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This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of The University of New Mexico in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

RANDOM EVOLUTIONS ON DIFFUSION PROCESSES

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RANDOM EVOLUTIONS ON DIFFUSION PROCESSES

BY

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B.S., Bethel College, 1966

DISSERTATION

Submitted in Partial Fulfillment of the
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RANDOM EVOLUTIONS ON DIFFUSION PROCESSES

BY

DONALD QUIRING

ABSTRACT OF DISSERTATION

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Abstract

Let $\{V(t, \omega), t \geq 0, \omega \in \Omega\}$ be a diffusion process on the real line with infinitesimal operator $\frac{1}{2} \sigma^2(\cdot) D^2 + m(\cdot) D$. Markov processes $\{V_n, n = 1, 2, \dots\}$ on the real line are constructed in such a way that the paths of V_n are step functions with jump size $n^{-1/2}$ and

$$P_0 \left[\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} |V_n(s) - V(s)| = 0 \right] = 1,$$

where P_0 assigns probability one to paths starting at the origin at $t = 0$.

Let $\{T_v(t), t \geq 0, v \in R\}$ be a family of linear contraction operators on a Banach space B . Suppose $T_v(t)f$ is continuous in v for all $t \geq 0$, $f \in B$, and $T_v(t)T_w(s) = T_w(s)T_v(t)$ for all $t, s \geq 0$, $v, w \in R$. Let A_v be the infinitesimal operator of T_v . Suppose the set of elements $f \in \bigcap_{v \in R} D(A_v)$ for which $A_v f$ is continuous in v is a dense subset of B .

Define an operator $M_n(t, \omega)$ on B by

$$M_n(t) = T_{V_n(0)}(\tau_1) T_{V_n(\tau_1)}(\tau_2 - \tau_1) \cdots T_{V_n(\tau_{N(t)})}(t - \tau_{N(t)}) f$$

where $\tau_i(\omega)$ is the time of the i^{th} jump of $V_n(\cdot, \omega)$ and $N(t, \omega)$ is the number of jumps before time t .

It is shown that for all $t \geq 0$, $f \in B$, and almost all ω , $M(t, \omega)f = \lim_{n \rightarrow \infty} M_n(t, \omega)f$ exists.

Let $\tilde{B}$ be the Banach space of bounded continuous functions from R to B . The family of operators (indexed by t) defined on $\tilde{B}$ by

$$(\tilde{T}(t)\tilde{f})(v) = E_v[M(t)\tilde{f}(V(t))]$$

is shown to be a semigroup of linear contractions and $\tilde{A}$, the infinitesimal operator of $\tilde{T}$, is shown to have the form

$$(\tilde{A}\tilde{f})(v) = A_v\tilde{f}(v) + \frac{1}{2}\sigma^2(v)D^2\tilde{f}(v) + m(v)D\tilde{f}(v) .$$

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Introduction.

The random process $X(t) = \int_0^t (-1)^{p(s)} v \, ds$, where p is a Poisson process with intensity a , describes the motion of a particle moving on a line with speed v and changing direction at the jump times of p . This process was studied by Kac [8] who showed that it is governed by the pair of partial differential equations

$$\frac{\partial}{\partial t} u_1(x, t) = v \frac{\partial}{\partial x} u_1(x, t) - au_1(x, t) + au_2(x, t)$$

$$\frac{\partial}{\partial t} u_2(x, t) = -v \frac{\partial}{\partial x} u_2(x, t) + au_1(x, t) - au_2(x, t)$$

which can be combined into the telegraph equation

$$\frac{1}{v} \frac{\partial^2}{\partial t^2} F(x, t) = v \frac{\partial^2}{\partial x^2} F(x, t) - \frac{2a}{v} \frac{\partial}{\partial t} F(x, t) .$$

The first order system is the backward equation of the Markov process $\xi(t) = (X(t), (-1)^{p(t)})$ whose state space is the union of two lines in the plane.

Pinsky [9] considered a Markov process on n lines, $\xi(t) = (X(t), V(t))$, where $V(t)$ is a finite Markov chain on $\{v_1, \dots, v_n\}$ with stationary transition probabilities and $X(t)$ follows the motion of a particle moving on a line with velocity $V(t)$ at time t . The backward equation for this process is

$$\frac{\partial}{\partial t} u(t, x, v_i) = v_i \frac{\partial}{\partial x} u(t, x, v_i) + \sum_{j=1}^n q_{ij} u(t, x, v_j)$$

where $Q = (q_{ij})$ is the infinitesimal matrix of V . Kac's process corresponds to $Q = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$, $v_1 = v$, $v_2 = -v$.

Hersh and Griego [3 and 5] consider the same type of process V defined on $\{1, \dots, n\}$ along with a family of strongly continuous semigroups $\{T_i(t), t \geq 0, i = 1, \dots, n\}$ of linear operators on a Banach space B . They construct a semigroup $\tilde{T}(t)$ on the n -fold product $B \times \dots \times B$ in such a way that $\tilde{A}$, the infinitesimal generator of $\tilde{T}$, has the form

$$(\tilde{A}\tilde{f})_i = A_i \tilde{f}_i + \sum_{j=1}^n q_{ij} \tilde{f}_j.$$

If B is a space of bounded measurable functions from the real line to itself and each T_i is a transition semigroup of Markov process on the real line, then $\tilde{T}$ is the transition semigroup of a Markov process on n lines. In particular, if T_i is the semigroup corresponding to the motion of a particle with constant velocity v_i , then $(A_i f)(x) = v_i \frac{\partial}{\partial x} f(x)$ and $\tilde{T}$ is the semigroup of the process ξ mentioned above.

Our purpose here is to perform a construction analogous to that of Hersh and Griego in which the Markov chain V is replaced by a diffusion process on the real line. In this case it is necessary to replace the family of semigroups $\{T_i(t), t \geq 0, i = 1, \dots, n\}$ with a family $\{T_v(t), t \geq 0, v \in R\}$, and it is convenient to define $\tilde{B}$ as the space of bounded continuous functions from R to B . We will construct a semigroup $\tilde{T}$ on $\tilde{B}$ whose infinitesimal generator has the form

$$\tilde{A}\tilde{f}(v) = A_v \tilde{f}(v) + \frac{1}{2} \sigma^2(v) \frac{d^2}{dv^2} \tilde{f}(v) + m(v) \frac{d}{dv} \tilde{f}(v) .$$

The functions $\sigma^2(\cdot)$ and $m(\cdot)$ are determined by the properties of the diffusion V .

If B is a space of bounded, measurable functions from R to R and each T_v is the transition semigroup of a Markov process on the line, then $\tilde{T}$ will be the transition semigroup of a Markov process on the plane. In particular if T_v corresponds to motion with velocity v , then $\tilde{T}$ is the semigroup of $\xi(t) = (X(t), V(t))$ where $X(t)$ follows the motion of a particle whose velocity at time t is $V(t)$. That is, $X(t) = X(0) + \int_0^t V(s)ds$. In this case, designating $\tilde{f}(v) \in B$ by $f(\cdot, v)$

$$(\tilde{A}f)(x, v) = v \frac{\partial}{\partial x} f(x, v) + \frac{1}{2} \sigma^2(v) \frac{\partial^2}{\partial v^2} f(x, v) + m(v) \frac{\partial}{\partial v} f(x, v) .$$

1. Preliminaries.

Let $X = \{X(t, \omega); t \geq 0, \omega \in \Omega\}$ be a stochastic process with stationary transition probabilities and with state space the real line. Denote by $\mathcal{F}(X(t); \tau \leq t)$ the smallest σ -field on Ω with respect to which the random variables $\{X(\tau, \cdot), 0 \leq \tau \leq t\}$ are measurable.

X is called a Markov process if

$$P[X(t+\tau) \in S | X(s), s \leq t] = P[X(t+\tau) \in S | X(t)] \quad \text{a.e.}$$

where $t, \tau \geq 0$ and S is a Lebesgue measurable set.

A random variable t^* is called a Markov time for the Markov process X if for every $t \geq 0$

$$\{t^* \leq t\} \in \mathcal{F}(X(t); \tau \leq t) .$$

X is called a strong Markov process if

$$P[X(t+t^*) \in S | X(s), s \leq t^*] = P[X(t+t^*) \in S | X(t^*)] \quad \text{a.e.}$$

A family of bounded linear operators $\{T(t), t \geq 0\}$ on a Banach space is called a semigroup if for all $s \geq 0, t \geq 0, T(s+t) = T(s)T(t)$. T is called a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$. The strong infinitesimal operator A of the semigroup T is defined by

$$Af = \lim_{t \rightarrow 0} \frac{1}{t} [T(t)f - f]$$

where f is any element of the Banach space for which this limit exists.

Let B be the Banach space of bounded measurable functions from the state space of X to the real line with norm defined by $\|f\| = \sup_{x \in R} |f(x)|$. Associated with X is the semigroup T on B defined by

$$(T(t)f)(x) = E_x[f(X(t))] .$$

In the case of a Markov process with countable state space and stationary transition probabilities we define the transition matrices $\{P(t), t \geq 0\}$ by $P(t) = (p_{ij}(t))$ where $p_{ij}(t) = P[X(t)=j | X(0)=i]$ and we define the infinitesimal matrix Q by $Q = (q_{ij})$ where q_{ij} is $\frac{d}{dt} p_{ij}(t)$ evaluated at 0. In this case $P(t)$ is the matrix of the operator $T(t)$ and Q is the matrix of A .

The equation

$$\frac{\partial}{\partial t} u(x,t) = (Au(\cdot, t))(x)$$

with initial condition $u(x,0) = f(x)$ is solved by

$u(x,t) = (T(t)f)(x)$ if f is in the domain of A . This is the backwards equation for X .

2. Random Evolution on Jump Processes.

Let $V = \{V(t, \omega), t \geq 0, \omega \in \Omega\}$ be a Markov jump process with stationary transition probabilities and state space R , the real line. For each $t \geq 0$ denote by $\mathcal{F}_t$ the smallest σ -field on the probability space Ω with respect to which the random variables $\{V(u, \cdot), 0 \leq u \leq t\}$ are measurable. Let V have the property that $|V(\tau_{i+1}) - V(\tau_i)| = d$ where $\tau_i(\omega)$ is the time of the i^{th} jump of $V(\cdot, \omega)$ and d is a fixed positive number.

Given the initial position $V(0) = v$, the process V becomes a Markov chain with state space $\{v + kd: k = 0, \pm 1, \pm 2, \dots\}$. It has an infinitesimal matrix $Q = (q_{u,w})$ for which $q_{u,w} = 0$ whenever $|u - w| > d$. We write $q_{v,v} = q(v)$, $q_{v,v+d} = q^+(v)$, and $q_{v,v-d} = q^-(v)$. These three functions depend on the initial value $v = V(0)$; thus they are defined everywhere in R . We impose on the process V the condition that q , q^+ , and q^- be continuous.

For each real number v let $\{T_v(t), t \geq 0\}$ be a strongly continuous semigroup of linear contraction operators on a fixed Banach space B . A_v is the strong infinitesimal operator of T_v . Let $T_v(t)f$ be a continuous function of v for each $t \geq 0$, $f \in B$.

Definition. A random evolution $\{M(t, \omega), t \geq 0, \omega \in \Omega\}$ is defined on V by the product

$$M(t) = T_{V(0)}(\tau_1) T_{V(\tau_1)}(\tau_2 - \tau_1) \cdots T_{V(\tau_{N(t)})}(t - \tau_{N(t)}),$$

where $N(t, \omega)$ is the number of jumps $V(\cdot, \omega)$ makes before time t .

Lemma. If $g: \Omega \rightarrow B$ is Bochner P_V -integrable for a fixed $v \in R$, then for each $t \geq 0$ the function $\omega \rightarrow M(t, \omega)g(\omega)$ is Bochner P_V -integrable and

$$E_V[M(t)g|\mathcal{F}_t](\omega) = M(t, \omega)E_V[g|\mathcal{F}_t](\omega) \quad (2.1)$$

for almost all ω with respect to P_V .

Proof. P_V is the probability measure corresponding to $V(0) = v$, in which case V has countable state space. The proof is analogous to that given in [5] for V with finite state space.

By theorem 3.7.4 of [6], $M(t, \cdot)g(\cdot)$ is integrable if and only if it is strongly measurable and $E_V[\|M(t)g\|] < \infty$. Since g is integrable, $E_V[\|g\|] < \infty$. Then $E_V[\|M(t)g\|] < \infty$ because for each ω $M(t, \omega)$ is a contraction operator.

In order to establish the strong measurability we must show that there exists a sequence $\{h_n\}$ of countably B -valued measurable functions on Ω so that $\lim_{n \rightarrow \infty} \|h_n - M(t)g\| = 0$ a.e. P_V . Such a sequence $\{g_n\}$ converging to g exists by the strong measurability of g . In addition, by approximating the random variables involved in the definition of $M(t)$ by discrete random variables we obtain a sequence of countably valued functions $M^n(t, \cdot)$ so that $h_n = M^n(t)g_n$ is countably B -valued and measurable. The strong continuity of the T_V 's then yields the required convergence of h_n to $M(t)g$.

To prove (2.1) we again use the approximation $M^n(t)$ which, since it is countably valued, can be written

$$M^n(t, \omega) = \sum_{k=1}^{\infty} L_k I_{\Lambda_k}(\omega)$$

where L_k is a linear contraction operator, $\Lambda_k \in \mathcal{F}_t$, and I_{Λ} is the indicator function of the set Λ .

For each $\Lambda \in \mathcal{F}_t$ we have

$$\begin{aligned} \int_{\Lambda} M^n(t) E_V[g | \mathcal{F}_t] dP_V &= \sum_k L_k \int_{\Lambda \cap \Lambda_k} E_V[g | \mathcal{F}_t] dP_V \\ &= \sum_k L_k \int_{\Lambda \cap \Lambda_k} g dP_V \\ &= \int_{\Lambda} \left(\sum_k L_k I_{\Lambda_k} \right) g dP_V \\ &= \int_{\Lambda} M^n(t) g dP_V . \end{aligned}$$

Since M^n converges strongly to M as $n \rightarrow \infty$ we have by the bounded convergence theorem

$$\int_{\Lambda} M(t) E_V[g | \mathcal{F}_t] dP_V = \int_{\Lambda} M(t) g dP_V .$$

The integrand on the left is $\mathcal{F}_t$ -measurable, so (1.1) follows by the definition of conditional expectation.

Let $\tilde{B}$ be the Banach space of bounded continuous functions $\tilde{f}: R \rightarrow B$ with norm defined by $\|\tilde{f}\|_{\tilde{B}} = \sup_{v \in R} \|\tilde{f}(v)\|_B$. Hereafter norms in both $\tilde{B}$ and B will be designed by $\|\cdot\|$ without subscript. The words "limit" and "continuity" are to be interpreted with regard to the appropriate norm topology.

For $t \geq 0$, the expectation semigroup $\tilde{T}$ corresponding to V is defined pointwise on $\tilde{B}$ by

$$\tilde{T}(t)\tilde{f}(v) = E_v[M(t)\tilde{f}(V(t))].$$

The expectation exists because of the above lemma. We write $\tilde{T}(t)\tilde{f}(v)$ instead of $(\tilde{T}(t)\tilde{f})(v)$ because only one interpretation of the expression without parentheses makes sense.

Theorem 1. $\tilde{T}$ is a semigroup of linear contraction operators on $\tilde{B}$.

Proof. It must be shown that $\tilde{T}(t+s)\tilde{f}(v) = \tilde{T}(t)\tilde{T}(s)\tilde{f}(v)$ and that $\tilde{T}(t)\tilde{f} \in \tilde{B}$ for all $t \geq 0, s \geq 0, \tilde{f} \in \tilde{B}$.

For the first part we follow [5] again. Let θ_t be defined on Ω by the requirement that $V(u, \theta_t \omega) = V(u+t, \omega)$. That is, θ_t shifts paths. Define $g \circ \theta_t$ by $(g \circ \theta_t)(\omega) = g(\theta_t \omega)$. Then the Markov property of V is expressed by the formula $E_v[g \circ \theta_t | \mathcal{F}_t](\omega) = E_{V(t, \omega)}[g]$ a.e. P_v . We can omit ω and write

simply $E_v[g \circ \Theta_t | \mathcal{F}_t] = E_{V(t)}[g]$. It is easy to check that M satisfies the multiplicative formula $M(t+s, \omega) = M(t, \omega)M(s, \Theta_t \omega)$ or

$$M(t+s) = M(t)(M(s) \circ \Theta_t) . \quad (2.2)$$

For fixed v we have

$$\begin{aligned} \tilde{T}(t+s)\tilde{f}(v) &= E_v[M(t+s)\tilde{f}(V(t+s))] \\ &= E_v[E_v[M(t+s)\tilde{f}(V(t+s)) | \mathcal{F}_t]] \\ &= E_v[E_v[M(t)(M(s) \circ \Theta_t)(\tilde{f}(V(s)) \circ \Theta_t) | \mathcal{F}_t]] \quad (\text{by (2.2)}) \\ &= E_v[M(t)E_v[M(s) \circ \Theta_t](\tilde{f}(V(s)) \circ \Theta_t) | \mathcal{F}_t] \quad (\text{by the lemma}) \\ &= E_v[M(t)E_{V(t)}[M(s)\tilde{f}(V(s))]] \quad (\text{by Markov property of } V) \\ &= E_v[M(t)\tilde{T}(s)\tilde{f}(V(t))] \\ &= \tilde{T}(t)\tilde{T}(s)\tilde{f}(v). \end{aligned}$$

Now we show that $\tilde{T}(t)\tilde{f}(v)$ is a bounded, continuous function of v .

Let $X^n(\omega) = (V(0), V(\tau_1), \dots, V(\tau_n))$. For x an $n+1$ -tuple of real numbers let $C_n(x)$ be the event $\{X^n(\omega) = x; \tau_n \leq t < \tau_{n+1}\}$. Given $V(0) = v$ there are 2^n values $x = (v, x_1, \dots, x_n)$ which X^n may take. Call this set of values $S_n(v)$.

$$\Omega = \bigcup_{n=0} \bigcup_{x \in S_n(v)} C_n(x) .$$

Thus Ω is decomposed into a countable union of P_v -measurable sets.

$$\begin{aligned}
\tilde{T}(t)\tilde{f}(v) &= E_v[M(t)\tilde{f}(V(t))] \\
&= \sum_{n=0}^{\infty} \sum_{x \in S_n(v)} E_v[M(t)\tilde{f}(V(t)); C_n(x)] \\
&= \sum_{n=0}^{\infty} \sum_{x \in S_n(v)} E_v[M(t)\tilde{f}(V(t)) | C_n(x)] P_v[C_n(x)] \\
&= \sum_{n=0}^{\infty} \sum_{x \in S_n(v)} E_v[T_v(\tau_1)T_{x_k}(\tau_2 - \tau_1) \cdots T_{x_n}(t - \tau_n)\tilde{f}(x_n)] P_v[C_n(x)] .
\end{aligned} \tag{2.3}$$

Let $x + h = (x_0 + h, x_1 + h, \dots, x_n + h)$.

$$\begin{aligned}
\tilde{T}(t)\tilde{f}(v+h) &= \\
&\sum_{n=0}^{\infty} \sum_{x \in S_n(v)} E_{v+h}[T_{v+h}(\tau_1)T_{x_1+h}(\tau_2 - \tau_1) \cdots T_{x_n+h}(t - \tau_n)\tilde{f}(x_n+h)] P_{v+h}[C_n(x+h)] .
\end{aligned} \tag{2.4}$$

Since $\tilde{f}(v)$, $q(v)$, $q^+(v)$, and $q^-(v)$ are continuous and $T_v(t)f$ is continuous in v for all $t \geq 0$, $f \in B$, we have by the bounded convergence theorem that any term in the sum in (2.4) converges as $h \rightarrow 0$ to the corresponding term in the sum in (2.3).

Since the T_v 's are contraction operators and

$$\sum_{n=0}^{\infty} \sum_{x \in S_n(v)} P_v[C_n(x)] = 1,$$

the sum in (2.3) converges uniformly in v

and $\|\tilde{T}(t)\tilde{f}\| \leq \|\tilde{f}\|$. We have shown that $\tilde{T}(t)\tilde{f}$ is a uniformly convergent sum of continuous functions, so it is continuous.

Theorem 2. If $f(v) \in \mathcal{D}(A_v)$, the domain of A_v , for all $v \in \mathbb{R}$ then

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} [\tilde{T}(t)\tilde{f}(v) - \tilde{f}(v)] &= \\ &= A_v \tilde{f}(v) + q(v)\tilde{f}(v) + q^+(v)\tilde{f}(v+d) + q^-(v)\tilde{f}(v-d). \end{aligned} \quad (2.5)$$

Proof. If τ is the first jump time of V then

$$\tilde{T}(t)\tilde{f}(v) = E_v[M(t)\tilde{f}(V(t)); \tau > t] + E_v[M(t)\tilde{f}(V(t)); \tau \leq t].$$

$$E_v[M(t)\tilde{f}(V(t)); \tau > t] = T_v(t)\tilde{f}(v)P_v[\tau > t].$$

$$\begin{aligned} E_v[M(t)\tilde{f}(V(t)); \tau \leq t] &= \int_0^t E_v[M(t)\tilde{f}(V(t)) | \tau=r] P_v[\tau \in dr] \\ &= \int_0^t E_v[T_v(r)E_{V(r)}[M(t-r)\tilde{f}(V(t-r)) | \tau=r] P_v[\tau \in dr]]. \end{aligned}$$

Given $\tau = r$ and $V(0) = v$, $V(r)$ is equal to either $v + d$ or $v - d$ and the last equation can be rewritten

$$\begin{aligned} E_v[M(t)\tilde{f}(V(t)); \tau \leq t] &= \int_0^t T_v(r)E_{v+d}[M(t-r)\tilde{f}(V(t-r))]P_v[V(\tau)=v+d] \cdot \\ &\cdot P_v[\tau \in dr] + \int_0^t T_v(r)E_{v-d}[M(t-r)\tilde{f}(V(t-r))]P_v[V(\tau)=v-d]P_v[\tau \in dr]. \end{aligned}$$

Let $p_{v,v}(r)$ denote the conditional probability that the process V starting at v is at v at time r . Then $P_v[\tau \in dr] = -q(v)p_{v,v}(r)dr$, $P_v[V(\tau)=v+d] = q^+(v)/-q(v)$, and $P_v[V(\tau)=v-d] = q^-(v)/-q(v)$. Therefore

$$E_v[M(t)\tilde{f}(V(t)); \tau \leq t] = \int_0^t T_v(r)E_{v+d}[M(t-r)\tilde{f}(V(t-r))]q^+(v)p_{v,v}(r)dr \\ + \int_0^t T_v(r)E_{v-d}[M(t-r)\tilde{f}(V(t-r))]q^-(v)p_{v,v}(r)dr.$$

Applying the definition of $\tilde{T}$ in both integrands we get

$$E_v[M(t)\tilde{f}(V(t)); \tau \leq t] = \int_0^t T_v(r)\tilde{T}(t-r)\tilde{f}(v+d)q^+(v)p_{v,v}(r)dr \\ + \int_0^t T_v(r)\tilde{T}(t-r)\tilde{f}(v-d)q^-(v)p_{v,v}(r)dr .$$

Thus,

$$\lim_{t \rightarrow 0} \frac{1}{t}[\tilde{T}(t)\tilde{f}(v) - \tilde{f}(v)] = \lim_{t \rightarrow 0} P_v[\tau > t] \frac{T_v(t)\tilde{f}(v) - \tilde{f}(v)}{t} \\ + \lim_{t \rightarrow 0} \frac{p_{v,v}(t) - 1}{t} \tilde{f}(v) \\ + \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t T_v(r)\tilde{T}(t-r)\tilde{f}(v+d)q^+(v)p_{v,v}(r)dr \\ + \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t T_v(r)\tilde{T}(t-r)\tilde{f}(v-d)q^-(v)p_{v,v}(r)dr .$$

This reduces to (2.5).

We say $\tilde{f} \in \mathcal{D}(\tilde{A})$ and define $\tilde{A}$ by

$$\tilde{A}\tilde{f}(v) = A_v\tilde{f}(v) + q(v)\tilde{f}(v) + q^+(v)\tilde{f}(v+d) + q^-(v)\tilde{f}(v-d)$$

if $\tilde{f}(v) \in \mathcal{D}(A_v)$ for all $v \in R$ and the right hand side of this equation is a bounded continuous function of v , and, furthermore,

$\| \frac{1}{t}[\tilde{T}(t)\tilde{f}-\tilde{f}] \| \leq M < \infty$ for sufficiently small t . In other words,
 $\mathcal{D}(\tilde{A})$ is the set of $\tilde{f} \in \tilde{B}$ for which $\frac{1}{t}[\tilde{T}(t)\tilde{f}-\tilde{f}]$ converges bounded
 pointwise to an element of $\tilde{B}$.

We let $\tilde{A}_0$ denote the strong infinitesimal operator of $\tilde{T}$
 defined by the limit in the B -norm $\lim_{t \rightarrow 0} \frac{1}{t}[\tilde{T}(t)\tilde{f}-\tilde{f}]$. $\tilde{A}$ is an
 extension of $\tilde{A}_0$.

3. Approximating Brownian Motion with Jump Processes.

Suppose $V = \{V(t, \omega), t \in [0, \infty), \omega \in \Omega\}$ is a Brownian motion normalized so that $E_V[V(t)] = v$ for all $t \in [0, \infty)$ and $E_V[(V(t) - v)^2] = t$ for all $t \in [0, \infty)$. Choose Ω in such a way that it is possible to construct a family $\{I_{k,n}; k = 1, 2, \dots; n = 1, 2, \dots\}$ of independent random variables independent of the process V and exponentially distributed with $E[I_{k,n}] = 1/n$.

Define

$$s_{k,n} = \sum_{j=1}^k I_{j,n}, \quad s_{0,n} = 0$$

and

$$e_{k,n} = \inf(t: |V(t) - V(e_{k-1,n})| = n^{-1/2}), \quad e_{0,n} = 0.$$

Define a jump process V_n by

$$V_n(t) = V(e_{k,n}) \quad \text{for } s_{k,n} \leq t < s_{k+1,n}.$$

Theorem 3. $P_O[\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |V_n(t) - V(t)| = 0] = 1$.

Proof.

$$\begin{aligned} |V_n(t) - V(t)| &\leq |V_n(t) - V_n(s_{k,n})| \\ &\quad + |V_n(s_{k,n}) - V(s_{k,n})| + |V(s_{k,n}) - V(t)|. \end{aligned}$$

If k is chosen so that $s_{k-1,n} \leq t < s_{k,n}$, this becomes

$$|v_n(t) - V(t)| \leq n^{-1/2} + |V(e_{k,n}) - V(s_{k,n})| + |V(s_{k,n}) - V(t)|. \quad (3.1)$$

Let $\delta_n = \sup_{k \leq n} (|e_{k,n} - s_{k,n}|, |s_{k,n} - t|)$. $V(t)$ is uniformly continuous on $[0,1]$ with probability 1 [7, p. 36], so to prove the theorem it suffices to show $P_0[\lim_{n \rightarrow \infty} \delta_n = 0] = 1$. This is true in spite of the fact that $s_{n,n}$ may be less than 1 requiring $k > n$ in (3.1) for $t > s_{n,n}$. It is enough that $\lim_{n \rightarrow \infty} s_{n,n} = 1$ with probability 1.

Now, $|s_{k,n} - t| \leq |s_{k,n} - s_{k-1,n}| = I_{k,n}$. If we can show

$$\sum_{n=1}^{\infty} P_0[\sup_{k \leq n} |e_{k,n} - s_{k,n}| > n^{-1/5}] < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} P[\sup_{k \leq n} I_{k,n} > n^{-1/5}] < \infty$$

then by the Borel-Cantelli lemma $P_0[\delta_n > n^{-1/5} \text{ infinitely often}] = 0$ and so $P_0[\lim_{n \rightarrow \infty} \delta_n = 0] = 1$ as desired.

We have $E[s_{k,n} | s_{k-1,n}] = s_{k-1,n} + n^{-1}$ because $E[I_{k,n}] = n^{-1}$.

For the Brownian motion V starting at an arbitrary point x , the expectation of the exit time from $[x-a, x+a]$ is a^2 . It follows by letting $x = V(e_{k-1,n})$ and $a = n^{-1/2}$ that $E[e_{k,n} | e_{k-1,n}] = e_{k-1,n} + n^{-1}$. The $s_{k,n}$'s were chosen independent of V so $E[e_{k,n} - s_{k,n} | e_{k-1,n} - s_{k-1,n}] = (e_{k-1,n} + n^{-1}) - (s_{k-1,n} + n^{-1})$ and $E[e_{k,n} - s_{k,n} | e_{k-1,n} - s_{k-1,n}] = e_{k-1,n} - s_{k-1,n}$. Therefore

$e_{k,n} - s_{k,n}$ is a martingale; hence, $(e_{k,n} - s_{k,n})^4$ is a submartingale.

By Doob's submartingale inequality

$$P_0[\max_{k \leq n} |e_{k,n} - s_{k,n}| > n^{-1/5}] \leq n^{4/5} E_0[(e_{n,n} - s_{n,n})^4] . \quad (3.2)$$

Now $(e_{n,n} - s_{n,n}) = \sum_{k=1}^n (e_{k,n} - e_{k-1,n} - I_{k,n})$ and since the summands are independent random variables with zero first moments, $E_0[(e_{n,n} - s_{n,n})^4]$ can be calculated by finding the second and fourth moments of these summands. This can be done because $I_{k,n}$ has an exponential distribution and $e_{k,n} - e_{k-1,n}$ has moment generating function [7, p. 30]

$$E_0[\exp(\lambda e_{1,n})] = \left(\cos \sqrt{\frac{2\lambda}{n}} \right)^{-1} .$$

The result is that $E_0[(e_{n,n} - s_{n,n})^4] = \text{constant} \times n^{-2}$ because

$$E_0[(e_{n,n} - s_{n,n})^4] = nE_0[I_{1,n}^4] + nE_0[e_{1,n}^4] + 6\binom{n}{2}E_0[I_{1,n}^2]E_0[e_{1,n}^2]$$

where each second moment is a constant multiple of n^{-2} and each fourth moment is a constant multiple of n^{-4} .

From (3.2)

$$P_0[\sup_{k \leq n} |e_{k,n} - s_{k,n}| > n^{-1/5}] \leq \text{constant} \times n^{-6/5} .$$

Thus,

$$\sum_{n=1}^{\infty} P_0[\sup_{k \leq n} |e_{k,n} - s_{k,n}| > n^{-1/5}] < \infty .$$

It remains to show

$$\sum_{n=1}^{\infty} P[\sup_{k \leq n} I_{k,n} > n^{-1/5}] < \infty .$$

Since $I_{k,n}$ is exponentially distributed with mean n^{-1}

$$P[I_{k,n} > n^{-1/5}] = e^{-n^{4/5}}$$

$$P[\sup_{k \leq n} I_{k,n} > n^{-1/5}] = P[\bigcup_{k \leq n} \{I_{k,n} > n^{-1/5}\}]$$

$$\leq \sum_{k \leq n} P[I_{k,n} > n^{-1/5}] = n e^{-n^{4/5}} .$$

The series $\sum_{n=1}^{\infty} n e^{-n^{4/5}}$ converges if the integral $\int_0^{\infty} x e^{-x^{4/5}} dx$

converges. Set $y = x^{4/5}$ and this integral becomes

$$\frac{5}{4} \int_0^{\infty} y^{3/2} e^{-y} dy < \infty .$$

4. Random Evolution on Brownian Motion.

V_n is a jump process of the type discussed in section 2. It corresponds to $q_n(v) = -n$, $q_n^+(v) = \frac{1}{2}n$, $q_n^-(v) = \frac{1}{2}n$, and $d_n = n^{-1/2}$. The functions are constant for fixed n because the process V_n is spatially homogeneous.

With respect to the process V_n we can define the random evolution M_n , the semigroup $\tilde{T}_n$, and the pointwise infinitesimal operator $\tilde{A}_n$ as in section 2. According to Theorem 2

$$\begin{aligned}\tilde{A}_n \tilde{f}(v) &= A_v \tilde{f}(v) + q_n(v) \tilde{f}(v) + q_n^+(v) \tilde{f}(v+d) + q_n^-(v) \tilde{f}(v-d) \\ &= A_v \tilde{f}(v) + \frac{1}{2}n [\tilde{f}(v+n^{-1/2}) + \tilde{f}(v-n^{-1/2}) - 2\tilde{f}(v)] .\end{aligned}$$

If $\tilde{f}$ has a continuous second derivative $D^2 \tilde{f}$, then $\lim_{n \rightarrow \infty} \tilde{A}_n \tilde{f}(v) = A_v \tilde{f}(v) + \frac{1}{2} D^2 \tilde{f}(v)$. If $\tilde{f} \in \mathcal{D}(\tilde{A}_n)$ and $D^2 \tilde{f} \in B$ we say $\tilde{f} \in \mathcal{D}(\tilde{A})$, and we define $\tilde{A}$ on $\mathcal{D}(\tilde{A})$ by

$$\tilde{A} \tilde{f}(v) = A_v \tilde{f}(v) + \frac{1}{2} D^2 \tilde{f}(v) . \quad (4.1)$$

Theorem 4. Suppose the operators $\{T_v(t), t \geq 0, v \in R\}$ all commute and the set of elements $f \in \bigcap_{v \in R} \mathcal{D}(A_v)$ for which $A_v f$ is a continuous function of v is a dense subset of B . Then for fixed $t \in [0, \infty)$ and $f \in B$, $M(t, \omega)f = \lim_{n \rightarrow \infty} M_n(t, \omega)f$ exists for almost all ω . In particular the limit exists for all ω where

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} |V_n(s, \omega) - V(s, \omega)| = 0.$$

Proof.

$$M_n(t)f = T_v(\tau_1)T_{V_n(\tau_1)}(\tau_2 - \tau_1) \cdots T_{V_n(\tau_N(t))}(t - \tau_N(t))f.$$

By the semigroup property we can introduce new points to the sequence $\{\tau_i\}$ increasing the number of factors without changing the product.

Therefore

$$M_n(t)f - M_m(t)f = \prod_{j=0}^{p-1} T_{V_n(s_j)}(s_{j+1} - s_j)f - \prod_{j=0}^{p-1} T_{V_m(s_j)}(s_{j+1} - s_j)f$$

where $s_0 = 0$, $s_p = t$, and $\{s_j\}$ is an increasing sequence including at least all the jump times before time t of the paths $V_n(\cdot, \omega)$ and $V_m(\cdot, \omega)$. This equation is equivalent to

$$M_n(t)f - M_m(t)f = \sum_{j=1}^{p-1} \prod_{i=0}^{j-1} T_{V_n(s_i)}(s_{i+1} - s_i) \cdot [T_{V_n(s_j)}(s_{j+1} - s_j) - T_{V_m(s_j)}(s_{j+1} - s_j)] \prod_{i=j+1}^{p-1} T_{V_m(s_i)}(s_{i+1} - s_i)f$$

because the sum telescopes.

Because of commutativity and the fact that these are contraction operators,

$$\|M_n(t)f - M_m(t)f\| \leq \sum_{j=1}^{p-1} \|T_{V_n(s_j)}(s_{j+1} - s_j)f - T_{V_m(s_j)}(s_{j+1} - s_j)f\|.$$

Suppose $f \in \bigcap_{v \in R} \mathcal{D}(A_v)$. $A_v f$ is continuous in v and uniformly continuous on a bounded interval containing the ranges of the paths

$\{V(\cdot, \omega), V_n(\cdot, \omega), n = 1, 2, \dots\}$ between 0 and t . Choose $\delta > 0$ so that $|v-w| < \delta$ implies $\|A_v f - A_w f\| < \epsilon$ and choose m, n large enough that $\sup_{0 \leq s \leq t} |V_n(s) - V_m(s)| < \delta$.

If $s_{j+1} - s_j$ is sufficiently small we have

$$\left\| \frac{T_{V_n(s_j)}^{(s_{j+1}-s_j)f} - T_{V_m(s_j)}^{(s_{j+1}-s_j)f}}{s_{j+1} - s_j} - [A_{V_n(s_j)}^f - A_{V_m(s_j)}^f] \right\| < \epsilon$$

by the definition of the strong infinitesimal operator. If $s_{j+1} - s_j$ is not sufficiently small more points may be added to the sequence $\{s_j\}$. Since $V_n(s)$ and $V_m(s)$ are constant on a time interval containing no jump points, we need only add finitely many points from such an interval. The enlarged sequence will therefore be finite.

Now

$$\|M_n(t)f - M_m(t)f\| \leq \sum_{j=1}^{p-1} (s_{j+1} - s_j) (\|A_{V_n(s_j)}^f - A_{V_m(s_j)}^f\| + \epsilon)$$

and $\|M_n(t)f - M_m(t)f\| < 2t\epsilon$ since $\|A_{V_n(s_j)}^f - A_{V_m(s_j)}^f\| < \epsilon$ for all j .

Therefore $M_n(t)f$ is a Cauchy sequence and we have proved $\lim_{n \rightarrow \infty} M_n(t)f$ exists for all f in a dense subset of B .

If f is not in this subset, choose g such that $\|f-g\| < \epsilon$ and $\|M_n(t)g - M_m(t)g\| < 2t\epsilon$. Then $\|M_n(t)f - M_m(t)f\| \leq \|M_n(t)f - M_n(t)g\| + \|M_n(t)g - M_m(t)g\| + \|M_m(t)g - M_m(t)f\| < 2\epsilon + 2t\epsilon$ because $M_n(t)$ and $M_m(t)$ are contraction operators. $M(t)f = \lim_{n \rightarrow \infty} M_n(t)f$ exists for all $f \in B$.

$M_n(t)f$ is continuous in t because $T_v(t)f$ is continuous in t for all $v \in \mathbb{R}$. From the above proof it can be seen that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} [M_n(s)f - M(s)f] = 0.$$

Therefore $M(\cdot)f$ is a uniform limit of continuous functions and is continuous. In particular, $\lim_{t \rightarrow 0} M(t)f = f$.

We regard M as the random evolution on the Brownian motion V . We will show that $\tilde{A}$ defined in (4.1) is the pointwise infinitesimal operator of a semigroup $\tilde{T}$ defined by $\tilde{T}(t)\tilde{f}(v) = E_v[M(t)\tilde{f}(V(t))]$.

Theorem 5. $\tilde{T}$ defined by $\tilde{T}(t)\tilde{f}(v) = E_v[M(t)\tilde{f}(V(t))]$ is a semigroup of linear contraction operators on $\tilde{B}$.

Proof. By the bounded convergence theorem,

$$\lim_{n \rightarrow \infty} E_v[M_n(t)\tilde{f}(V_n(t))] = E_v[\lim_{n \rightarrow \infty} M_n(t)\tilde{f}(V_n(t))].$$

That is, $\lim_{n \rightarrow \infty} \tilde{T}_n(t)\tilde{f}(v) = \tilde{T}(t)\tilde{f}(v)$. It follows that $\tilde{T}(s+t) = \tilde{T}(s)\tilde{T}(t)$ and $\|\tilde{T}(t)\tilde{f}\| \leq \|\tilde{f}\|$ because of the corresponding properties of $\tilde{T}_n$. It remains to prove $\tilde{T}(t)\tilde{f}(v)$ is continuous in v .

Let τ_v denote the hitting time of the point v for the process V starting at $v+h$. V has the property that $\lim_{h \rightarrow 0} P_{v+h}[\tau_v < t] = 1$. Now,

$$\begin{aligned} \tilde{T}(t)\tilde{f}(v+h) &= E_{v+h}[M(t)\tilde{f}(V(t))] \\ &= E_{v+h}[M(t)\tilde{f}(V(t)); \tau_v < t] + E_{v+h}[M(t)\tilde{f}(V(t)); \tau_v \geq t]. \end{aligned}$$

Thus,

$$\begin{aligned}\lim_{h \rightarrow 0} \tilde{T}(t) \tilde{f}(v+h) &= \lim_{h \rightarrow 0} E_{v+h} [M(t) \tilde{f}(V(t)); \tau_v < t] \\ &= \lim_{h \rightarrow 0} E_{v+h} [E[M(t) \tilde{f}(V(t)) | \mathcal{F}_{\tau_v}]; \tau_v < t] .\end{aligned}$$

By (2.1)

$$\begin{aligned}\lim_{h \rightarrow 0} \tilde{T}(t) \tilde{f}(v+h) &= \lim_{h \rightarrow 0} E_{v+h} [M(\tau_v) E_v [M(t-\tau_v) \tilde{f}(V(t-\tau_v))]]; \tau_v < t] \\ &= \lim_{h \rightarrow 0} E_{v+h} [M(\tau_v) \tilde{T}(t-\tau_v) \tilde{f}(v); \tau_v < t] .\end{aligned}$$

We know from the remark following the proof of theorem 4 that

$$\lim_{s \rightarrow 0} M(s) \tilde{T}(t-s) \tilde{f}(v) = \tilde{T}(t) \tilde{f}(v). \text{ For sufficiently small } h,$$

$P_{v+h}[\tau_v < \epsilon] > 1 - \epsilon$, and we can conclude that

$$\lim_{h \rightarrow 0} \tilde{T}(t) \tilde{f}(v+h) = \tilde{T}(t) \tilde{f}(v) .$$

5. Infinitesimal Operator of $\tilde{T}$.

Now define $\tilde{G}$ by

$$\tilde{G}\tilde{f}(v) = \lim_{t \rightarrow 0} \frac{1}{t} [\tilde{T}(t)\tilde{f}(v) - \tilde{f}(v)]$$

and let its domain $\mathcal{D}(\tilde{G})$ be the set of $\tilde{f} \in \tilde{B}$ for which the right hand side is bounded and continuous in v and for which

$$\| \frac{1}{t} [\tilde{T}(t)\tilde{f} - \tilde{f}] \| \leq M < \infty \quad \text{when } t \text{ is sufficiently small.}$$

Theorem 6. If $\tilde{f} \in \tilde{B}$, then $\tilde{R}_\lambda \tilde{f} = \int_0^\infty e^{-\lambda t} \tilde{T}(t) \tilde{f} dt$ is in $\mathcal{D}(\tilde{G})$ for $\lambda > 0$ and $(\lambda I - \tilde{G}) \tilde{R}_\lambda \tilde{f} = \tilde{f}$. If $\tilde{f} \in \mathcal{D}(\tilde{G})$, then $\tilde{R}_\lambda (\lambda I - \tilde{G}) \tilde{f} = \tilde{f}$.

Proof. Let $\tilde{g} = \tilde{R}_\lambda \tilde{f} = \int_0^\infty e^{-\lambda t} \tilde{T}(t) \tilde{f} dt$. Then

$$\begin{aligned} \tilde{T}(h)\tilde{g} &= \int_0^\infty e^{-\lambda t} \tilde{T}(t+h) \tilde{f} dt \\ &= \int_h^\infty e^{-\lambda(t-h)} \tilde{T}(t) \tilde{f} dt \\ &= e^{\lambda h} \int_h^\infty e^{-\lambda t} \tilde{T}(t) \tilde{f} dt \\ &= e^{\lambda h} [\tilde{g} - \int_0^h e^{-\lambda t} \tilde{T}(t) \tilde{f} dt] . \end{aligned}$$

Then,

$$\frac{1}{h} [\tilde{T}(h)\tilde{g} - \tilde{g}] = \frac{1}{h} (e^{\lambda h} - 1) \tilde{g} - \frac{1}{h} e^{\lambda h} \int_0^h e^{-\lambda t} \tilde{T}(t) \tilde{f} dt \quad (5.1)$$

which implies

$$\lim_{h \rightarrow 0} \frac{1}{h} [\tilde{T}(h)\tilde{g}(v) - \tilde{g}(v)] = \lambda \tilde{g}(v) - \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h e^{-\lambda t} \tilde{T}(t) \tilde{f}(v) dt .$$

Since $\lim_{t \rightarrow 0} \tilde{T}(t) \tilde{f}(v) = \tilde{f}(v)$ we have

$$\lim_{h \rightarrow 0} \frac{1}{h} [\tilde{T}(h)\tilde{g}(v) - \tilde{g}(v)] = \lambda \tilde{g}(v) - \tilde{f}(v)$$

or $(\lambda I - \tilde{G}) \tilde{R}_\lambda \tilde{f} = \tilde{f}$. We have $\tilde{g} \in \mathcal{D}(\tilde{G})$ because

$$\|\tilde{g}\| \leq \int_0^\infty e^{-\lambda t} \|\tilde{f}\| dt = \frac{1}{\lambda} \|\tilde{f}\| \quad \text{and, by (5.1),}$$

$$\left\| \frac{1}{h} [\tilde{T}(h)\tilde{g} - \tilde{g}] \right\| \leq \frac{1}{h} (e^{\lambda h} - 1) \|\tilde{g}\| + e^{\lambda h} \|\tilde{f}\|.$$

Now take $\tilde{f} \in \mathcal{D}(\tilde{G})$. By the bounded convergence theorem

$$\tilde{R}_\lambda (\lambda I - \tilde{G}) \tilde{f}(v) = \lim_{t \rightarrow 0} \tilde{R}_\lambda (\lambda I - \frac{1}{t} (\tilde{T}(t) - I)) \tilde{f}(v) .$$

$\tilde{R}_\lambda$ commutes with $(\lambda I - \frac{1}{t} (\tilde{T}(t) - I))$, so

$$\tilde{R}_\lambda (\lambda I - \tilde{G}) \tilde{f}(v) = \lim_{t \rightarrow 0} (\lambda I - \frac{1}{t} (\tilde{T}(t) - I)) \tilde{R}_\lambda \tilde{f}(v) .$$

Since by the first part of the theorem $\tilde{R}_\lambda \tilde{f} \in \mathcal{D}(\tilde{G})$, $\tilde{R}_\lambda (\lambda I - \tilde{G}) \tilde{f}(v) = (\lambda I - \tilde{G}) \tilde{R}_\lambda \tilde{f}(v) = \tilde{f}(v)$, and the theorem is proved.

We can write $(\lambda I - \tilde{G})^{-1} = \tilde{R}_\lambda$. Furthermore, $\lim_{n \rightarrow \infty} (\lambda I - \tilde{A}_n)^{-1} \tilde{f}(v) = (\lambda I - \tilde{G})^{-1} \tilde{f}(v)$, because $(\lambda I - \tilde{A}_n)^{-1} \tilde{f}(v) = \int_0^\infty e^{-\lambda t} \tilde{T}_n(t) \tilde{f}(v) dt$ and the bounded convergence theorem applies.

Theorem 7. If $\tilde{f} \in \mathcal{D}(A)$, then $\tilde{\tilde{A}}\tilde{f} = \tilde{\tilde{G}}\tilde{f}$.

Proof.

$$\begin{aligned}\tilde{f}(v) - (\lambda I - \tilde{G})^{-1}(\lambda I - \tilde{A})\tilde{f}(v) &= \lim_{n \rightarrow \infty} (\lambda I - \tilde{A}_n)^{-1}[(\lambda I - \tilde{A}_n)\tilde{f}(v) - (\lambda I - \tilde{A})\tilde{f}(v)] \\ &= \lim_{n \rightarrow \infty} (\lambda I - \tilde{A}_n)^{-1}[\tilde{\tilde{A}}\tilde{f}(v) - \tilde{A}_n\tilde{f}(v)] .\end{aligned}\quad (5.2)$$

If $D^2\tilde{f}(v)$ is uniformly continuous, then $\lim_{n \rightarrow \infty} \tilde{A}_n\tilde{f} = \tilde{\tilde{A}}\tilde{f}$ and $\lim_{n \rightarrow \infty} (\lambda I - \tilde{A}_n)^{-1}[\tilde{A}_n\tilde{f} - \tilde{\tilde{A}}\tilde{f}] = 0$, because $\|(\lambda I - \tilde{A}_n)^{-1}\| \leq \frac{1}{\lambda}$. In this case, by (5.2)

$$\tilde{f}(v) - (\lambda I - \tilde{G})^{-1}(\lambda I - \tilde{A})\tilde{f}(v) = 0$$

for all $v \in R$. Hence $(\lambda I - \tilde{G})\tilde{f} = (\lambda I - \tilde{A})\tilde{f}$, and $\tilde{\tilde{G}}\tilde{f} = \tilde{\tilde{A}}\tilde{f}$.

If $D^2\tilde{f}(v)$ is merely continuous we select a function $\tilde{f}_0$ with $D^2\tilde{f}_0(v)$ uniformly continuous for which $\tilde{f}$ is equal to $\tilde{f}_0$ on a neighborhood of v . Then

$$\tilde{\tilde{A}}\tilde{f}(v) = \tilde{\tilde{A}}\tilde{f}_0(v) = A_v\tilde{f}(v) + \frac{1}{2} D^2\tilde{f}(v) .$$

Likewise $\tilde{\tilde{G}}\tilde{f}(v)$ depends only on the behavior of $\tilde{f}$ near v so $\tilde{\tilde{G}}\tilde{f}(v) = \tilde{\tilde{A}}\tilde{f}(v)$ because $\tilde{\tilde{G}}\tilde{f}_0 = \tilde{\tilde{A}}\tilde{f}_0$.

6. Extension to Diffusion Processes.

We wish to prove the same theorems with the Brownian motion replaced by a diffusion process V having the property that for any two real numbers v and w , $P_v[V(t) = w \text{ for some } t < \infty] > 0$. Call this property regularity.

Say that functions $\phi_t(v)$ converge bounded pointwise on a subset J of the real line to $\phi(v)$ as $t \rightarrow 0$ if $\lim_{t \rightarrow 0} \phi_t(v) = \phi(v)$ for all $v \in J$ and $\sup_{v \in J} \phi_t(v) \leq M < \infty$ for sufficiently small t .

Let V be a regular Feller process for which there exist real-valued functions $\sigma^2(v)$ and $m(v)$ defined and continuous on R with

- i) $\frac{1}{t} P_v(|V(t)-v| \geq \epsilon) \xrightarrow{bp} 0$
- ii) $\frac{1}{t} \int_{|V(t)-v| < \epsilon} (V(t)-v) dP_v \xrightarrow{bp} m(v)$
- iii) $\frac{1}{t} \int_{|V(t)-v| < \epsilon} (V(t)-v)^2 dP_v \xrightarrow{bp} \sigma^2(v)$

as $t \rightarrow 0$. Convergence is bounded pointwise on all finite intervals of the real line.

The properties of such processes are described in Breiman [1].

We list some of them here without proofs.

The infinitesimal operator associated with V takes the form

$$Af(v) = \frac{1}{2} \sigma^2(v) D^2 f(v) + m(v) Df(v) .$$

If u is a solution of

$$\frac{1}{2} \sigma^2(v) D^2 u(v) + m(v) Du(v) = 0 ,$$

then $\bar{V}$ defined by $\bar{V}(t) = u(V(t))$ is a diffusion with $\bar{\sigma}^2(u(v)) = \sigma^2(v)[u'(v)]^2$ and $\bar{m}(v) \equiv 0$. The function u is one-to-one and unique up to a linear transformation.

Let J be an open interval $J = (a,b)$, and let τ_J be the first time that $\bar{V}$ leaves the interval J . Define a function G_J on $R \times R$ by

$$G_J(x,y) = \begin{cases} \frac{2(x-a)(b-y)}{b-a} & \text{if } x \leq y \quad x,y \in J \\ \frac{2(y-a)(b-x)}{b-a} & \text{if } x \geq y \quad x,y \in J \\ 0 & \text{otherwise .} \end{cases}$$

Then

$$E_V[\tau_J] = \int G_J(v,y) \frac{1}{\bar{\sigma}^2(y)} dy . \quad (6.1)$$

Define $e_{k,n} = \inf(t: |V(t) - V(e_{k-1,n})| = n^{-1/2})$, $e_{0,n} = 0$. Let $\{I_{k,n}, k = 1, 2, \dots, n = 1, 2, \dots\}$ be independent exponentially distributed random variables with $E[I_{k,n}] = E[e_{k,n} | V(e_{k-1,n}) = v] = E_V[e_{1,n}]$. Thus $I_{k,n}$ does depend on $V(e_{k-1,n})$. Define $s_{k,n} = \sum_{j=1}^k I_{j,n}$, $s_{0,n} = 0$.

Define a jump process V_n by

$$V_n(t) = V(e_{k,n}) \quad \text{for } s_{k,n} \leq t < s_{k+1,n} .$$

Since $V_n(s_{k,n}) = V(e_{k,n})$ and V is a strong Markov process

$$P[V_n(s_{k+1,n}) = v+n^{-1/2} | V_n(s_{k,n})=v] = P_v[V(e_{1,n})=v+n^{-1/2}]$$

and

$$P[V_n(s_{k+1,n}) = v-n^{-1/2} | V_n(s_{k,n})=v] = P_v[V(e_{1,n})=v-n^{-1/2}] .$$

Call these probabilities respectively $p_n^+(v)$ and $p_n^-(v)$ and define $q_n(v)$, $q_n^+(v)$, and $q_n^-(v)$ by $q_n(v) = \frac{1}{E_v[e_{1,n}]}$, $q_n^+(v) = -q_n(v)p_n^+(v)$, and $q_n^-(v) = -q_n(v)p_n^-(v)$.

V_n is a jump process of the type dealt with in section 2, and we can define a random evolution M_n on V_n and construct the semi-group $\tilde{T}_n$ on $\tilde{B}$ with pointwise infinitesimal operator $\tilde{A}_n$ given by

$$\tilde{A}_n \tilde{f}(v) = A_v \tilde{f}(v) + q_n(v) \tilde{f}(v) + q_n^+(v) \tilde{f}(v+n^{-1/2}) + q_n^-(v) \tilde{f}(v-n^{-1/2}) .$$

The domain of $\tilde{A}_n$ is the set of $\tilde{f}$ for which the right hand side of this equation is bounded and continuous in v . If q_n is not bounded, it is not sufficient that $A_v \tilde{f}(v)$ be bounded and continuous in v . If $\tilde{f} \in \mathcal{D}(\tilde{A}_n)$ for all n , $D^2 \tilde{f}$ and $D \tilde{f}$ exist as elements of $\tilde{B}$, and $\frac{1}{2} \sigma^2(v) D^2 \tilde{f}(v) + m(v) D \tilde{f}(v)$ is bounded, then we say $\tilde{f} \in \mathcal{D}(\tilde{A})$ and define $\tilde{A}$ by

$$\tilde{A} \tilde{f}(v) = A_v \tilde{f}(v) + \frac{1}{2} \sigma^2(v) D^2 \tilde{f}(v) + m(v) D \tilde{f}(v) .$$

We will show that $\lim_{n \rightarrow \infty} \tilde{A}_n \tilde{f}(v) = \tilde{A} \tilde{f}(v)$ for all $\tilde{f} \in \mathcal{D}(\tilde{A})$.

The process V starting at v leaves the interval $(v-n^{-1/2}, v+n^{-1/2})$ at the same time as the process $\bar{V}$ starting at $u(v)$ leaves the interval $J_n = (u(v-n^{-1/2}), u(v+n^{-1/2}))$. We can take u to be increasing because it is continuous, one-to-one, and unique only up to a linear transformation. Thus by (6.1)

$$E_v[e_{1,n}] = \int G_{J_n}(u(v), y) \frac{1}{\bar{\sigma}^2(y)} dy$$

and

$$\frac{\int G_{J_n}(u(v), y) dy}{\sup_{y \in J_n} \bar{\sigma}^2(y)} \leq E_v[e_{1,n}] \leq \frac{\int G_{J_n}(u(v), y) dy}{\inf_{y \in J_n} \bar{\sigma}^2(y)} \quad (6.2)$$

Now,

$$\begin{aligned} \int G_{J_n}(u(v), y) dy &= \int_{u(v-n^{-1/2})}^{u(v)} \frac{2[y-u(v-n^{-1/2})][u(v+n^{-1/2})-u(v)]}{u(v+n^{-1/2})-u(v-n^{-1/2})} dy \\ &\quad + \int_{u(v)}^{u(v+n^{-1/2})} \frac{2[u(v)-u(v-n^{-1/2})][u(v+n^{-1/2})-y]}{u(v+n^{-1/2})-u(v-n^{-1/2})} dy. \end{aligned}$$

These integrands are linear in y and it is easily calculated that

$$\int G_{J_n}(u(v), y) dy = [u(v+n^{-1/2})-u(v)][u(v)-u(v-n^{-1/2})].$$

So $\lim_{n \rightarrow \infty} n \int_{J_n} G_J(u(v), y) dy = [u'(v)]^2$. Since

$$\bar{\sigma}^2(y) = \sigma^2(x)[u'(x)]^2 \quad \text{where } x = u^{-1}(y), \quad \lim_{n \rightarrow \infty} \inf_{y \in J_n} \bar{\sigma}^2(y) =$$

$\lim_{n \rightarrow \infty} \sup_{y \in J_n} \bar{\sigma}^2(y) = \sigma^2(v)[u'(v)]^2$. Applying these facts in (6.2) and

substituting $\frac{-1}{q_n(v)}$ for $E_v[e_{1,n}]$ we get $\lim_{n \rightarrow \infty} -q_n(v)/n = \sigma^2(v)$.

The process V starting at v leaves the interval $(v-n^{-1/2}, v+n^{-1/2})$ on the left or right as the process $\bar{V}$ starting at $u(v)$ leaves the interval $(u(v-n^{-1/2}), u(v+n^{-1/2}))$ on the left or right. Since $\bar{V}$ is driftless the probability that it exits on the right is

$$\begin{aligned} p_n^+(v) &= \frac{u(v) - u(v-n^{-1/2})}{u(v+n^{-1/2}) - u(v-n^{-1/2})} \\ &= \frac{1}{2} + \frac{u(v) - \frac{1}{2}u(v+n^{-1/2}) - \frac{1}{2}u(v-n^{-1/2})}{u(v+n^{-1/2}) - u(v-n^{-1/2})} \\ &= \frac{1}{2} - \frac{1}{4} n^{-1/2} \left[\frac{u(v+n^{-1/2}) + u(v-n^{-1/2}) - 2u(v)}{n^{-1}} \right] \left[\frac{2n^{-1/2}}{u(v+n^{-1/2}) - u(v-n^{-1/2})} \right] \\ &= \frac{1}{2} - \frac{1}{4} n^{-1/2} \frac{u''(v)}{u'(v)} (1 + \delta_n) \end{aligned}$$

where $\lim_{n \rightarrow \infty} \delta_n = 0$.

Since $\frac{1}{2} \sigma^2(v) u''(v) + m(v) u'(v) = 0$, we have $u''(v)/u'(v) = -2m(v)/\sigma^2(v)$ and

$$p_n^+(v) = \frac{1}{2} + \frac{1}{2} n^{-1/2} \frac{m(v)}{\sigma^2(v)} (1 + \delta_n)$$

$$p_n^-(v) = 1 - p_n^+(v) = \frac{1}{2} - \frac{1}{2} n^{-1/2} \frac{m(v)}{\sigma^2(v)} (1 + \delta_n) .$$

We now calculate $\tilde{A}$.

$$\begin{aligned} \tilde{A}_n \tilde{f}(v) &= A_v \tilde{f}(v) + q_n(v) \tilde{f}(v) + q_n^+(v) \tilde{f}(v+n^{-1/2}) + q_n^-(v) \tilde{f}(v-n^{-1/2}) \\ &= A_v \tilde{f}(v) + q_n(v) \tilde{f}(v) - q_n(v) p_n^+(v) \tilde{f}(v+n^{-1/2}) - q_n(v) p_n^-(v) \tilde{f}(v-n^{-1/2}) \\ &= A_v \tilde{f}(v) + \frac{1}{2} q_n(v) [2\tilde{f}(v) - \tilde{f}(v+n^{-1/2}) - \tilde{f}(v-n^{-1/2})] \\ &\quad - \frac{1}{2} n^{-1/2} q_n(v) \frac{m(v)}{\sigma^2(v)} [\tilde{f}(v+n^{-1/2}) - \tilde{f}(v-n^{-1/2})] (1 + \delta_n) \\ &= A_v f(v) + \frac{1}{2} \frac{-q_n(v)}{n} \left[\frac{\tilde{f}(v+n^{-1/2}) + \tilde{f}(v-n^{-1/2}) - 2\tilde{f}(v)}{n^{-1}} \right] \\ &\quad + \frac{-q_n(v)}{n} \frac{m(v)}{\sigma^2(v)} \left[\frac{\tilde{f}(v+n^{-1/2}) - \tilde{f}(v-n^{-1/2})}{2n^{-1/2}} \right] (1 + \delta_n) . \end{aligned}$$

Now if $\tilde{f} \in \mathcal{D}(\tilde{A})$

$$\lim_{n \rightarrow \infty} \tilde{A}_n \tilde{f}(v) = A_v \tilde{f}(v) + \frac{1}{2} \sigma^2(v) D^2 \tilde{f}(v) + m(v) D \tilde{f}(v) .$$

We show

$$P_0 \left[\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |v_n(t) - v(t)| = 0 \right] = 1 .$$

In theorem 3 we proved this for the case where V is a normalized Brownian motion. In order to use the same method of proof we need the following:

- i) Almost all paths of V are uniformly continuous on $[0,1]$.
- ii) $e_{k,n} - s_{k,n}$ is a martingale.
- iii) $\sup_{v \in R} E_v[e_{1,n}^2] \leq \text{constant} \times n^{-2}$ and
 $\sup_{v \in R} E_v[e_{1,n}^4] \leq \text{constant} \times n^{-4}$.
- iv) $\sum_{n=1}^{\infty} P[\sup_{k \leq n} I_{k,n} > n^{-1/5}] < \infty$.

Proofs:

i) Almost all sample paths of a Feller process are continuous and $[0,1]$ is compact.

ii) $E[e_{k,n} | e_{k-1,n}] = e_{k-1,n} + E_{V(e_{k-1,n})}[e_{1,n}]$, and
 $E[s_{k,n} | s_{k-1,n}] = E[s_{k-1,n} + I_{k,n} | s_{k-1,n}] = s_{k-1,n} + E[I_{k,n} | s_{k-1,n}] =$
 $s_{k-1,n} + E_{V(e_{k-1,n})}[e_{1,n}]$. Therefore $E[e_{k,n} - s_{k,n} | e_{k-1,n} - s_{k-1,n}] =$
 $e_{k-1,n} - s_{k-1,n}$.

Now we prove iii) and iv) for almost all ω in the set C_m defined by

$$C_m = \left\{ \sup_{0 \leq t \leq 1} (|V(t)|, |V_n(t)|, n = 1, 2, \dots) < m \right\}.$$

iii) Since $\sigma^2(\cdot)$ is positive and continuous $\inf_{|v|>m} \sigma^2(v) = \sigma^2 > 0$.

If V is driftless (i.e., $m(\cdot) \equiv 0$), then $\sigma^2 e_{1,n}$ has smaller moments than those of the exit time of normalized Brownian motion from an interval of radius $n^{-1/2}$.

If V is not driftless, V starting at v leaves $(v-n^{-1/2}, v+n^{-1/2})$ when $\bar{V}$ starting at $u(v)$ leaves $(u(v-n^{-1/2}), u(v+n^{-1/2}))$. $u(v)$ is not in the center of this interval, but the further endpoint is at a distance less than $n^{-1/2} \cdot \sup_{|v|<m} u'(v)$. Therefore $\bar{\sigma}^2 e_{1,n}$ has smaller moments than those of the exit time of normalized Brownian motion from an interval of radius $n^{-1/2} \cdot \sup_{|v|<m} u'(v)$.

iv) The random variable $I_{k,n}$ is still exponentially distributed but its mean depends on the starting point $V_n(s_{k-1,n}) = V(e_{k-1,n})$. On C_m we have $\sup_{|v|<m} E[I_{k,n} | V(e_{k-1,n}) = v] = \sup_{|v|<m} E_v[e_{1,n}] \leq M/n < \infty$, where $M = (\sup_{|v|<m} u'(v))^2 / \bar{\sigma}^2$.

We can conclude

$$\sum_{n=1}^{\infty} P[\sup_{k \leq n} I_{k,n} > Mn^{-1/5}] < \infty$$

which is good enough for our purpose.

We now have

$$P_0[C_m \cap \{\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |V_n(t) - V(t)| = 0\}] = P_0[C_m]$$

and since $P_0[\bigcup_{m=1}^{\infty} C_m] = 1$ we are done.

In proving theorem 4, the only property of the Brownian motion we used was theorem 3. We can now conclude that theorem 4 is true with the Brownian motion replaced by a diffusion process.

In theorem 5 we used the strong Markov property and the property that $\lim_{h \rightarrow 0} P_{v+h}[\tau_v < t] = 1$. All Feller processes have the strong Markov property. V has the property $\lim_{h \rightarrow 0} P_{v+h}[\tau_v < t] = 1$ if $\bar{V}$ does; so we may assume $m(\cdot) \equiv 0$. We may also take $h > 0$. Suppose $\lim_{h \rightarrow 0} P_{v+h}[\tau_v < t] = \alpha < 1$. Then $\lim_{h \rightarrow 0} E_{v+h}[\tau_v] \geq (1-\alpha)t$. Choose w small enough that $\tau_{(v,w)}$, the exit time from the interval (v,w) , satisfies for $x \in (v,w)$

$$E_x[\tau_{(v,w)}] = \int_v^w G_{(v,w)}(x,y) \frac{1}{\sigma^2(y)} dy < (1-\alpha)t.$$

This can be done because $\inf_{v \leq y \leq w} \sigma^2(y) > 0$ and $G_{(v,w)}$ is bounded.

Since V is driftless $P_x[V(\tau_{(v,w)}) = v] = P_x[\tau_{(v,w)} = \tau_v] = \frac{w-x}{w-v}$, and

$$\lim_{h \rightarrow 0} E_{v+h}[\tau_v] = \lim_{x \rightarrow v} E_x[\tau_{(v,w)}] < (1-\alpha)t$$

contradicting an earlier statement derived from the assumption

$$\lim_{h \rightarrow 0} P_{v+h}[\tau_v < t] = \alpha < 1.$$

Theorem 5 can now be applied to the general diffusion processes, and we have a semigroup $\tilde{T}$ defined on $\tilde{B}$. By theorems 6 and 7 the operator $\tilde{G}$ can be defined and $\tilde{G}$ is an extension of $\tilde{A}$.

BIBLIOGRAPHY

1. Breiman, Leo, Probability, Addison-Wesley Publishing Co., Reading, Mass., 1968.
2. Dynkin, E.B., Markov Processes, Springer-Verlag, Berlin, 1965.
3. Griego, R.J. and R. Hersh, "Markov Chains and Systems of Partial Differential Equations," Proc. Nat. Acad. Sci., February, 1969.
4. Heath, David C., "Probabilistic Analysis of Certain Hyperbolic Systems of Partial Differential Equations," Dissertation, Univ. of Illinois, 1969.
5. Hersh, R. and R. Griego, "Random Evolutions--Theory and Applications," Technical Report 180, Department of Mathematics and Statistics, University of New Mexico, 1969.
6. Hille, E. and R. Phillips, Functional Analysis and Semigroups, Am. Math. Soc., Providence, R.I., 1957.
7. Ito, K. and H.P. McKean, Diffusion Processes and their Sample Paths, Academic Press, New York, 1965.
8. Kac, M., Some Stochastic Problems in Physics and Mathematics, Magnolia Petroleum Co., Lectures in Pure and Applied Science, No. 2, 1956.
9. Pinsky, M., "Differential Equations with a Small Parameter and the Central Limit Theorem for Functions Defined on a Finite Markov Chain," Z. Wahrs. verw. Geb., 1968.

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