6-28-2006

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Recommended Citation

Abdallah, Chaouki T. and Ivan Lopez. "Recent Advances on Linear Control Theory under Communication Constraints: A Survey." *14th Mediterranean Conference on Control and Automation* (2006): 1-6. doi:10.1109/MED.2006.328758.

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Recent Advances on Linear Control Theory under Communication Constraints: A Survey

Ivan Lopez and Chaouki T. Abdallah

Abstract—Recent developments in Networked Control Systems have focused on the issue of communications constraints. In this paper, we summarize the main results for stabilization and performance of networked control systems from an information theoretic point of view.

I. INTRODUCTION

Feedback control systems wherein the control loops are closed through a real-time network are called networked control systems (NCS) [7]. An architecture of a typical NCS is shown in Figure 1. The primary advantage of a NCS is that reduced system components and connections are achieved, resulting in easier maintenance and diagnosis of the system. However, when controlling across networks, the assumptions of classical theory of control must be revisited. In recent years, much research and development have been expanded in this area and, because of the attractive benefits of remote industrial control, several reliable protocols have been developed for robust for real-time control purposes. With the decrease in cost and with the steady investment in infrastructure, the Internet is in fact becoming a suitable network for control applications. However, without dedicated protocols, a

Fig. 1. Closed-loop network control system

new theory is needed for control design. In addition to introducing both delay and quantization, the finite data rate channel raises the issue of how to best determine the usefulness of the sensed and control bits [9]. Back in 1999, Wong and Brockett [27] considered a digital channel with a finite capacity and found that unstable systems can never be asymptotically stabilized and thus the concept of containability was introduced. From there on, several researchers have looked into these problems. Mitter et al [10] have recently contributed to the development of a new theory that matches classical control theory with traditional information theory. This survey does also include the so-called packet-based theory whose underlying concept is that the control loop can no longer be thought of as several signal processors that interchange signals in a feedback configuration. A more accurate description

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Fig. 2. Closed-loop with communication channel

is that several signals are transformed into packets of information, and the design of a network control system should then follow a packet-network approach [11].

II. LIMITATIONS FOR STABILIZABILITY

A. Feedback Scheme including a Noiseless Digital Communication Channel.

The initial problem of state estimation and stabilization of an LTI system was introduced by Wong and Brockett [27], but it was Tatikonda and Mitter who generalized some of these ideas [21], [22]. In [21], limits were established for the channel data rate to achieve observability and stabilizability in a NCS. That work considered both a noiseless communication channel as well as a noisy one. The system with a noiseless digital communication channel is shown in figure 2. The communication channel can transmit at each time, 2^R symbols, i.e., R bits of information per second without error (data rate). Consider then the discrete linear time-invariant system:

$$\begin{aligned} X_{t+1} &= AX_t + Bu_t \\ Y_t &= CX_t \end{aligned} \quad (1)$$

where $X_t \in \mathbb{R}^d$ is the state of the system and $U_t \in \mathbb{R}^m$ is the control input, and $Y_t \in \mathbb{R}^l$ is the output of the process, all at time t . Assuming, also, that A, B and C have proper dimensions, while $C = I_{n \times n}$, i.e. the state is available for measurement.

Definition 2.1: [21] System 1 is asymptotically stabilizable if there exist an encoder, decoder and controller such that the following holds:

- 1) Stability: $\forall \epsilon > 0, \exists \delta(\epsilon)$ such that $\|X_0\|_2 \leq \delta(\epsilon)$ implies $\|X_t\|_2 \leq \epsilon, \forall t \geq 0$.
- 2) Uniform attractivity: $\forall \epsilon > 0, \forall \delta > 0 \exists T(\epsilon, \delta)$ such that $\|X_0\|_2 \leq \delta$ implies $\|X_t\|_2 \leq \epsilon, \forall t \geq T$.

The first part of the definition implies that the state cannot grow unbounded for any bounded initial state, X_0 , while the attractivity property implies that the state decreases uniformly to zero. Considering this definition, the following result holds:

Proposition 2.1: [21] Assuming (A, B) is a stabilizable pair, a necessary condition for system (1) to be asymptotically stabilizable

is that the rate R satisfies

$$R \geq \sum_{\lambda(A)} \max \{ 0, \log |\lambda(A)| \}. \quad (2)$$

where $\lambda(A)$ are the eigenvalues of matrix A .

The importance of this result is that the stabilizability property can only be achieved if we have a minimum data rate which is related to the dynamics of the plant, i.e., if the system has fast (unstable) dynamics, then the data rate must be fast enough to overcome their effects. Moreover, Tatikonda and Mitter also proved that proposition 2.1 holds even if there exist additive process disturbances. It was also shown that this result provides exactly the same conditions for asymptotic observability. As stated before, these ideas were extended by the same authors [22], for noisy communication channels.

B. Feedback Scheme including a Noisy Digital Communication Channel.

When a noisy channel is present in the feedback loop given in figure 2, there must be a restatement of the concept of asymptotic stabilizability, since the deterministic definition no longer holds. There have been three main approaches in the stochastic case, the first one involving a mean-square convergence criterion, the second using the almost-sure convergence criterion while the third considers the m -th moment convergence criterion (see [28]). The mean-square criterion is a good candidate for situations where large deviations occur and must be penalized, whereas the almost-sure convergence is more appropriate when almost all realizations are typical [5]. The resulting conditions are not the same because one convergence criterion does not necessarily imply any of the two others [28].

1) *Mean-Square Convergence Criterion:* Recalling from [24], the definition of Shannon's channel capacity as (see [28]):

Definition 2.2: [24] Given a channel $\{P(W_t|v^t, w^{t-1})\}$, the Shannon capacity over a time horizon of length T is defined as the supremum of the mutual information over all channel input distributions $P(V^{T-1})$. Specifically, $C_T^{cap} = \sup_{P(V^{T-1})} I(V^{T-1}; W^{T-1})$, where $I(\cdot; \cdot)$ is the mutual information, V is the input alphabet and W is the output alphabet.

When the concepts of information theory are used for NCS explanations, it is interesting to know if the traditional Shannon's classical channel capacity still works. The work of Sahai in [19], [20] suggests that Shannon capacity is not enough for stability. He then introduces the concept of *anytime capacity* as a candidate to replace the more traditional Shannon's channel capacity concept, and demonstrates that this is a sufficient concept for stability of a closed-loop when using a noisy communication link. The model used by Sahai is described by $X_{t+1} = AX_t + U_t + W_t, k \geq 0$, where X_t is the state and W_t is a bounded noise process such that $\|W_t\| \leq \frac{w}{2}$. Then, the concept of Anytime Capacity is defined as:

Definition 2.3: [20] The α -anytime capacity, $C_{\text{anytime}}(\alpha)$, is the maximum rate at which the channel can be used to transmit data with a probability of error that decays at a rate α , i.e., $C_{\text{anytime}}(\alpha) = \sup \{R : E^R, K > 0, \forall N, \exists D_N^R, \mathbb{P}_{\text{error}}(E^R, D_N^R) < K2^{-\alpha N}\}$.

It is important to point out that this definition requires a fixed encoding scheme and the decoder to work at all delays. For this reason it is called anytime capacity (C_{anytime}). The value α specifies the exponential rate at which the answers need to be improved. In other words, if the decoding process stops at anytime, the answer has to be increasingly meaningful. Also, note that:

$\forall \alpha, C_o \leq C_{\text{anytime}}(\alpha) \leq C$, where C is Shannon's capacity and C_o is the zero-error capacity of the channel, i.e., the maximal rate at which the channel can be used to transmit data without error. Under this less conservative definition of channel capacity, Sahai proved the following theorem for a *scalar* system:

Theorem 2.1: [19] An unstable scalar linear system with intrinsic rate $\log_2 A$ is stabilizable over a noisy channel if and only if $C_{\text{anytime}}(2 \log_2 A) > \log_2 A$ for the channel with feedback.

This result is limited, along with others in Sahai's work [19], to the mean-square stability case and it can be extended to m -th moment stability, but C_{anytime} is usually hard to find. Later, it will be shown how to overcome this problem for m -th moment stability. Almost-sure stabilizability is also a convenient way to express stability and the next subsection will focus on such results.

2) *Almost-sure convergence criteria:* In [22], almost-sure convergence is introduced as follows:

Definition 2.4: [22] System 1 is asymptotically stabilizable if there exist an encoder, a decoder and a controller such that $\|X_t\| \rightarrow 0$ almost surely.

For definition 2.4, the bound for R given by equation (2) can no longer be assumed valid. A new framework is needed to guarantee almost sure asymptotic stabilizability. The approach used in [22] was the addition of the Shannon's channel capacity concept explained in definition 2.2 as a measure of channel quality. The main difference between the deterministic case and the stochastic idea is that the noiseless channel, with data rate R , has a capacity $C_T^{cap} = TR$ over a time horizon T , while for the noisy channel we have different situations depending on the features of the channel. For a delay noisy channel we have now $C_T^{cap} = (T - \Delta)R$ where Δ is the channel delay; for an erasure channel with erasure probability α we have $C_T^{cap} = (1 - \alpha)TR$ and for a memoryless Gaussian channel with power ρ we have $C_T^{cap} = (T/2) \log(1 + \rho)$. However, in [22] the conclusions were obtained regardless of the type of noisy channel, since they used limit $C^{cap} = \liminf_{t \rightarrow \infty} (1/t) C_t^{cap}$ instead of the channel type-dependent quantity C_T^{cap} . A result from [22] is given by the following proposition:

Proposition 2.2: [22] For system (1) with (A, B) a stabilizable pair, a necessary condition on the channel capacity for almost surely asymptotic stabilizability is that $C^{cap} \geq \sum_{\lambda(A)} \max \{ 0, \log |\lambda(A)| \}$.

Although we do not include in this survey, the proofs of these results, the reader may notice that these conditions make no assumptions on specific architectures for the encoder, decoder, or controller. A fictitious difference between the noisy and the noiseless cases is that for the noiseless channel, the results are given in terms of R , while in the noisy case, the conditions are given in terms of C^{cap} . Relations in terms of R may however be obtained, depending on the type of noisy channel.

3) *M-moment Stabilization:* The previous subsections have shown necessary conditions for the stabilization of unstable LTI systems in the presence of communications constraints (data rate). We have seen that in those approaches there is a limit (a critical value) on the data rate of the closed-loop for stabilizing the system. We conclude that the minimum data rate $R_{\min}$ depends on the unstable dynamics of the plant as shown by equation 2. However, when the system is uncertain and external disturbances are present, the limit on $R_{\min}$ is no longer enough in the 2-nd moment stabilization sense as Sahai's work showed with the introduction of the anytime capacity concept. Martins, Dahleh and Elia [15] found complementary results to the ones in [19]. Their

work provides results for tighter ideas of stability such as the m -th moment stability, that will be defined later. Their framework includes a first-order linear system with uncertainties and external disturbances as well as a stochastic link. The conditions that they obtained provide new results to achieve robust stability. We now present the following definition of stochastic link:

Definition 2.5: [15] Consider a link that, at every instant k , transmits $r(k)$ bits. It is defined to be a stochastic link, provided that $r(k) \in \{r_{\min}, \dots, \bar{r}\}$ is a random process satisfying, $r(k) = R - r^\delta(k)$, where $r^\delta(k)$ is an i.i.d. zero-mean process. More specifically, the link is a stochastic truncation operator $F_k^r: \{0, 1\}^r \rightarrow \bigcup_{i=0}^r \{0, 1\}^i$ defined as $F_k^r(b_1, \dots, b_r) = (b_1, \dots, b_{r(k)})$, where R is the average data-rate of the feedback loop and $b_i \in \{0, 1\}$.

They also considered a stochastic first-order system assuming that its nominal version states as follows: Given $x(0) \in [-\frac{1}{2}, \frac{1}{2}]$, $d \geq 0$ and $x(k+1) = a(k)x(k) + u(k) + d(k)$ with $|d(k)| \leq d$ and $x(i) = 0$ for $i < 0$. However, the results can be extended to a certain class of finite-dimensional LTI systems using real Jordan forms [17]. Finally, the full stochastic model, including the uncertainties, is given by $x(k+1) = a(k)(1 + z_a(k))x(k) + u(k) + z_f(k) + d(k)$ with the perturbation processes z_a and $z_f = G_f(x)$ that satisfy: $|z_a(k)| \leq \bar{z}_a$ and G_f causal and $\|G_f\|_{\infty(\rho)} \leq \bar{z}_f$, where $\bar{z}_a \in [0, 1)$ and $\bar{z}_f \in [0, 1)$. This system actually covers a wide class of possible stochastic models. Mainly because z_a is the uncertainty around the nominal $a(k)$, while $z_f(k)$ is the output of an uncertain block $G_f(x)$ placed at the feedback. Other assumptions were that the process a_k was i.i.d and independent of the link $r(k)$ and the initial state $x(0)$. Moreover, the statistical description satisfies $\log(|a(k)|) = R_{\det} + I^\delta(k)$, where $I^\delta(k)$ is a zero mean and i.i.d. sequence and $R_{\det} = \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ is the critical bound for the rate in the deterministic case given by equation (2). In order to create a feedback loop around the system, a stochastic controller and a particular encoder were built. The encoder was a function $F_k^e: \mathbb{R}^{k+1} \rightarrow \{0, 1\}^r$ given by $F_k^e(x(0), \dots, x(k)) = (b_1, \dots, b_r)$. Similarly, the controller $F_k^c: \bigcup_{i=0}^r \{0, 1\}^i \rightarrow \mathbb{R}$ was: $u(k) = F_k^c(\bar{b}(k))$. Finally, the m -th moment stability was defined as follows:

Definition 2.6: [15] Let $m > 0$, $\rho \in \mathbb{N}_+ \cup \{\infty\}$, $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$ and $\bar{d} \geq 0$ be given. The system (II-B.3), under a given feedback scheme, is m -th moment (robustly) stable provided that the following holds:

$$\begin{cases} \lim_{k \rightarrow \infty} E[\bar{x}(k)^m] = 0 & \text{if } \bar{z}_f = \bar{d} = 0, \\ \exists b \in \mathbb{R} \text{ s.t. } \limsup_{k \rightarrow \infty} E[\bar{x}(k)^m] < b & \text{otherwise} \end{cases} \quad (3)$$

Basically, this is a double stability definition: the first limit in (3) is the m -moment internal stability, while the second limit is the m -moment external stability. Having explained the configuration of the stochastic feedback loop and under the assumptions mentioned so far, let us present the analysis to achieve m -th moment stability for both the deterministic (in order to follow the chronological sequence, this case was not included in the noiseless channel case) and stochastic cases.

Deterministic Case: The deterministic case occurs when $r(k) = R$ and $\log|a(k)| = R_{\det}$. A sufficient condition for the existence of a stabilizing feedback scheme in such a case is given in the next theorem:

Theorem 2.2: [15] Let $\rho \in \mathbb{N}_+ \cup \{\infty\}$, $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$ and $\bar{d} \geq 0$ be given and $h(k)$ be defined as $h(k) = 2^{k(R_{\det} - C_e)}$, $k \geq 0$, where $C_e = r_e = -\log(2^{-R} + \bar{z}_a)$. Consider that $x(k)$ is the solution of (II-B.3) under the feedback scheme as well as the following

Fig. 3. $\alpha(m)$

conditions: $C_e > R_{\det}$ and $\bar{z}_f \|h\|_1 < 1$. If conditions 1 and 2 are satisfied then the following holds for all $|d(t)| \leq \bar{d}$, $G_f \in \Delta_{f,\rho}$ and $|z_a(k)| \leq \bar{z}_a$: $\bar{x}(k) \leq \|h\|_1 \left(\bar{z}_f \frac{\|h\|_1 \bar{d} + \frac{1}{2}}{1 - \|h\|_1 \bar{z}_f} + \bar{d} \right) + h(k) \frac{1}{2}$, where $\Delta_{f,\rho} = \{G_f: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty : \|G_f\|_{\infty(\rho)} \leq \bar{z}_f\}$.

Stochastic Case: Similarly to the deterministic case, there are sufficient conditions for m -moment stability in the stochastic case. These conditions are given by the following theorem:

Theorem 2.3: [15] Let $m, \rho \in \mathbb{N}_+$, $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$ and $\bar{d} \geq 0$ be given along with the quantities: $\beta(m) = \frac{1}{m} \log E[2^{mI^\delta(k)}]$ and $h_m(k) = 2^{k(R_{\det} + \beta(m) + \alpha_e(m) - C_e)}$, $k \geq 0$, where $r_e(k) = -\log(2^{-r(k)} + \bar{z}_a) = C_e - r_e^\delta(k)$ and $\alpha_e(m) = \frac{1}{m} \log(E[2^{mI^\delta(k)}])$ and $r_e^\delta(k)$ is a zero-mean i.i.d. process. Consider that $x(k)$ is the solution of II-B.3 under the feedback scheme as well as the following conditions: $C_e > R_{\det} + \beta(m) + \alpha_e(m)$ and $\rho \frac{1}{m} \bar{z}_f \|h_m\|_1 < 1$.

If conditions 1 and 2 are satisfied, then the following holds for all $|d(t)| \leq \bar{d}$, $G_f \in \Delta_{f,\rho}$ and $|z_a(k)| \leq \bar{z}_a$: $E[\bar{x}(k)^m]^{\frac{1}{m}} \leq \|h_m\|_1 \left(\rho \frac{1}{m} \bar{z}_f \frac{\|h_m\|_1 \bar{d} + \frac{1}{2}}{1 - \rho \frac{1}{m} \bar{z}_f \|h_m\|_1} + \bar{d} \right) + h_m(k) \frac{1}{2}$, where $\Delta_{f,\rho} = \{G_f: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty : \|G_f\|_{\infty(\rho)} \leq \bar{z}_f\}$.

The last two theorems are derived for scalar systems. With some extra assumptions, reference [15] provides necessary conditions for the stabilizability of a general LTI system. If $\bar{z}_a = \bar{z}_f = \bar{d} = 0$, then the following theorem is true:

Theorem 2.4: [15] Let $x(k)$ be the solution of the following LTI system: $x(k+1) = Ax(k) + Bu(k)$. If the state satisfies

$$\sup_k E \left[\sup_{x(0) \in [-\frac{1}{2}, \frac{1}{2}]^n} \|x(k)\|_m^m \right] < \infty$$

then the following must hold: $R \geq \alpha \left(\frac{m}{n_{\text{unstable}}} \right) + \sum_{i=1}^n \max\{0, \log|\lambda_i(A)|\}$, where $\alpha(m) = \frac{1}{m} \log(E[2^{mI^\delta(k)}])$ and n_{unstable} is the number of unstable eigenvalues.

This provides analogous conditions to equation (2), but a higher R is now needed as expected, since the m -moment stability is a more robust stability concept. However, it is interesting that because of the form of $\alpha(m)$ (Figure 3) then any almost-sure stable system, is also m -th stable for m sufficiently small.

C. Feedback Scheme including a Packet-Based Network.

The previous results taken from [15], [21], [22] and [23], considered noisy and noiseless discrete communication channels. However, recent work by Shi and Murray [25] took initial steps towards the development of a packet-based control theory. They worked in the idea of stabilizing an unstable, but controllable and observable, discrete linear time-invariant system when the closed-loop includes a packet-based network. The following assumptions for system (1) were made: matrix A has at least one eigenvalue in

the right half plane (unstable), and C is also assumed to be equal to $I_{n \times n}$, therefore, $Y_t = X_t$, and the pair (A, B) is controllable while the pair (A, C) is observable. It was also assumed that the packet network has a finite data rate R bits/s and no packets are lost, no reordering of packets may occur, and a packet length of l bits was considered. Therefore, the transmission delay was $\delta = l/C$. Finally, it was assumed that the total delay induced by the network, in addition to the transmission delay, is constant and equal to D . Limits were obtained for the minimum data rate needed to stabilize the closed-loop system in three bit allocation schemes.

1) *Equal bit allocation*: In the first case, an equal bit allocation was assumed, i.e., the allocation of bits in a packet was such that l/n bits were used for the i^{th} component of $Y_t \forall t$, i.e. all components $Y_i, i = 1, \dots, n$ received the same bit allocation in the packet. For this scheme, the following condition was obtained in order to guarantee exponential stability of the closed-loop system: $R > \frac{l \log(\|e^A\|)}{l - 1 - D \log(\|e^A\|)}$, where $\|e^A\|$ is the induced L_2 norm or the largest singular value of e^A .

2) *Proportional bit allocation*: Assuming that $A = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. A proportional bit allocation is intuitively a smarter scheme, because instead of using the same number of bits for each component, $\frac{\lambda_i}{\lambda} l$ bits are used for the i^{th} component of Y_t , where $\lambda = \sum_{i=1}^n \lambda_i$. Therefore, the bit allocation is proportional to the size of the eigenvalues and more bits in a packet are thus allocated to the more unstable modes. For this scheme, Shi and Murray showed that the limit for the data rate to achieve exponential stability is given by: $R > \frac{l \lambda \log e}{l - \lambda D \log e - \frac{\lambda}{\lambda_n}}$. However, this result is counter-intuitive since a proportional bit allocation scheme is expected to give a dependence of R on the largest eigenvalue (the most unstable) but here, the smallest eigenvalue is the one that determines the rate R . This contradictory condition pointed Shi and Murray towards a third approach, where they looked for an optimal bit allocation.

3) *Optimal bit allocation*: The idea here is to give variable portions for each individual subsystem (i^{th} component), and then perform an optimization algorithm on those variable portions. The optimization problem is stated as follows: Denote $\beta = [\beta_1, \dots, \beta_n]$, then find: $\min R_{\min}$ subject to $\sum_{i=1}^n \beta_i = 1, \beta_i > 0 \forall i$, and $\beta_i l > 1 + \lambda_i D \log e \forall i$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ are given. The disadvantage of this approach is that there is no analytical solution to the bits allocation in terms of the size of the eigenvalues since the answer is based on the solution of a LMI. However, the scheme tries to give more weight to the most unstable eigenvalue and less weight to the least unstable eigenvalue. For this allocation scheme, [25] shows that if $l \ll n$ then the following approximation holds: $R_{\text{in f optimal}} \approx \sum_{i=1}^n \lambda_i \log e$. The concepts discussed so far provide initial steps towards a deterministic theory of packet-based control. However, there is also a more robust result for a stochastic case where the model considered is: $x_{k+1} = (A + \Delta_k)x_k + \gamma_k B u_k$ and $y_k = \lambda_k C x_k$, where λ_k and γ_k (used to model packet drops) are Bernoulli i.i.d. random variables with parameters $E[\lambda_k] = \lambda$ and $E[\gamma_k] = \gamma$ for all k , respectively. Also Δ_k satisfies $\Delta_k^T \Delta_k \leq K^2 I$ for all k , and it is used to model the uncertainty on A . The network in the right side of figure 1 has a data rate $R_1 + n$ and the network at the left side of figure has a data rate $R_2 + n$. Where R_i bits are used to allocate the magnitude of the state and n bits are for the sign of the n state signals. The problem was formulated to guarantee almost-sure stability and the result is given by the following theorem:

Theorem 2.5: [26] Assume B, C are invertible and the system dimension is n . Then a sufficient condition for the closed loop almost sure stability (if there are no packet drops, i.e., $\lambda = 1$ and $\gamma = 1$, change this notion to exponential stability) is that the network parameters and system parameters satisfy the inequality $(|A| + K)^{1-\lambda\gamma} \left(|A| 2^{-\frac{R_1}{n}} + |B| |B^{-1}A| 2^{-\frac{R_2}{n}} + K \right)^{\lambda\gamma} < 1$, where $|\cdot|$ is the induced matrix Euclidean norm.

The importance of this result is that it can be proved to be equivalent to the one of Tatikonda, equation 2, but considering a packet-based network (let in equation 2.5: $R_2 = \infty, \lambda = 1, \gamma = 1$ and $K = 0$). Future work is expected that includes disturbances in the model.

III. EQUIVALENCE OF FEEDBACK STABILIZATION AND COMMUNICATION SCHEMES.

In the previous sections we have sufficiently shown that in the NCS theory, there are important relationships between information theory, communications theory and control theory. Elia [14] has demonstrated that a communication with feedback is limited by control theory ideas. He actually provided confirmation that the achievable information transmission rates of a feedback communication scheme is limited by the well-known Bode's integral formula, which is a fundamental limitation of causal linear feedback control systems (see [12]). The system considered by Elia was the same that has been exposed in this survey, separated into unstable and stable modes:

$$K : \begin{bmatrix} x_s^+ \\ x_u^+ \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} y$$

$$u = \begin{bmatrix} C_s & C_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} \quad (4)$$

where A_u are the eigenvalues outside the unit circle and A_s are the eigenvalues strictly inside the unit circle. Also, for simplicity, A_s and A_u are in Jordan form. Based on this model, Elia showed the equivalence between the feedback stabilization and the feedback communication schemes and relate the achievable rates to the fundamental limitation given by the Bode's integral. The first theorem in this approach is the following:

Theorem 3.1: [14] The stable feedback interconnection of an discrete-time LTI SISO unstable system K , with an intersymbol interference additive Gaussian channel F is equivalent to a feedback communication system, which reliably (in the sense of Shannon) transmits at a rate $R = \sum_i \ln |\lambda_i(A_u)| - \epsilon$, for any $\epsilon > 0$, the initial state, $x_u(0) = x_{u0}$.

What this implies is that a communication system with the rate given by the theorem can be built from a closed-loop system with the same degree of instability and with an encoder (see figures 4 and 5). An interesting comment is that Elia used this equivalence as an explanation of the communication of biological systems, which apparently do not have encoding/decoding communication systems, but multiple regulatory feedback systems. Now, the main result of [14] is the equivalence of the Bode's integral formula for the sensitivity function, S , from w to y in figure 4, and the average directed information, $\bar{I}(U \rightarrow Y)$, of figure 5. The idea is summarized by the following theorem:

Theorem 3.2: [14]

$$\lim_{T \rightarrow \infty} \frac{1}{T} \bar{I}(U^T \rightarrow Y^T) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |S(e^{j2\pi\theta})| d\theta = \sum_{i=1}^m \ln |\lambda_i(A_u)|$$

Fig. 4. Feedback Stabilization Block Diagram

Fig. 5. Feedback Communication Block Diagram

If we compare this theorem with the condition given by equation 2 we see how the problem of stabilization for a feedback system over analog communication channels is closely related to the problem of communication over the same channel, when the encoder has access to noiseless feedback from a decoder.

IV. LIMITATIONS ON A CASUAL AND FINITE CAPACITY FEEDBACK SCHEME

Recently, Martins and Dahleh [16] developed exciting ideas on fundamental limitations in the performance of a finite capacity feedback. This is a relevant achievement for the NCS theory similar to the the Bode Integral for the classic control theory. The feedback structure of Martins and Dahleh is shown in figure 6, where d is the disturbance and the transfer function between d and $\tilde{e} = \tilde{u} + d$ is the sensitivity represented by $S(w)$. Let us use $e = G^{-1}\tilde{e}$ and $u = G^{-1}\tilde{u}$. Under this feedback scheme two limitations were found for the control loop performance and their causes were two issues: causality and finite feedback capacity.

A. Fundamental limitations caused by causality

The analysis is conducted considering a single input plant with the following state-space structure:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} x_u(k+1) \\ x_s(k+1) \end{bmatrix} = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} x(k) + \begin{bmatrix} b_u \\ b_s \end{bmatrix} e(k) \\ y(k) &= Cx(k), |\lambda_i(A_u)| \geq 1, |\lambda_i(A_s)| < 1 \text{ and } k \geq 0 \end{aligned} \quad (5)$$

The following notation is given in order to simplify the expression:

Fig. 6. Feedback Scheme

- The expectation of $\mathbf{a}$ is given by $E[\mathbf{a}]$ and $a_{k_{min}}^{k_{max}}$ is a finite segment of a sequence $\mathbf{a}$.
- The auto-covariance function of a given stochastic process $\mathbf{a}$ is given by: $R_a(k, l) = E[(a(k) - E[a(k)])(a(l) - E[a(l)])^T]$.
- If $\mathbf{a}$ is stationary, its power spectral density is $\hat{F}_a(w) = \sum_{k=-\infty}^{\infty} R_a(k, 0)e^{-iwk}$.
- If $\mathbf{a}$ is a stochastic process, then its covariance matrix is: $\left[\sum_{k_{min}}^{k_{max}} (a(i) - E[a(i)])(a(j) - E[a(j)])^T \right]_{(i-k_{min}+1), (j-k_{min}+1)} = E[(a(i) - E[a(i)])(a(j) - E[a(j)])^T]$, where $i, j \in \{k_{min}, \dots, k_{max}\}$.
- For $a \in \mathbb{R}$, we have: $[a]_- = \min\{a, 0\}$ and $[a]_+ = \max\{a, 0\}$.
- The eigenvalues of a matrix A are denoted by $\lambda_i(A)$.
- The log-density of the eigenvalues with magnitude smaller than 1, of a covariance matrix is $L_-(a_{k_{min}}^{k_{max}}) = \frac{\sum_{i=1}^{k_{max}-k_{min}+1} |\log(\lambda_i(\sum_{k_{min}}^{k_{max}}))|_-}{k_{max}-k_{min}+1}$.
- The log-density of the eigenvalues with magnitude larger than 1, of a covariance matrix is $L_+(a_{k_{min}}^{k_{max}}) = \frac{\sum_{i=1}^{k_{max}-k_{min}+1} |\log(\lambda_i(\sum_{k_{min}}^{k_{max}}))|_+}{k_{max}-k_{min}+1}$.

With these definitions, the next step is to determine a relation between the concepts of causality and stability and the logarithmic sum of the eigenvalues of $\Sigma(e_1^k)$. Mathematically, the relationship is expressed in the following theorem:

Theorem 4.1: [16] Let $x(k)$ be the solution of the state-space equation 5. If the system is stable, i.e., $\sup_k E[x^T(k)x(k)] < \infty$ holds, then $\liminf_{k \rightarrow \infty} (L_-(e_1^k) + L_+(e_1^k)) \geq 2 \sum_i \max\{0, \log(|\lambda_i(A)|)\}$.

This is a fundamental limitation of the eigenvalues distribution and it demonstrates that not all of the eigenvalues of the covariance matrix of e_1^k can be small and that the reduction of some necessarily imply the increase of others, like the water-bed effect given by the Bode integral formula. Assuming that e is stationary, then according to the definition of L and the power spectrum $\hat{F}_e(w)$ we know that $\lim_{k \rightarrow \infty} L(e_1^k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\hat{F}_e(w)) dw$ and, therefore, the following corollary holds:

Corollary 4.1: [16] Let $x(k)$ be the solution of the state space equation 5 and $g(k)$ be an arbitrary sequence satisfying $\frac{g(k)}{k} \rightarrow 0$. If the system is stable and $\mathbf{e}$ a stationary process, where $0 < m < \hat{F}_e(w) < \infty$ is Lebesgue integrable, then $\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(S(w)) dw \geq \sum_i \max\{0, \log(|\lambda_i(A)|)\}$, where $S(w) = \sqrt{\hat{F}_e(w)}$.

B. Fundamental limitations caused by finite feedback capacity

It is important to note that the limitation expressed by theorem 4.1 and corollary 4.1 do not consider the channel capacity. However, if we note that $\bar{I}_\infty(v \rightarrow z)$ is the directed information rate at the channel (see [28]), then inequality $\bar{I}_\infty(v \rightarrow z) \leq C_{channel}$ which causes the shaping of the eigenvalues distribution. The following theorem explain how there is a trade-off between the disturbance attenuation and the directed information rate, measured by $L_-(e_1^k)$ and $\bar{I}_\infty(v \rightarrow z)$, respectively:

Theorem 4.2: [16] Let $x(k)$ be the solution of the state-space equation 5. If the system is stable, i.e., $\sup_k E[x^T(k)x(k)] < \infty$ holds, then $\bar{I}_\infty(v \rightarrow z) + \frac{1}{2} \liminf_{k \rightarrow \infty} L_-(e_{g(k)}^k) \geq \sum_i \max\{0, \log(|\lambda_i(A)|)\}$.

From this theorem, it can be shown that if $C_{channel}$ is close to the deterministic stabilization rate given by equation 2, it is impossible to reject the disturbances. This result is given by the following corollary:

Fig. 7. Feedback Scheme

Corollary 4.2: [16] Let $x(k)$ be the solution of the state space equation 5 and $g(k)$ be an arbitrary sequence satisfying $\frac{g(k)}{k} \rightarrow 0$. If the system is stable, then $\frac{1}{2} \liminf_{k \rightarrow \infty} L_-(e^k_{g(k)}) + C_{channel} \geq \sum_i \max \{0, \log(|\lambda_i(A)|)\}$.

As noted before, the corollary implies that a reduction of the eigenvalues of $\Sigma(e^k)$, for values below one, must come at the expense of information flow in the channel ($\bar{I}_\infty(v \rightarrow z)$). Moreover, if we assume stationarity, the theorem 4.2 can be expressed with an integral term as following:

Corollary 4.3: [16] Let $x(k)$ be the solution of the state space equation 5. If the system is stable, i.e., $\sup_k E[x^T(k)x(k)] < \infty$ holds, and e is stationary, where $0 < m < \hat{F}_e(w) < M < \infty$ is Lebesgue integrable, then $\frac{1}{2\pi} \int_{-\pi}^{\pi} [\log S(w)]_- dw + \bar{I}_\infty(v \rightarrow z) \geq \sum_i \max \{0, \log|\lambda_i(A)|\}$, where $S(w) = \sqrt{\hat{F}_e(w)} = \sqrt{\frac{\hat{F}_e(w)}{\hat{F}_d(w)}} = \sqrt{\frac{\hat{F}_e(w)}{|G(e^{jw})|^2}}$.

This is a universal result with theoretic importance [18]. But, in order to generalize the idea of the bound on disturbance attenuation caused by finite feedback capacity, let us define feedback capacity.

Definition 4.1: [24] Consider a communication channel inserted in a strictly causal feedback loop. Given a set of stochastic processes $\mathbb{V}$, a channel has a well defined quantity denoted as feedback capacity, which is represented as C_f . Feedback capacity is the least upper bound C_f satisfying: $\sup_{k \in \mathbb{N}_+} \frac{I((x(0), d^k; z^k))}{k} \leq C_f$.

For the memoryless channel, $C_{channel} = C_f$, but in general the result can be extended. In [18] the feedback scheme in figure 7 was considered as a modified version of the feedback scheme in figure 6 and the following generalized result was proved:

Theorem 4.3: [18] Consider the scheme of figure 7, where e and d are assumed asymptotically stationary, with d Gaussian auto-regressive. If the state of the plant satisfies $\sup_k E[x(k)x(k)^T] < \infty$ then the following holds: $\frac{1}{2\pi} \int_{-\pi}^{\pi} [\log S(w)]_- dw \geq \sum_i \max \{0, \log|\lambda_i(A)|\} - C_f$.

We notice from the assumptions that this bound in the disturbance attenuation is not predicted from the Bode formula, nor is it not implied by it. The limitation is a direct effect of the finite feedback capacity consideration.

V. CONCLUSIONS

This paper has reviewed recent results pertaining to the analysis of NCS from an information theoretic angle. Limitations for stabilization in the deterministic sense and in several stochastic frameworks have been presented. The most recent results on fundamental limitations on the performance of a feedback scheme have also been presented as well as the new Bode Integral Formula interpretation for these limitations. The incorporation of the Bode Integral has also shown the equivalence of feedback control schemes and feedback communications schemes, bringing

new explanations for other areas such as biology. Future work is expected in order to extend these ideas to distributed systems with multiple channel communication schemes.

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