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RANDOMNESS AND OPTIMAL ESTIMATION IN DATA SAMPLING

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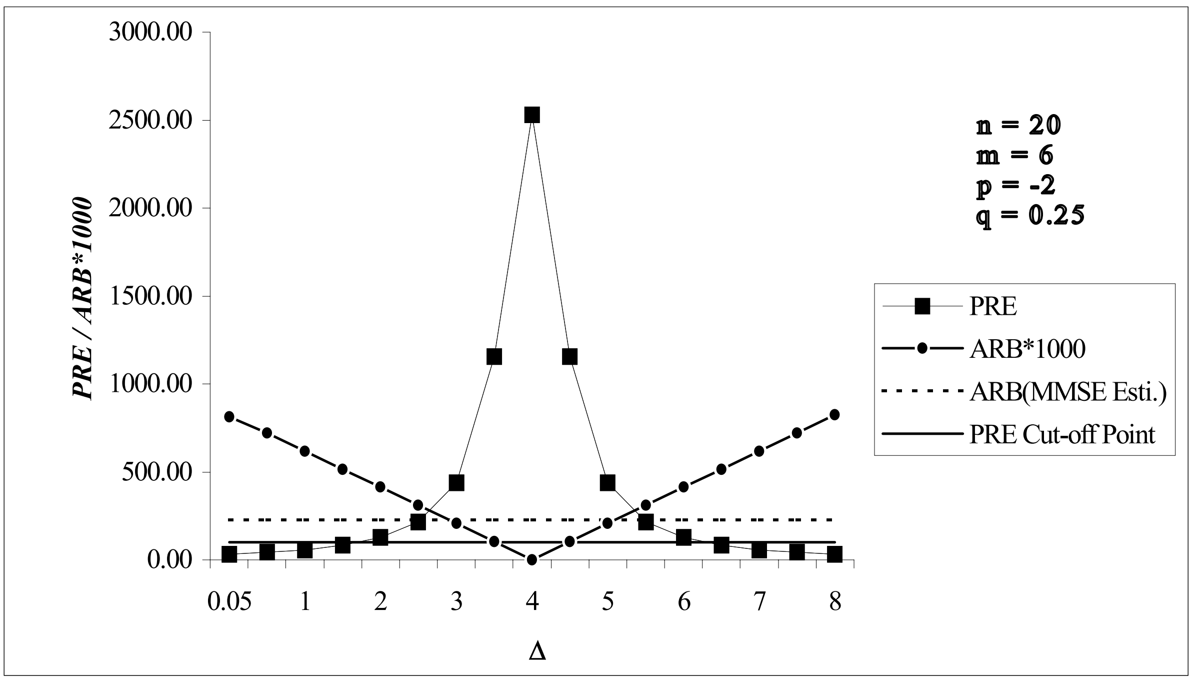

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Forward

The purpose of this book is to postulate some theories and test them numerically. Estimation is often a difficult task and it has wide application in social sciences and financial market. In order to obtain the optimum efficiency for some classes of estimators, we have devoted this book into three specialized sections:

Part 1. In this section we have studied a class of shrinkage estimators for shape parameter beta in failure censored samples from two-parameter Weibull distribution when some 'apriori' or guessed interval containing the parameter beta is available in addition to sample information and analyses their properties. Some estimators are generated from the proposed class and compared with the minimum mean squared error (MMSE) estimator. Numerical computations in terms of percent relative efficiency and absolute relative bias indicate that certain of these estimators substantially improve the MMSE estimator in some guessed interval of the parameter space of beta, especially for censored samples with small sizes. Subsequently, a modified class of shrinkage estimators is proposed with its properties.

Part2. In this section we have analyzed the two classes of estimators for population median $\mathbf{M_Y}$ of the study character Y using information on two auxiliary characters X and Z in double sampling. In this section we have shown that the suggested classes of estimators are more efficient than the one suggested by Singh *et al* (2001). Estimators based on estimated optimum values have been also considered with their properties. The optimum values of the first phase and second phase sample sizes are also obtained for the fixed cost of survey.

Part3. In this section, we have investigated the impact of measurement errors on a family of estimators of population mean using multiauxiliary information. This error minimization is vital in financial modeling whereby the objective function lies upon minimizing over-shooting and undershooting.

This book has been designed for graduate students and researchers who are active in the area of estimation and data sampling applied in financial survey modeling and applied statistics. In our future research, we will address the computational aspects of the algorithms developed in this book.

The Authors

Estimation of Weibull Shape Parameter by Shrinkage Towards An Interval Under Failure Censored Sampling

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Abstract

This paper is speculated to propose a class of shrinkage estimators for shape parameter β in failure censored samples from two-parameter Weibull distribution when some 'apriori' or guessed interval containing the parameter β is available in addition to sample information and analyses their properties. Some estimators are generated from the proposed class and compared with the minimum mean squared error (MMSE) estimator. Numerical computations in terms of percent relative efficiency and absolute relative bias indicate that certain of these estimators substantially improve the MMSE estimator in some guessed interval of the parameter space of β, especially for censored samples with small sizes. Subsequently, a modified class of shrinkage estimators is proposed with its properties.

Key Words & Phrases:

Two-parameter Weibull distribution, Shape parameter, Guessed interval, Shrinkage estimation technique, Absolute relative bias, Relative mean square error, Percent relative efficiency.

2000 MSC: 62E17

1. INTRODUCTION

Identical rudiments subjected to identical environmental conditions will fail at different and unpredictable times. The 'time of failure' or 'life length' of a component, measured from some specified time until it fails, is represented by the continuous random variable *X*. One distribution that has been used extensively in recent years to deal with such problems of reliability and life-testing is the Weibull distribution introduced by Weibull(1939), who proposed it in connection with his studies on strength of material.

The Weibull distribution includes the exponential and the Rayleigh distributions as special cases. The use of the distribution in reliability and quality control work was advocated by many authors following Weibull(1951), Lieblin and Zelen(1956), Kao(1958,1959), Berrettoni(1964) and Mann(1968 A). Weibull(1951) showed that the distribution is useful in describing the 'wear-out' or fatigue failures.

Kao(1959) used it as a model for vacuum tube failures while Lieblin and Zelen(1956) used it as a model for ball bearing failures. Mann(1968 A) gives a variety of situations in which the distribution is used for other types of failure data. The distribution often becomes suitable where the conditions for "strict randomness" of the exponential distribution are not satisfied with the shape parameter β having a characteristic or predictable value depending upon the fundamental nature of the problem being considered.

1.1 The Model

Let x_1, x_2, …, x_n be a random sample of size n from a two-parameter Weibull distribution, probability density function of which is given by :

$$f\left(x;\alpha,\beta\right)=\beta\alpha^{-\beta}x^{\beta-1}\exp\left\{-\left(x/\alpha\right)^{\beta}\right\};\ x>0,\alpha>0,\beta>0$$

(1.1)

where α being the characteristic life acts as a scale parameter and β is the shape parameter.

The variable $Y=\ln x$ follows an extreme value distribution, sometimes called the log-Weibull distribution [e.g. White(1969)], cumulative distribution function of which is given by :

$$F\left(y\right)=1-\exp\left\{-\exp\left(\frac{y-u}{b}\right)\right\};-\infty<y<\infty,-\infty<u<\infty,b>0$$

(1.2)

where $b = 1/\beta$ and $u = \ln\alpha$ are respectively the scale and location parameters.

The inferential procedures of the above model are quite complex. Mann(1967 A,B, 1968 B) suggested the generalised least squares estimator using the variances and covariances of the ordered observations for which tables are available up to $n = 25$ only.

1.2 Classical Estimators

Suppose x_1, x_2, …, x_m be the m smallest ordered observations in a sample of size n from Weibull distribution. Bain(1972) defined an unbiased estimator for b as

$$\hat{b}_u=-\sum_{i=1}^{m-1}\left\{\frac{y_i-y_m}{nK_{(m,n)}}\right\},$$

(1.3)

where $$K_{(m,n)}=-\left(\frac{1}{n}\right)E\left[\sum_{i=1}^{m-1}\left(v_i-v_m\right)\right],$$

(1.4)

and $v_i = \dfrac{y_i - u}{b}$ are ordered variables from the extreme value distribution with $u = 0$ and $b =$ 1.The estimator $\hat{b}_u$ is found to have high relative efficiency for heavily censored cases. Contrary to this, the asymptotic relative efficiency of $\hat{b}_u$ is zero for complete samples.

Engelhardt and Bain(1973) suggested a general form of the estimator as

$$\hat{b}_g = -\sum_{i=1}^{m} \left\{ \frac{|y_i - y_m|}{nK_{(g,m,n)}} \right\},$$

(1.5)

where g is a constant to be chosen so that the variance of $\hat{b}_g$ is least and $K_{(g,m,n)}$ is an unbiasing constant. The statistic $\dfrac{h\hat{b}_g}{b}$ has been shown to follow approximately χ^2 - distribution with h degrees of freedom, where $h = 2 \Big/ Var\left(\hat{b}_g / b\right)$. Therefore, we have

$$E\left[\hat{\beta}^{-jp}\right] = \frac{1}{\beta^{jp}} \left(\frac{2}{h-2}\right)^{jp} \frac{\Gamma\left[(h/2) + jp\right]}{\Gamma(h/2)} \quad ; \; j = 1,2$$

(1.6)

where $\hat{\beta} = \dfrac{h-2}{t}$ is an unbiased estimator of β with $\mathrm{Var}(\hat{\beta}) = \dfrac{2\beta^2}{(h-4)}$ and $t = h\hat{b}_g$ having density

$$f(t) = \frac{1}{\Gamma(h/2)} \left(\frac{\beta}{2}\right)^{h/2} \exp\left(\frac{-\beta t}{2}\right) t^{(h/2)-1} \; ; t > 0 .$$

The MMSE estimator of β, among the class of estimators of the form $C\hat{\beta}$; C being a constant for which the mean square error (MSE) of $C\hat{\beta}$ is minimum, is

$$\hat{\beta}_M = \frac{h-4}{t},$$

(1.7)

having absolute relative bias and relative mean squared error as

$$\mathrm{ARB}\left\{\hat{\beta}_M\right\} = \left|\frac{2}{h-2}\right|,$$

(1.8)

and $$\mathrm{RMSE}\left\{\hat{\beta}_M\right\} = \frac{2}{h-2},$$

(1.9)

respectively.

1.3 Shrinkage Technique of Estimation

Considerable amount of work dealing with shrinkage estimation methods for the parameters of the Weibull distribution has been done since 1970. An experimenter involved in life-testing experiments becomes quite familiar with failure data and hence may often develop knowledge about some parameters of the distribution. In the case of Weibull distribution, for example, knowledge on the shape parameter β can be utilised to develop improved inference for the other parameters. Thompson(1968 A,B) considered the problem of shrinking an unbiased estimator $\hat{\xi}$ of the parameter ξ either towards a natural origin ξ_0 or towards an interval $\left(\xi_1, \xi_2\right)$ and suggested the shrunken estimators $h\hat{\xi} + (1-h)\xi_0$ and $h\hat{\xi} + (1-h)\left(\frac{\xi_1 + \xi_2}{2}\right)$, where $0 < h < 1$ is a constant. The relevance of such type of shrunken estimators lies in the fact that, though perhaps they are biased, has smaller MSE than $\hat{\xi}$ for ξ in some interval around ξ_0 or $\left(\frac{\xi_1 + \xi_2}{2}\right)$, as the case may be. This type of shrinkage estimation of the Weibull parameters has been discussed by various authors, including Singh and Bhatkulikar(1978), Pandey(1983), Pandey and Upadhyay(1985,1986) and Singh and Shukla(2000). For example, Singh and Bhatkulikar(1978) suggested performing a significance test of the validity of the prior value of β (which they took as 1). Pandey(1983) also suggested a similar preliminary test shrunken estimator for β.

In the present investigation, it is desired to estimate β in the presence of a prior information available in the form of an interval $\left(\beta_1, \beta_2\right)$ and the sample information contained in $\hat{\beta}$. Consequently, this article is an attempt in the direction of obtaining an efficient class of shrunken estimators for the scale parameter β. The properties of the suggested class of estimators are also discussed theoretically and empirically. The proposed class of shrunken estimators is furthermore modified with its properties.

2. THE PROPOSED CLASS OF SHRINKAGE ESTIMATORS

Consider a class of estimators $\overset{*}{\beta}_{(p,q)}$ for β in model (1.1) defined by

$$\overset{*}{\beta}_{(p,q)} = \left(\frac{\beta_1+\beta_2}{2}\right)\left[q + w\left(\frac{\beta_1+\beta_2}{2\hat{\beta}}\right)^p\right],$$

(2.1)

where p and q are real numbers such that $p \neq 0$ and $q > 0$, w is a stochastic variable which may in particular be a scalar, to be chosen such that MSE of $\overset{*}{\beta}_{(p,q)}$ is minimum.

Assuming w a scalar and using result (1.6), the MSE of $\overset{*}{\beta}_{(p,q)}$ is given by

$$\text{MSE}\left\{\overset{*}{\beta}_{(p,q)}\right\} = \beta^2\left[\{q\Delta - 1\}^2 + w^2\Delta^{2(p+1)}\left(\frac{2}{h-2}\right)^{2p}\frac{\Gamma[(h/2)+2p]}{\Gamma(h/2)}\right.$$

$$\left. + \{q\Delta - 1\}w\Delta^{(p+1)}\left(\frac{2}{h-2}\right)^{p}\frac{\Gamma[(h/2)+2p]}{\Gamma(h/2)}\right]$$

(2.2)

where $$\Delta = \left(\frac{\beta_1+\beta_2}{2\beta}\right).$$

Minimising (2.2) with respect to w and replacing β by its unbiased estimator $\hat{\beta}$, we get

$$\hat{w} = \frac{-\left\{q\left(\frac{\beta_1+\beta_2}{2}\right) - \hat{\beta}\right\}\hat{\beta}^{\,p}}{\left(\frac{\beta_1+\beta_2}{2}\right)^{(p+1)}} w(p).$$

(2.3)

where $$w(p) = \left(\frac{h-2}{2}\right)^p \frac{\Gamma[(h/2)+p]}{\Gamma[(h/2)+2p]},$$

(2.4)

lies between 0 and 1, {i.e., $0 < w(p) \leq 1$} provided gamma functions exist, i.e., $p > (-h/2)$.

Substituting (2.3) in (2.1) yields a class of shrinkage estimators for β in a more feasible form as

$$\hat{\beta}_{(p,q)} = \left(\frac{h-2}{t}\right)w(p) + q\left(\frac{\beta_1+\beta_2}{2}\right)\{1 - w(p)\}.$$

(2.5)

2.1 Non-negativity

Clearly, the proposed class of estimators (2.5) is the convex combination of $\{(h-2)/t\}$ and $\{q(\beta_1+\beta_2)/2\}$ and hence $\hat{\beta}_{(p,q)}$ is always positive as $\{(h-2)/t\} > 0$ and $q > 0$.

2.2 Unbiasedness

If $w(p) = 1$, the proposed class of shrinkage estimators $\hat{\beta}_{(p,q)}$ turns into the unbiased estimator $\hat{\beta}$, otherwise it is biased with

$$\text{Bias}\left\{\hat{\beta}_{(p,q)}\right\} = \beta\ \{q\Delta - 1\}\,[1 - w(p)] \tag{2.6}$$

and thus the absolute relative bias is given by

$$\text{ARB}\left\{\hat{\beta}_{(p,q)}\right\} = \left|\ \{q\Delta - 1\}\,[1 - w(p)]\ \right|.$$

(2.7)

The condition for unbiasedness that $w(p) = 1$, holds iff, censored sample size m is indefinitely large, i.e., $m \to \infty$. Moreover, if the proposed class of estimators $\hat{\beta}_{(p,q)}$ turns into $\hat{\beta}$ then this case does not deal with the use of prior information.

A more realistic condition for unbiasedness without damaging the basic structure of $\hat{\beta}_{(p,q)}$ and utilises prior information intelligibly can be obtained by (2.7). The ARB of $\hat{\beta}_{(p,q)}$ is zero when $q = \Delta^{-1}$ (or $\Delta = q^{-1}$).

2.3 Relative Mean Squared Error

The MSE of the suggested class of shrinkage estimators is derived as

$$\text{MSE}\left\{\hat{\beta}_{(p,q)}\right\} = \beta^2\left[\{q\Delta - 1\}^2\{1 - w(p)\}^2 + \frac{2\{w(p)\}^2}{(h-4)}\right], \tag{2.8}$$

and relative mean square error is therefore given by

$$\text{RMSE}\left\{\hat{\beta}_{(p,q)}\right\} = \{q\Delta - 1\}^2\{1 - w(p)\}^2 + \frac{2\{w(p)\}^2}{(h-4)}.$$

(2.9)

It is obvious from (2.9) that RMSE $\{\hat{\beta}_{(p,q)}\}$ is minimum when $q = \Delta^{-1}$ (or $\Delta = q^{-1}$).

2.4 Selection of the Scalar '*p*'

The convex nature of the proposed statistic and the condition that gamma functions contained in $w(p)$ exist, provides the criterion of choosing the scalar p. Therefore, the acceptable range of value of p is given by

$$\{p \mid 0 < w(p) \le 1 \text{ and } p > (-h/2)\},\ \forall\, n, m. \tag{2.10}$$

2.5 Selection of the Scalar '*q*'

It is pointed out that at $q = \Delta^{-1}$, the proposed class of estimators is not only unbiased but renders maximum gain in efficiency, which is a remarkable property of the proposed class of estimators. Thus obtaining significant gain in efficiency as well as proportionately small magnitude of bias for fixed Δ or for fixed (β_1/β) and (β_2/β), one should choose q in the vicinity of $q = \Delta^{-1}$. It is interesting to note that if one selects smaller values of q then higher values of Δ leads to a large gain in efficiency (along with appreciable smaller magnitude of bias) and vice-versa. This implies that for smaller values of q, the proposed class of estimators allows to choose the guessed interval much wider, i.e., even if the experimenter is less experienced the risk of estimation using the proposed class of estimators is not higher. This is legitimate for all values of p.

2.3 Estimation of Average Departure: A Practical Way of selecting *q*

The quantity $\Delta = \{(\beta_1 + \beta_2)/2\beta\}$, represents the average departure of natural origins β_1 and β_2 from the true value β. But in practical situations it is hardly possible to get an idea about Δ. Consequently, an unbiased estimator of Δ is proposed, namely

$$\hat{\Delta} = \left\{\frac{t\,(\beta_1 + \beta_2)}{4}\right\}\frac{\Gamma(h/2)}{\Gamma[(h/2)+1]}.$$

(2.12)

In section 2.5 it is investigated that, if $q = \Delta^{-1}$, the suggested class of estimators yields favourable results. Keeping in view of this concept, one may select q as

$$q = \hat{\Delta}^{-1} = \left\{\frac{4}{t\,(\beta_1 + \beta_2)}\right\}\frac{\Gamma[(h/2)+1]}{\Gamma(h/2)}. \tag{2.13}$$

Here this is fit for being quoted that this is the criterion of selecting q numerically and one should

carefully notice that this doesn't mean q is replaced by (2.13) in $\hat{\beta}_{(p,q)}$.

3. COMPARISION OF ESTIMATORS AND EMPIRICAL STUDY

James and Stein(1961) reported that minimum MSE is a highly desirable property and it is therefore used as a criterion to compare different estimators with each other. The condition under which the proposed class of estimators is more efficient than the MMSE estimator is given below.

MSE $\left\{\hat{\beta}_{(p,q)}\right\}$ does not exceed the MSE of MMSE estimator $\hat{\beta}_M$ if -

$$\left(1-\sqrt{G}\right)q^{-1} < \Delta < \left(1+\sqrt{G}\right)q^{-1}$$

(3.1)

where $$G = \frac{2}{\{1-w(p)\}^2}\left[\frac{1}{(h-2)} - \frac{\{w(p)\}^2}{(h-4)}\right].$$

Besides minimum MSE criterion, minimum bias is also important and therefore should be incorporated under study. Thus, ARB $\left\{\hat{\beta}_{(p,q)}\right\}$ is less than ARB $\left\{\hat{\beta}_M\right\}$ if -

$$\left\{1-\frac{2}{(h-2)\left(1-w_{(p)}\right)}\right\}q^{-1} < \Delta < \left\{1+\frac{2}{(h-2)\left(1-w_{(p)}\right)}\right\}q^{-1}$$

(3.2)

3.1 The Best Range of Dominance of Δ

The intersection of the ranges of Δ in (3.1) and (3.2) gives the best range of dominance of Δ denoted by Δ_{Best}. In this range, the proposed class of estimators is not only less biased than the MMSE estimator but is more efficient than that. The four possible cases in this regard are:

(i) if $\left\{1-\frac{2}{(h-2)[1-w(p)]}\right\} < \left(1-\sqrt{G}\right)$ and $\left\{1+\frac{2}{(h-2)[1-w(p)]}\right\} < \left(1+\sqrt{G}\right)$ then

$$\Delta_{Best} = \left(\left\{1-\sqrt{G}\right\}q^{-1}\ ,\ \left\{1+\frac{2}{(h-2)[1-w(p)]}\right\}q^{-1}\right)$$

(ii) if $\left\{1-\frac{2}{(h-2)[1-w(p)]}\right\} < \left(1-\sqrt{G}\right)$ and $\left(1+\sqrt{G}\right) < \left\{1+\frac{2}{(h-2)[1-w(p)]}\right\}$ then

Δ_{Best} is the same as defined in (3.1).

(iii) if $\left(1-\sqrt{G}\right)<\left\{1-\frac{2}{(h-2)\left[1-w(p)\right]}\right\}$ and $\left(1+\sqrt{G}\right)<\left\{1+\frac{2}{(h-2)\left[1-w(p)\right]}\right\}$ then

$$\Delta_{Best}=\left(\left\{1-\frac{2}{(h-2)\left[1-w(p)\right]}\right\}q^{-1}\ ,\ \left\{1+\sqrt{G}\right\}q^{-1}\right)$$

(iv) if $\left(1-\sqrt{G}\right)<\left\{1-\frac{2}{(h-2)\left[1-w(p)\right]}\right\}$ and $\left\{1+\frac{2}{(h-2)\left[1-w(p)\right]}\right\}<\left(1+\sqrt{G}\right)$ then

Δ_{Best} is the same as defined in (3.2).

3.2 Percent Relative Efficiency

To elucidate the performance of the proposed class of estimators $\hat{\beta}_{(p,q)}$ with the MMSE estimator $\hat{\beta}_M$, the Percent Relative Efficiencies (PREs) of $\hat{\beta}_{(p,q)}$ with respect to $\hat{\beta}_M$ have been computed by the formula:

$$\text{PRE}\left\{\hat{\beta}_{(p,q)},\hat{\beta}_M\right\}=\frac{2(h-4)}{(h-2)\left[(q\Delta-1)^2\{1-w(p)\}^2(h-4)+2\{w(p)\}^2\right]}\times 100$$

(3.5) The PREs of $\hat{\beta}_{(p,q)}$ with respect to $\hat{\beta}_M$ and ARBs of $\hat{\beta}_{(p,q)}$ for fixed n = 20 and different values of p, q, m $\Delta_1\left(=\beta_1/\beta\right)$ and $\Delta_2\left(=\beta_2/\beta\right)$ or Δ are compiled in Table 3.1 with corresponding values of h [which can be had from Engelhardt(1975)] and $w(p)$. The first column in every m corresponds to PREs and the second one corresponds to ARBs of $\hat{\beta}_{(p,q)}$. The last two rows of each set of q includes the range of dominance of Δ and Δ_{Best}. The ARBs of $\hat{\beta}_M$ has also been given at the end of each set of table.

Table 3.1

PREs of proposed estimator $\hat{\beta}_{(p,q)}$ with respect to MMSE estimator $\hat{\beta}_m$ and ARBs of $\hat{\beta}_{(p,q)}$

$q\downarrow$			p = -2								
	$\Delta_1\downarrow$	$\Delta_2\downarrow$	$m\rightarrow$	6		8		10		12	
			$h\rightarrow$	10.8519		15.6740		20.8442		26.4026	
			$\Delta\downarrow$ $w(p)\rightarrow$	0.1750		0.3970		0.5369		0.6305	
0.25	0.1	0.2	0.15	35.33	0.7941	40.20	0.5804	45.57	0.4457	50.60	0.3556
	0.4	0.6	0.50	42.62	0.7219	47.90	0.5276	53.49	0.4052	58.53	0.3233
	0.4	1.6	1.00	57.66	0.6188	63.18	0.4522	68.54	0.3473	72.99	0.2771
	1.0	2.0	1.50	82.21	0.5156	86.53	0.3769	89.95	0.2894	92.27	0.2309
	1.6	2.4	2.00	126.15	0.4125	124.06	0.3015	120.83	0.2315	117.72	0.1847
	2.0	3.0	2.50	215.89	0.3094	187.20	0.2261	164.84	0.1737	149.86	0.1386
	2.5	3.5	3.00	438.90	0.2063	294.12	0.1507	222.82	0.1158	186.17	0.0924
	3.5	3.5	3.50	1154.45	0.1031	447.47	0.0754	282.42	0.0579	217.84	0.0462
	3.8	4.2	4.00	2528.52	0.0000	541.60	0.0000	310.07	0.0000	230.93	0.0000
	Range of $\Delta\rightarrow$			(1.74, 6.25)	(2.90, 5.09)	(1.70, 6.29)	(3.02, 4.97)	(1.68, 6.31)	(3.08, 4.91)	(1.66, 6.33)	(3.11, 4.88)
	$\Delta_{Best}\rightarrow$			(2.90, 5.09)		(3.02, 4.97)		(3.08, 4.91)		(3.11, 4.88)	
0.50	0.1	0.2	0.15	38.21	0.7632	43.26	0.5577	48.75	0.4284	53.81	0.3418
	0.4	0.6	0.50	57.66	0.6188	63.18	0.4522	68.54	0.3473	72.99	0.2771
	0.4	1.6	1.00	126.15	0.4125	124.06	0.3015	120.83	0.2315	117.72	0.1847
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	3.8	4.2	4.00	32.76	0.8250	37.45	0.6030	42.68	0.4631	47.65	0.3695
	Range of $\Delta\rightarrow$			(0.87, 3.13)	(1.45, 2.55)	(0.85, 3.15)	(1.51, 2.49)	(0.84, 3.16)	(1.54, 2.46)	(0.83, 3.17)	(1.56, 2.44)
	$\Delta_{Best}\rightarrow$			(1.45, 2.55)		(1.51, 2.49)		(1.54, 2.46)		(1.56, 2.44)	
0.75	0.1	0.2	0.15	41.45	0.7322	46.67	0.5351	52.25	0.4110	57.30	0.3279
	0.4	0.6	0.50	82.21	0.5156	86.53	0.3769	89.95	0.2894	92.27	0.2309
	0.4	1.6	1.00	438.90	0.2063	294.12	0.1507	222.82	0.1158	186.17	0.0924
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	2.0	3.0	2.50	42.62	0.7219	47.90	0.5276	53.49	0.4052	58.53	0.3233
	2.5	3.5	3.00	21.07	1.0313	24.58	0.7537	28.74	0.5789	32.94	0.4619
	3.5	3.5	3.50	12.51	1.3407	14.82	0.9798	17.67	0.7525	20.70	0.6004
	3.8	4.2	4.00	8.27	1.6501	9.87	1.2059	11.90	0.9262	14.09	0.7390
	Range of $\Delta\rightarrow$			(0.58, 2.09)	(0.97, 1.70)	(0.57, 2.10)	(1.01, 1.66)	(0.56, 2.11)	(1.03, 1.64)	(0.56, 2.11)	(1.04, 1.63)
	$\Delta_{Best}\rightarrow$			(0.97, 1.70)		(1.01, 1.66)		(1.03, 1.64)		(1.04, 1.63)	
ARB of MMSE Estimator$\rightarrow$				0.2259		0.1463		0.1061		0.0820	

Table 3.1 continued …

q↓	Δ_1↓	Δ_2↓		**p** = -1							
			m→	**6**		**8**		**10**		**12**	
			h→	10.8519		15.6740		20.8442		26.4026	
			Δ↓ ***w***(***p***)→	0.7739		0.8537		0.8939		0.9180	
0.25	0.1	0.2	0.15	101.69	0.2176	101.09	0.1408	100.79	0.1022	100.61	0.0789
	0.4	0.6	0.50	105.60	0.1978	103.55	0.1280	102.55	0.0929	101.96	0.0718
	0.4	1.6	1.00	110.98	0.1696	106.84	0.1097	104.87	0.0796	103.73	0.0615
	1.0	2.0	1.50	115.99	0.1413	109.79	0.0914	106.91	0.0663	105.27	0.0513
	1.6	2.4	2.00	120.43	0.1130	112.32	0.0731	108.65	0.0531	106.56	0.0410
	2.0	3.0	2.50	124.13	0.0848	114.38	0.0549	110.04	0.0398	107.59	0.0308
	2.5	3.5	3.00	126.91	0.0565	115.89	0.0366	111.05	0.0265	108.34	0.0205
	3.5	3.5	3.50	128.65	0.0283	116.82	0.0183	111.67	0.0133	108.79	0.0103
	3.8	4.2	4.00	129.23	0.0000	117.13	0.0000	111.87	0.0000	108.94	0.0000
	Range of Δ→			(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)	(0.00, 8.00)
	Δ_{Best} →			(0.00, 8.00)		(0.00, 8.00)		(0.00, 8.00)		(0.00, 8.00)	
0.50	0.1	0.2	0.15	103.38	0.2091	102.16	0.1353	101.56	0.0982	101.20	0.0759
	0.4	0.6	0.50	110.98	0.1696	106.84	0.1097	104.87	0.0796	103.73	0.0615
	0.4	1.6	1.00	120.43	0.1130	112.32	0.0731	108.65	0.0531	106.56	0.0410
	1.0	2.0	1.50	126.91	0.0565	115.89	0.0366	111.05	0.0265	108.34	0.0205
	1.6	2.4	2.00	129.23	0.0000	117.13	0.0000	111.87	0.0000	108.94	0.0000
	2.0	3.0	2.50	126.91	0.0565	115.89	0.0366	111.05	0.0265	108.34	0.0205
	2.5	3.5	3.00	120.43	0.1130	112.32	0.0731	108.65	0.0531	106.56	0.0410
	3.5	3.5	3.50	110.98	0.1696	106.84	0.1097	104.87	0.0796	103.73	0.0615
	3.8	4.2	4.00	100.00	0.2261	100.00	0.1463	100.00	0.1061	100.00	0.0820
	Range of Δ→			(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
	Δ_{Best} →			(0.00, 4.00)		(0.00, 4.00)		(0.00, 4.00)		(0.00, 4.00)	
0.75	0.1	0.2	0.15	105.05	0.2006	103.21	0.1298	102.31	0.0942	101.77	0.0728
	0.4	0.6	0.50	115.99	0.1413	109.79	0.0914	106.91	0.0663	105.27	0.0513
	0.4	1.6	1.00	126.91	0.0565	115.89	0.0366	111.05	0.0265	108.34	0.0205
	1.0	2.0	1.50	128.65	0.0283	116.82	0.0183	111.67	0.0133	108.79	0.0103
	1.6	2.4	2.00	120.43	0.1130	112.32	0.0731	108.65	0.0531	106.56	0.0410
	2.0	3.0	2.50	105.60	0.1978	103.55	0.1280	102.55	0.0929	101.96	0.0718
	2.5	3.5	3.00	88.71	0.2826	92.40	0.1828	94.37	0.1327	95.59	0.1025
	3.5	3.5	3.50	72.93	0.3674	80.65	0.2377	85.17	0.1725	88.13	0.1333
	3.8	4.2	4.00	59.57	0.4521	69.50	0.2925	75.85	0.2123	80.24	0.1640
	Range of Δ→			(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)	(0.00, 2.67)
	Δ_{Best} →			(0.00, 2.67)		(0.00, 2.67)		(0.00, 2.67)		(0.00, 2.67)	
ARB of MMSE Estimator→				0.2259		0.1463		0.1061		0.0820	

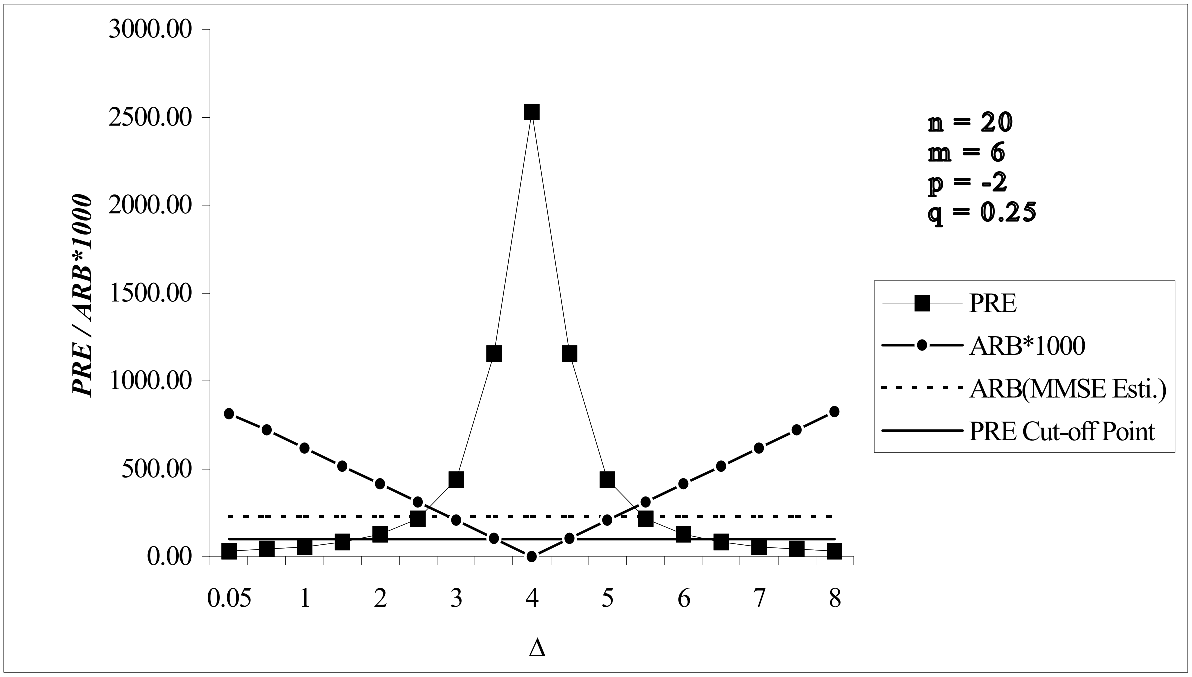
Table 3.1 continued …

$q\downarrow$				$p = 1$							
	$\Delta_1\downarrow$	$\Delta_2\downarrow$	$m\rightarrow$	**6**		**8**		**10**		**12**	
			$h\rightarrow$	10.8519		15.6740		20.8442		26.4026	
			$\Delta\downarrow$ $w(p)\rightarrow$	0.6888		0.7737		0.8251		0.8779	
0.25	0.1	0.2	0.15	99.00	0.2996	97.51	0.2178	97.21	0.1684	99.20	0.1175
	0.4	0.6	0.50	106.26	0.2723	103.17	0.1980	101.80	0.1531	102.17	0.1069
	0.4	1.6	1.00	117.09	0.2334	111.34	0.1697	108.25	0.1312	106.18	0.0916
	1.0	2.0	1.50	128.15	0.1945	119.34	0.1415	114.39	0.1093	109.82	0.0763
	1.6	2.4	2.00	138.88	0.1556	126.79	0.1132	119.95	0.0875	113.00	0.0611
	2.0	3.0	2.50	148.56	0.1167	133.27	0.0849	124.67	0.0656	115.60	0.0458
	2.5	3.5	3.00	156.33	0.0778	138.31	0.0566	128.27	0.0437	117.53	0.0305
	3.5	3.5	3.50	161.41	0.0389	141.52	0.0283	130.54	0.0219	118.72	0.0153
	3.8	4.2	4.00	163.17	0.0000	142.63	0.0000	131.31	0.0000	119.12	0.0000
	Range of $\Delta\rightarrow$			(0.20, 7.80)	(0.00, 8.00)	(0.30, 7.70)	(0.00, 8.00)	(0.36, 7.64)	(0.00, 8.00)	(0.24, 7.76)	(0.00, 8.00)
				(0.20, 7.80)		(0.30, 7.70)		(0.36, 7.64)		(0.24, 7.76)	
0.50	0.1	0.2	0.15	102.07	0.2879	99.92	0.2093	99.18	0.1618	100.49	0.1130
	0.4	0.6	0.50	117.09	0.2334	111.34	0.1697	108.25	0.1312	106.18	0.0916
	0.4	1.6	1.00	138.88	0.1556	126.79	0.1132	119.95	0.0875	113.00	0.0611
	1.0	2.0	1.50	156.33	0.0778	138.31	0.0566	128.27	0.0437	117.53	0.0305
	1.6	2.4	2.00	163.17	0.0000	142.63	0.0000	131.31	0.0000	119.12	0.0000
	2.0	3.0	2.50	156.33	0.0778	138.31	0.0566	128.27	0.0437	117.53	0.0305
	2.5	3.5	3.00	138.88	0.1556	126.79	0.1132	119.95	0.0875	113.00	0.0611
	3.5	3.5	3.50	117.09	0.2334	111.34	0.1697	108.25	0.1312	106.18	0.0916
	3.8	4.2	4.00	96.01	0.3112	95.12	0.2263	95.25	0.1749	97.90	0.1221
	Range of $\Delta\rightarrow$			(0.10, 3.90)	(0.55, 3.45)	(0.15, 3.85)	(0.71, 3.29)	(0.18, 3.82)	(0.79, 3.21)	(0.12, 3.88)	(0.66, 3.34)
	$\Delta_{\mathbf{Best}}\rightarrow$			(0.55, 3.45)		(0.71, 3.29)		(0.79, 3.21)		(0.66, 3.34)	
0.75	0.1	0.2	0.15	105.20	0.2762	102.36	0.2009	101.15	0.1553	101.75	0.1084
	0.4	0.6	0.50	128.15	0.1945	119.34	0.1415	114.39	0.1093	109.82	0.0763
	0.4	1.6	1.00	156.33	0.0778	138.31	0.0566	128.27	0.0437	117.53	0.0305
	1.0	2.0	1.50	161.41	0.0389	141.52	0.0283	130.54	0.0219	118.72	0.0153
	1.6	2.4	2.00	138.88	0.1556	126.79	0.1132	119.95	0.0875	113.00	0.0611
	2.0	3.0	2.50	106.26	0.2723	103.17	0.1980	101.80	0.1531	102.17	0.1069
	2.5	3.5	3.00	77.96	0.3891	80.11	0.2829	82.50	0.2187	88.98	0.1526
	3.5	3.5	3.50	57.31	0.5058	61.51	0.3678	65.66	0.2843	75.76	0.1984
	3.8	4.2	4.00	42.96	0.6225	47.58	0.4526	52.22	0.3499	63.80	0.2442
	Range of $\Delta\rightarrow$			(0.07, 2.60)	(0.37, 2.30)	(0.10, 2.57)	(0.47, 2.20)	(0.12, 2.55)	(0.52, 2.14)	(0.08, 2.59)	(0.44, 2.23)
	$\Delta_{\mathbf{Best}}\rightarrow$			(0.37, 2.30)		(0.47, 2.20)		(0.52, 2.14)		(0.44, 2.23)	
ARB of MMSE Estimator$\rightarrow$				0.2259		0.1463		0.1061		0.0820	

Table 3.1 continued …

q↓	Δ_1↓	Δ_2↓	m→	6		8		10		12	
			h→	10.8519		15.6740		20.8442		26.4026	
			Δ↓ $w(p)$→	0.3131		0.4385		0.5392		0.6816	
0.25	0.1	0.2	0.15	48.51	0.6612	45.00	0.5405	45.90	0.4435	60.53	0.3065
	0.4	0.6	0.50	57.95	0.6011	53.31	0.4913	53.85	0.4032	68.81	0.2786
	0.4	1.6	1.00	76.84	0.5152	69.55	0.4211	68.94	0.3456	83.20	0.2388
	1.0	2.0	1.50	106.11	0.4293	93.70	0.3509	90.35	0.2880	101.08	0.1990
	1.6	2.4	2.00	154.14	0.3435	130.87	0.2808	121.15	0.2304	122.65	0.1592
	2.0	3.0	2.50	237.92	0.2576	189.27	0.2106	164.85	0.1728	147.06	0.1194
	2.5	3.5	3.00	388.87	0.1717	277.82	0.1404	222.08	0.1152	171.43	0.0796
	3.5	3.5	3.50	627.92	0.0859	386.26	0.0702	280.49	0.0576	190.36	0.0398
	3.8	4.2	4.00	789.74	0.0000	444.03	0.0000	307.45	0.0000	197.63	0.0000
	Range of Δ→			(1.41, 6.59)	(2.68, 5.32)	(1.60, 6.40)	(2.96, 5.04)	(1.68, 6.32)	(3.08, 4.92)	(1.47, 6.53)	(2.97, 5.03)
	Δ_{Best} →			(2.68, 5.32)		(2.96, 5.04)		(3.08, 4.92)		(2.97, 5.03)	
0.50	0.1	0.2	0.15	52.26	0.6354	48.32	0.5194	49.09	0.4262	63.91	0.2946
	0.4	0.6	0.50	76.84	0.5152	69.55	0.4211	68.94	0.3456	83.20	0.2388
	0.4	1.6	1.00	154.14	0.3435	130.87	0.2808	121.15	0.2304	122.65	0.1592
	1.0	2.0	1.50	388.87	0.1717	277.82	0.1404	222.08	0.1152	171.43	0.0796
	1.6	2.4	2.00	789.74	0.0000	444.03	0.0000	307.45	0.0000	197.63	0.0000
	2.0	3.0	2.50	388.87	0.1717	277.82	0.1404	222.08	0.1152	171.43	0.0796
	2.5	3.5	3.00	154.14	0.3435	130.87	0.2808	121.15	0.2304	122.65	0.1592
	3.5	3.5	3.50	76.84	0.5152	69.55	0.4211	68.94	0.3456	83.20	0.2388
	3.8	4.2	4.00	45.14	0.6869	42.00	0.5615	42.99	0.4608	57.36	0.3184
	Range of Δ→			(0.71, 3.29)	(1.34, 2.66)	(0.80, 3.20)	(1.48, 2.52)	(0.84, 3.16)	(1.54, 2.46)	(0.74, 3.26)	(1.49, 2.51)
	Δ_{Best} →			(1.34, 2.66)		(1.48, 2.52)		(1.54, 2.46)		(1.49, 2.51)	
0.75	0.1	0.2	0.15	56.45	0.6096	52.00	0.4983	52.60	0.4090	67.54	0.2826
	0.4	0.6	0.50	106.11	0.4293	93.70	0.3509	90.35	0.2880	101.08	0.1990
	0.4	1.6	1.00	388.87	0.1717	277.82	0.1404	222.08	0.1152	171.43	0.0796
	1.0	2.0	1.50	627.92	0.0859	386.26	0.0702	280.49	0.0576	190.36	0.0398
	1.6	2.4	2.00	154.14	0.3435	130.87	0.2808	121.15	0.2304	122.65	0.1592
	2.0	3.0	2.50	57.95	0.6011	53.31	0.4913	53.85	0.4032	68.81	0.2786
	2.5	3.5	3.00	29.50	0.8587	27.83	0.7019	28.97	0.5760	41.00	0.3980
	3.5	3.5	3.50	17.73	1.1163	16.90	0.9125	17.83	0.7488	26.50	0.5175
	3.8	4.2	4.00	11.79	1.3739	11.30	1.1230	12.01	0.9216	18.33	0.6369
	Range of Δ→			(0.47, 2.20)	(0.89, 1.77)	(0.53, 2.13)	(0.99, 1.68)	(0.56, 2.11)	(1.03, 1.64)	(0.49, 2.18)	(0.99, 1.68)
	Δ_{Best} →			(0.89, 1.77)		(0.99, 1.68)		(1.03, 1.64)		(0.99, 1.68)	
ARB of MMSE Estimator→				0.2259		0.1463		0.1061		0.0820	

Table header: p = 2



It has been observed from Table 3.1, that on keeping m, p, q fixed, the relative efficiencies of the proposed class of shrinkage estimators increases up to $\Delta = q^{-1}$, attains its maximum at this point and then decreases symmetrically in magnitude, as Δ increases in its range of dominance for all n, p and q. On the other hand, the ARBs of the proposed class of estimators decreases up to $\Delta = q^{-1}$, the estimator becomes unbiased at this point and then ARBs increases symmetrically in magnitude, as Δ increases in its range of dominance. Thus it is interesting to note that, at $q = \Delta^{-1}$, the proposed class of estimators is unbiased with largest efficiency and hence in the vicinity of $q = \Delta^{-1}$ also, the proposed class not only renders the massive gain in efficiency but also it is marginally biased in comparison of MMSE estimator. This implies that q plays an important role in the proposed class of estimators. The following figure illustrates the discussion.

Figure 3.1

The effect of change in censored sample size m is also a matter of great interest. For fixed p, q and Δ, the gain in relative efficiency diminishes, and ARB also decreases, with increment in m. Moreover, it appears that to get better estimators in the class, the value of $w(p)$ should be as small as possible in the interval (0,1]. Thus, to choose p one should not consider the smaller values of $w(p)$ in isolation, but also the wider length of the interval of Δ.

4. MODIFIED CLASS OF SHRINKAGE ESTIMATORS AND ITS PROPERTIES

The proposed class of estimators $\hat{\beta}_{(p,q)}$ is not uniformly better than $\hat{\beta}$. It will be better if β_1 and β_2 are in the vicinity of true value β. Thus, the centre of the guessed interval $(\beta_1+\beta_2)/2$ is of much importance in this case. If we partially violate this, i.e., only the centre of the guessed interval is not of much importance, but the end points of the interval β_1 and β_2 are itself equally important then we can propose a new class of shrinkage estimators for the shape parameter β by using the suggested class $\hat{\beta}_{(p,q)}$ as

$$\tilde{\beta}_{(p,q)} = \begin{cases} \beta_1 & , \text{ if } \quad t > [(h-2)/\beta_1] \\ \left(\dfrac{h-2}{t}\right) w(p) + q\left(\dfrac{\beta_1+\beta_2}{2}\right)\{1-w(p)\} & , \text{ if } \quad [(h-2)/\beta_2] \le t \le [(h-2)/\beta_1] \\ \beta_2 & , \text{ if } \quad t < [(h-2)/\beta_2] \end{cases}$$

(4.1)

which has

$$\text{Bias}\{\tilde{\beta}_{(p,q)}\} = \beta\left[\Delta_1\left\{1-I\left(\eta_1,\frac{h}{2}\right)\right\} + w(p)\left\{I\left(\eta_1,\frac{h}{2}-1\right)-I\left(\eta_2,\frac{h}{2}-1\right)\right\}\right.$$
$$\left. + q\,\Delta\{1-w(p)\}\left\{I\left(\eta_1,\frac{h}{2}\right)-I\left(\eta_2,\frac{h}{2}\right)\right\} + \Delta_2\, I\left(\eta_2,\frac{h}{2}\right) - 1\right]$$

(4.2)

and

$$\text{MSE}\{\tilde{\beta}_{(p,q)}\} = \beta^2\left[(\Delta_1-1)^2 - \Delta_1(\Delta_1-2)\; I\left(\eta_1,\frac{h}{2}\right) + \Delta_2(\Delta_2-2)\; I\left(\eta_2,\frac{h}{2}\right)\right.$$
$$+\{w(p)\}^2\left(\frac{h-2}{h-4}\right)\left\{I\left(\eta_1,\frac{h}{2}-2\right)-I\left(\eta_2,\frac{h}{2}-2\right)\right\}$$
$$+q\,\Delta\{1-w(p)\}\left\{I\left(\eta_1,\frac{h}{2}\right)-I\left(\eta_2,\frac{h}{2}\right)\right\}\{q\,\Delta\{1-w(p)\}-2\}$$
$$\left.+2\,w(p)\left\{I\left(\eta_1,\frac{h}{2}-1\right)-I\left(\eta_2,\frac{h}{2}-1\right)\right\}\{q\,\Delta\{1-w(p)\}-1\}\right]$$

(4.3)

where $\eta_1 = \left(\frac{h}{2}-1\right)\Delta_1^{-1}$, $\quad \eta_2 = \left(\frac{h}{2}-1\right)\Delta_2^{-1}$ and $\quad I(\eta,\omega) = \frac{1}{\Gamma(\omega)}\int_0^{\eta} e^{-u}u^{\omega-1}du$.

This modified class of shrinkage estimators is proposed in accordance with Rao(1973) and it seems to be more realistic than the previous one as it deals with the case where the whole interval is taken as apriori information.

5. NUMERICAL ILLUSTRATIONS

The percent relative efficiency of the proposed estimator $\widetilde{\beta}_{(p,q)}$ with respect to MMSE estimator $\hat{\beta}_m$ has been defined as

$$\text{PRE}\left\{\widetilde{\beta}_{(p,q)}, \hat{\beta}_m\right\} = \frac{\text{MSE}\left\{\hat{\beta}_m\right\}}{\text{MSE}\left\{\widetilde{\beta}_{(p,q)}\right\}} \times 100$$

(5.1)

and it is obtained for n = 20 and different values of p, q, m, Δ_1 and Δ_2 (or Δ). The findings are summarised in Table 5.1 with corresponding values of h and $w(p)$.

Table 5.1

PREs of proposed estimator $\widetilde{\beta}_{(p,q)}$ with respect to MMSE estimator $\hat{\beta}_m$

n = 20											
$q\downarrow$			***p*** $\rightarrow$	-1				1			
	$\Delta_1\downarrow$	$\Delta_2\downarrow$	***m*** $\rightarrow$	**6**	**8**	**10**	**12**	**6**	**8**	**10**	**12**
			h $\rightarrow$	10.8519	15.6740	20.8442	26.4026	10.8519	15.6740	20.8442	26.4026
			$\Delta\downarrow$ ***w(p)*** $\rightarrow$	0.7739	0.8537	0.8939	0.9180	0.6888	0.7737	0.8251	0.8779
0.25	0.2	0.3	0.25	50.80	41.39	34.91	30.59	49.84	40.10	34.66	31.15
	0.4	0.6	0.50	117.60	81.01	67.45	63.17	113.90	79.57	65.63	61.55
	0.6	0.9	0.75	261.72	227.42	203.08	172.06	227.59	191.97	172.31	156.69
	0.8	1.2	1.00	548.60	426.98	342.54	286.06	454.93	355.31	293.42	262.79
	1.0	1.5	1.25	649.95	470.44	375.91	314.98	636.21	504.49	427.74	353.74
	1.2	1.8	1.50	268.31	189.82	150.17	125.21	286.06	210.91	168.38	135.01
	1.5	2.0	1.75	80.46	53.66	39.90	31.38	82.35	55.10	40.79	31.74
0.50	0.2	0.3	0.25	50.84	41.32	34.76	30.39	49.90	40.03	34.45	30.87
	0.4	0.6	0.50	120.81	82.01	67.97	63.49	118.31	81.13	66.48	62.03
	0.6	0.9	0.75	298.17	253.12	221.74	184.38	271.73	225.47	198.40	173.57
	0.8	1.2	1.00	642.86	473.19	368.65	303.15	583.65	433.16	344.05	292.64
	1.0	1.5	1.25	626.09	435.87	345.16	289.53	658.77	481.87	390.95	317.87
	1.2	1.8	1.50	247.90	175.97	140.57	118.43	264.16	191.09	152.66	124.73
	1.5	2.0	1.75	78.41	52.66	39.39	31.11	79.96	53.72	40.02	31.36
0.75	0.2	0.3	0.25	50.89	41.24	34.60	30.19	49.97	39.95	34.23	30.59
	0.4	0.6	0.50	124.02	83.01	68.50	63.81	122.74	82.68	67.32	62.50
	0.6	0.9	0.75	339.92	282.24	242.46	197.73	325.66	266.36	229.58	192.68
	0.8	1.2	1.00	723.50	510.42	389.34	316.87	710.96	504.67	388.35	317.53
	1.0	1.5	1.25	566.19	392.47	312.16	263.77	597.64	421.61	337.17	278.26

	1.2	1.8	1.50	224.67	161.95	131.14	111.81	233.41	169.19	136.65	114.63
	1.5	2.0	1.75	76.05	51.59	38.85	30.83	76.93	52.14	39.17	30.95

Table 5.1 continued ...

q↓	Δ_1↓	Δ_2↓	$p \rightarrow$	-2				2			
			$m \rightarrow$	**6**	**8**	**10**	**12**	**6**	**8**	**10**	**12**
			$h \rightarrow$	10.8519	15.6740	20.8442	26.4026	10.8519	15.6740	20.8442	26.4026
			Δ↓ $w(p)\rightarrow$	0.7739	0.8537	0.8939	0.9180	0.6888	0.7737	0.8251	0.8779
0.25	0.2	0.3	0.25	46.04	34.18	30.92	30.53	46.77	34.81	30.96	31.23
	0.4	0.6	0.50	92.48	72.59	59.44	53.42	98.00	73.36	59.48	54.88
	0.6	0.9	0.75	106.83	95.44	92.75	90.11	128.68	102.24	93.16	100.45
	0.8	1.2	1.00	145.02	131.16	126.15	122.15	191.47	145.23	126.97	144.22
	1.0	1.5	1.25	220.29	243.10	282.54	320.74	305.32	273.81	284.60	368.42
	1.2	1.8	1.50	208.14	211.32	202.36	179.81	250.20	220.57	202.56	175.49
	1.5	2.0	1.75	82.08	57.89	43.07	33.36	84.21	57.95	43.06	33.12
0.50	0.2	0.3	0.25	46.28	34.31	30.86	30.24	46.95	34.91	30.90	30.87
	0.4	0.6	0.50	103.18	76.82	61.54	54.80	107.21	77.31	61.57	56.08
	0.6	0.9	0.75	157.81	135.64	127.02	118.59	181.60	142.94	127.44	128.23
	0.8	1.2	1.00	267.16	228.67	207.62	190.69	331.58	246.71	208.58	212.20
	1.0	1.5	1.25	445.44	443.06	448.55	438.38	541.60	467.49	449.42	432.21
	1.2	1.8	1.50	289.70	240.03	198.56	163.98	298.93	238.16	198.30	156.40
	1.5	2.0	1.75	84.92	57.28	42.13	32.67	84.44	57.03	42.12	32.44
0.75	0.2	0.3	0.25	46.50	34.43	30.78	29.92	47.13	34.99	30.82	30.50
	0.4	0.6	0.50	114.64	81.04	63.59	56.13	116.87	81.23	63.61	57.24
	0.6	0.9	0.75	247.11	202.90	181.31	160.85	266.60	209.00	181.65	167.34
	0.8	1.2	1.00	543.26	418.40	345.15	293.90	596.79	430.93	345.67	302.22
	1.0	1.5	1.25	704.42	541.77	447.06	381.03	696.36	532.12	446.25	358.48
	1.2	1.8	1.50	280.39	203.46	160.74	132.95	269.47	199.82	160.55	129.07
	1.5	2.0	1.75	81.39	54.49	40.40	31.66	80.35	54.26	40.39	31.52

It has been observed from Table 5.1 that likewise $\hat{\beta}_{(p,q)}$ the PRE of $\tilde{\beta}_{(p,q)}$ with respect to $\hat{\beta}_m$ decreases as censoring fraction (m/n) increases. For fixed m, p and q the relative efficiency increases up to a certain point of Δ, procures its maximum at this point and then starts decreasing as Δ increases. It seems from the expression in (4.3) that the point of maximum efficiency may be a point where either any one of the following holds or any two of the following holds or all the following three holds-

(i) the lower end point of the guessed interval, i.e., β_1 coincides exactly with the true value β, i.e., $\Delta_1 = 1$.

(ii) the upper end point of the guessed interval, i.e., β_2 departs exactly two times from the true value β, i.e., $\Delta_2 = 2$.

(iii) $\Delta = q^{-1}$

This leads to say that on contrary to $\hat{\beta}_{(p,q)}$, there is much importance of Δ_1 and Δ_2 in addition to Δ. The discussion is also supported by the illustrations in Table 5.1. As well, the range of dominance of

average departure Δ is smaller than that is obtained for $\hat{\beta}_{(p,q)}$ but this does not humiliate the merit of $\tilde{\beta}_{(p,q)}$ because still the range of dominance of Δ is enough wider.

6. CONCLUSION AND RECOMMENDATIONS

It has been seen that the suggested classes of shrunken estimators have considerable gain in efficiency for a number of choices of scalars comprehend in it, particularly for heavily censored samples, i.e., for small *m*. Even for buoyantly censored samples, i.e., for large *m*, so far as the proper selection of scalars is concerned, some of the estimators from the suggested classes of shrinkage estimators are more efficient than the MMSE estimators subject to certain conditions. Accordingly, even if the experimenter has less confidence in the guessed interval (β_1, β_2) of β, the efficiency of the suggested classes of shrinkage estimators can be increased considerably by choosing the scalars *p* and *q* appropriately.

While dealing with the suggested class of shrunken estimators $\hat{\beta}_{(p,q)}$ it is recommended that one should not consider the substantial gain in efficiency in isolation, but also the wider range of dominance of Δ, because enough flexible range of dominance of Δ will leads to increase the possibility of getting better estimators from the proposed class. Thus it is recommended to use the proposed class of shrunken estimators in practice.

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A General Class of Estimators of Population Median Using Two Auxiliary Variables in Double Sampling

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Abstract:
In this paper we have suggested two classes of estimators for population median M_Y of the study character Y using information on two auxiliary characters X and Z in double sampling. It has been shown that the suggested classes of estimators are more efficient than the one suggested by Singh *et al* (2001). Estimators based on estimated optimum values have been also considered with their properties. The optimum values of the first phase and second phase sample sizes are also obtained for the fixed cost of survey.

Keywords: Median estimation, Chain ratio and regression estimators, Study variate, Auxiliary variate, Classes of estimators, Mean squared errors, Cost, Double sampling.

2000 MSC: 60E99

1. INTRODUCTION

In survey sampling, statisticians often come across the study of variables which have highly skewed distributions, such as income, expenditure etc. In such situations, the estimation of median deserves special attention. Kuk and Mak (1989) are the first to introduce the estimation of population median of the study variate Y using auxiliary information in survey sampling. Francisco and Fuller (1991) have also considered the problem of estimation of the median as part of the estimation of a finite population distribution function. Later Singh *et al* (2001) have dealt extensively with the problem of estimation of median using auxiliary information on an auxiliary variate in two phase sampling.

Consider a finite population $U=\{1,2,\ldots,i,\ldots,N\}$. Let Y and X be the variable for study and auxiliary variable, taking values Y_i and X_i respectively for the i-th unit. When the two variables are strongly related but no information is available on the population median M_X of X, we seek to estimate the population median M_Y of Y from a sample S_m, obtained through a two-phase selection. Permitting simple random sampling without replacement (SRSWOR) design in each phase, the two-phase sampling scheme will be as follows:

(i) The first phase sample $S_n(S_n \subset U)$ of fixed size n is drawn to observe only X in order to furnish an estimate of M_X.

(ii) Given S_n, the second phase sample $S_m(S_m \subset S_n)$ of fixed size m is drawn to observe Y only.

Assuming that the median M_X of the variable X is known, Kuk and Mak (1989) suggested a ratio estimator for the population median M_Y of Y as

$$\hat{M}_1 = \hat{M}_Y \frac{M_X}{\hat{M}_X} \tag{1.1}$$

where $\hat{M}_Y$ and $\hat{M}_X$ are the sample estimators of M_Y and M_X respectively based on a sample S_m of size m. Suppose that $y_{(1)}, y_{(2)}, \ldots, y_{(m)}$ are the y values of sample units in ascending order. Further, let t be an integer such that $Y_{(t)} \le M_Y \le Y_{(t+1)}$ and let p=t/m be the proportion of Y, values in the sample that are less than or equal to the median value M_Y, an unknown population parameter. If $\hat{p}$ is a predictor of p, the sample median $\hat{M}_Y$ can be written in terms of quantities as $\hat{Q}_Y(\hat{p})$ where $\hat{p} = 0.5$. Kuk and Mak (1989) define a matrix of proportions ($P_{ij}(x,y)$) as

	$Y \le M_Y$	$Y > M_Y$	Total
$X \le M_X$	$P_{11}(x,y)$	$P_{21}(x,y)$	$P_{\cdot 1}(x,y)$
$X > M_X$	$P_{12}(x,y)$	$P_{22}(x,y)$	$P_{\cdot 2}(x,y)$
Total	$P_{1\cdot}(x,y)$	$P_{2\cdot}(x,y)$	1

and a position estimator of M_y given by

$$\hat{M}_Y^{(p)} = \hat{Q}_Y(\hat{p}_Y) \tag{1.2}$$

$$\text{where} \quad \hat{p}_Y = \frac{1}{m}\left(\frac{m_x \hat{p}_{11}(x,y)}{\hat{p}_{\cdot 1}(x,y)} + \frac{(m-m_x)\hat{p}_{12}(x,y)}{\hat{p}_{\cdot 2}(x,y)}\right)$$
$$\approx 2\left(\frac{m_x \hat{p}_{11}(x,y) + (m-m_x)\hat{p}_{12}(x,y)}{m}\right)$$

with $\hat{p}_{ij}(x,y)$ being the sample analogues of the $P_{ij}(x,y)$ obtained from the population and m_x the number of units in S_m with $X \le M_X$.

Let $\tilde{F}_{YA}(y)$ and $\tilde{F}_{YB}(y)$ denote the proportion of units in the sample S_m with $X \le M_X$, and $X>M_X$, respectively that have Y values less than or equal to y. Then for estimating M_Y, Kuk and Mak (1989) suggested the 'stratification estimator' as

$$\hat{M}_Y^{(St)} = \inf\left\{y : \tilde{F}_Y^{(y)} \ge 0.5\right\} \tag{1.3}$$

where $\hat{F}_Y(y) \cong \frac{1}{2}\left[\tilde{F}_{YA}^{(y)} + \tilde{F}_{YB}^{(y)}\right]$

It is to be noted that the estimators defined in (1.1), (1.2) and (1.3) are based on prior knowledge of the median M_X of the auxiliary character X. In many situations of practical importance the population median M_X of X may not be known. This led Singh *et al* (2001) to discuss the problem of estimating the population median M_Y in double sampling and suggested an analogous ratio estimator as

$$\hat{M}_{1d} = \hat{M}_Y \frac{\hat{M}_X^1}{\hat{M}_X} \tag{1.4}$$

where $\hat{M}_X^1$ is sample median based on first phase sample S_n.

Sometimes even if M_X is unknown, information on a second auxiliary variable Z, closely related to X but compared X remotely related to Y, is available on all units of the population. This type of situation has been briefly discussed by, among others, Chand (1975), Kiregyera (1980, 84), Srivenkataramana and Tracy (1989), Sahoo and Sahoo (1993) and Singh (1993). Let M_Z be the known population median of Z. Defining

$$e_0 = \left(\frac{\hat{M}_Y}{M_Y} - 1\right), e_1 = \left(\frac{\hat{M}_X}{M_X} - 1\right), e_2 = \left(\frac{\hat{M}_X{}^1}{M_X} - 1\right), e_3\left(\frac{\hat{M}_Z}{M_Z} - 1\right) \text{and } e_4 = \left(\frac{\hat{M}_Z{}^1}{M_Z} - 1\right)$$

such that $E(e_k) \cong 0$ and $|e_k| < 1$ for k=0,1,2,3; where $\hat{M}_2$ and $\hat{M}_2^1$ are the sample median estimators based on second phase sample S_m and first phase sample S_n. Let us define the following two new matrices as

	$Z \le M_Z$	$Z > M_Z$	Total
$X \le M_X$	$P_{11}(x,z)$	$P_{21}(x,z)$	$P_{\cdot 1}(x,z)$
$X > M_X$	$P_{12}(x,z)$	$P_{22}(x,z)$	$P_{\cdot 2}(x,z)$
Total	$P_{1\cdot}(x,z)$	$P_{2\cdot}(x,z)$	1

and

	$Z \le M_Z$	$Z > M_Z$	Total
$Y \le M_Y$	$P_{11}(y,z)$	$P_{21}(y,z)$	$P_{\cdot 1}(y,z)$
$Y > M_Y$	$P_{12}(y,z)$	$P_{22}(y,z)$	$P_{\cdot 2}(y,z)$
Total	$P_{1\cdot}(y,z)$	$P_{2\cdot}(y,z)$	1

Using results given in the Appendix-1, to the first order of approximation, we have

$E(e_0^2) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{M_Y f_Y(M_Y)\}^{-2}$,

$E(e_1^2) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{M_X f_X(M_X)\}^{-2}$,

$E(e_2^2) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{M_X f_X(M_X)\}^{-2}$,

$E(e_3^2) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{M_Z f_Z(M_Z)\}^{-2}$,

$E(e_4^2) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{M_Z f_Z(M_Z)\}^{-2}$,

$E(e_0 e_1) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{4P_{11}(x,y)-1\}\{M_X M_Y f_X(M_X) f_Y(M_Y)\}^{-1}$,

$E(e_0 e_2) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{4P_{11}(x,y)-1\}\{M_X M_Y f_X(M_X) f_Y(M_Y)\}^{-1}$,

$E(e_0 e_3) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{4P_{11}(y,z)-1\}\{M_Y M_Z f_Y(M_Y) f_Z(M_Z)\}^{-1}$,

$E(e_0 e_4) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{4P_{11}(y,z)-1\}\{M_Y M_Z f_Y(M_Y) f_Z(M_Z)\}^{-1}$,

$E(e_1 e_2) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{M_X f_X(M_X)\}^{-2}$,

$E(e_1 e_3) = \left(\frac{N-m}{N}\right)(4m)^{-1}\{4P_{11}(x,z)-1\}\{M_X M_Z f_X(M_X) f_Z(M_Z)\}^{-1}$,

$E(e_1e_4) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{4P_{11}(x,z)-1\}\{M_XM_Zf_X(M_X)f_Z(M_Z)\}^{-1}$,

$E(e_2e_3) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{4P_{11}(x,z)-1\}\{M_XM_Zf_X(M_X)f_Z(M_Z)\}^{-1}$,

$E(e_2e_4) = \left(\frac{N-n}{N}\right)(4n)^{-1}\{4P_{11}(x,z)-1\}\{M_XM_Zf_X(M_X)f_Z(M_Z)\}^{-1}$,

$E(e_3e_4) = \left(\frac{N-n}{N}\right)(4n)^{-1}(f_Z(M_Z)M_Z)^{-2}$

where it is assumed that as $N\to\infty$ the distribution of the trivariate variable (X,Y,Z) approaches a continuous distribution with marginal densities $f_X(x)$, $f_Y(y)$ and $f_Z(z)$ for X, Y and Z respectively. This assumption holds in particular under a superpopulation model framework, treating the values of (X, Y, Z) in the population as a realization of N independent observations from a continuous distribution. We also assume that $f_Y(M_Y)$, $f_X(M_X)$ and $f_Z(M_Z)$ are positive.

Under these conditions, the sample median $\hat{M}_Y$ is consistent and asymptotically normal (Gross, 1980) with mean M_Y and variance

$$\left(\frac{N-m}{N}\right)(4m)^{-1}\{f_Y(M_Y)\}^{-2}$$

In this paper we have suggested a class of estimators for M_Y using information on two auxiliary variables X and Z in double sampling and analyzes its properties.

2. SUGGESTED CLASS OF ESTIMATORS

Motivated by Srivastava (1971), we suggest a class of estimators of M_Y of Y as

$$g = \left\{\hat{M}_Y^{(g)} : \hat{M}_Y^{(g)} = M_Y g(u,v)\right\} \tag{2.1}$$

where $u = \frac{\hat{M}_X}{\hat{M}_X^1}, v = \frac{\hat{M}_Z^1}{\hat{M}_Z}$ and g(u,v) is a function of u and v such that g(1,1)=1 and such that it satisfies the following conditions.

1. Whatever be the samples (S_n and S_m) chosen, let (u,v) assume values in a closed convex sub-space, P, of the two dimensional real space containing the point (1,1).
2. The function g(u,v) is continuous in P, such that g(1,1)=1.
3. The first and second order partial derivatives of g(u,v) exist and are also continuous in P.

Expanding g(u,v) about the point (1,1) in a second order Taylor's series and taking expectations, it is found that

$$E\left(\hat{M}_Y^{(g)}\right) = M_Y + 0(n^{-1})$$

so the bias is of order n^{-1}.

Using a first order Taylor's series expansion around the point (1,1) and noting that g(1,1)=1, we have

$$\hat{M}_Y^{(g)} \cong M_Y[1 + e_0 + (e_1 - e_2)g_1(1,1) + e_4 g_2(1,1) + 0(n^{-1})]$$

or

$$\left(M_Y^{(g)} - M_Y\right) \cong M_Y\left[e_0 + (e_1 - e_2)g_1(1,1) + e_4 g_2(1,1)\right] \quad (2.2)$$

where $g_1(1,1)$ and $g_2(1,1)$ denote first order partial derivatives of g(u,v) with respect to u and v respectively around the point (1,1).

Squaring both sides in (2.2) and then taking expectations, we get the variance of $\hat{M}_Y^{(g)}$ to the first degree of approximation, as

$$Var\left(\hat{M}_Y^{(g)}\right) = \frac{1}{4\left(f_Y(M_Y)\right)^2}\left[\left(\frac{1}{m} - \frac{1}{N}\right) + \left(\frac{1}{m} - \frac{1}{n}\right)A + \left(\frac{1}{n} - \frac{1}{N}\right)B\right], \quad (2.3)$$

where

$$A = \left(\frac{M_Y f_Y(M_Y)}{M_X f_X(M_X)}\right)g_1(1,1)\left[\left(\frac{M_Y f_Y(M_Y)}{M_X f_X(M_X)}\right)g_1(1,1) + 2\left(4P_{11}(x,y) - 1\right)\right] \quad (2.4)$$

$$B = \left(\frac{M_Y f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)g_Z(1,1)\left[\left(\frac{M_Y f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)g_2(1,1) + 2\left(4P_{11}(y,z) - 1\right)\right] \quad (2.5)$$

The variance of $\hat{M}_Y^{(g)}$ in (2.3) is minimized for

$$\begin{aligned} g_1(1,1) &= -\left(\frac{M_X f_X(M_X)}{M_Y f_Y(M_Y)}\right)\left(4P_{11}(x,y) - 1\right) \\ g_2(1,1) &= -\left(\frac{M_Z f_Z(M_Z)}{M_Y f_Y(M_Y)}\right)\left(4P_{11}(y,z) - 1\right) \end{aligned} \quad (2.6)$$

Thus the resulting (minimum) variance of $M_Y^{(g)}$ is given by

$$\min.\ \mathrm{Var}\left(\hat{M}_Y^{(g)}\right) = \frac{1}{4\left(f_Y(M_Y)\right)^2}\left[\left(\frac{1}{m} - \frac{1}{N}\right) - \left(\frac{1}{m} - \frac{1}{n}\right)\left(4P_{11}(x,y) - 1\right)^2 - \left(\frac{1}{n} - \frac{1}{N}\right)\left(4P_{11}(y,z) - 1\right)\right]$$

(2.7)

Now, we proved the following theorem.

Theorem 2.1 - Up to terms of order n^{-1},

$$\mathrm{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{g}\right) \geq \frac{1}{4\left(f_y(M_Y)\right)^2}\left[\left(\frac{1}{m} - \frac{1}{N}\right) - \left(\frac{1}{m} - \frac{1}{n}\right)\left(4P_{11}(x,y) - 1\right)^2 - \left(\frac{1}{n} - \frac{1}{N}\right)\left(4P_{11}(y,z) - 1\right)^2\right]$$

with equality holding if

$$g_1(1,1) = -\left(\frac{M_x f_x(M_x)}{M_Y f_Y(M_Y)}\right)(4P_{11}(x,y)-1)$$

$$g_2(1,1) = -\left(\frac{M_z f_z(M_z)}{M_Y f_Y(M_Y)}\right)(4P_{11}(y,z)-1)$$

It is interesting to note that the lower bound of the variance of $\hat{M}_y^{(g)}$ at (2.1) is the variance of the linear regression estimator

$$\hat{M}_Y^{(l)} = \hat{M}_Y + \hat{d}_1\left(\hat{M}_X^1 - \hat{M}_X\right) + \hat{d}_2\left(M_Z - \hat{M}_Z^1\right) \tag{2.8}$$

where

$$\hat{d}_1 = \frac{\hat{f}_X\left(\hat{M}_x\right)}{\hat{f}_Y\left(\hat{M}_y\right)}\left(4\hat{p}_{11}(x,y)-1\right),$$
$$\hat{d}_2 = \frac{\hat{f}_Z\left(\hat{M}_Z\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}\left(4\hat{p}_{11}(y,z)-1\right),$$

with $\hat{p}_{11}(x,y)$ and $\hat{p}_{11}(y,z)$ being the sample analogues of the $p_{11}(x,y)$ and $p_{11}(y,z)$ respectively and $\hat{f}_Y\left(\hat{M}_Y\right), \hat{f}_X(M_X)$ and $\hat{f}_Z(M_Z)$ can be obtained by following Silverman (1986).

Any parametric function g(u,v) satisfying the conditions (1), (2) and (3) can generate an asymptotically acceptable estimator. The class of such estimators are large. The following simple functions g(u,v) give even estimators of the class

$$g^{(1)}(u,v) = u^{\alpha} v^{\beta}, g^{(2)}(u,v) = \frac{1+\alpha(u-1)}{1-\beta(v-1)},$$
$$g^{(3)}(u,v) = 1+\alpha(u-1)+\beta(v-1), g^{(4)}(u,v) = \{1-\alpha(u-1)-\beta(v-1)\}^{-1}$$
$$g^{(5)}(u,v) = w_1 u^{\alpha} + w_2 v^{\beta}, w_1 + w_2 = 1$$
$$g^{(6)}(u,v) = \alpha u + (1-\alpha)v^{\beta}, g^{(7)}(u,v) = \exp\{\alpha(u-1)+\beta(v-1)\}$$

Let the seven estimators generated by $g^{(i)}$(u,v) be denoted by $\hat{M}_{Yi}^{(g)} = \hat{M}_Y g^{(i)}(u,v), (i = 1 \text{ to} 7)$. It is easily seen that the optimum values of the parameters α,β,w_i(i-1,2) are given by the right hand sides of (2.6).

3. A WIDER CLASS OF ESTIMATORS

The class of estimators (2.1) does not include the estimator

$$\hat{M}_{Yd} = \hat{M}_Y + d_1\left(\hat{M}_X^1 - M_X\right) + d_2\left(M_Z - \hat{M}_Z^1\right), (d_1, d_2)$$

being constants.

However, it is easily shown that if we consider a class of estimators wider than (2.1), defined by

$$\hat{M}_Y^{(G)} = G_1\left(\hat{M}_Y, u, v\right) \tag{3.1}$$

of M_Y, where G(·) is a function of $\hat{M}_Y$, u and v such that $G(M_Y,1,1) = M_Y$ and $G_1(M_Y,1,1) = 1$. $G_1(M_Y,1,1)$ denoting the first partial derivative of G(·) with respect to $\hat{M}_Y$.

Proceeding as in Section 2 it is easily seen that the bias of $\hat{M}_Y^{(G)}$ is of the order n^{-1} and up to this order of terms, the variance of $\hat{M}_Y^{(G)}$ is given by

$$\begin{aligned}\operatorname{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(G)}\right) = \frac{1}{4\left(f_Y(M_Y)\right)^2} \quad & [\left(\frac{1}{m}-\frac{1}{N}\right) + \left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{f_Y(M_Y)}{M_X f_X(M_X)}\right) \\ & G_2(M_Y,1,1)\left\{\left(\frac{f_Y(M_Y)}{M_X f_X(M_X)}\right)G_2(M_Y,1,1) + 2\left(4P_{11}(x,y)-1\right)\right\} \\ & + \left(\frac{1}{n}-\frac{1}{N}\right)\frac{f_Y(M_Y)}{f_Z(M_Z)M_Z}\left\{\left(\frac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)G_3(M_Y,1,1) + 2\left(4P_{11}(y,z)-1\right)\right\}]\end{aligned} \tag{3.2}$$

where $G_2(M_Y1,1)$ and $G_3(M_Y1,1)$ denote the first partial derivatives of u and v respectively around the point $(M_Y,(1,1)$.

The variance of $\hat{M}_Y^{(G)}$ is minimized for

$$\begin{aligned} G_2(M_Y,1,1) &= -\left(\frac{M_X f_X(M_X)}{f_Y(M_Y)}\right)\left(4P_{11}(x,y)-1\right) \\ G_3(M_Y,1,1) &= -\left(\frac{M_Z f_Z(M_Z)}{f_Y(M_Y)}\right)\left(4P_{11}(y,z)-1\right)\end{aligned} \tag{3.3}$$

Substitution of (3.3) in (3.2) yields the minimum variance of $\hat{M}_Y^{(G)}$ as

$$\begin{aligned}\min.\operatorname{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(G)}\right) &= \frac{1}{4\left(f_Y(M_Y)\right)^2}[\left(\frac{1}{m}-\frac{1}{N}\right) - \left(\frac{1}{m}-\frac{1}{n}\right)\left(4P_{11}(x,y)-1\right)^2 - \left(\frac{1}{n}-\frac{1}{N}\right)\left(4P_{11}(y,z)-1\right)^2] \\ &= \min.\operatorname{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(g)}\right)\end{aligned} \tag{3.4}$$

Thus we established the following theorem. Theorem 3.1 - Up to terms of order n^{-1},

$$\mathrm{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(G)}\right) \geq \frac{1}{4\left(f_Y\left(M_Y\right)\right)^2}\left[\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{m}-\frac{1}{n}\right)\left(4P_{11}(x,y)-1\right)^2-\left(\frac{1}{n}-\frac{1}{N}\right)\left(4P_{11}(y,z)-1\right)^2\right]$$

with equality holding if

$$G_2\left(M_Y,1,1\right) = -\left(\frac{f_x\left(M_X\right)M_X}{f_Y\left(M_Y\right)}\right)\left(4P_{11}(x,y)-1\right)$$
$$G_3\left(M_Y,1,1\right) = -\left(\frac{M_Z f_Z\left(M_Z\right)}{f_Y\left(M_Y\right)}\right)\left(4P_{11}(y,z)-1\right)$$

If the information on second auxiliary variable z is not used, then the class of estimators $\hat{M}_Y^{(G)}$ reduces to the class of estimators of M_Y as

$$\hat{M}_Y^{(H)} = H\left(\hat{M}_Y, u\right) \tag{3.5}$$

where $H\left(\hat{M}_Y, u\right)$ is a function of $\left(\hat{M}_Y, u\right)$ such that $H\left(M_Y, 1\right) = M_Y$ and $H_1\left(M_Y, 1\right) = 1$, $H_1\left(M_Y,1\right) = \left.\frac{\partial H(\cdot)}{\partial \hat{M}_Y}\right]_{(M_Y,1)}$. The estimator $\hat{M}_Y^{(H)}$ is reported by Singh *et al* (2001).

The minimum variance of $\hat{M}_Y^{(H)}$ to the first degree of approximation is given by

$$\mathrm{min.Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(H)}\right) = \frac{1}{4\left(f_Y\left(M_Y\right)\right)^2}\left[\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{m}-\frac{1}{n}\right)\left(4P_{11}(x,y)-1\right)^2\right] \tag{3.6}$$

From (3.4) and (3.6) we have

$$\mathrm{minVar}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(H)}\right) - \mathrm{min.Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(G)}\right) = \left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{4\left(f_Y\left(M_Y\right)\right)^2}\left(4P_{11}(y,z)-1\right)^2 \tag{3.7}$$

which is always positive. Thus the proposed class of estimators $\hat{M}_Y^{(G)}$ is more efficient than the estimator $\hat{M}_Y^{(H)}$ considered by Singh *et al* (2001).

4. ESTIMATOR BASED ON ESTIMATED OPTIMUM VALUES

We denote

$$\begin{aligned} \alpha_1 &= \frac{M_X f_X\left(M_X\right)}{M_Y f_Y\left(M_Y\right)}\left(4P_{11}(x,y)-1\right) \\ \alpha_2 &= \frac{M_Z f_Z\left(M_Z\right)}{M_Y f_Y\left(M_Y\right)}\left(4P_{11}(y,z)-1\right) \end{aligned} \tag{4.1}$$

In practice the optimum values of $g_1(1,1)(=-\alpha_1)$ and $g_2(1,1)(=-\alpha_2)$ are not known. Then we use to find out their sample estimates from the data at hand. Estimators of optimum value of $g_1(1,1)$ and $g_2(1,1)$ are given as

$$\begin{aligned}\hat{g}_1(1,1) &= -\hat{\alpha}_1 \\ \hat{g}_2(1,1) &= -\hat{\alpha}_2\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\hat{\alpha}_1 &= \frac{\hat{M}_X \hat{f}_X(\hat{M}_X)}{\hat{M}_Y \hat{f}_Y(\hat{M}_Y)}\left(4\hat{p}_{11}(x,y)-1\right) \\ \hat{\alpha}_2 &= \frac{\hat{M}_Z \hat{f}_Z(\hat{M}_Z)}{\hat{M}_Y \hat{f}_Y(\hat{M}_Y)}\left(4p_{11}(y,z)-1\right)\end{aligned} \tag{4.3}$$

Now following the procedure discussed in Singh and Singh (19xx) and Srivastava and Jhajj (1983), we define the following class of estimators of M_Y (based on estimated optimum) as

$$\hat{M}_Y^{(g*)} = \hat{M}_Y g*\left(u,v,\hat{\alpha}_1,\hat{\alpha}_2\right) \tag{4.4}$$

where $g*(\cdot)$ is a function of $(u,v,\hat{\alpha}_1,\hat{\alpha}_2)$ such that

$$g*(1,1,\alpha_1\alpha_2)=1$$

$$g_1^*(1,1,\alpha_1,\alpha_2)=\left.\frac{\partial g*(\cdot)}{\partial u}\right|_{(1,1,\alpha_1,\alpha_2)} = -\alpha_1$$

$$g_2^*(1,1,\alpha_1,\alpha_2)=\left.\frac{\partial g*(\cdot)}{\partial v}\right|_{(1,1,\alpha_1,\alpha_2)} = -\alpha_2$$

$$g_3^*(1,1,\alpha_1,\alpha_2)=\left.\frac{\partial g*(\cdot)}{\partial \hat{\alpha}_1}\right|_{(1,1,\alpha_1,\alpha_2)} = 0$$

$$g_4^*(1,1,\alpha_1,\alpha_2)=\left.\frac{\partial g*(\cdot)}{\partial \hat{\alpha}_2}\right|_{(1,1,\alpha_1,\alpha_2)} = 0$$

and such that it satisfies the following conditions:

1. Whatever be the samples (S_n and S_m) chosen, let $u,v,\hat{\alpha}_1\hat{\alpha}_2$ assume values in a closed convex sub-space, S, of the four dimensional real space containing the point $(1,1,\alpha_1,\alpha_2)$.

2. The function $g*(u,v,\alpha_1,\alpha_2)$ continuous in S.

3. The first and second order partial derivatives of $g*\left(u,v,\hat{\alpha}_1,\hat{\alpha}_2\right)$ exst. and are also continuous in S.

Under the above conditions, it can be shown that

$$E\left(\hat{M}_Y^{(g*)}\right) = M_Y + 0\left(n^{-1}\right)$$

and to the first degree of approximation, the variance of $\hat{M}_Y^{(g*)}$ is given by

$$\operatorname{Var}\left(\hat{M}_Y^{(g*)}\right) = \min.\operatorname{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{g}\right) \tag{4.5}$$

where $\min.\operatorname{Var}\left(\hat{M}_Y^{(g)}\right)$ is given in (2.7).

A wider class of estimators of M_Y based on estimated optimum values is defined by

$$\hat{M}_Y^{(G*)} = G*\left(\hat{M}_Y, u, v, \hat{\alpha}_1^*, \hat{\alpha}_2^*\right) \tag{4.6}$$

where

$$\begin{aligned} \hat{\alpha}_1^* &= \frac{\hat{M}_X \hat{f}_X\left(\hat{M}_X\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}\left(4\hat{p}_{11}(x,y)-1\right) \\ \hat{\alpha}_2^* &= \frac{\hat{M}_Z \hat{f}_Z\left(\hat{M}_Z\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}\left(4\hat{p}_{11}(y,z)-1\right) \end{aligned} \tag{4.7}$$

are the estimates of

$$\begin{aligned} \alpha_1^* &= \frac{M_x f_X\left(M_X\right)}{f_Y\left(M_Y\right)}\left(4P_{11}(x,y)-1\right) \\ \alpha_2^* &= \frac{M_Z f_Z\left(M_Z\right)}{f_Y\left(M_Y\right)}\left(4P_{11}(y,z)-1\right) \end{aligned} \tag{4.8}$$

and G*(·) is a function of $\left(\hat{M}_Y, u, v, \alpha_1^*, \hat{\alpha}_2^*\right)$ such that

$$G*\left(M_Y, 1, 1, \alpha_1^*, \alpha_2^*\right) = M_Y$$

$$G_1^*\left(M_Y, 1, 1, \alpha_1^*, \alpha_2^*\right) = \left.\frac{\partial G*(\cdot)}{\partial \hat{M}_Y}\right|_{\left(M_Y, 1, 1, \alpha_1^*, \alpha_2^*\right)} = 1$$

$$G_2^*\left(M_Y 1, 1, \alpha_1^*, \alpha_2^*\right) = \left.\frac{\partial G*(\cdot)}{\partial u}\right|_{\left(M_Y, 1, 1, \alpha_1^*, \alpha_2^*\right)} = -\alpha_1^*$$

$$G_3^*\left(M_Y 1, 1, \alpha_1^*, \alpha_2^*\right) = \left.\frac{\partial G*(\cdot)}{\partial v}\right|_{\left(M_Y 1, 1, \partial_1^* \alpha_2^*\right)} = -\alpha_2^*$$

$$G_4^*\left(M_Y 1, 1, \alpha_1^*, \alpha_2^*\right) = \left.\frac{\partial G*(\cdot)}{\partial \hat{\alpha}_1^*}\right|_{\left(M_Y, 1, 1, \alpha_1^*, \alpha_2^*\right)} = 0$$

$$G_5^*\left(M_Y 1,1,\alpha_1^*,\alpha_2^*\right)=\left.\frac{\partial G*(\cdot)}{\partial\hat{\alpha}_2^*}\right|_{\left(M_Y,1,1,\alpha_1^*,\alpha_2^*\right)}=0$$

Under these conditions it can be easily shown that

$$E\left(\hat{M}_Y^{(G*)}\right)=M_Y+0\left(n^{-1}\right)$$

and to the first degree of approximation, the variance of $\hat{M}_Y^{(G*)}$ is given by

$$\operatorname{Var}\left(\hat{M}_Y^{G*}\right)=\min.\operatorname{Var}\left(\hat{M}_Y^{(G)}\right) \tag{4.9}$$

where $\min.\operatorname{Var}\left(\hat{M}_Y^{G}\right)$ is given in (3.4).

It is to be mentioned that a large number of estimators can be generated from the classes $\hat{M}_Y^{(g*)}$ and $\hat{M}_Y^{(G*)}$ based on estimated optimum values.

5. EFFICIENCY OF THE SUGGESTED CLASS OF ESTIMATORS FOR FIXED COST

The appropriate estimator based on on single-phase sampling without using any auxiliary variable is $\hat{M}_Y$, whose variance is given by

$$\operatorname{Var}\left(\hat{M}_Y\right)=\left(\frac{1}{m}-\frac{1}{N}\right)\frac{1}{4\left(f_Y\left(M_Y\right)\right)^2} \tag{5.1}$$

In case when we do not use any auxiliary character then the cost function is of the form C_0-mC_1, where C_0 and C_1 are total cost and cost per unit of collecting information on the character Y.

The optimum value of the variance for the fixed cost C_0 is given by

$$\text{Opt.}\left[\operatorname{Var}\left(\hat{M}_Y\right)=V_0\left(\frac{G}{C_0}-\frac{1}{N}\right)\right] \tag{5.2}$$

where

$$V_0\frac{1}{4\left(f_Y\left(M_Y\right)\right)^2} \tag{5.3}$$

When we use one auxiliary character X then the cost function is given by

$$C_0=Gm+C_2n, \tag{5.4}$$

where C_2 is the cost per unit of collecting information on the auxiliary character Z.

The optimum sample sizes under (5.4) for which the minimum variance of $\hat{M}_Y^{(H)}$ is optimum, are

$$\mathrm{m}_{\mathrm{opt}}=\frac{C_0\sqrt{(V_0-V_1)/C_1}}{\left[\sqrt{(V_0-V_1)C_1}+\sqrt{V_1C_2}\right]} \tag{5.5}$$

$$\mathrm{n}_{\mathrm{opt}}=\frac{C_0\sqrt{V_1/C_2}}{\left[\sqrt{(V_0-V_1)C_1}+\sqrt{V_1C_2}\right]}$$

where $V_1=V_0(4P_{11}(x,y)-1)^2$.

Putting these optimum values of m and n in the minimum variance expression of $\hat{M}_Y^{(H)}$ in (3.6), we get the optimum $\min.\mathrm{Var}\left(\hat{M}_Y^{(H)}\right)$ as

$$\mathrm{Opt.}\left[\min.\mathrm{Var}\left(\hat{M}_Y^{(H)}\right)\right]=\left[\frac{\left(\sqrt{(V_0-V_1)C_1}+\sqrt{V_1C_2}\right)^2}{C_0}-\frac{V_0}{N}\right] \tag{5.7}$$

Similarly, when we use an additional character Z then the cost function is given by

$$C_0=C_1m+(C_2+C_3)n \tag{5.8}$$

where C_3 is the cost per unit of collecting information on character Z.

It is assumed that $C_1>C_2>C_3$. The optimum values of m and n for fixed cost C_0 which minimizes the minimum variance of $\hat{M}_Y^{(g)}\left(\mathrm{or}\hat{M}_Y^{(G)}\right)$ (2.7) (or (3.4)) are given by

$$\mathrm{m}_{\mathrm{opt}}=\frac{C_0\sqrt{(V_0-V_1)/C_1}}{\left[\sqrt{(V_0-V_1)C_1}+\sqrt{(C_2+C_3)(V_1-V_2)}\right]} \tag{5.9}$$

$$\mathrm{n}_{\mathrm{opt}}=\frac{C_0\sqrt{(V_1-V_2)/C_2+C_3}}{\left[\sqrt{(V_0-V_1)C_1}+\sqrt{(C_2+C_3)(V_1-V_2)}\right]} \tag{5.10}$$

where $V_2=V_0(4P_{11}(y,z)-1)^2$.

The optimum variance of $\hat{M}_Y^{(g)}\left(\mathrm{or}\hat{M}_Y^{(G)}\right)$ corresponding to optimal two-phase sampling strategy is

$$\mathrm{Opt}\left[\min.\mathrm{Var}\left(\hat{M}_Y^{(g)}\right)\mathrm{or}\ \min.\mathrm{Var}\left(\hat{M}_Y^{(G)}\right)\right]=\left[\frac{[\sqrt{(V_0-V_1)C_1}+\sqrt{(C_2+C_3)(V_1-V_2)}]^2}{C_0}-\frac{V_2}{N}\right] \tag{5.11}$$

Assuming large N, the proposed two phase sampling strategy would be profitable over single phase sampling so long as

$$\left[\mathrm{Opt.Var}\left(\hat{M}_Y\right)\right]>\mathrm{Opt.}\left[\min.\mathrm{Var}\left(\hat{M}_Y^{(g)}\right)\mathrm{or}\ \min.\mathrm{Var}\left(\hat{M}_Y^{(G)}\right)\right]$$

$$\text{i.e.}\frac{C_2+C_3}{C_1}<\left[\frac{\sqrt{V_0}-\sqrt{V_0-V_1}}{\sqrt{V_1-V_2}}\right] \tag{5.12}$$

When N is large, the proposed two phase sampling is more efficient than that Singh *et al* (2001) strategy if

$$\text{Opt}\left[\min.\text{Var}\left(\hat{M}_Y{}^{(g)}\right)\text{or } \min.\text{Var}\left(\hat{M}_Y{}^{(G)}\right)\right]<\text{Opt}\left[\min.\text{Var}\left(\hat{M}_Y{}^{(H)}\right)\right]$$

$$\text{i.e.}\frac{C_2+C_3}{C_1}<\frac{V_1}{V_1-V_2} \tag{5.13}$$

6. GENERALIZED CLASS OF ESTIMATORS

We suggest a class of estimators of M_Y as

$$\Im=\left\{\hat{M}_Y{}^{(F)}:\hat{M}_Y{}^{(F)}=F\left(\hat{M}_Y,u,v,w\right)\right\} \tag{6.1}$$

where $u=\hat{M}_X/\hat{M}'_X$, $v=\hat{M}'_Z/M_Z$, $w=\hat{M}_Z/M_Z$ and the function F(·) assumes a value in a bounded closed convex subset $W\subset\Re_4$, which contains the point $(M_Y,1,1,1)=T$ and is such that $F(T)=M_Y\Rightarrow F_1(T)=1$, $F_1(T)$ denoting the first order partial derivative of F(·) with respect to $\hat{M}_Y$ around the point $T=(M_Y,1,1,1)$. Using a first order Taylor's series expansion around the point T, we get

$$\hat{M}_Y{}^{(F)}=F(T)+\left(\hat{M}_Y=M_Y\right)F_1(T)+(u-1)F_2(T)+(v-1)F_3(T)+(w-1)F_4(T)+0(n^{-1}) \tag{6.2}$$

where $F_2(T)$, $F_3(T)$ and $F_4(T)$ denote the first order partial derivatives of $F\left(\hat{M}_Y,u,v,w\right)$ with respect to u, v and w around the point T respectively. Under the assumption that $F(T)=M_Y$ and $F_1(T)=1$, we have the following theorem.

Theorem 6.1. Any estimator in $\Im$ is asymptotically unbiased and normal.

Proof: Following Kuk and Mak (1989), let P_Y, P_X and P_Z denote the proportion of Y, X and Z values respectively for which $Y\le M_Y$, $X\le M_X$ and $Z\le M_Z$; then we have

$$\hat{M}_Y-M_Y=\frac{1}{2f_Y(M_Y)}(1-2P_Y)+0_p\left(n^{-1/2}\right),$$

$$\hat{M}_X-M_X=\frac{1}{2f_X(M_X)}(1-2P_X)+0_p\left(n^{-1/2}\right),$$

$$\hat{M}'_x-M_X=\frac{1}{2f_X(M_X)}(1-2P_X)+0_p\left(n^{-1/2}\right)$$

$$\hat{M}_z-M_Z=\frac{1}{2f_Z(M_Z)}(1-2P_Z)+0_p\left(n^{-1/2}\right)$$

and

$$\hat{M}'_Z - M_Z = \frac{1}{2f_Z(M_z)}(1-2P_Z) + 0_p\left(n^{-1/2}\right)$$

Using these expressions in (6.2), we get the required results.

Expression (6.2) can be rewritten as

$$\hat{M}_Y^{(F)} - M_Y \cong \left(\hat{M}_Y - M_Y\right) + (u-1)F_2(T) + (v-1)F_3(T) + (w-1)F_4(T)$$

or

$$\hat{M}_Y^{(F)} - M_Y \cong M_Y e_0 + (e_1 - e_2)F_2(T) + e_4 F_3(T) + e_3 F_4(T) \tag{6.3}$$

Squaring both sides of (6.3) and then taking expectation, we get the variance of $\hat{M}_Y^{(F)}$ to the first degree of approximation, as

$$\mathrm{Var}\left(\hat{M}_Y^{(F)}\right) = \frac{1}{4\left(f_Y(M_Y)\right)^2}\left[\left(\frac{1}{m}-\frac{1}{N}\right)A_1 + \left(\frac{1}{m}-\frac{1}{n}\right)A_2 + \left(\frac{1}{n}-\frac{1}{N}\right)A_3\right], \tag{6.4}$$

where

$$A_1 = \left[1 + \left(\frac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)^2 F_4^{\,2}(T) + 2\left(4P_{11}(y,z)-1\right)\left(\frac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)F_4(T)\right]$$

$$A_2 = \left(\frac{f_Y(M_Y)}{M_X f_X(M_X)}\right)\left[\begin{array}{c}\left(\dfrac{f_Y(M_Y)}{M_X f_X(M_X)}\right)F_2^{\,2}(T) + 2(4P_{11}(x,y)-1)F_2(T) \\ +\,2(4P_{11}(x,z)-1\left(\dfrac{f_Y(M_Y)}{M_z f_Z(M_Z)}\right)F_2(T)F_4(T)\end{array}\right]$$

$$A_3 = \left(\frac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)\left[\begin{array}{c}\left(\dfrac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)F_3^{\,2}(T) + 2(4P_{11}(y,z)-1)F_3(T) \\ +\,2\left(\dfrac{f_Y(M_Y)}{M_Z f_Z(M_Z)}\right)F_3(T)F_4(T)\end{array}\right]$$

The $\mathrm{Var}\left(\hat{M}_Y^{(F)}\right)$ at (6.4) is minimized for

$$F_2(T) = -\frac{[(4P_{11}(x,y)-1)-(4P_{11}(x,z)-1)(4P_{11}(y,z)-1)]}{[1-(4P_{11}(x,z)-1)^2]} \cdot \frac{M_X f_X(M_X)}{f_Y(M_Y)} \cdot$$

$$= -a_2 \text{ (say)}$$

(6.5)

$$F_3(T) = -\frac{(4P_{11}(x,z)-1)[(4P_{11}(x,y)-1)-(4P_{11}(y,z)-1)(4P_{11}(x,z)-1)]}{[1-(4P_{11}(x,z)-1)^2]} \cdot \frac{M_Z f_Z(M_Z)}{f_Y(M_Y)} \cdot$$

$$= -a_2 \text{ (say)}$$

$$F_4(T) = -\frac{[(4P_{11}(y,z)-1)-(4P_{11}(x,y)-1)(4P_{11}(x,z)-1)]}{[1-(4P_{11}(x,z)-1)^2]} \cdot \frac{M_Z f_Z(M_Z)}{f_Y(M_Y)} \cdot$$

$$= -a_3 \text{ (say)}$$

Thus the resulting (minimum) variance of $\hat{M}_Y^{(F)}$ is given by

$$\text{minVar}\left(\hat{M}_Y^{(F)}\right) = \frac{1}{4(\text{f}_\text{Y}(M_Y))^2}\left[\begin{array}{c}\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{m}-\frac{1}{n}\right)\left\{\frac{D^2}{1-(4P_{11}(x,z)-1)^2}+(4P_{11}(x,y)-1)\right\}^2 \\ -\left(\frac{1}{n}-\frac{1}{N}\right)(4P_{11}(y,z)-1)^2\end{array}\right]$$

$$= \text{min.Var}\left(\hat{M}_Y^{(G)}\right) - \left(\frac{1}{m}-\frac{1}{n}\right)\frac{1}{4(f_Y(M_Y))^2}\frac{D^2}{1-[4P_{11}(x,z)-1^2]}$$

(6.6)

where

$$D = [(4P_{11}(y,z)-1)-(4P_{11}(x,y)-1)(4P_{11}(x.z)-1)] \tag{6.7}$$

and $\text{min.Var}\left(\hat{M}_Y^{(G)}\right)$ is given in (3.4)

Expression (6.6) clearly indicates that the proposed class of estimators $\hat{M}_Y^{(F)}$ is more efficient than the class of estimator $\hat{M}_Y^{(G)}$ or $\left(\hat{M}_Y^{(g)}\right)$ and hence the class of estimators $\hat{M}_Y^{(H)}$ suggested by Singh *et al* (2001) and the estimator $\hat{M}_Y$ at its optimum conditions.

The estimator based on estimated optimum values is defined by

$$p^* = \left\{\hat{M}_Y^{(F^*)} : \hat{M}_Y^{F^*} = F^*\left(\hat{M}_Y, u, v, w, \hat{a}_1, \hat{a}_2, \hat{a}_3\right)\right\} \tag{6.8}$$

where

$$\hat{a}_1 = \frac{\left[\left(4\hat{p}_{11}(x,y)-1\right)-\left(4\hat{p}_{11}(x,z)-1\right)\left(4\hat{p}_{11}(y,z)-1\right)\right]}{\left[1-\left(4\hat{p}_{11}(x,z)-1\right)^2\right]}\frac{\hat{M}_x\hat{f}_x\left(\hat{M}_x\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}.$$

$$\hat{a}_2 = \frac{\left(4\hat{p}_{11}(x,z)-1\right)\left[\left(4\hat{p}_{11}(x,y)-1\right)-\left(4\hat{p}_{11}(y,z)-1\right)\left(4\hat{p}_{11}(x,z)-1\right)\right]}{\left[1-\left(4\hat{p}_{11}(x,z)-1\right)^2\right]}\frac{\hat{M}_Z\hat{f}_Z\left(\hat{M}_Z\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}.$$

$$a_3 = \frac{\left[\left(4\hat{p}_{11}(y,z)-1\right)-\left(4\hat{p}_{11}(x,y)-1\right)\left(4\hat{p}_{11}(x,z)-1\right)\right]}{\left[1-\left(4\hat{p}_{11}(x,z)-1\right)^2\right]}\frac{\hat{M}_Z\hat{f}_Z\left(\hat{M}_Z\right)}{\hat{f}_Y\left(\hat{M}_Y\right)}.$$

(6.9)

are the sample estimates of a_1, a_2 and a_3 given in (6.5) respectively, $F^*(\cdot)$ is a function of $\left(\hat{M}_Y, u, v, w, \hat{a}_1, \hat{a}_2, \hat{a}_3\right)$ such that

$$F^*(T^*) = M_Y$$

$$\Rightarrow F_1^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial \hat{M}_Y}\right|_{T^*} = 1$$

$$F_2^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial u}\right|_{T^*} = -a_1$$

$$F_3^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial v}\right|_{T^*} = -a_2$$

$$F_4^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial w}\right|_{T^*} = -a_3$$

$$F_5^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial \hat{a}_1}\right|_{T^*} = 0$$

$$F_6^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial \hat{a}_2}\right|_{T^*} = 0$$

$$F_7^*(T^*) = \left.\frac{\partial F^*(\cdot)}{\partial \hat{a}_3}\right|_{T^*} = 0$$

where $T^* = (M_Y,1,1,1,a_1,a_2,a_3)$

Under these conditions it can easily be shown that

$$E\left(\hat{M}_Y^{(F^*)}\right) = M_Y + 0\left(n^{-1}\right)$$

and to the first degree of approximation, the variance of $\hat{M}_Y^{(F*)}$ is given by

$$\operatorname{Var}\left(\hat{M}_Y^{(F*)}\right) = \min.\operatorname{Var}\left(\hat{M}_Y^{F}\right) \tag{6.10}$$

where $\min.\operatorname{Var}\left(\hat{M}_Y^{(F)}\right)$ is given in (6.6).

Under the cost function (5.8), the optimum values of m and n which minimizes the minimum variance of $\hat{M}_Y^{(F)}$ is (6.6) are given by

$$\mathrm{m}_{\mathrm{opt}} = \frac{C_0\sqrt{(V_0 - V_1 - V_3)/C_1}}{[\sqrt{(V_0 - V_1 - V_3)C_1} + \sqrt{(V_1 - V_2 - V_3)(C_2 + C_3)}]} \tag{6.11}$$

$$\mathrm{n}_{\mathrm{opt}} = \frac{C_0\sqrt{(V_1 - V_2 - V_3)}/C_2}{[\sqrt{(V_0 - V_1 - V_3)C_1} + \sqrt{(V_1 - V_2 + V_3)(C_2 + C_3)}]}$$

where

$$V_3 = \frac{D^2 V_0}{\left[1 - (4P_{11}(x,z) - 1)^2\right]} \tag{6.12}$$

for large N, the optimum value of $\min.\operatorname{Var}\left(\hat{M}_Y^{(F)}\right)$ is given by

$$\operatorname{Opt.}\left[\min.\operatorname{Var}\left(\hat{M}_Y^{(F)}\right)\right] = \frac{\left[\sqrt{(V_0 - V_1 - V_3)C_1} + \sqrt{(V_1 - V_2 + V_3)(C_2 + C_3)}\right]}{C_0} \tag{6.13}$$

The proposed two-phase sampling strategy would be profitable over single phase-sampling so long as $\operatorname{Opt.}\left[\operatorname{Var}\left(\hat{M}_Y\right)\right] > \operatorname{Opt.}\left[\min.\operatorname{Var}\left(\hat{\mathrm{M}}_{\mathrm{Y}}^{(F)}\right)\right]$

$$\text{i.e.}\quad \frac{C_2 + C_3}{c_1} < \left[\frac{\sqrt{V_0} - \sqrt{V_0 - V_1 - V_3}}{\sqrt{V_1 - V_2 + V_3}}\right]^2 \tag{6.14}$$

It follows from (5.7) and (6.13) that

$$\operatorname{Opt.}\left[\min.\operatorname{Var}\left(\hat{M}_Y^{(F)}\right)\right] < \operatorname{Opt.}\left[\min.\operatorname{Var}\left(\hat{M}_Y^{H}\right)\right]$$

$$\text{if}\quad \left(\frac{\sqrt{\mathrm{V}_0 - V_1} - \sqrt{V_0 - V_1 - V_3}}{\sqrt{V_1 - V_2 + V_3}}\right) > \left[\sqrt{\frac{C_2 + C_3}{C_1}} - \sqrt{\frac{V_1}{(V_1 - V_2 + V_3)C_1}\frac{C_2}{C_1}}\right] \tag{6.15}$$

for large N.

Further we note from (5.11) and (6.13) that

$$\text{Opt.}\left\lfloor \text{min.Var}\left(\hat{M}_Y{}^{(F)}\right)\right\rfloor < \text{Opt.}\left\lfloor \text{min.Var}\left(\hat{M}_Y{}^{(g)} \text{or} \hat{M}_Y{}^{G}\right)\right\rfloor$$

$$\text{if} \quad \frac{C_2 + C_3}{C_1} < \left[\frac{\sqrt{(V_0 - V_1)} - \sqrt{(V_0 - V_1 - V_3)}}{\sqrt{(V_1 - V_2 + V_3)} - \sqrt{V_1 - V_2}}\right]^2 \tag{6.16}$$

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A Family of Estimators of Population Mean Using Multiauxiliary Information in Presence of Measurement Errors

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Abstract

This paper proposes a family of estimators of population mean using information on several auxiliary variables and analyzes its properties in the presence of measurement errors.

Keywords: Population mean, Study variate, Auxiliary variates, Bias, Mean squared error, Measurement errors.

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1. INTRODUCTION

The discrepancies between the values exactly obtained on the variables under consideration for sampled units and the corresponding true values are termed as measurement errors. In general, standard theory of survey sampling assumes that data collected through surveys are often assumed to be free of measurement or response errors. In reality such a supposition does not hold true and the data may be contaminated with measurement errors due to various reasons; see, e.g., Cochran (1963) and Sukhatme *et al* (1984).

One of the major sources of measurement errors in survey is the nature of variables. This may happen in case of qualitative variables. Simple examples of such variables are intelligence, preference, specific abilities, utility, aggressiveness, tastes, etc. In many sample surveys it is recognized that errors of measurement can also arise from the person being interviewed, from the interviewer, from the supervisor or leader of a team of interviewers, and from the processor who transmits the information from the recorded interview on to the punched cards or tapes that will be analyzed, for instance, see Cochran (1968). Another source of measurement error is when the variable is conceptually well defined but observations can be obtained on some closely related substitutes termed as proxies or surrogates. Such a situation is

encountered when one needs to measure the economic status or the level of education of individuals, see Salabh (1997) and Sud and Srivastava (2000). In presence of measurement errors, inferences may be misleading, see Biemer *et al* (1991), Fuller (1995) and Manisha and Singh (2001).

There is today a great deal of research on measurement errors in surveys. An attempt has been made to study the impact of measurement errors on a family of estimators of population mean using multiauxiliary information.

2. THE SUGGESTED FAMILY OF ESTIMATORS

Let Y be the study variate and its population mean μ_0 to be estimated using information on p(>1) auxiliary variates $X_1, X_2, ..., X_p$. Further, let the population mean row vector $\underset{\sim}{\mu}' = (\mu_1, \mu_2, \cdots, \mu_p)$ of the vector $\underset{\sim}{X}' = (X_1, X_2, X_p)$. Assume that a simple random sample of size n is drawn from a population, on the study character Y and auxiliary characters $X_1, X_2, ..., X_p$. For the sake of simplicity we assume that the population is infinite. The recorded fallible measurements are given by

$$
\begin{aligned}
y_j &= Y_j + E_j \\
x_{ij} &= X_{ij} + \eta_{ij}, \quad i = 1,2,\cdots,p; \\
& \qquad\qquad\qquad\; j = 1,2,\cdots,n.
\end{aligned}
$$

where Y_j and X_{ij} are correct values of the characteristics Y and X_i (i=1,2,..., p; j=1,2,..., n).

For the sake of simplicity in exposition, we assume that the error E_j's are stochastic with mean 'zero' and variance $\sigma_{(0)}^2$ and uncorrelated with Y_j's. The errors η_{ij} in x_{ij} are distributed independently of each other and of the X_{ij} with mean 'zero' and variance $\sigma_{(i)}^2$ (i=1,2,...,p). Also E_j's and η_{ij}'s are uncorrelated although Y_j's and X_{ij}'s are correlated.

Define

$$u_i = \frac{\bar{x}_i}{\mu_i}, \left(i = 1,2,\cdots,p\right)$$
$$u^T = \left(u_1, u_2, \cdots u_p\right)_{1\times p},\; e^T = \left(1,1,\cdots,1\right)_{1\times p}$$
$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j$$
$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}$$

With this background we suggest a family of estimators of μ_0 as

$$\hat{\mu}_g = g\left(\bar{y}, u^T\right) \tag{2.1}$$

where $g\left(\bar{y}, u^T\right)$ is a function of $\bar{y}, u_1, u_2, \cdots, u_p$ such that

$$g_{\left(u_0, e^T\right)} = \mu_0$$
$$\Rightarrow \left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)} = 1$$

and such that it satisfies the following conditions:

1. The function $g\left(\bar{y}, u^T\right)$ is continuous and bounded in Q.

2. The first and second order partial derivatives of the function $g\left(\bar{y}, u^T\right)$ exist and are continuous and bounded in Q.

To obtain the mean squared error of $\hat{\mu}_g$, we expand the function $g\left(\bar{y}, u^T\right)$ about the point (μ_0, e^T) in a second order Taylor's series. We get

$$\hat{\mu}_g = g\left(\mu_0, e^T\right) + \left(\bar{y} - \mu_0\right)\left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)} + \left(u - e\right)^T g^{(1)}{}_{\left(\mu_0, e^T\right)}$$
$$+ \frac{1}{2}\left\{\left(\bar{y} - \mu_0\right)^2 \left.\frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\right|_{\left(\bar{y}^*, u^{*T}\right)} + 2\left(\bar{y} - \mu_0\right)\left(u - e\right)^T \left.\frac{\partial g^{(1)}(\cdot)}{\partial \bar{y}}\right|_{\left(\bar{y}^*, u^{7T}\right)}\right.$$
$$\left. + \left(u - e\right)^T g^{(2)}\left(\vec{y}*, u*^T\right)\left(u - e\right)\right\} \tag{2.2}$$

where

$$\bar{y}^* = \mu_0 + \theta\left(\bar{y} - \mu_0\right), u^* = e + \theta\left(u - e\right), 0 < \theta < 1; \quad g^{(1)}(\cdot)$$

denote the p element column vector of first partial derivatives of $g(\cdot)$ and $g^{(2)}(\cdot)$ denotes a p×p matrix of second partial derivatives of $g(\cdot)$ with respect to u.

Noting that $g(\mu_0,e^T)= \mu_0$, it can be shown that

$$E\left(\hat{\mu}_g\right) = \mu_0 + O(n^{-1}) \tag{2.3}$$

which follows that the bias of $\hat{\mu}_g$ is of the order of n^{-1}, and hence its contribution to the mean squared error of $\hat{\mu}_g$ will be of the order of n^{-2}.

From (2.2), we have to terms of order n^{-1},

$$\begin{aligned}\mathrm{MSE}\left(\hat{\mu}_g\right) &= E\left\{\left(\bar{y} - \mu_0\right) + \left(u - e\right)^T g^{(1)}\left(\mu_0, e^T\right)\right\}^2 \\ &= E\Big[\left(\bar{y} = \mu_0\right)^2 + 2\left(\bar{y} - \mu_0\right)\left(u - e\right)^T g^{(1)}\left(\mu_0, e^T\right) \\ &\qquad + \left(g^{(1)}\left(\mu_0, e^T\right)\right)^T \left(u - e\right)\left(u - e\right)^T \left(g^{(1)}\left(\mu_0, e^T\right)\right)\Big] \\ &= \frac{1}{n}\left[\mu_0^2\left(C_0^2 + C_{(0)}^2\right) + 2\mu_0 b^T g^{(1)}\left(\mu_0, e^T\right) + \left(g^{(1)}\left(\mu_0, e^T\right)\right)^T A\left(g^{(1)}\left(\mu_0, e^T\right)\right)\right]\end{aligned} \tag{2.4}$$

where $b^T=(b_1,b_2,\ldots,b_p)$, $b_i,=\rho_{0i}C_0C_i,(i=1,2, \ldots,p)$;

$C_i=\sigma_i/\mu_i$, $C_{(i)}= \sigma_i/\mu_i$, $(i=1,2, \ldots,p)$ and $C_{0=}\sigma_0/\mu_0$,

$$\mathrm{A} = \begin{bmatrix} C_1^2 + C_{(1)}^2 & \rho_{12}C_1C_2 & \rho_{13}C_1C_3 & \cdots & \rho_{1p}C_1C_p \\ \rho_{12}C_1C_2 & C_2^2 + C_{(2)}^2 & \rho_{23}C_2C_3 & \cdots & \rho_{2p}C_2C_p \\ \rho_{13}C_1C_3 & \rho_{23}C_2C_3 & C_3^2 + C_{(3)}^2 & \cdots & \rho_{3p}C_3C_p \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{1p}C_1C_p & \rho_{2p}C_2C_p & \rho_{3p}C_3C_p & \cdots & C_p^2 + C_{(p)}^2 \end{bmatrix}_{p\times p}$$

The $\mathrm{MSE}\left(\hat{\mu}_{\mathrm{g}}\right)$ at (2.4) is minimized for

$$g^{(1)}\left(\mu_0, e^T\right) = -\mu_0 A^{-1} b \tag{2.5}$$

Thus the resulting minimum MSE of $\hat{\mu}_g$ is given by

$$\text{min.MSE}(\hat{\mu}_g) = (\mu_0^2 / n)\left[C_0^2 + C_{(0)}^2 - b^T A^{-1} b\right] \tag{2.6}$$

Now we have established the following theorem.

Theorem 2.1 = Up to terms of order n^{-1},

$$MSE(\hat{\mu}_g) \geq (\mu_0^2 / n)\left[C_0^2 + C_{(0)}^2 - b^T A^{-1} b\right] \tag{2.7}$$

with equality holding if

$$g^{(1)}(\mu_0, e^T) = -\mu_0 A^{-1} b$$

It is to be mentioned that the family of estimators $\hat{\mu}_g$ at (2.1) is very large. The following estimators:

$$\hat{\mu}_g^{(1)} = \bar{y} \sum_{i=1}^{p} \omega_i \left(\frac{\mu_i}{\bar{x}_i} \right); \quad \sum_{i=1}^{p} \omega_i = 1,$$ [Olkin (1958)]

$$\hat{\mu}_g^{(2)} = \bar{y} \sum_{i=1}^{p} \omega_i \left(\frac{\bar{x}_i}{\mu_i} \right), \quad \sum_{i=1}^{p} \omega_i - 1,$$ [Singh (1967)]

$$\hat{\mu}_g^{(3)} = \bar{y} \frac{\sum_{i=1}^{p} \omega_i \mu_i}{\sum_{i=1}^{p} \omega_i \bar{x}_i}, \quad \sum_{i=1}^{p} \omega_i = 1,$$ [Shukla (1966) and John (1969)]

$$\hat{\mu}_g^{(4)} = \bar{y} \frac{\sum_{i=1}^{p} \omega_i \bar{x}_i}{\sum_{i=1}^{p} \omega_i \mu_i}; \quad \sum_{i=1}^{p} \omega_i = 1,$$ [Sahai *et al* (1980)]

$$\hat{\mu}_g^{(5)} = \bar{y} \prod_{i=1}^{p} \left(\frac{\mu_i}{\bar{x}_i} \right)^{\omega_i}, \quad \sum_{i=1}^{p} \omega_i = 1,$$ [Mohanty and Pattanaik (1984)]

$$\hat{\mu}_g^{(6)} = \bar{y} \left(\sum_{i=1}^{p} \frac{\omega_i \bar{x}_i}{\mu_i} \right)^{-1}, \quad \sum_{i=1}^{p} \omega_i = 1,$$ [Mohanty and Pattanaik (1984)]

$$\hat{\mu}_g^{(7)} = \bar{y}\prod_{i=1}^{p}\left(\frac{\bar{x}_i}{\mu_i}\right)^{\omega_i}, \quad \sum_{i=1}^{p}\omega_i = 1,$$ [Tuteja and Bahl (1991)]

$$\hat{\mu}_g^{(8)} = \bar{y}\left[\sum_{i=1}^{p}\frac{\omega_i\mu_i}{\bar{x}_i}\right]^{-1}, \quad \sum_{i=1}^{p}\omega_i = 1,$$ [Tuteja and Bahl (1991)]

$$\hat{\mu}_g^{(9)} = \bar{y}\left[\omega_{p+1} + \sum_{i=1}^{p}\omega_i\left(\frac{\mu_i}{\bar{x}_i}\right)\right], \quad \sum_{i=1}^{p+1}\omega_i = 1.$$

$$\hat{\mu}_g^{(10)} = \bar{y}\left[\omega_{p+1} + \sum_{i=1}^{p}\omega_i\left(\frac{\bar{x}_i}{\mu_i}\right)\right], \quad \sum_{i=1}^{p+1}\omega_i = 1.$$

$$\hat{\mu}_g^{(11)} = \bar{y}\left[\sum_{i=1}^{q}\omega_i\left(\frac{\mu_i}{x_i}\right) + \sum_{i=q+1}^{p}\left(\frac{\hat{x}_i}{\mu_i}\right)\right]; \quad \left(\sum_{i=1}^{q}\omega_i + \sum_{i=q+1}^{p}\omega_i\right)^{=1};$$ [Srivastava (1965) and Rao and Mudhalkar (1967)]

$$\hat{\mu}_g^{(12)} = \bar{y}\prod_{i=1}^{p}\left(\frac{\bar{x}_i}{\mu_i}\right)^{\alpha_i} (\alpha_i\text{'s are suitably constants})$$ [Srivastava (1967)]

$$\hat{\mu}_g^{(13)} = \bar{y}\prod_{i=1}^{p}\left\{2 - \left(\frac{\bar{x}_i}{\mu_i}\right)^{\alpha_i}\right\}$$ [Sahai and Rey (1980)]

$$\hat{\mu}_g^{(14)} = \bar{y}\prod_{i=1}^{p}\frac{\bar{x}_i}{\{\mu_i + \alpha_i(\bar{x}_i - \mu_i)\}}$$ [Walsh (1970)]

$$\hat{\mu}_g^{(15)} = \bar{y}\exp\left\{\sum_{i=1}^{p}\theta_i \log u_i\right\}$$ [Srivastava (1971)]

$$\hat{\mu}_g^{(16)} = \bar{y}\exp\left\{\sum_{i=1}^{p}\theta_i(u_i - 1)\right\}$$ [Srivastava (1971)]

$$\hat{\mu}_g^{(17)} = \bar{y}\sum_{i=1}^{p}\omega_i \exp\{(\theta_i/\omega_i)\log u_i\}; \quad \sum_{i=1}^{p}\omega_i = 1,$$ [Srivastava (1971)]

$$\hat{\mu}_g^{(18)} = \bar{y} + \sum_{i=1}^{p}\alpha_i(\bar{x}_i - \mu_i)$$

etc. may be identified as particular members of the suggested family of estimators $\hat{\mu}_g$. The MSE of these estimators can be obtained from (2.4).

It is well known that

$$\mathrm{V}(\bar{y}) = \left(\mu_0^2 / n\right)\left(C_0^2 + C_{(0)}^2\right) \tag{2.8}$$

It follows from (2.6) and (2.8) that the minimum variance of $\hat{\mu}_g$ is no longer than conventional unbiased estimator $\bar{y}$.

On substituting $\sigma_{(0)}{}^2=0$, $\sigma_{(i)}{}^2=0$ $\forall$i=1,2,…,p in the equation (2.4), we obtain the no-measurement error case.

In that case, the MSE of $\hat{\mu}_g$, is given by

$$\begin{aligned}\mathrm{MSE}\left(\hat{\mu}_g\right) &= \frac{1}{n}\left[C_0^2\mu_0^2 + 2\mu_0 b^T g^{*(1)}\left(\mu_0, e^T\right) + \left(g^{*(1)}\left(\mu_0, e^T\right)\right)^T A^*\left(g^{*(1)}\left(\mu_0, e^T\right)\right)\right] \\ &= \mathrm{MSE}\left(\hat{\mu}_g{}^*\right)\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}\hat{\mu}_g &= g^*\left(\bar{Y}, \frac{\bar{X}_1}{\mu_1}, \frac{\bar{X}_2}{\mu_2}, \cdots, \frac{\bar{X}_p}{\mu_p}\right) \\ &= g^*\left(\bar{Y}, U^T\right)\end{aligned} \tag{2.10}$$

and $\overline{\mathrm{Y}}$ and $\overline{\mathrm{X}}_{\mathrm{i}}\left(i = 1,2,\cdots,p\right)$ are the sample means of the characteristics Y and X_i based on true measurements. (Y_j,X_{ij}, i=1,2,…,p; j=1,2,…,n). The family of estimators $\hat{\mu}_g{}^*$ at (2.10) is a generalized version of Srivastava (1971, 80).

The MSE of $\hat{\mu}_g{}^*$ is minimized for

$$g^{*(1)}\left(\mu_0, e^T\right) = -A^{*-1}\, b\mu_0 \tag{2.11}$$

Thus the resulting minimum MSE of $\hat{\mu}_g{}^*$ is given by

$$\min.\text{MSE}\left(\hat{\mu}_g *\right) = \frac{\mu_0{}^2}{n}\left[C_0^2 - b^T A*^{-1} b\right]$$
$$= \frac{\sigma_0{}^2}{n}\left(1 - R^2\right)$$

(2.12)

where A*=[a*$_{ij}$] be a p×p matrix with a*$_{ij}$ = $\rho_{ij}C_iC_j$ and R stands for the multiple correlation coefficient of Y on $X_1, X_2, \ldots, X_p$.

From (2.6) and (2.12) the increase in minimum MSE $\left(\hat{\mu}_g\right)$ due to measurement errors is obtained as

$$\min.\text{MSE}\left(\hat{\mu}_g\right) - \min.\text{MSE}\left(\hat{\mu}_g *\right) = \left(\frac{\mu_0^2}{n}\right)\left[C_{(0)}^2 + b^T A*^{-1} b - b^T A^{-1} b\right]$$
$$> 0$$

This is due to the fact that the measurement errors introduce the variances fallible measurements of study variate Y and auxiliary variates X_i. Hence there is a need to take the contribution of measurement errors into account.

3. BIASES AND MEAN SQUARE ERRORS OF SOME PARTICULAR ESTIMATORS IN THE PRESENCE OF MEASUREMENT ERRORS.

To obtain the bias of the estimator $\hat{\mu}_g$, we further assume that the third partial derivatives of $g\left(\bar{y}, u^T\right)$ also exist and are continuous and bounded. Then expanding $g\left(\bar{y}, u^T\right)$ about the point $\left(\bar{y}, u^T\right) = \left(\mu_0, e^T\right)$ in a third-order Taylor's series we obtain

$$\hat{\mu}_g = g\left(\mu_0, e^T\right) + \left(\bar{y} - \mu_0\right)\left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)} + (u - e)^T g^{(1)}{}_{\left(\mu_0, e^T\right)}$$
$$+ \frac{1}{2}\left\{\left(\bar{y} - \mu_0\right)^2 \left.\frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\right|_{\left(\mu_0, u^T\right)} + 2\left(\bar{y} - \mu_0\right)(u - e)^T g^{(1)}{}_{\left(\mu_0, e^T\right)}\right.$$
$$\left. + (u - e)^T \left(g^{(2)}{}_{\left(\mu_0, e^T\right)}\right)(u - e)\right\}$$

$$+\frac{1}{6}\left\{(\bar{y}-\mu_0)\frac{\partial}{\partial \bar{y}}+(u-e)\frac{\partial}{\partial u}\right\}^3 g\left(\bar{y}^*, u^{*T}\right)$$

(3.1)

where $g^{(12)}(\mu_0, e^T)$ denotes the matrix of second partial derivatives of $g\left(\bar{y}, u^T\right)$ at the point $\left(\bar{y}, u^T\right)=\left(\mu_0, e^T\right)$.

Noting that

$$g\left(u_0 e^T\right)=\mu_0$$

$$\left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)}=1$$

$$\left.\frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\right|_{\left(\mu_0, e^T\right)}=0$$

and taking expectation we obtain the bias of the family of estimators $\hat{\mu}_g$ to the first degree of approximation,

$$B\left(\hat{\mu}_g\right)=\frac{1}{2}\left[E\left\{(u-e)^T\left(g^{(2)}\left(\mu_0, e^T\right)\right)(u-e)\right\}+2\left(\frac{\mu_0}{n}\right)b^T g^{(12)}\left(\mu_0, e^T\right)\right]$$

(3.2)

where $b^T=(b_1, b_2, \ldots, b_p)$ with $bi=\rho_{oi}C_0C_i$; $(i=1,2,\ldots, p)$. Thus we see that the bias of $\hat{\mu}_g$ depends also upon the second order partial derivatives of the function on $g\left(\bar{y}, u^T\right)$ at the point (μ_0, e^T), and hence will be different for different optimum estimators of the family.

The biases and mean square errors of the estimators $\hat{\mu}_g^{(i)}; i=1 \text{ to } 18$ up to terms of order n^{-1} along with the values of $g^{(1)}(\mu_0, e^T)$, $g^{(2)}(\mu_0, e^T)$ and $g^{(12)}(\mu_0, e^T)$ are given in the Table 3.1.

Table 3.1 Biases and mean squared errors of various estimators of μ_0

ESTIMATOR	$g^{(1)}(\mu_0, e^T)$	$g^{(2)}(\mu_0, e^T)$	$g^{(12)}(\mu_0, e^T)$	BIAS	MSE
$\hat{\mu}_g^{(1)}$	$-\mu_0 \underset{\sim}{\omega}$	$2\mu_0 \underset{\sim}{W}_{p\times p}$ where $W_{pxp}=dig(\omega_1, \omega_2, \ldots, \omega_p)$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left(C^T \underset{\sim}{W}_{p\times p} - b^T \underset{\sim}{\omega}\right)$ where $C^T = \left(C_1^2 + C_{(1)}^2, C_2^2 + C_{(2)}^2, \cdots, C_p^2 + C_{(p)}^2\right)$	$\left(\frac{{\mu_0}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(2)}$	$\mu_0 \underset{\sim}{\omega}$	$\underset{\sim}{O}_{p\times p}$ (null matrix)	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right) b^T \underset{\sim}{\omega}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(3)}$	$-\frac{\mu_0 \underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}$ where $\underset{\sim}{\omega}*^T =$ $(\omega_1,*\omega_2,*\ldots,\omega_p*)$ with $(\omega_i,*=\omega_i \mu_i)$ $(i=1,2,\ldots,p)$	$\frac{2\mu_0 \underset{\sim}{\omega}* \underset{\sim}{\omega}*^T}{\underset{\sim}{\omega}^T \underset{\sim}{\mu} \underset{\sim}{\mu}^T \underset{\sim}{\omega}}$	$-\frac{\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}$	$\left(\frac{\mu_0}{n}\right)\left(\frac{\underset{\sim}{\omega}*^T A \underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu} \underset{\sim}{\mu}^T \underset{\sim}{\omega}} - \frac{b^T \underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}\right)$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 - \frac{2b^T \underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}} + \frac{\underset{\sim}{\omega}*^T A \underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T \underset{\sim}{\mu} \underset{\sim}{\mu}^T \underset{\sim}{\omega}}\right]$
$\hat{\mu}_g^{(4)}$	$\frac{\mu_0 \underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}$	$\underset{\sim}{O}_{p\times p}$ (null matrix)	$\frac{\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}$	$\left(\frac{\mu_0}{n}\right)\frac{b^T \underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + \frac{2b^T \underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu}} + \frac{\underset{\sim}{\omega}^T A \underset{\sim}{\omega}}{\underset{\sim}{\omega}^T \underset{\sim}{\mu} \underset{\sim}{\mu}^T \underset{\sim}{\omega}}\right]$
$\hat{\mu}_g^{(5)}$	$-\mu_0 \underset{\sim}{\omega}$	$\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T + \underset{\sim}{W}_{p\times p}\right)$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\omega}^T A \underset{\sim}{\omega} + C^T \underset{\sim}{W}_{p\times p} - 2b^T \underset{\sim}{\omega}\right]$	$\left(\frac{{\mu_0}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$

$\hat{\mu}_g^{(6)}$	$-\mu_0 \underset{\sim}{\omega}$	$2\mu_0 \underset{\sim}{\omega}^T \underset{\sim}{\omega}$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\omega}^T A \underset{\sim}{\omega} - b^T \underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(7)}$	$\mu_0 \underset{\sim}{\omega}$	$\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T - \underset{\sim}{W}_{p\times p}\right)$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\omega}^T A \underset{\sim}{\omega} - C^T \underset{\sim}{W}_{p\times p} + 2b^T \underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(8)}$	$\mu_0 \underset{\sim}{\omega}$	$2\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T - \underset{\sim}{W}_{p\times p}\right)$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\omega}^T A \underset{\sim}{\omega} - C^T \underset{\sim}{W}_{p\times p} + b^T \underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$

Table 3.1 Biases and mean squared errors of various estimators of μ_0

ESTIMATOR	$g^{(1)}(\mu_0,e^T)$	$g^{(2)}(\mu_0,e^T)$	$g^{(12)}(\mu_0,e^T)$	BIAS	MSE
$\hat{\mu}_g^{(9)}$	$-\mu_0\underset{\sim}{\omega}$	$2\mu_0\underset{\sim}{W}_{p\times p}$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left(C^T\underset{\sim}{W}_{p\times p}-b^T\underset{\sim}{\omega}\right)$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2+C_{(0)}^2+2b^T\underset{\sim}{\omega}+\underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(10)}$	$\mu_0\underset{\sim}{\omega}$	$\underset{\sim}{O}$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)b^T\underset{\sim}{\omega}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2+C_{(0)}^2+2b^T\underset{\sim}{\omega}+\underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(11)}$	$\underset{\sim}{\omega}_{(1)}\mu_0$ where $\underset{\sim}{\omega}_{(1)}=(-\omega_1,-\omega_2,\ldots,-\omega_q,-\omega_{q+1},\ldots,\omega_p)_{1\times p}$ $\underset{\sim}{W}_{(1)p\times p}=\begin{bmatrix}\omega_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \omega_2 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \omega_3 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \omega_q & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{bmatrix}_{p\times p}$,	$2\underset{\sim}{W}_{(1)p\times p}\mu_0$	$\underset{\sim}{\omega}_{(1)}$	$\left(\frac{\mu_0}{n}\right)\left(C^{*T}\underset{\sim}{W}_{(1)}-b^T\underset{\sim}{\omega}_{(1)}\right)$ $C^{*T}=(C_1^2+C_{(1)}^2,\ldots,$ $C_q^2+C_{(q)}^2;\ldots 0)$	$\left(\frac{\mu_0{}^2}{n}\right)\left[C_0^2+C_{(o)}^2-2b^T\underset{\sim}{\omega}_{(1)}+\underset{\sim}{\omega}^T{}_{(1)}A\underset{\sim}{\omega}_{(1)}\right]$

$\hat{\mu}_g^{(12)}$	$\underset{\sim}{\alpha}\,\mu_0$	$\mu_0\left(\underset{\sim}{\alpha}\,\underset{\sim}{\alpha}^T - \underset{\sim}{\propto}_{p\times p}\right)$ where $\underset{\sim}{\alpha}^T=(\alpha_1,\alpha_2,\ldots,\alpha_p)_{1\times p}$ $\underset{\sim}{\propto}=\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_p)$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\alpha}^T A\underset{\sim}{\alpha} - C^T\underset{\sim}{\propto}_{p\times p} + 2b^T\underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0{}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A\,\underset{\sim}{\alpha}\right]$
$\hat{\mu}_g^{(13)}$	$-\underset{\sim}{\alpha}\,\mu_0$	$-\mu_0\left(\underset{\sim}{\alpha}\,\underset{\sim}{\alpha}^T - \underset{\sim}{\propto}_{p\times p}\right)$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{2n}\right)\left[C^T\underset{\sim}{\propto}_{p\times p} - \underset{\sim}{\alpha}^T A\underset{\sim}{\alpha} - 2b^T\underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0{}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A\,\underset{\sim}{\alpha}\right]$
$\hat{\mu}_g^{(14)}$	$-\underset{\sim}{\alpha}\,\mu_0$	$2\mu_0\,\underset{\sim}{\alpha}\,\underset{\sim}{\alpha}$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\alpha}^T A\underset{\sim}{\alpha} - C^T - b^T\underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0{}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A\,\underset{\sim}{\alpha}\right]$

Table 3.1 Biases and mean squared errors of various estimators of μ_0

ESTIMATOR	$g^{(1)}(\mu_0,e^T)$	$g^{(2)}(\mu_0,e^T)$	$g^{(12)}(\mu_0,e^T)$	BIAS	MSE
$\hat{\mu}_g^{(15)}$	$\mu_0 \underset{\sim}{\theta}$, where $\underset{\sim}{\Theta}_{p\times p}$ =diag{$\theta_1,\theta_2,\ldots \theta_p$}	$\mu_0\left(\underset{\sim}{\theta}\underset{\sim}{\theta}^T - \underset{\sim}{\Theta}_{p\times p}\right)$,	v 0 $\underset{\sim}{\theta}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\theta}^T A \underset{\sim}{\theta} - C^T \underset{\sim}{\Theta}_{p\times p} + 2b^T \underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T \underset{\sim}{\theta} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}\right]$
$\hat{\mu}_g^{(16)}$	$\mu_0 \underset{\sim}{\theta}$	$\mu_0 \underset{\sim}{\theta}\underset{\sim}{\theta}^T$	$\underset{\sim}{\theta}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\theta}^T A \underset{\sim}{\theta} + 2b^T \underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T \underset{\sim}{\theta} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}\right]$
$\hat{\mu}_g^{(17)}$	$\mu_0 \underset{\sim}{\theta}$, where $\underset{\sim}{\Theta}^*_{p\times p}$ =diag{$\theta_1\left(\frac{\theta_1}{\omega_1}-1\right)\ldots,\theta_p\left(\frac{\theta_p}{\omega_p}-1\right)$}	$\underset{\sim}{\Theta}^*_{p\times p}\ \mu_0$,	$\underset{\sim}{\theta}$	$\left(\frac{\mu_0}{2n}\right)\left[C^T \underset{\sim}{\Theta}^*_{p\times p} + 2b^T \underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0{}^2}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T \underset{\sim}{\theta} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}\right]$
$\hat{\mu}_g^{(18)}$	$\underset{\sim}{\alpha}^*$, where $\underset{\sim}{\alpha}^{*T}$ =($\alpha_{1*},\alpha_{2*},\ldots,\alpha_{p*}$) with $\underset{\sim}{\alpha}^*_i$ =(α_i,μ_i, i=1,2,...,p)	$\underset{\sim}{O}_{p\times p}$	$\underset{\sim}{O}_{p\times p}$	Unbiased	$\left(\frac{1}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2\mu_0 b^T \underset{\sim}{\alpha}^* + \underset{\sim}{\alpha}^{*T} A \underset{\sim}{\alpha}^*\right]$

4. ESTIMATORS BASED ON ESTIMATED OPTIMUM

It may be noted that the minimum MSE (2.6) is obtained only when the optimum values of constants involved in the estimator, which are functions of the unknown population parameters μ_0, b and A, are known quite accurately.

To use such estimators in practice, one has to use some guessed values of the parameters μ_0, b and A, either through past experience or through a pilot sample survey. Das and Tripathi (1978, sec.3) have illustrated that even if the values of the parameters used in the estimator are not exactly equal to their optimum values as given by (2.5) but are close enough, the resulting estimator will be better than the conventional unbiased estimator $\bar{y}$. For further discussion on this issue, the reader is referred to Murthy (1967), Reddy (1973), Srivenkataramana and Tracy (1984) and Sahai and Sahai (1985).

On the other hand if the experimenter is unable to guess the values of population parameters due to lack of experience, it is advisable to replace the unknown population parameters by their consistent estimators. Let $\hat{\phi}$ be a consistent estimator of $\phi = A^{-1}b$. We then replace ϕ by $\hat{\phi}$ and also μ_0 by $\bar{y}$ if necessary, in the optimum $\hat{\mu}_g$ resulting in the estimator $\hat{\mu}_{g(est)}$, say, which will now be a function of $\bar{y}$, u and ϕ. Thus we define a family of estimators (based on estimated optimum values) of μ_0 as

$$\hat{\mu}_{g(est)} = g^{**}\left(\bar{y}, u^T, \hat{\phi}^T\right) \tag{4.1}$$

where g**(·) is a function of $\left(\bar{y}, u^T, \hat{\phi}^T\right)$ such that

$$g^{**}\left(\mu_0, e^T, \phi^T\right) = \mu_0 \text{ for all } \mu_0,$$
$$\Rightarrow \left.\frac{\partial g^{**}(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T, \phi^T\right)} = 1$$

$$\left.\frac{\partial g^{**}(\cdot)}{\partial u}\right|_{\left(\mu_0, e^T \phi^T\right)} = \left.\frac{\partial g(\cdot)}{\partial u}\right|_{(\mu_0, e)^T} = -\mu_0 A^{-1} b = -\mu_0 \phi \tag{4.2}$$

and

$$\left.\frac{\partial g^{**}(\cdot)}{\partial \hat{\phi}}\right|_{\left(\mu_0, e^T, \phi^T\right)} = 0$$

With these conditions and following Srivastava and Jhajj (1983), it can be shown to the first degree of approximation that

$$\begin{aligned} \mathrm{MSE}\left(\hat{\mu}_{g(est)}\right) &= \min.\mathrm{MSE}\left(\hat{\mu}_{\mathrm{g}}\right) \\ &= \left(\frac{{\mu_0}^2}{n}\right)\left[C_0^2 + C_{(0)}^2 - b^T A^{-1} b\right] \end{aligned}$$

Thus if the optimum values of constants involved in the estimator are replaced by their consistent estimators and conditions (4.2) hold true, the resulting estimator $\hat{\mu}_{g(est)}$ will have the same asymptotic mean square error, as that of optimum $\hat{\mu}_g$. Our work needs to be extended and future research will explore the computational aspects of the proposed algorithm.

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CONTENTS

The purpose of this book is to postulate some theories and test them numerically. Estimation is often a difficult task and it has wide application in social sciences and financial market. In order to obtain the optimum efficiency for some classes of estimators, we have devoted this book into three specialized sections.

	$Y \le M_Y$	$Y > M_Y$	Total
$X \le M_X$	$P_{11}(x,y)$	$P_{21}(x,y)$	$P_{\cdot 1}(x,y)$
$X > M_X$	$P_{12}(x,y)$	$P_{22}(x,y)$	$P_{\cdot 2}(x,y)$
Total	$P_{1\cdot}(x,y)$	$P_{2\cdot}(x,y)$	1

$ 9.95