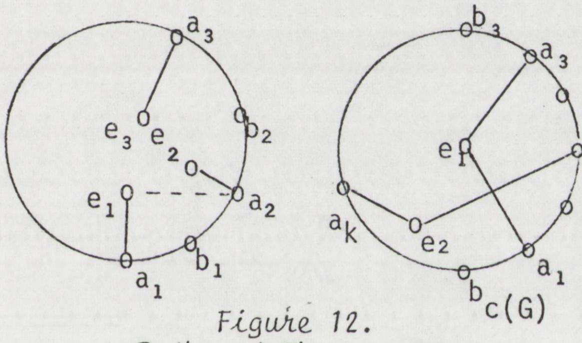


Figure 12.
Paths of Theorem IV

DEFINITION 4:

A bigraph which is a complete bigraph on a maximum cycle C is a <u>complete cycle bigraph</u>. Points not on C are <u>interior points</u> of G.

THEOREM V:

A complete cycle bigraph G with length of C = 2m and with color sets A and B such that $|A| = m$, and $|B| = m + 2$ and such that the two interior points u,v satisfy

1. u is adjacent to exactly one point a_1 of A
2. v is adjacent to every point of $A - a_1$

is isopathic of length 2m.

Figure 13 shows this configuration.

PROOF:

Let P be a maximal path in G. Either u is an endpoint of P or it is not. If so, $P = u\text{-}a_1\text{-}b_j\text{-}\ldots$, where $b_j \neq v$. After b_j, there are $m - 1$ A-points, all of which are reachable from any B-point. Because G is bipartite, any path must be an alternating A-B path, so that only $m - 1$ B-points can be used in P after b_j. The total number of points in P is therefore $3 + 2(m - 1) = 2m + 1$. Hence, the length of P is 2m.

Note that a_1 cannot be an endpoint to a maximal path because it can be extended to u.

Any path which does not have a_1 as an endpoint cannot include u anywhere. This being the present case, (u not an endpoint of P) and since G-u is an alternating random hamiltonian graph, P must have $2m + 1$ points. G is therefore isopathic. QED.

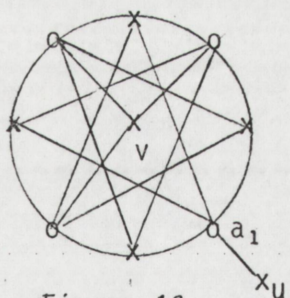


Figure 13.
Complete Cycle Bigraph with Two Interior Points

From Theorem V we observe that u and v together cover A. This generalizes to the next theorem.

THEOREM VI:

A complete cycle bigraph G is isopathic if, and only if, its interior points pairwise cover A.

Note that if there are zero or one interior points, the resulting graphs are the isopathic hamiltonian graphs $K_{m,m}$ and $K_{m,m+1}$ respectively.

Also note that if the interior is nonempty, M = 2m.

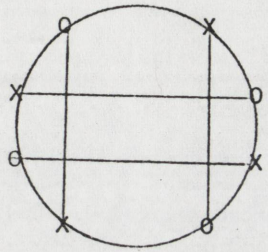


Figure 14.
Isopathic Hamiltonian Bigraph

The cycle B-points will, as usual, be referred to as "b's" and the interior B-points will be denoted by "u's".

PROOF:

(if) Let G be a complete cycle bigraph with the property of pairwise coverage. A maximum path is attained when all m of the A-points, necessarily from the maximal cycle C of G, and m + 1 of the B-points are used alternately with two of the B points as endpoints. Thus M = 2m.

Because all paths are alternating and because $|A| \leq |B|$ with the inequality strict if there are any interior points, the endpoints of any maximal path must be points from B.

Suppose P is any path. If all the A-points are in P, it must be a maximal path and it has 2m + 1 points. If not all the A-points have been used in P, we proceed by induction on $|A|$. We can always make a one point extension from a path with one A-point. Assume, therefore, that P contains m - 1 points from A. If the left endpoint of P is a b-point, P can be extended to the left to pick up the missing A-point, a_j, because b-points are adjacent to every A-point. Then a_j is adjacent to some unused

B-point for the same reason, or because A-points have extensions in the interior of G. If the left endpoint of P is a u-point, and the right endpoint is a b-point b_r, P can be extended to the right as before to pick up the missing a_j, and then another B-point. If both endpoints of P are of the u type, say u_ℓ and u_r, we have, because the interior points pairwise cover A, that at least one of these interior points is adjacent to a_j which in turn is adjacent to some B-point. We have, therefore, proved by induction that every A-point belongs to every maximal path making all maximal paths of length 2 m. Hence G is isopathic.

(only if) This part is proved by contradiction. Assume that G is an isopathic complete cycle bigraph and that the pairwise covering property fails to hold. There are then two interior points u,v and some point a_j from C such that neither u nor v is adjacent to a_j. Both u and v must be adjacent to some A-points a_k and a_i respectively. If $a_k \neq a_i$, any path from a_k to a_i which uses every A-point except a_j can be extended to u and v. P can easily be formed because G is complete on C. The extended P is maximal and has only 2m - 1 points to it. This contradicts the assumed isopathic condition of G. If there is no i,k such that $a_k \neq a_i$, the degree u = deg v = 1. Then the path u-a_k-v is maximal and of length two, another contradiction. Therefore, pairwise coverage must be true. QED.

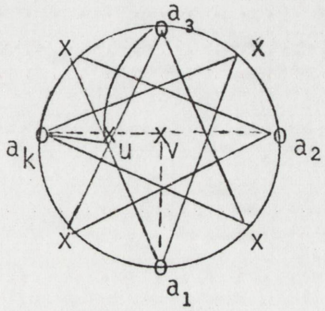


Figure 15.

Theorem VII gives a sufficient condition for any graph to be isopathic.

THEOREM VII:

If $G = (p,q)$ is any connected graph such that any two points of G cover G, then G is isopathic, and the length of any maximal path is P-1.

NOTE:

By u,v covering $G = (p,q)$ is meant that they are together adjacent to all the points of $G - \{u,v\}$.

PROOF:

Let P be any path in G and call its endpoints u and v. If P contains every point of G, it is maximal with p points. If not, either u or v must be adjacent by hypothesis to some point not in P. Extending the length of P by one we have a new path P'. We can obviously continue this process until the point set V of G is exhausted. QED.

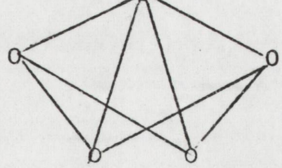


Figure 16.
A Pairwise Covering of a 5-Point Graph

COROLLARY VII-A

If the points of a connected graph (p,q) with $p \geq 5$ pairwise cover (G), it is hamiltonian.

PROOF:

By Theorem VII the graph is isopathic, and by Theorem III it is a tree, bipartite or hamiltonian. For $p \geq 5$ at least 3 points must be in some supposed color set, so the covering condition forces an ajacency within this set precluding the bipartite possibility. Since the graph cannot be either of the first two types, it must be hamiltonian. QED.

DEFINITION 5:

A graph $G = (p,q) \neq K_p$ is nearly complete if for every u from G, $\deg u \geq p - 2$. The notation NK_p shall mean a nearly complete graph on p points.

THEOREM VIII:

The points of G pairwise cover V(G) if, and only if, G is nearly complete.

PROOF:

Suppose the points of G pairwise cover G. If there is some point u of G such that $\deg u < p - 2$, there are two points v,w which are neither one adjacent to u. But then the points of G do not pairwise cover V, a contradiction.

On the other hand suppose G to be nearly complete. If G were not to have the pairwise covering property, there would be points u,v,w such that v and w would neither one be adjacent to u so that $\deg u < p - 2$, another contradiction.

Hence, the pairwise covering property is equivalent to being nearly complete. QED.

COROLLARY VIII-A

$G = NK_p$ implies G is isopathic and $M = p - 1$.

PROOF:

By Theorem VIII G has the pairwise covering property, and by Theorem VII G is then isopathic with $M = p - 1$.

We now turn our attention to graphs which have isomorphic images under cyclic permatations.

DEFINITION 6:

Suppose the p points of a graph G are arranged in come order on a circle. Impose a counterclockwise numbering of V(G), so that G becomes a labled graph. A <u>chord</u> is a line between any two points. The <u>length</u> of a chord is the number of lines in the corresponding subtended smaller arc of the circle.

The greatest common divisor of a and b will be denoted (a,b) in this paper.

THEOREM IX:

Let k be the length of any chord of G for which all chords of length K which could belong to G, do in fact, belong to G. Then $(p,k) = 1$ implies that G is hamiltonian.

PROOF:

$p < \infty$ implies that there exists a,b such that

$$ak \cong bk \pmod p \tag{6}$$

so that

$$(a - b)k \cong 0 \pmod p \tag{7}$$

We may assume that a is the least a such that $ak \cong bk \pmod{p}$ and that $a \geq b$.

$$(p,k) = 1 \text{ implies } p|(a - b). \qquad (8)$$

Therefore, $p = a - b$.

If we now trace around $V(G)$ along chords of length k, all of thich are present by hypothesis, we must reach each point of G before returning to the initial point. Therefore, G is hamiltonian. QED.

DEFINITION 7:

G is a starred polygon if there exists some labeling of its points such that G is isomorphic under all cyclic permutations of its labeled points.

NOTATION:

The following sequential subscripting of the points from G assumes such labeling has already been assigned.

THEOREM X:

If $G = (p,q)$ is an isopathic hamiltonian starred polygon, then it is complete or nearly complete, or complete bipartite $K_{m,m}$.

LEMMA X-A

If G has a chord of length k, then it has all possible chords of length $k - 2n$, $n = 1,2,\ldots..$

PROOF:

Note that $M = p - 1$, and that $p \geq 6$ in order for there to be a chord of length $k - 2$. Also observe that there are no chords if $p = 3$, and if

$p = 4$ or $p = 5$, the respective graphs with a chord of length 2 are K_4 and K_5.

Under the conditions of the theorem, if $p = 6$, the smallest K is 3 and the theorem is trivial since $G = K_{a,3}$. We may therefore assume without loss of generality that $p \geq 7$ and $k \geq 4$.

Select a point $u_i \in G$. There are all chords of length k and we may assume that $k \leq p/2$. The path $u_{k-2}-u_1-u_{p-k+1}-u_{k-1}-u_{p-1}-u_{p-k+2}$ leaves out u_p. Because G is isopathic, we must have one or the other of the endpoints adjacent to u_p. But this is a chord of length $k - 2$, so that under cyclic permutations we have all chords of length $k - 2$. Repeat the analogous process with chord length $k - 4$, $k - 6$, etc., to establish the Lemma.

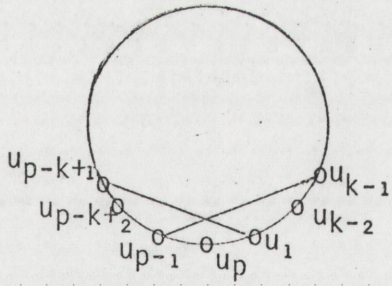


Figure 17.
Diagram for Lemma X-A

COROLLARY X-A:

The shortest chord in any isopathic hamiltonian starred polygon has length two of three.

NOTATION:

The shortest chord length of G shall be denoted by "ch"; the longest chord length shall be represented by "CH". Thus, we have ch = 2 or ch = 3 for the conclusion of Corollary X-A.

LEMMA X-B:

The longest or the next to the longest possible chord is always present in G.

PROOF:

There are two cases:

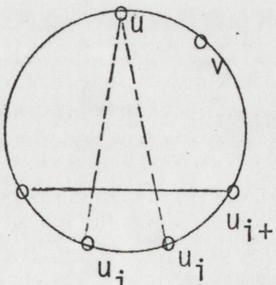
case 1 ch = 2

case 2 ch = 3

Case 1: ch = 2. Pick a point u_i, and two adjacent consecutive points u,v at maximum chord distance from u_i. The path $u-u_{i+1}-u_{i-1}-v$ omits u_i. Thus either u adj u_i or v adj u_i.

Case 2: ch = 3. Select u_i and u_{i+1} and two points u,v at as nearly maximum chord distance from u_i and u_{i+1} as possible. Consider the path $u-u_{i+2}-u_{i-1}-v$. The graph being symmetric, we may assume without loss of generality that the forced (because of isopathicity) adjacency is from u. If p is even, either u adj u_i, or u adj u_{i+1}, the same adjacencies are true if p is odd, but the chord pattern is different (see *Figure 18*). Thus we have in G either the longest or next longest chord. QED.

p odd



p even

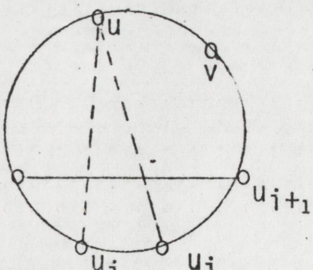


Figure 18.
Diagram for Proog of Lemma X-B

LEMMA X-C:

If ch = 2, then G is complete or nearly complete.

PROOF:

Let k be the length of a shortest missing chord. Note that the Lemma is true for $p \leq 8$. We may arbitrarily center our attention on u_p. The path $u_k - u_1 - u_{k-1} - u_{k+1}\, u_{p-k-1} - u_{p-k+1} - u_{p-1} - u_{p-k}$ leaves out u_p, so that one of the endpoints must be adjacent to u_p at a chord length of k. This is a contradiction unless $u_k = u_{p-k}$ ($p = 8$), so G is either k_p or NK_p. QED.

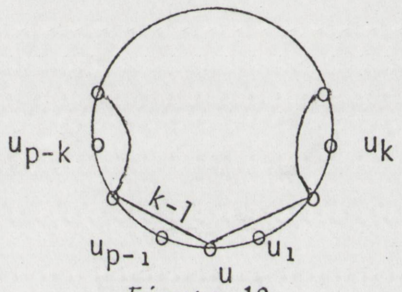


Figure 19.
Diagram for Proof of Lemma X-C

LEMMA X-D:

If ch = 3, then p is even.

PROOF:

Suppose p were odd. We may focus our attention on u_p. Consider now the path $u_2 - u_1 - u_4 - u_3 - u_6 - u_5 - u_8 - u_7 - \ldots\ldots$. This path uses points in pairs so that the last step would be to $u_{p-1} - u_{p-2}$, because p is odd. This path has its endpoints at a chord distance of two from u_p, a contradiction of ch = 3, because the isopathicity forces either u_2 adj u_p or u_{p-2} adj u_p. Thus p must be even. QED.

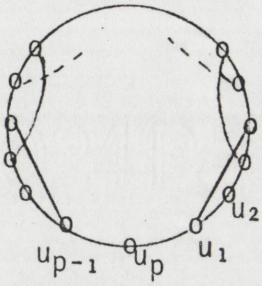


Figure 20.

LEMMA X-E:

If ch = 3, then G is complete bipartite $k_{m,m}$.

PROOF:

By Lemma X-D, p is even and by Lemma X-A, there can be no chords of even length, for then there would be a chord of length 2. By Lemma X-B, either there must be a chord of longest possible length or of next to the longest possible length, and since there are no even chords, all the odd ones must be present. Thus, G is complete bipartite $K_{m,m}$ where p = 2m.

The above Lemmas, X-A through X-E, established Theorem X.

An examination of the proof to Lemma X-D shows that the only use which was made of the starred polygon condition was to guarantee that all of the possible 3-chords were present in G. We can by means of the same proof arrive at the following more general theorem.

THEOREM XI:

Let G be an isopathic hamiltonian graph with ch = 3, and suppose that G has all possible 3-chords. Then p is even.

COROLLARY:

An isopathic hamiltonian graph which has all 3-chords and an odd number of points must have a 2-chord.

THEOREM XII:

Let G be an isopathic hamiltonian graph, and let all possible 2-chords belong to G. Then G has a chord of length k, $k < p/2$.

PROOF:

Without loss of generality we may focus our attention on the point u_p. Furthermore, we may assume $p \geq 6$ in order for there to be something to talk about. For any $k < p/2$ the path $P = u_k - u_{k-2} - \ldots u_1 - (u_2$ or u_3, depending on whether k is odd or even)-around the cycle and back to u_p in the same way as the other side of the cycle, forces u_k adj u_p or u_{p-k} adj u_p.

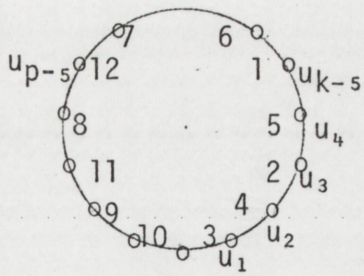


Figure 21.
Diagram for Theorem XII

The path P is valid unless $k = p/2$. In that case when p is odd, the desired endpoints of P are adjacent, and there is no guarantee that there is a chord from u_{k-1} to u_{k+2}. If p is even, both desired endpoints of p are the same point. Hence, the longest possible chords of G might be missing. QED.

We have seen some of the natures of isopathic graphs which have

all possible 2-chords, or all possible 3-chords. We now show that there are no isopathic graphs with all possible k-chords, where ch = k > 3.

THEOREM XIII:

Let G be isopathic hamiltonian with k = ch > 3. All possible k-chords cannot be in G.

PROOF:

As before we center attention on u_p. Then the path $P = u_3\text{-}u_{k+1}\text{-}u_1\text{-}u_2\text{-}u_{k+2}\text{-}\ldots u_{p-k-2}\text{-}u_{p-2}\text{-}u_{p-1}\text{-}u_{p-k-1}\text{-}u_{p-3}$ forces a chord of length 3, a contradiction.

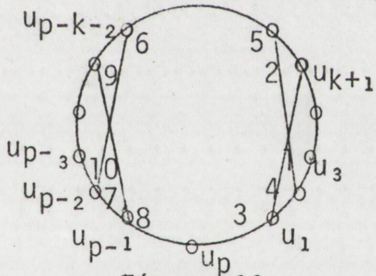


Figure 22.
Diagram for Theorem XIII

The following principle is shared by all isopathic hamiltonian graphs.

THEOREM XIV: (Crossing Theorem)

If G is hamiltonian, isopathic, not c_p, and has a chord F of length ≥ 2, then from every pair u,v of consecutive adjacent points on one side of F, either u or v, is adjacent to some point on the other side of F.

PROOF:

G is hamiltonian so M = p - 1. Suppose $F = u_i u_j$. The path $u\text{-}u_i\text{-}u_j\text{-}v$ uses fewer than p points because the length of F $\geq$ 2, so by the iso-

pathicity of G either u or v must be adjacent to some point across F. QED.

Under the hypothesis of Theorem XIV, a point of degree 2 considerably constrains the possibilities for G to be isopathic as we shall see in Theorem XVII.

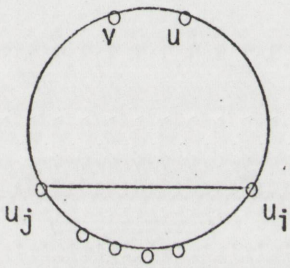


Figure 23.
Diagram for Theorem XIV

THEOREM XV:

G is isopathic and hamiltonian if the shortest chord of G has length $r \geq 4$, then G has all chords of length r.

PROOF:

There is no loss of generalization in the following proof if we assume that r = 4 and that $u_1u_5 \varepsilon$ G. The proof shows that the next chord u_2u_6 is also in G. Note that $r \geq 4 \Longrightarrow p \geq 8$. Assume that r = 4 and that $u_1u_5 \varepsilon$ G. We show that u_2u_6 is also in G. By crossing Theorem, there must be a chord from one of the endpoints of the path u_6-u_5-u_1-u_7 to u_2,u_3, or u_4. u_4 is too close to either endpoint. If $u_3u_7 \varepsilon$ G, the path u_2-u_3-u_7-u_1-u_5-u_6 is too short since the endpoints are to close to u_4. If u_2u_7 ε G, u_4-u_5-u_1-u_7-u_2-u_3 leaves out u_6, and either extension puts in a chord which contradicts the minimality of r. Hence, u_2u_6 ε G. QED.

Theorem XIII and Theorem XV imply:

THEOREM XVI:

If G is isopathic, hamiltonian and not c_p, then G has a chord of length 2 or 3.

THEOREM XVII:

If G is isopathic, hamiltonian, not c_p, and has a point u of degree 2, then G is one of the following graphs:

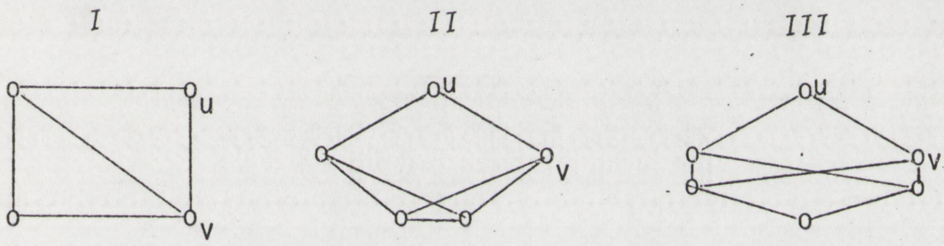


Figure 24.
Isopathic Hamiltonian Graphs With a Point of Degree 2

PROOF:

The proof is via the next three lemmas.

LEMMA XVII-A:

No pair of consecutive adjacent points can have degree 2.

PROOF:

The proof is by contradiction. Suppose there were two consecutive adjacent points $u,v \in G$ of degree 2. Since G is not a cycle, it must have a chord $F = u_i u_j$ and u,v must lie on one side of the chord. By Theorem XIV the path $u\text{-}u_i\text{-}u_j\text{-}v$ must have a chord crossing F from either u or v. Thus, either u or v must have degree ≥ 3, a contradiction

of the assumption, completing the proof of Lemma XVII-A

LEMMA XVII-B

In addition to $F = u_i u_j$, let vw be a chord from v crossing F. There then can be no point t on the small arc uw.

PROOF:

Suppose there were such a point, then there must be a point t adjacent to w on C. The path $t-u_i-u_j-v-w-u_{j-1}$ is maximal and of length p - 2. It is maximal because neither endpoint can be adjacent to u, a point of degree 2. This contradiction of G's isopathic condition completes the proof of Lemma XVII-B.

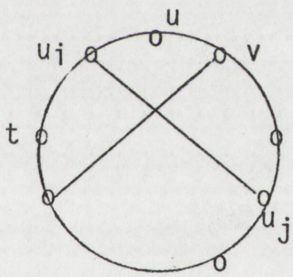


Figure 25.
Diagram of Lemma XVII-B

LEMMA XVII-C:

There is at most one point on the arc wu_j.

PROOF:

Suppose there were two (or more) such points, u_{j-2} and u_{j-1}, on arc wu_j. The path $u_{j-2}-w-v-u_{j-1}$ must be extended across vw. Because the degree of u is 2, we must have either $u_{j-1}u_i \in G$, or $u_{j-2}u_i \in G$. Assume $u_{j-1}u_i \in G$, then the path $u_{j-1}-u_i-u_j-v-w-u_{j-2}$ is maximal and of length p - 2, a contradiction. If $u_{j-2}u_i \in G$, the path $u_{j-1}-u_{j-2}-u_i-u_j-v-w-u_{j-3}$ is again maximal and of length p - 2. This contradiction completes the

proof of Lemma XVII-C

In order to satisfy the hypothesis of having a chord we must have $p \geq 4$. If $p = 4$, the above lemmas give Graph *I* in *Figure 25*. If there are no points between w and u_j, Graph *II* results; if there is one point between w and u_j, Graph *III* is obtained. QED.

ISOPATHIC DIGRAPHS

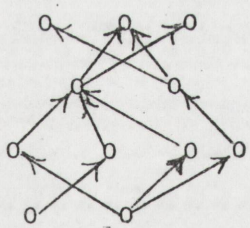
The forgoing discussion prompts the question "what can be said about isopathicity of digraphs?" The answer to this question for weakly connected acyclic digraphs is given in the following:

THEOREM XVIII:

An acyclic weakly connected digraph G is isopathic if, and only if, there exists a partition of the point set V(G) into subsets S_i, $0 \leq i \leq M$ where M is a maximum path length, so that for $j \neq k - 1$, $u \varepsilon S_j$ and $v \varepsilon S_k$ implies $uv \notin X(G)$.

Note that the theorem says that all arcs are directed upward in unit steps, if the S_i are arranged in levels as in the figure below.

S_M



S

Figure 26.
Isopathic Directed Tree

PROOF:

Let S_0 - $\{u \varepsilon G : id(u) = 0\}$. S_0 is nonempty because G is acyclic. Let $S_1 = \{u \varepsilon G : u$ is adjacent from some point in $S_0\}$. Then define $S_j = \{u \varepsilon G :$ the least index of adjacency is from $S_{j-1}\}$. Let $S_M = \{u \varepsilon G : od(u) = 0\}$. Again, because G is acyclic S_M is nonvoid. Furthermore, all S_j are nonempty.

Assume G to be isopathic. Any path from a point of S_0 to a point of S_M is maximal, and because G is isopathic, all such paths must have

length M. If there were an arc uv from S_k to S_j, $k \geq j$, we would then have one of the following contradictions.

1. $k > j$ and $vu \notin G$ because G has no cycles and {uv,vu} constitutes a cycle. There must be a path P_1 from S_0 to v because of the above construction. Now either $v \in S_M$ or $v \notin S_M$. If $v \in S_M$, the length of P_1 is less than an S_0-u-v path = $P_2 \cup uv$. Hence, $v \notin S_M$. v then must be adjacent to some point $w \in S_{j+1}$ by the construction of the S_j. w cannot be any point of P_2 without creating a cycle and w cannot be adjacent to any point, except u, of P_2 for the same reason. If $wu \in G$, the triangle inequality makes the length of an S_0-v-w-u path greater than the length of P_2. Now we can extend these two unequal paths to maximality in the same number of steps because no points can be reused.

In the same way we can extend the $P_1 \cup vw$ and $P_2 \cup uv \cup vw$ to maximal paths of unequal lengths. This contradiction tells us that $uv \notin G$.

Therefore, $k \not> j$.

2. $k = j$. Any S_0-u-v-maximal path has lengthone more than a S_0-v-maximal path; hence $k \neq j$.

The necessity of the condition is established.

If on the otherhand, G meets the conditions of the theorem, every path can be extended only by going to a point in the next higher (lower) set, so that any maximal path must consist of precisely one point from each set S_j. $0 \leq j \leq M$. Thus, G is isopathic. QED.

The last theorem shows that G cannot even have a semicycle.

Suppose we consider a slightly more complicated graph which, like the last type of graph, is an oriented graph. Let O be an oriented graph with a cycle; this forces $p \geq 3$. If we suppose further that O has

a cycle which includes all p points, we have the following facts about O, for which we shall find convenient the following.

DEFINITION 8:

Let the graph O be laid out in a hamiltonian cyclic C. Any arc which has the same sense as the cycle will be called a positively directed arc, parc for short. A negatively directed arc, narc, has the opposite sense as a parc. Note that a parc (narc) can be adjacent to a point at most halfway around the cycle C. The length of a parc A, denoted L(A), is the number of arcs it subtends on the cycle C of O. If p is even and L(A) = p/2, we adopt the convention of calling A a parc.

LEMMA IXX-A:

If O has one parc of length 2 it has all possible 2-parcs.

PROOF:

Let the orientation of C be counterclockwise and assume without loss of generality that the 2-parc A is from u_1-u_3. M is clearly p - 1. Consider the path u_1-u_3...-u_p = P. P has length M - 1 so that u_pu_2 or u_2u_1 must be in O. But C is oriented, with $u_1u_2 \in O$, so that $u_2u_1 \notin O$. Now reapply the forgoing argument to get $u_{p-1}u_1 \in O$. This process can evidently be continued until V(P) is exhausted. Hence, all possible 2-parcs belong to O. The configuration is shown below.

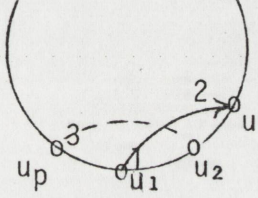


Figure 28.
Diagram for Lemma IXX-A

LEMMA IXX-B:

If 0 has all 2-parcs, it has all 3-parcs.

PROOF:

Consider the path $P = u_1 - u_2 - u_4 \ldots u_p$. As in Lemma IXX-A, $u_1 u_3 \varepsilon 0$ implies $u_3 u_1 \notin 0$ so that $u_p u_3$ is forced to be in 0. Because the starting point of P was arbitrarily chosen, we must have that all possible 3-parcs belong to 0. QED.

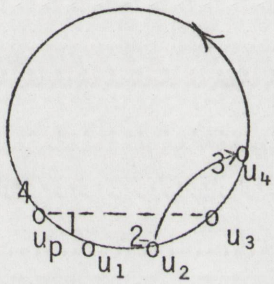


Figure 29.
Diagram for Lemma IXX-B

LEMMA IXX-C:

There cannot be two consecutive 3-parcs.

PROOF:

The path $P = u_1 - u_4 - \ldots u_p - u_3$ is maximal since neither $u_2 u_1$ nor $u_3 u_2$ can belong to 0. But P has length M - 1, which contradicts the isopathicity of 0. QED.

The Lemmas stated above together imply:

THEOREM IXX:

An oriented graph with a hamiltonian cycle cannot have a 2-parc.

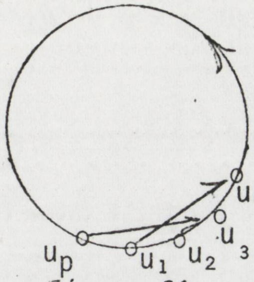


Figure 30.
Diagram for Lemma IXX-C

II

AIRPORT GRAPHS

SPECIAL POINTS:

In the algorithm[1] for determining whether a given partition is graphical, the graph G can be replaced by a graph G' which has the same partition as G, and has a further property. The graph G' has a special point u_1 which is adjacent to other points of G' in such a way that u_1 of degree d_1 is adjacent to d_1 points u_i of degrees d_i, and the d_i are in descending order for the partition of G. This motivates the following definition.

DEFINITION 9:

A point u is a special point of G, sometimes abbreviated "sp pt", iff whenever u is adjacent to a point of degree j, it is adjacent to every point from G of degree > j. Note that isolated points are vacuously special points.

THEOREM XX:

If u is a special point of G, then u is a special point of $\overline{G}$, the complement of G.

PROOF:

Let m be the minimum of the degrees of all points of G to which u is adjacent, and let v be a point from G of degree m. The degree of v in $\overline{G}$ is then p-m-1. u is adjacent to no point of degree < m in G, and therefore u is adjacent to every point in $\overline{G}$ with degree > p-m-1. Because u is adjacent to every point with degree > m in G, the minimum degree to which u can be adjacent in $\overline{G}$ is p-m-1, and u is adjacent to all points of

$\overline{G}$ of degree higher than p-m-1, so u is a special point of $\overline{G}$. QED.

NOTATION:

The number of points in a graph G of degree n will be denoted by #(n), or by #n when there is no ambiguity caused by dropping the parentheses.

The question arises from the aforementioned algorithm "When is it not necessary to rearrange the summands in successive modified partitions?" The answer to the question is given by the following:

THEOREM XXI:

The algorithm for showing that a partition $\Pi = (d_1, d_2, \ldots, d_p)$ is graphical does not have to be reordered at any stage if, and only if, Π is of the form

$$d_i = \begin{cases} p-1 & \text{if } 1 \leq i \leq k \text{ and } 1 \leq k \leq p \\ k & \text{if } k < i \leq p \end{cases} \tag{9}$$

PROOF:

It is clear that if Π has this form, (9), it will result in a partition of p-k zero summands in precisely k iterations.

On the other hand, if the algorithm is to proceed without rearrangement, it must be that d_1 is large enought to subtract one from each of the other p-1 summands at each iteration. If this were not true, we would have, because $d_1 \geq d_2 \geq d_3 \geq \ldots d_p$, no way to decrease the later entries of the sequence $\{d_i\}$ without reordering the entries. The process will result in a totally disconnected graph in k iterations. Thus Π has form (9). QED.

We note that because d_1 = p-1, G is connected. Note further that the theorem also says what kinds of graphs have partitions which need no rearrangement in the algorithm.

It is easy to construct graphs which have no special points. Among these there must be some graph with a fewest number of points and lines. It turns oft that (8,9) is a minimal pair for graphs having no special points.

THEOREM XXII:

Among all (P,q) graphs, (8,9) is minimal for having no special points.

PROOF:

The *Figure 31* below demonstrates the existence of an (8,9) graph with no special points.

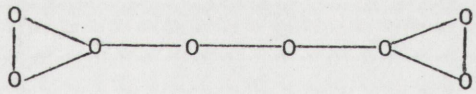


Figure 31.
(8,9) Graph With No Special Points

It is easy to verify by examining the graphs in the appendix of Harary's book, Graph Theory, that every graph on fewer than seven points has a special point. The search is much more tedious for graphs on seven points, but examination of the (7,q) graphs[2] $1 \leq q \leq 11$ together with Theorem XX shows that every graph on seven points has a special point. We thus arrive at the conclusion that p = 8 is the minimum number of points for a graph to have no special points.

Let us focus our attention on graphs with eight points. We shall

see that each 8-point graph with fewer than nine lines has a special point. The next five lemmas are about graphs on eight points. Each graph G is assumed to have no special points. Recall that isolated points are special points. C denotes a component: Δ_C is the maximum degree of all the points in C.

LEMMA XXII-A:

Each component must have atleast three points.

PROOF:

Suppose a component has exactly one point. Then it is an isolated point of G and therefore, a special point of G. If C has two non-isolated points, then each must be of degree one. G-C is then a graph on six points, and a special point of G-C is necessarily a special point of G. Therefore, $|C| \geq 3$. QED.

Since there are eight points in G and each component has at least three points, we have the following:

COROLLARY:

G has at most two components.

LEMMA XXII-B:

There cannot be a component on three points.

PROOF:

Suppose that C is a component of G on three points, and that C' is the other component of G on five points. Since C has three points, it must have some point of degree two. C' must then have a point u which is

special relative to C', but which is not special relative to G. The point u must be adjacent to a point v of degree 1 in order for u to not be special in G. Furthermore, the deg u < $\Delta_C = \Delta$ to keep v from being special in G. The figure below shows the relationship.

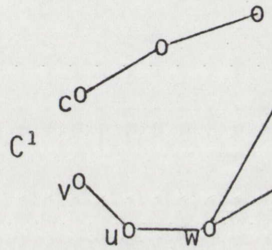


Figure 32.
Diagram for Lemma XXII-B

The deg u = 2, for if deg u > 2, there would have to be a point in C' of degree $\geq$ 4 in order to satisfy deg u < Δ_C. But this is impossible because there are only five points in C', and v adj u and deg v = 1. We may, therefore, conclude that deg u = 2. u is adjacent to some other than v point, say w, of C'. If deg w = 2, w must be adjacent to one of the two remaining points r or s of C', say w adj r. Now since C' is connected, we must have r adj s, and deg s = 1. But now u is not special relative to C' because it is not adjacent to r. Hence deg w = 3. So we have w adjacent to u, to r, and to s. If r is not adjacent to s, r becomes a point of degree 1 adjacent to a Δ-point, and thus special in G. If r is adjacent to s, u fails to be a special point of C' because it is not adjacent to r, a point of degree 2. This impossible dilemma forces us to reject the original hypothesis that there could be a component with three points. QED.

The last lemma leads us to consider the possibility of two components each of order four. We shall see that this is not possible either.

LEMMA XXII-C:

Let $|C| = |C'| = 4$. Then neither component can have a point of degree 1.

PROOF:

Suppose u is in C and that the deg u = 1. There are four points in each component of G so that $\Delta \leq 3$. If $\Delta = 2$, every point of degree 2 would be a special point if both components were paths, and every point of the cycle would be a special point of G if either component were a cycle. Since C_4 and P_4 are the only two possibilities when $\Delta = 2$, we conclude that $\Delta = 3$. We know that u must be adjacent to some point v and that deg $v \leq 3$ so deg v must be 2, since deg u = 1 and deg v = 3 implies that u is a sp pt ε G. This automatically forces the component C to be a path on four points. The other component C' then has a Δ-point, and can have no 1-degree point adjacent to this Δ-point. Thus C' is either K_4 or K_4-x. In either case the points of degree 3 are special in G. Therefore, the 1-degree point cannot belong to G. QED.

As we saw in the last lemma, there are only two connected graphs on four points which are permissible, and now we note that any combination of them results in a graph with a special point. Thus, we come to consider only connected eight point graphs. Consequently we have $q \geq 7$.

By examination of all 23 trees on eight points, we find that each has a special point. Thus we are led to (8,8) graphs which we shall show to have special points in each case, and this will establish the Theorem XXII.

LEMMA XXII-D:

G has at least two points of degree ≥ 3.

PROOF:

We already know that G must have at least one point of degree ≥ 3, for otherwise, it would be a cycle on eight points, or a path on eight points, each of which has special points. That there is another such point now follows quickly, because if there were not, the one point u of degree ≥ 3 would be adjacent to only point of degree 2 (a 1-degree point is forbidden from being adjacent to a Δ-point) and then u would be special in G. QED.

The (8,8) graphs fall into one of two classes:

class i $\quad \Delta = 3$

class ii $\quad \Delta > 3$

LEMMA XXII-E:

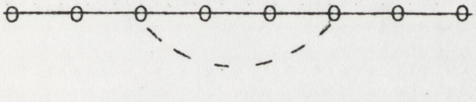
Every (8,8) graph has a special point.

PROOF:

The method of proof at this time will be to examine all possible eight point graphs with eight lines; these are of the form $T_8 + x$. In class i we may eliminate all 8-point trees which already have a point of degree > 3.

Class i $\quad \Delta = 3$.

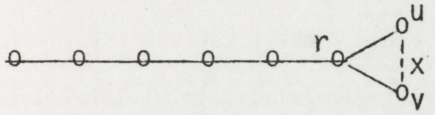
1\.



Two points r,s of degree 3 must be separated by at least two 2-degree points so that the 2-degree point t between r and s is not special, nor is r, in case no points separ-

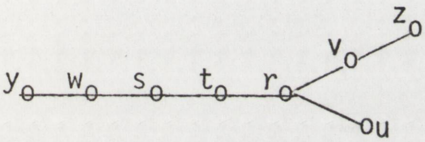
ate r and s. r cannot be adjacent to a 1-degree point, so there is just one remaining possibility (shown) and that alternative makes r special.

2.



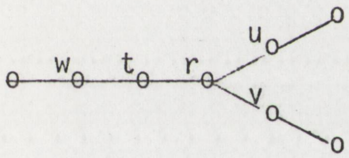
Both u and v must not have degree 1, but if x = uv, r is special.

3.



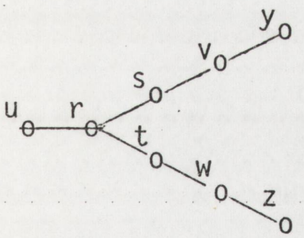
u must be adjacent to some point. v,w are out because of a 1-degpoint adj Δ-point. u adj s makes t special, and u adj t makes r special as does u adj y or z.

4.



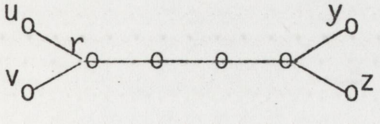
u, v, and w cannot be adjacent to any other point. If t were adjacent to any other possible point, r becomes special. Other possible adjacencies violate Lemma XXII-D.

5.



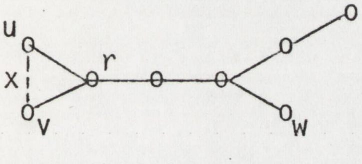
u must be adjacent to some point not v or w. If u adj s or t, u is special. If u adj y or z, r is special.

6.



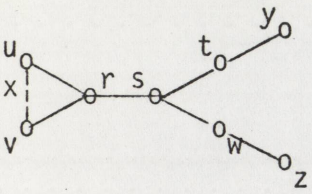
u, v, y, and z cannot have degree 1, and the addition of just one more line x fails to remedy this condition.

7.



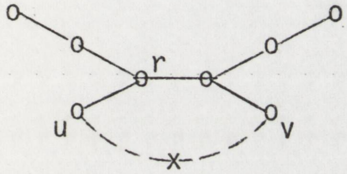
As in the previous cases, u,v, and w cannot remain of degree 1, and the addition of one more line does not correct the problem.

8.



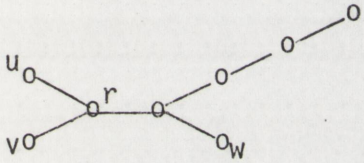
u and v must not have degree 1 and x = uv makes r special.

9.



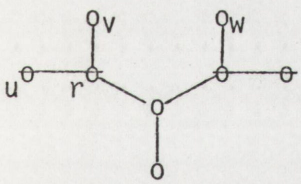
The same argument as in 8 holds.

10.



u, v, and w cannot remain of degree 1.

11.

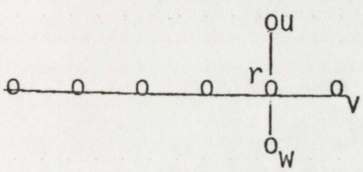


There are too many 1-degree points adjacent to Δ = 3-points.

We have now completed our search of class i and have found that every (8,8) graph in this class has a special point. It remains only to exhaust class ii in a similar manner. We first examine the trees which already have a point of degree 4 or higher, and then we take another look at those trees from class i which could be made to have a 4-degree point by the addition of a single line.

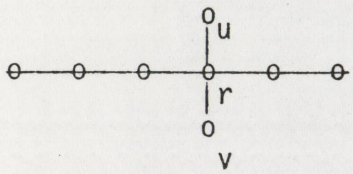
Class ii $\Delta > 3$.

12.



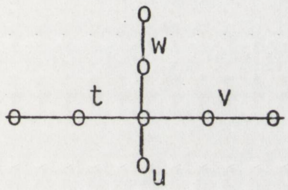
u, v, and w have degree 1.

13.



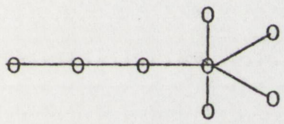
u must be adjacent to v, and then r becomes special.

14.



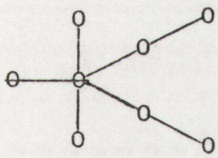
In order for G to have at least two points of degree 3, u must be adjacent to v, w, or t, and any of these adjacencies makes r special.

15.



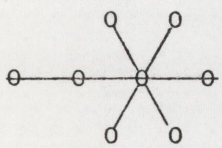
There are too many 1-degree points adjacent to a Δ-point.

16.



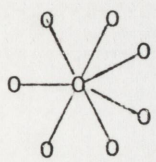
There are too many 1-degree points adjacent to a Δ-point.

17.



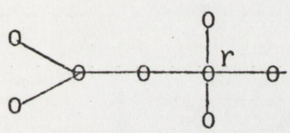
There are too many 1-degree points adjacent to a Δ-point.

18.



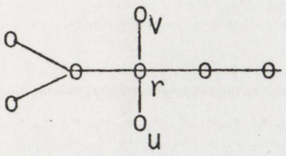
There are too many 1-degree points adjacent to a Δ-point.

19.



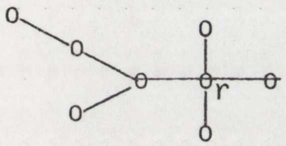
Too many 1-degree points are adjacent to r.

20.



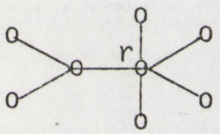
u adj v forces r to be special.

21.



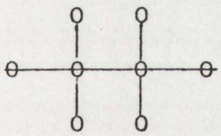
Too many 1-degree points are adjacent to r.

22.



There are too many 1-degree points adjacent to r.

23.



There are too many 1-degree points. adjacent to a Δ-point.

Let us now examine those graphs from class i which already have point(s) of degree 3 so that G can have a point of degree > 3 by the addition of a single line. Tree 1 has no possibility. Trees 2, 3, 5, 6, 7, 9, 10, and 11 are impossible because of the 1-degree point adjacent to a Δ-point property. The only possibility in 4 is r adj w, and this makes t special. As for 8, x cannot be incident with r. x incident with s forces x = sy or x = sz, either of which makes s special.

It is easy to verify that every nine point tree fails to not have a special point.

There are no other (8,8) graphs, so we have completed the proof to the theorem.

We are now led to a consideration of a special type of graph in which every point is a special point.

DEFINITION 10:

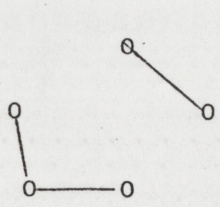
An airport graph is a graph every point of which is a special point.

It was determined that airlines do not adhere to airport graphs, so a better name might be "snob graphs". This name stems from the possibility that in some sociological setting, a person might well want to make friends with all people of "high rank" before he would make friends with a person of "lower rank".

NOTATION:

Denote by A the class of all airport graphs.

We have already seen that isolated points are special points. With another look at all graphs on fewer than five points we notice that they are all airport graphs. The minimal non-airport graph is the (5,3) which is shown in *Figure 33.*



not special

Figure 33.
Minimal Non-Airport Graph

It is convenient to sketch airport graphs with a level for each degree, and with all points of a given degree on the same level. Level (k) will designate the set of points on level k.

The first theorem about airport graphs follows easily as a corollary to Theorem XX.

THEOREM XXIII:

If $G \varepsilon A$, then $\overline{G} \varepsilon A$.

DEFINITION 11:

We shall refer to the graphs like *Figure 34* as twin stars. They are, in fact, Siamese twins. The tree consists of two $K_{1,n}$ graphs whose intersection is the two n-points and the line between them.

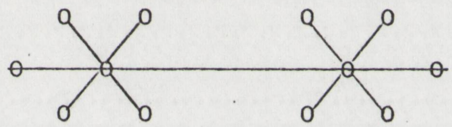


Figure 34.
A Twin Star

The following theorem characterizes trees in A.

THEOREM XXIV:

A tree $T \varepsilon A$ iff T is a twin star, or $K_{1,n}$.

PROOF:

The condition is obviously sufficient, so we need be concerned only with the necessity. Let T be a tree in A. If T is a path, it cannot have more than four points. If it were to have ≥ 5 points, the two points adjacent to the endpoints would fail to be special. P_4 can be thought of as the twin star from two $K_{1,2}$ graphs. Suppose then that T is not a path. It must then have a Δ-point of $\Delta \geq 3$.

If there is only one Δ-point r, we choose any point u adjacent to

r, and note that if u is adjacent to any other point v, v fails to be special because it is not adjacent to r, and it is adjacent to a point of degree < Δ. Thus, the degree of u is 1. Since u was arbitrary, $T = K_{1,n}$.

Suppose that there are two or more Δ-points. Actually, more than two Δ-points is impossible because any two such points which are not adjacent to each other, as is the case for trees, cannot be special points. For the same reason, the two Δ-points must be adjacent to each other. Let r,s be the two Δ-points. If u is any point such that u adj r and 1 < deg u < Δ, v is not special, so we may conclude that deg u = 1. Thus T is a twin star. QED.

The definition of an airport graph does not require that G be connected. The following theorem shows that this does not make the class A appreciably larger than it would have been if connectedness were required.

THEOREM XXV:

If $G \varepsilon A$ has more than one component, then

1. A component is an isolated point or
2. Every non-trivial component C_i is a complete subgraph and $|C_i| = k$ for some fixed k and for every i.

PROOF:

An isolated point is vacuously a special point. Consider any $i \neq j$ for which $\Delta_{C_i} \neq \Delta_{C_j}$. We may assume without loss of generality that $\Delta_{C_i} < \Delta_{C_j}$. Then every point of C_i fails to be a special of G. Thus, $\Delta_{C_i} = \Delta_{C_j} = \Delta$. If there is a point u in any component C_i such that degree

$u < \Delta$, then every Δ-point of C_i adjacent to u fails to be special in G. We now have that every non-trivial component can have only Δ-points. Therefore, every non-trivial component must be a complete graph on a fixed number, $K > 1$, of points. QED.

Next we observe several facts about airport graphs. Recall that #k means the number of points of degree k.

FACT A-1:

Every 1-degree point is adjacent only to a Δ-point.

FACT A-2:

Let $\sum_{i=k+1}^{\Delta} \#1 = j$. Then no point u with deg $u \leq j$ can be adjacent to a point of degree $\leq k$.

PROOF:

If u were adjacent to a point of degree $\leq k$, it would have to have degree at least $j + 1$.

FACT A-3:

If any point u from level (n) is adjacent to a point v from level ($k < n$), then every n-point is adjacent to the same number of k-points as is u.

PROOF:

Let the notation be as in Fact 2. If w is another n-point, the contribution to its degree from levels higher than k is only j. But u has degree $n \geq j + 1$. Thus w must be adjacent to some k-point. If every n-point is adjacent to every k-point, the theorem is satisfied. If the

number of k-points to which u is adjacent $< \#k$, w must be adjacent to precisely this same number of k-points in order for deg $w = n$ because deg w, like the deg $u = j + m$ where $1 \leq m < \#k$.

As a consequence of the previous fact we have

FACT A-4:

Every Δ-point is adjacent to the same number of points of degree 1.

FACT A-5:

Let G be a connected airport graph such that $\Delta \neq \delta$. Then $\Delta \geq p - \#\delta$.

PROOF:

Some Δ-point u must be adjacent to some δ-point. Thus $\Delta \geq (p-1) - (\#\delta - 1) = p - \#\delta$.

COROLLARY:

All Δ-points are mutually adjacent if $\Delta \neq \delta$

FACT A-6:

Regular graphs belong to A. ($\Delta = \delta$)

Not only is the Δ - level mutually adjacent if $\Delta \neq \delta$, but a large subgraph of every airport graph is a complete subgraph.

THEOREM XXVI:

For every $G \in A$, let j be the smallest number such that a point u $\in$ level (j) is adjacent to some point $v \in$ level$(k < j)$. Then the subgraph from all levels $\geq$ j is a complete subgraph of G.

PROOF:

Figure 35 illustrates the nature of this proof, as well as the general structure of airport graphs.

Theorem XXV shows that Theorem XXVI is true if G has > 1 component. We therefore assume that G is connected. Let u be as in the hypothesis, but with the further requirement that the level k to which u is adjacent is the lowest level to which a j-point is adjacent. By fact A - 3, we know that all points from level(j) are adjacent to points from level(k). That

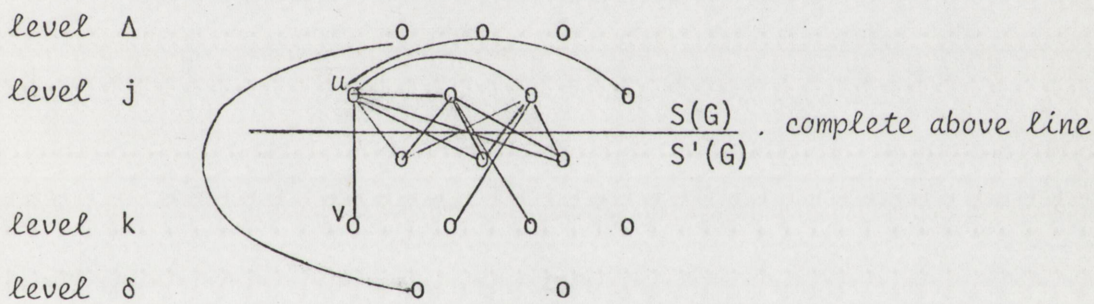


Figure 35.
Diagram for Theorem XXVI

forces u, and every other j-point to be adjacent to every point from levels higher than k, which includes level(j).

Thus the graph G is complete from level(j) up. QED.

Note that in the preceding theorem it would be possible for the complete subgraph to include the first level below level j. This would not be a violation of the way in which j was chosen.

DEFINITION 12:

The complete part of an airport graph G will be called the super-structure, S(G). S'(G) will be the substructure of G.

We notice further that no point from two or more levels below level (j) can be a part of S(G).

FACT A-7:

Let j,k be as in Theorem XXVI. Then if #j > #k, then $k \geq 2$.

PROOF:

If #j > #k, we see by the use of the pigeonhole principle, and by Fact A-3, that every j-point must be adjacent to at least two k points. QED.

We have seen the nature of the adjacencies among levels of G; the next theorem shows that there cannot be adjacencies within a level much below S(G).

THEOREM XXVII:

Let m be the degree of a level two levels below level(j). Then no point from level $n \leq m$ can be adjacent to any other point from level n.

PROOF:

Let u,v, ε level(n) and suppose u adj v. Then u must be adjacent to every point from levels > n, which includes a level < j. But this makes a point from a level < j have a downward adjacency, contrary to the way in which j was chosen. QED.

THEOREM XXVIII:

Let level(m) be the level immediately below S(G). Relative to level m, G is regular.

PROOF:

If $m = 0$, the theorem is vacuously true. Let u adj v where u,v ε level(n). u must be adjacent to every point of S(G) - level(m). Thus the degree of each m-point is

$$m = \sum_{k=m+1}^{\Delta} \#k + \text{number of adjacencies within level}(m); \qquad (10)$$

because there are no downward adjacencies from level(m). This implies that every point of level(m) is adjacent to the same number of points from level (m); i.e. relative to level(m), G is regular. QED.

The two previous theorems tell us a great deal about the structure of G. G must be complete above some level, j. The level immediately below j may or may not be part of the superstructure, but it is certainly regular in its own level. All levels below that can be adjacent only to points from level(j) or higher, so that G - {S(G) u level(m)} is totally disconnected.

The last two theorems, together with the remark, give us the following corollary.

COROLLARY:

G - S(G) ε A.

As a consequence of all the above structure of G, we can see that for arbitrary m and n, if u ε level(m), every point v from level(m) is adjacent to precisely the same number of n-points as is u.

We already know that not all graphs have special points, and therefore not all graphs are airport graphs. Next we shall see an algorithm

for constructing an airport graph for any arbitrary (p,q) graph.

THEOREM IXXX: (Construction Algorithm)

For every p,q ε {whole numbers} with $q \leq \frac{p(p-1)}{2}$, there exists a (p,q) graph G ε A.

PROOF: (Algorithm)

1. Label the points $u_1, u_2, \ldots, u_p$
2. Set $q_0 = q$ and go to 4.
3. Form $q_i = q_{i-1} - (p-i)$
4. If $q_i \leq 0$ stop.

   If $q_i > 0$ make u_{i+1} sequentially adjacent to as many of the remaining points as possible.
5. Return to 3.

From the very nature of the construction, no point is made adjacent to a point of degree k until it is made adjacent to all points of degree > k. Thus every point must be special. Because $q \leq \frac{p(p-1)}{2}$, the algortihm must terminate. QED.

Figure 36 shows the construction of an (6,10) airport graph.

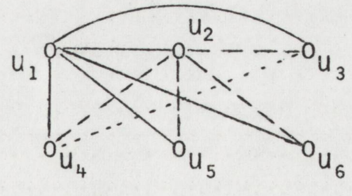


Figure 36.
A (6,10) *Constructed from Algorithm*

The line sequence, solid, then dashed, then dotted, indicates the order in which the lines of step four are filled in. Below is a rearrangement of the same graph to put it into the standard airport graph form.

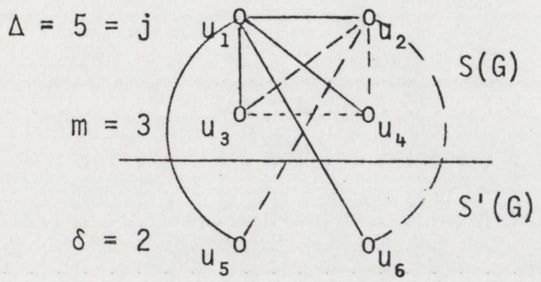


Figure 37.
Standard Airport Graph Form for Figure 36.

Unfortunately, for a given (p,q), an airport graph is not unique as the following figure shows for (6,10) graphs.

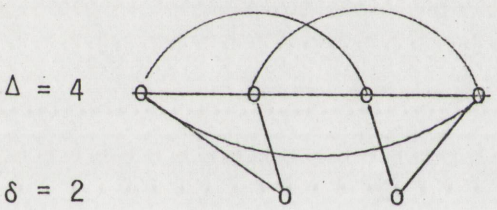


Figure 38.
Another (6,10) Airport Graph

III

THE DISTANCE OF A GRAPH

The concept of an airport graph leads to the idea of the distance of a point and of a graph. Unfortunately, there does not actually seem to be any relationship between the two concepts.

DEFINITION 13:

The distance of a point u in G, D(u), is

$$D(u) = \sum_{v \in G} d(u,v) \text{ where } d(u,v) \text{ means the distance} \qquad (11)$$

from u to v.

The distance of G is

$$D(G) = \frac{\sum_{u \in G} D(u)}{2} \qquad (12)$$

The following theorem gives the distance of several familiar graphs.

THEOREM XXX:

1. $D(P_n) = \frac{n^3-n}{6}$
2. $D(K_{1,n}) = n^2$
3. $D(K_{m,n}) = (m+n)^2 - (m+n+mn)$
4. $D(C_{2n}) = n^3$
5. $D(C_{2n+1}) = \frac{n(n+1)\ (2n+1)}{2}$
6. $D(K_p) = \frac{p(p-1)}{2}$

PROOF:

1. Observe first that for a tree on n points and for the same tree with an endpoint e removed, the following relationship holds:

$$D(T_n) = D(e) + D(T_{n-1}) \qquad (13)$$

The proof of 1. is by induction. The formula holds for n = 1, and n - 2. Assume therefore that the formula is true through n = 1. Then

$$\begin{aligned} D(P_n) &= D(e) + D(P_{n-1}) \\ &= \sum_{k=1}^{n-1} k + \frac{(n-1)^3 - (n-1)}{6} \\ &= \frac{3n^2 - 3n + n^3 - 3n^2 + 3n - 1 - n + 1}{6} \\ &= \frac{n^3 - n}{6} \end{aligned}$$

Thus $n \longrightarrow n + 1$, and the proof to 1. is complete.

2. In $K_{1,n}$ there are n points with distance two from the other n - 1 points, and one point at distance one from n points. Thus

$$\begin{aligned} D(K_{1,n}) &= \frac{2n(n-1)}{2} + n \\ &= n^2 \end{aligned}$$

3. A typical point in the m-set has distance 2 from the other m-1 m-points. A similar relationship holds for the n-set. In addition to those distances, there are mn units. Thus

$$\begin{aligned} K(K_{m,n}) &= \frac{2m(m-1)}{2} + \frac{2n(n-1)}{2} + mn \\ &= m^2 - m + n^2 - n + mn \\ &= (m+n)^2 - (m+n+mn) \end{aligned}$$

4. Consider any point u. Its distance is computed in two directions, then adding in the distance to the antipodal point. Thus

$$D(C_{2n}) = \frac{\left[2\sum_{k=1}^{n-1} k + n\right] 2n}{2}$$

$$= \left[\frac{2n(n-1)}{2} + n\right] n$$

$$= [n^2 - n + n]\, n$$

$$= n^3$$

5. For odd cycles, a typical point must have its distance determined along two equal arcs. Therefore

$$D(C_{2n+1}) = \frac{\left[2\sum_{k=1}^{n} k\right] (2n + 1)}{2}$$

$$= \frac{\frac{2n(n+1)}{2}\,(2n+1)}{2}$$

$$= \frac{n(n+1)\,(2n+1)}{2}$$

6. The complete graph K_p has each of its p points at distance one from every other point. Hence

$$D(K_p) = \frac{p(p-1)}{2}$$

We note that as a corollary to 3, we have

$$D(K_{m,m}) = 3m^2 - 2m = m(3m-2).$$

THEOREM XXXI:

The distance of a path, $D(P_n)$, is an upper bound for distances of connected graphs on n points.

PROOF:

Every connected graph G has a spanning tree T_G and it is clear from the triangle inequality that $D(G) \leq D(T_G)$, with strict inequality if G is not a tree.

We now show that $D(T_G) \leq D(P_n)$. The argument here is by induction, and is similar to that of part 1 of the previous theorem. It is clearly true for n = 1. Assume the inequality is true through n-1. Then we split off an endpoint e from T_G and maximize D(e). Because T_G is connected, a distance of k cannot be attained until a distance of k-1 has already been reached. Thus max D(e) = 1 + 2 + 3 + ... + (n-1). Therefore

$$D(T_G) \leq \max D(e) + D(T_{G-e})$$
$$= \sum_{k=1}^{n-1} k + \frac{(n-1)^3 - (n-1)}{6}$$
$$= \frac{n^3 - n}{6} = D(P_n) \quad \text{QED.}$$

We have seen that the removal of an endpoint from a path or from a tree has the following relationship:

$$D(T_n) = D(T_{n-1}) + D(e), \text{ where e is an endpoint}$$

This is a special case of the following more general theorem.

THEOREM XXXII:

$$D(G) \leq D(G-u) + D(u) \tag{14}$$

PROOF:

We may assume that G is connected, since the inequality is true otherwise. If G-u is disconnected, the inequality holds, so we assume further that u is not a cutpoint. Now examine D(G-u). Whatever this number

is, the addition of another point u may decrease the distance between some pair of points of G-u, but the addition can certainly not increase this distance. Thus the inequality is established.

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2. Crow, D. W., "Graphs on Seven Points," (unpublished manuscript, University of Wisconsin).