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Intrinsic Curvature For Schemes

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INTRINSIC CURVATURE OF SCHEMES

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DEDICATION

I dedicate this work to my mother, father, brother, and all of the wonderful people I have met along the way.

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ABSTRACT

This thesis develops an algebraic analog of psuedo-Riemannian geometry for relative schemes whose cotangent sheaf is finite locally free. It is a generalization of the algebraic differential calculus proposed by Dr. Ernst Kunz in an unpublished manuscript to the non-affine case. These analogs include the psuedo-Riemannian metric, Levi-Civitá connection, curvature, and various existence theorems.

CONTENTS

1 Introduction

In this paper, a notion of the psuedo-Riemannian metric is presented for relative schemes whose contangent sheaf is finite locally free. For example, a smooth algebraic variety V of dimension n over a field k has this property that the cotangent sheaf $\Omega_{X/k}$ is locally free of rank n. More generally, nonsingular varieties have this property as well. The motivation may be found in the fact that there lacks a well-developed analog of a metric tensor in algebraic geometry. However, this does not mean that there has not been any efforts made to develop this lack of.

For instance, there are two independent efforts that start to address this notion. On one hand, [1] is an unpublished work developing psuedo-Riemannian geometry in the affine case of commutative algebra, but this machinery does not consider glueing operations or a scheme-theoretic flavor. In particular, the analog of the Levi-Civitá connection is worked out only in the case where 2 is always a unit of the commutative ring. Yet, on the other hand, in [2] and [3] one does have a notion of a smooth psuedo-Riemannian variety over a field of characteristic zero and its geometry is studied. A primary objective of this paper is to blend as much as possible these two bodies of work and strive for the greatest generality. Algebraic constructions of differential calculus dates far back, and more at the foundational level, it can be thought to be started with this paper [4]. A very good account of this may also be found in the paper [5]. As scheme-theoretic methods were developed, soon followed generalizations of connections to vector bundles over a scheme. For instance, this may be found in [6], [7], and [8]. In what follows is a brief outline of this paper.

The first section of this paper constructs the desired analog of a metric in algebraic geometry. We start with an S-scheme X such that the cotangent sheaf $\Omega_{X/S}$ is locally free of (finite) rank n. A metric on X over S can be thought of as an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves whose global section induces a symmetric 2-tensor and satisfies a non-degeneracy condition. Similar to classical differential geometry, the existence of a metric determines an isomorphism of cotangent sheaf $\Omega_{X/S}$ and the tangent sheaf $\Theta_{X/S}$. However, it is known that such an isomorphism does not always exist, so this allows one to restrict to points of X for which the restriction of g to this set forms a metric, and hence the induced isomorphism of (restricted sheaves).

In the second section, we introduce the meaning of a connection in algebraic geometry and generalize the machinery introduced in [1] for our purposes in a scheme-theoretic approach. This gives a meaning to local constructions like connections forms, connection matrices, Christoffel symbols, and more. There is also a set of localization idea's of these constructions as we will see later on. The third section has one of the main theorems of this paper, i.e. the existence of the algebraic Levi-Civitá connection for a psuedo-Riemannian scheme. Similar to the second section, we will further carry over some of the constructions proposed in [1] to this non-affine case and have notions of Laplace operators, Hesse forms, and also what is called a residual of a psuedo-Riemannian scheme. These residuals can be thought of as a convenient quotient of the structure that is constructed from a sheaf of ideals. The last section will study the curvature properties of the developed framework with connections and metrics for this algebraic context.

2 Algebraic Metrics

Let X be an S-scheme where the cotangent sheaf $\Omega_{X/S}$ is finite locally free as an $\mathcal{O}_X$-module. Consider an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves. It gives rise naturally to a morphism of $\mathcal{O}_X$-modules

$$g_\otimes : \Theta_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S} \to \mathcal{O}_X$$

which is defined on simple local sections by $g_\otimes(v \otimes w) = g(v, w)$ and on arbitrary local sections by extending linearly to linear combinations of simple local sections. We have that the section g is symmetric if, and only if, $g_\otimes \circ \tau = g_\otimes$ where

$$\tau : \Theta_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S} \to \Theta_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S}$$

is the braiding isomorphism of abelian sheaves, i.e. on simple local sections $v \otimes w \mapsto w \otimes v$.

Lemma 1. *For an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian* *sheaves, there exists an induced morphism*

$$\delta_g : \Theta_{X/S} \to \Theta^*_{X/S}$$

of $\mathcal{O}_X$-modules defined on an open U as the map taking $l \in H^0(U, \Theta_{X/S})$ to the $\mathcal{O}_X(U)$-linear map $g(U)(l, -)$ where $(\delta_g(U)(l))(w) = g(U)(l, w)$.

Proof. The morphism δ_g is the natural transformation taking an open subset $U \subset X$ to the $\mathcal{O}_X(U)$-linear map $\delta_g(U)$ that acts on elements $l \in \Theta_{X/S}$ by sending it to the map $g(U)(l, -)$. This makes sense as $g(U)$ is an $\mathcal{O}_X(U)$-bilinear morphism $g(U) : \Theta_{X/S}(U) \times \Theta_{X/S}(U) \to \mathcal{O}_X(U)$. □

Lemma 2. *If $\mathcal{E}$ is a locally free sheaf on a ringed space $(X, \mathcal{O}_X)$, then $(E \otimes_{\mathcal{O}_X} E)^*$ and $E^* \otimes_{\mathcal{O}_X} E^*$ are isomorphic as $\mathcal{O}_X$-modules.*

Proof. The isomorphism is defined by the natural bilinear map $(E \otimes_{\mathcal{O}_X} E) \times (E^* \otimes_{\mathcal{O}_X} E^*) \to \mathcal{O}_X$, which is shown to be an isomorphism looking at bases locally. □

Remark 1. *Under suitable circumstances, when the cotangent sheaf $\Omega_{X/S}$ of an S-scheme X is locally free of finite rank n, then we have an isomorphism of the $\mathcal{O}_X$-modules $\Omega_{X/S} \otimes_{\mathcal{O}_X} \Omega_{X/S}$ and the dual $\mathcal{O}_X$ of $\Theta_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S}$. Therefore, we may interchangeably consider sections of one as sections of the other without loss of generality.*

Definition 1. *Given an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves, then we say that it is a* ***psuedo-Riemannian metric*** *(or* ***metric*** *for short) on X when the following conditions hold:*

1. *(symmetry) $g_{\otimes}(X)$ is a global section of the sheaf $Sym^2(\Omega_{X/S})$;*

2. *(non-degeneracy) the map $\delta_g : \Theta_{X/S} \to \Theta^*_{X/S}$ is an isomorphism of $\mathcal{O}_X$-modules.*

Notation 1. *When evaluating the metric g and map δ_g to some open set $U \subset X$, we will still write them respectively as g and δ_g rather than $g(U)$ and $\delta_g(U)$ when the open set U is clear from context.*

Lemma 3. *Consider an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves. The stalks of the map of sheaves $\delta_g : \Theta_{X/S} \to \Theta^*_{X/S}$ are all isomorphisms if, and only if, it is non-degenerate.*

Proof. This follows from elementary properties of sheaves of $\mathcal{O}_X$-modules. $\square$

Proposition 1. *For an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves to be non-degenerate, then following hold and are equivalent:*

1. *the map of determinants $det(\delta_g) : det(\Theta_{X/S}) \to det(\Theta^*_{X/S})$ being an isomorphism;*

2. *the stalks of the map of determinants $det(\delta_g) : det(\Theta_{X/S}) \to det(\Theta^*_{X/S})$ are all isomorphisms.*

Proof. This follows from the fact that $\Theta_{X/S}$ and $\Omega_{X/S}$ are isomorphic finite locally free $\mathcal{O}_X$-modules, and also from the fact that the operator det preserves short exact sequences. $\square$

Proposition 2. *(Existence By Removal) Consider an S-scheme X where $\Omega_{X/S}$ is finite locally free as an $\mathcal{O}_X$-module, and a $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves. If $R(g)$ denotes the closure of the subset of X for which the stalks of the $\mathcal{O}_{X,\mathfrak{p}}$-linear map $\delta_{g,\mathfrak{p}} : \Theta_{X/S,\mathfrak{p}} \to \Theta^*_{X/S,\mathfrak{p}}$ fails to be an isomorphism, then $X(g) := X \setminus R(g)$ is an open subscheme of X and $g|_{X(g)}$ is a metric on $\Theta_{X/S}|_{X(g)}$.*

Proof. Clearly, open subsets are open subschemes. Since at each point $\mathfrak{p} \in X(g)$, the map $\delta_{g,\mathfrak{p}}$ is an isomorphism by the construction of the space $X(g)$, we have that g is non-degenerate. $\square$

Definition 2. *Suppose that X is an S-scheme. Given a metric g on X, we call the pair (X, g) a* ***psuedo-Riemannian scheme over*** *S. For sections V, W of $\Theta_{X/S}$, $g(V, W)$ is called the* ***scalar product*** *of V, W.*

Example 1. *For R a commutative unital ring, the polynomial ring $R[x_1, \ldots, x_n]$ has the property that $\Omega_{Y/X}$ is a free $\mathcal{O}_X$-module of rank n trivialized by $dx_1, \ldots, dx_n$ where $Y = \mathbb{A}^n_X$ and $X = Spec(R)$. Writing $\delta x_1, \ldots, \delta x_n$ for the dual basis of $\Theta_{Y/X}$, we see that any $\mathcal{O}_Y$-bilinear morphism $g : \Theta_{Y/X} \times \Theta_{Y/X} \to \mathcal{O}_Y$ of abelian sheaves* *may be written as the sum*

$$\sum_{i,j=1}^{n} g_{ij}\delta x_i \otimes \delta x_j$$

where g_{ij} are elements of $R[x_1, \ldots, x_n]$. It follows that g is symmetric if, and only if, the matrix $(g_{ij})_{i,j=1}^{n}$ is symmetric in $Mat_n(R[x_1, \ldots, x_n])$. By identifying $\Theta^*_{Y/X}$ with $\Omega_{Y/X}$ via the canonical $\mathcal{O}_Y$-isomorphism, we see that the map $\delta_g : \Theta_{Y/X} \to \Omega_{Y/X}$ is given by $\delta_g(dx_i) = \sum_{j=1}^{n} g_{ij}\delta x^j$ for $1 \leq i \leq n$. Therefore, we see that g is non-degenerate if, and only if, $det(g_{ij})_{i,j=1}^{n}$ is a unit in $R[x_1, \ldots, x_n]$.

Example 2. *Suppose that X is a nonsingular variety of dimension n over a field k. The cotangent sheaf $\Omega_{X/k}$ is an $\mathcal{O}_X$-module locally free of rank n. It may very well hold in general that $\Omega_{X/k}$ is not isomorphic to $\Theta_{X/k}$ as an $\mathcal{O}_X$-module. In fact, even if it were the case, we do not necessarily have that $\Omega_{X/k}$ is free and of finite rank unlike the example above. However, observe that for each symmetric matrix*

$$G = (g_{ij})_{i,j=1}^{n} \in Mat_n(\mathcal{O}_X(X)),$$

we can consider the set $R(G)$ of points $\mathfrak{p} \in X$ for which the determinant of G is not a unit of the local ring $\mathcal{O}_{X,\mathfrak{p}}$. It can be seen that this set $R(g)$ is closed and we have that $X \setminus R(g)$ is an open subscheme of X. Applying the Existence by Removal, we have that there exists a metric on the scheme $X \setminus R(g)$. More generally, any smooth algebraic variety X over k of dimension n has the property that the cotangent sheaf $\Omega_{X/k}$ is locally free of rank n. Applying the same idea *above, we can restrict to particular open subschemes that surely have a metric by studying the symmetric matrices with entries in the ring of global sections.*

Example 3. *Suppose that $S \subset T$ are fields where $\{x_i\}_{i=1}^l$ is a collection of elements. Assume that either char$(S) = 0$ and $\{x_i\}_{i=1}^l$ is a transcendence basis of T over S, or that char$(S) = p > 0$ and $\{x_i\}_{i=1}^l$ is a p-basis for T over S. Then the global sections of $\Omega_{T/S}$ form a vector space over T of dimension l with basis $\{x_i\}_{i=1}^l$. As a result, the contangent sheaf a free $\mathcal{O}_T$-sheaf of rank l. Writing $x^1, \ldots, x^l$ as the dual basis of $\{x_i\}_{i=1}^l$, then each $\mathcal{O}_T$-bilinear morphism $g : \Theta_{T/S} \times \Theta_{T/S} \to \mathcal{O}_T$ of abelian sheaves may be written in the form $\sum_{i,j=1}^l g_{ij} x^i \otimes x^j$. From observations in the above examples, we see that any such $\mathcal{O}_T$-bilinear morphism g is a metric if, and only if, $(g_{ij})_{i,j=1}^l$ is a symmetric matrix in Mat$_l(T)$ whose determinant is nonzero.*

Example 4. *Suppose that $K \subset L$ are field with L a separable finitely generated over K. Let r be the transcendence degree of L over K. Then the global sections of $\Omega_{L/K}$ is a L-vector space of dimension r. By the discussions in the examples above, we have that $\mathcal{O}_L$-bilnear forms g written in the form $\sum_{i,j=1}^r g_{ij} x^i \otimes x^j$ relative to a dual basis $x^1, \ldots, x^r$ are metrics if, and only if, $(g_{ij})_{i,j=1}^r$ is a symmetric matrix in Mat$_r(T)$ whose determinant is nonzero.*

Proposition 3. *(Correspondence of Algebraic Metrics) Let X be an S-cheme over S where $\Omega_{X/S}(X)$ is a free $\mathcal{O}_X(X)$-module of rank n. There exists a one-to-one* *correspondence between symmetric non-degenerate $\mathcal{O}_X$-bilinear morphisms $\Theta_{X/S} \times \Theta_{S/S} \to \mathcal{O}_X$ of abelian sheaves and symmetric matrices $(g_{ij}) \in Mat_n(\mathcal{O}_X(X))$ with the property that $det(g_{ij}) \in \mathcal{O}_X(X)^\times$.*

Proof. Suppose that $\Omega_{X/S}(X)$ has a basis $b^1, \ldots, b^n$ and $b_1, \ldots, b_n$ forms the dual basis to $b^1, \ldots, b^n$ for $\Theta_{X/S}(X)$. We can write an arbitrary tensor

$$g \in H^0(X, \Omega_{X/S} \otimes_{\mathcal{O}_X} \Omega_{X/S})$$

as the sum

$$\sum_{i,j=1}^{n} g_{ij} b^i \otimes b^j$$

where $g_{ij} \in \mathcal{O}_X(X)$, then it can be seen that g is symmetric if, and only if, the matrix $(g_{ij})_{i,j=1}^n$ is symmetric. Let $l_1 = \sum_{i=1}^n x^i b_i$ and $l_2 = \sum_{i=1}^n y^i b_i$ be elements of $\Theta_{X/S}(X)$ where x^i and y^i are elements of $\mathcal{O}_X(X)$. The scalar product is given by

$$g(l_1, l_2) = \sum_{i,j=1}^{n} g_{ij} x^i y^j.$$

The map $\delta_g(b_i)$ sends b_j to g_{ij}. By identifying $\Omega^{**}_{X/S}(X)$ with $\Omega_{X/S}(X)$ by the canonical $\mathcal{O}_X(X)$-isomorphism, then $\delta_g : \Theta_{X/S}(X) \to \Omega_{X/S}(X)$ is given by $\delta_g(b_i) = \sum_{j=1}^n g_{ij} b^j$ for $1 \leq i \leq n$. Therefore, we see that g is non-degenerate if, and only if, $(g_{ij})_{i,j=1}^n$ is an element of $\mathcal{O}_X(X)^\times$. $\square$

Theorem 1. *(Local Existence of Psuedo-Riemannian Schemes) If X is a smooth scheme over S and $U \subset X$ is an open for which $\Omega_{X/S}|_U$ is a free $\mathcal{O}_X|_U$-module of* *rank n, then there exists an algebraic metric g on $\Omega_{X/S}|_U$ making the pair (U, g) a psuedo-Riemannian scheme.*

Proof. Since $\Gamma(U, \Omega_{X/S}|_U)$ is a free $\mathcal{O}_X(U)$-module of finite rank, then the above proposition ensures the existence of the desired algebraic metric. Surely, for any choice of $a_1, \ldots, a_n \in \mathcal{O}_X(U)^\times$, the diagonal matrix A given by

$$A = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

is symmetric and the determinant of A is $\det(A) = a_1 \cdots a_n$, which is in the multiplicative subgroup of units $\mathcal{O}_X(U)^\times$. If $b^1, \ldots, b^n$ for a basis for $\Theta_{X/S}|_U$, then there exists the following algebraic metric given by $g = \sum_{i=1}^n a_i b^i \otimes b^i$ by the correspondence above. $\square$

Consider an S-scheme X where the cotangent sheaf $\Omega_{X/S}$ is locally free of rank n. Given a metric g on X and a point $\mathfrak{p} \in X$, there exists a basis $b_1, \ldots, b_n$ of $\Omega_{X/S,\mathfrak{p}}$ as a free $\mathcal{O}_{X,p}$-module, and we write its dual basis by $b^1, \ldots, b^n$. In this case, we call $(g_{ij})_{i,j=1}^n$ the **fundamental matrix at $\mathfrak{p}$** of the metric with respect to the basis $b^1, \ldots, b^n$. The canonical image $g_\mathfrak{p}$ of g at the germ $\mathfrak{p}$ induces an

$\mathcal{O}_{X,\mathfrak{p}}$-bilinear form $\Theta_{X/S,\mathfrak{p}} \times \Theta_{X/S,\mathfrak{p}} \to \mathcal{O}_{X,\mathfrak{p}}$ given by

$$g_{\mathfrak{p}} : (\frac{l_1}{v_1}, \frac{l_2}{v_2}) \mapsto \frac{g(l_1, l_2)}{v_1 v_2}.$$

As well as, the $\mathcal{O}_{X,\mathfrak{p}}$-linear map $\delta_{g,\mathfrak{p}} : \Theta_{X/S,\mathfrak{p}} \to \Theta^*_{X/S,\mathfrak{p}}$ is the isomorphism via the stalk of δ_g at $\mathfrak{p}$. Thus, $g_{\mathfrak{p}}$ is a algebraic metric on $\mathcal{O}_{X,\mathfrak{p}}$-module $\Theta_{X/S,\mathfrak{p}}$, which we call the **localization of g at $\mathfrak{p}$**.

Given the localization $g_{\mathfrak{p}}$, it induces a $\kappa(\mathfrak{p})$-bilinear mapping

$$g(\mathfrak{p}) : \Theta_{X/S}(\mathfrak{p}) \times \Theta_{X/S}(\mathfrak{p}) \to \kappa(\mathfrak{p})$$

where $\Theta_{X/S}(\mathfrak{p})$ is the quotient $\kappa(\mathfrak{p})$-module $\Theta_{X/S,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\Theta_{X/S,\mathfrak{p}}$ for $\mathfrak{m}_{\mathfrak{p}}$ the maximal ideal of the local ring $\mathcal{O}_{X,\mathfrak{p}}$. Denoting $\bar{l}_i$ for the image of $l_i \in \Theta_{X/S,\mathfrak{p}}$ in $\Theta_{X/S}(\mathfrak{p})$, we have that mapping $g(\mathfrak{p})$ above is given by

$$g(\mathfrak{p})(\bar{l}_1, \bar{l}_2) = \overline{g_{\mathfrak{p}}(l_1, l_2)},$$

i.e. the image of $g_{\mathfrak{p}}(l_1, l_2)$ in the residue field $\kappa(\mathfrak{p})$, whose value is called the **scalar product of tangent vectors at $\mathfrak{p}$**. From this, it is easy to see that

$$g(\mathfrak{p})(l_1(\mathfrak{p}), l_2(\mathfrak{p})) = g(l_1, l_2)(\mathfrak{p})$$

for sections l_1, l_2 of $\Theta_{X/k}$ and $l_1(\mathfrak{p}), l_2(\mathfrak{p})$ are their respective images. Since $\Theta_{X/S,\mathfrak{p}}$ is a free $\mathcal{O}_{X,\mathfrak{p}}$-module of finite rank, it happens that $g(\mathfrak{p})$ is non-degenerate if, and only if, $g_{\mathfrak{p}}$ is so. Given a psuedo-Riemannian scheme (X, g), for each $\mathfrak{p} \in X$, we call the pair $(\mathcal{O}_{X,\mathfrak{p}}, g_{\mathfrak{p}})$ the **localization of (X, g) at $\mathfrak{p}$**. When X is a scheme over a field k, then for any psuedo-Riemannian structure (X, g) that may be imposed on X is called **psuedo-Riemannian scheme over** k.

Definition 3. *Given sections l_1 and l_2 of $\Theta_{X/S}$, we say that they are **orthogonal**, and write $l_1 \perp l_2$, if $g(l_1, l_2) = 0$.*

Proposition 4. *(Algebraic Metric Equivalence Principle) Suppose that $\mathcal{B}$ is a basis for an S-scheme X as a topological space. Given an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves, the following assertions are equivalent:*

1. *g defines a metric on $\Theta_{X/S}$;*

2. *$g(U)$ defines a metric on $\Theta_{X/S}(U)$ for each $U \in \mathcal{B}$;*

3. *$g_{\mathfrak{p}}$ defines a metric on $\Theta_{X/S,\mathfrak{p}}$ for each $\mathfrak{p} \in X$;*

4. *$g(\mathfrak{p})$ is non-degenerate for each $\mathfrak{p} \in X$.*

Proposition 5. *(Pasting For Algebraic Metrics) Suppose that $\mathcal{B} = \{U_i\}_{i \in I}$ is a basis for an S-scheme X. Consider an $\mathcal{O}_X$-bilinear morphism $g : \Theta_{X/S} \times \Theta_{X/S} \to \mathcal{O}_X$ of abelian sheaves. Let g_i be metrics on $\Theta_{X/S}|_{U_i}$ for $i \in I$ be give such that $res_{U_i, U_i \cap U_j}(g_i) = res_{U_j, U_i \cap U_j}(g_j)$ in*

$$P(U_i \cap U_i) \otimes_{\mathcal{O}_X(U_i \cap U_i)} \Omega_{X/S}(U_i \cap U_i)$$

for all $i, j \in I$. There exactly one metric g on $\Theta_{X/S}$ such that for all $i \in I$, $g_i = g(U_i)$.

Proposition 6. *(Existence of Local Orthogonal Bases) Suppose that* (X, g) *is a pseudo-Riemannian scheme over a field* k *of characteristic* $p \neq 2$*. Fix a* $\mathfrak{p} \in X$*. For any affine open* $U = Spec(S)$ *neighborhood of* $\mathfrak{p}$ *in* X *(i.e.* $\mathfrak{p} \in Spec(S)$*) where* $2 \in S^{\times}$ *such that* $\Omega_{X/k}(U)$ *is free of finite rank, there exists* $f \in S \setminus \mathfrak{p}$ *and local sections* $l_1, \ldots, l_n$ *of* $\Theta_{X/k}(D(f))$ *which form a basis of* $\Theta_{X/k}(D(f))$ *as an* $\mathcal{O}_X(D(f))$*-module such that*

$$g(D(f))(l_i, l_j) = \epsilon_i \delta_{ij}$$

where $i, j = 1, \ldots, n$ *and each* ϵ_i *is a unit of* $\mathcal{O}_X(D(f))$*.*

Proof. Since the characteristic of k is not equal to 2, we know that 2 is a unit of k. Furthermore, $\Omega_{X/k}$ being a finite locally free $\mathcal{O}_X$-module, its stalks are free modules of finite rank. Notice that the residue field $\kappa(\mathfrak{p})$ has characteristic not equal to 2. There exists a basis $(\zeta_1, \ldots, \zeta_n)$ of $\Theta_{X/k}(\mathfrak{p})$ satisfying the property that $g(\mathfrak{p})(\zeta_i, \zeta_j) = \alpha_i \delta_{ij}$ where $\alpha_i \in \kappa(\mathfrak{p})^{\times}$ and $1 \leq i, j \leq n$. Choose $l_i \in \Omega^{\times}_{X/k,\mathfrak{p}}$ with image ζ_i in $\Theta_{X/S}(\mathfrak{p})$ for each $1 \leq i \leq n$. Then $(l_1, \ldots, l_n)$ is a basis of $\Theta_{X/S,\mathfrak{p}}$ by Nakayama's Lemma. Thus, for each $1 \leq i, j \leq n$, we see that

$$g_{\mathfrak{p}}(l_i, l_j)(\mathfrak{p}) = g(\mathfrak{p})(\zeta_i, \zeta_j),$$

and as a result, $g_{\mathfrak{p}}(l_i, l_j)$ is a unit ϵ_i of $\mathcal{O}_{X,\mathfrak{p}}$ for $i = j$ and an element of $\mathfrak{m}_{\mathfrak{p}}\mathcal{O}_{X,\mathfrak{p}}$ otherwise. We may require that $g_{\mathfrak{p}}(l_i, l_j) = 0$ for $i \neq j$ by choosing suitable l_i. We proceed by an inductive argument in the sense that one can assume this condition has already been satisfied for $l_1, \ldots, l_m$ with $1 \le m < n$. Define an element $l^{'}_{m+1}$ as follows,

$$l^{'}_{m+1} := l_{m+1} - \sum_{k=1}^{m} g_{\mathfrak{p}}(l_{m+1}, l_k)\epsilon_k^{-1} l_k.$$

It follows that

$$g_{\mathfrak{p}}(l^{'}_{m+1}, l^{'}_{m+1}) = g_{\mathfrak{p}}(l_{m+1}, l_{m+1}) - 2\sum_{k=1}^{m} g_{\mathfrak{p}}(l_{m+1}, l_k)^2\epsilon_k^{-1} + \sum_{k+1}^{m} g_{\mathfrak{p}}(l_{m+1}, l_k)^2\epsilon_k^{-1}$$

is a unit of $\mathcal{O}_{X,\mathfrak{p}}$ and for $1 \le k \le m$,

$$g_{\mathfrak{p}}(l^{'}_{m+1}, l_k) = g_{\mathfrak{p}}(l_{m+1}, l_k) - g_{\mathfrak{p}}(l_{m+1}, l_k)\epsilon_k^{-1}\epsilon_k = 0.$$

By induction, we can find a basis $(l_1, \ldots, l_n)$ of $\Theta_{X/S,\mathfrak{p}}$ with the desired property that

$$g_{\mathfrak{p}}(l_i, l_j) = \epsilon_i \delta_{ij}$$

where $i, j = 1, \ldots, n$ and each ϵ_i is a unit of $\mathcal{O}_{X,\mathfrak{p}}$ for each $1 \le i \le n$. Therefore, for any affine open $\mathrm{Spec}(R)$ of $\mathfrak{p}$ in X, choosing a suitable $f \in R \setminus \mathfrak{p}$, there exists local sections $l_1, \ldots, l_n$ of $\Theta_{X/k}(D(f))$ which form a basis of $\Theta_{X/k}(D(f))$ as an $\mathcal{O}_X(D(f))$-module such that

$$g(D(f))(l_i, l_j) = \epsilon_i \delta_{ij}$$

where $i, j = 1, \ldots, n$ and each ϵ_i is a unit of $\mathcal{O}_X(D(f))$. $\square$

Definition 4. *For a psuedo-Riemannian scheme (X, g) over a scheme S, we see that for a section f of $\mathcal{O}_X$, the differential section df of $\Omega_{X/S}$ is given by $df(V) =$*

$V(f)$ for V a section of $\Theta_{X/S}$. The algebraic vector field $\delta_g^{-1}(df) =: Grad(f)$ is called the ***gradient of the section*** *f.*

Recall that when consider as sections, $\delta_g \circ \mathrm{Grad}, d : \mathcal{O}_X \to \Omega_{X/k}$ are equivalent, so d being a k-derivation of $\mathcal{O}_X$ ensures that Grad is also such. That is, for sections f, g of $\mathcal{O}_X$,

$$\mathrm{Grad}(fg) = f\mathrm{Grad}(g) + \mathrm{Grad}(f)g.$$

Further, this gives the following formula for V as a section of $\Theta_{X/k}$,

$$g(\mathrm{Grad}(f), V) = V(f).$$

Consider a psuedo-Riemannian scheme (X, g) over S. For any local sections $(\sigma^1, \dots, \sigma^n)$ generating a (local) basis for $\Omega_{X/S}$ and $(V_1, \dots, V_n)$ its (local) dual basis for $\Theta_{X/S}$, we can write g as follows:

$$g = \sum_{i,j=1}^{n} g_{ij} \sigma^i \otimes \sigma^j$$

where g_{ij} are sections of $\mathcal{O}_X$. Then $(g_{ij})_{i,j=1}^n$ is called the **local fundamental matrix of** (X, g) **relative to** $(\sigma^1, \dots, \sigma^n)$. Its inverse is denoted by $(g^{ij})_{i,j=1}^n$, and we have that

$$\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$$

for $1 \leq i, k \leq n$. The map δ_g is given by $\delta_g(V_i) = \sum_{j=1}^n g_{ij}\sigma^j$ for $1 \leq i \leq n$. Writing $\mathrm{Grad}(f) = \sum_{i=1}^n \phi^i V_i$ where each ϕ^i is a section of $\mathcal{O}_X$, then

$$V_j(f) = g(\mathrm{Grad}(f), V_j) = \sum_{i=1}^{n} \phi^i g_{ij},$$

hence, we see that $\phi^i = \sum_{j=1}^n g^{ij} V_j(f)$ and the following result is obtained,

$$\mathrm{Grad}(f) = \sum_{i=1}^{n} \big(\sum_{j=1}^{n} g^{ij} V_j(f) \big) V_i.$$

Consider a psuedo-Riemannian schemes (X, g) and (Y, g) over a field k. If there exists a k-isomorphism of schemes $\phi : X \to Y$, then there is an isomorphism of $\mathcal{O}_X$-modules $\phi^* : \Theta_{X/k} \to \Omega^*_{Y/k}$ where $\Omega_{Y/k}$ is considered as an $\mathcal{O}_X$-module via the map ϕ and $\phi^* : V \mapsto \phi V \phi^{-1}$ for a section V of $\Theta_{X/k}$. Denote V^ϕ to mean $\phi V \phi^{-1}$. We know that both $\Omega_{X/k}$ and $\Omega_{Y/k}$ are finite locally free as modules over their respective schemes. If $x^1, \ldots, x^n$ is a collection of local sections generating $\Theta_{X/k}$ as a free $\mathcal{O}_X$-module of finite rank which are dual to a collection of local sections $x_1, \ldots, x_n$ generating $\Omega_{X/k}$ as a free $\mathcal{O}_X$-module of finite rank, then the section V_i^ϕ is equal to y^i for $1 \le i \le n$ where the collection of sections $y_i = \phi(x_i)$ form a basis and the collection of y^i is its dual. When ϕ is a k-isomorphism of schemes over k as above, we say that ϕ is an **isometry over** k if the following diagram commutes:

$$\begin{tikzcd}
\Theta_{X/k} \otimes_{\mathcal{O}_X} \Theta_{X/k} \arrow[r, "g"] \arrow[d, "\phi^* \otimes_{\mathcal{O}_X} \phi^*"'] & \mathcal{O}_X \arrow[d, "\phi"] \\
\Theta_{Y/k} \otimes_{\mathcal{O}_X} \Theta_{Y/k} \arrow[r, "h"] & \mathcal{O}_Y
\end{tikzcd}$$

It follows that ϕ is an isometry if, and only if, for each local fundamental matrix G of (X, g) with respect to a collection local sections $V_1, \ldots, V_n$ forming a (local) basis of $\Theta_{X/k}$ as an $\mathcal{O}_X$-module, we have that $\phi(G)$ is the fundamental matrix of (Y, g) with respect to the sections $V_1^\phi, \ldots, V_n^\phi$ when they form a basis for $\Theta_{Y/k}$ as an $\mathcal{O}_Y$-module. When $(X, g) = (Y, h)$, the isometries $\phi : X \to X$ over k of X form the **group of isometric automorphisms of** (X, g) **over** k.

3 Connections

Let (X, g) be psuedo-Riemannian scheme over a scheme S where $\Omega)X/S$ is locally free of rank n. This section will study the properties of what are called connections on X. We wish to create constructions so that certain finiteness conditions are met for the desired analogs of psuedo-Riemannian geometry.

Definition 5. *Let X be an S-scheme. A **connection on** $\Theta_{X/S}$ is an $\mathcal{O}_S$-bilinear morphism of abelian sheaves*

$$\nabla : \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S},\ (V, W) \in \Theta_{X/S}(U) \times \Theta_{X/S}(U) \mapsto \nabla_V^U(W)$$

satisfying the properties on local sections f of $\mathcal{O}_X$ on some open set $U \subset X$,

1. $\nabla_{fV}^U(W) = f\nabla_V^U(W)$,
2. $\nabla_V^U(fW) = f\nabla_V^U(W) + V(f)W$.

*We say that $\nabla_V^U(W)$ is the **covariant derivative of** W **with respect to** V.*

Notation 2. *When the open subset $U \subset X$ is clear from context, we will write ∇^U as just ∇, i.e. $\nabla_V^U(W) \equiv \nabla_V(W)$.*

Definition 6. *Given an X be an S-scheme and connection ∇ on $\Theta_{X/S}$, it is* ***integrable*** *when the following identity holds on all sections V_1, V_2, W of $\Theta_{X/S}$,*

$$\nabla_{V_1}(\nabla_{V_2}(W)) - \nabla_{V_1}(\nabla_{V_1}(W)) = \nabla_{[V_1,V_2]}(W).$$

Remark 2. *If ∇_1 and ∇_2 are two connections on $\Theta_{X/S}$, then $\nabla_1 - \nabla_2$ is an $\mathcal{O}_X$-bilinear map:*

$$\begin{aligned}(\nabla_1 - \nabla_2)_{fV}(W) &= \nabla_{1,fV}(W) - \nabla_{2,fV}(W)\\ &= f\nabla_{1,V}(W) - f\nabla_{2,V}(W)\\ &= f(\nabla_1 - \nabla_2)_V(W);\end{aligned} \tag{1}$$

$$\begin{aligned}(\nabla_1 - \nabla_2)_V(fW) &= f\nabla_{1,V}(W) + V(f)W - f\nabla_{2,V}(W) - V(f)W\\ &= f\nabla_{1,V}(W) - f\nabla_{2,V}(W)\\ &= f(\nabla_1 - \nabla_2)_V(W).\end{aligned} \tag{2}$$

On the other hand, given a connection ∇ on $\Theta_{X/S}$ and any $\mathcal{O}_X$-bilinear map $\beta : \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S}$, the sum $\nabla \times \beta$ is an connection on $\Theta_{X/S}$. This correspondence allows one to say that the class of connections on $\Theta_{X/S}$ is either empty or can be identified with $\boldsymbol{Mod}_{\mathcal{O}_X}(\Theta_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S}, \Theta_{X/S})$.

Definition 7. *Consider local sections $b_1, \ldots, b_n \in H^0(U, \Theta_{X/S})$ generating $\Theta_{X/S}$ as a finite free $\mathcal{O}_X(U)$-module. For each local section V of $H^0(U, \Theta_{X/S})$ and each*

$1 \leq i \leq n$, *we have that*

$$\nabla_V(b_i) = \sum_{k=1}^{n} \omega_i^k(V) b_k$$

where $\omega_i^k(V) \in \Gamma(X, \mathcal{O}_X)$. *The maps* ω_i^k *are local sections of* $\Omega_{X/S}(U)$, *and we call them the* ***local connection forms of*** ∇. *Further, the matrix* $(\omega_i^k)_{i,k=1}^n$ *is called the* ***local connection matrix of*** ∇ *with respect to the local sections* $b_1, \dots, b_n$.

Remark 3. *Consider local sections* $b_1, \dots, b_n \in H^0(U, \Theta_{X/S})$ *generating* $\Theta_{X/S}$ *as a finite free* $\mathcal{O}_X(U)$*-module. Let* $m = \sum_{i=1}^n f^i b_i$ *be a local section of* $\Theta_{X/S}(U)$ *with each* f^i *sections of* $\mathcal{O}_X(U)$, *and we have that*

$$\nabla_V(m) = \sum_{i,k=1}^{n} f^i \omega_i^k(V) b_k + \sum_{k=1}^{n} V(f^k) b_k.$$

Consequently, we see that ∇ *is uniquely determined by its local connection matrix* $(\omega_i^k)_{i,k=1}^n$. *Yet, on the other hand, for an arbitrary* $n \times n$ *matrix* $(\omega_i^k)_{i,k=1}^n$ *with entries sections of* $\Theta_{X/S}(U)$, *the equation on* $\nabla_V(m)$ *defines an connection on* $\Theta_{X/S}|_U$ *with a global connection matrix* $(\omega_i^k)_{i,k=1}^n$ *in the sense of entries being global sections of* $\Omega_{X/S}|_U$.

Example 5. *For* R *a commutative unital ring, the polynomial ring* $S = R[x_1, \dots, x_n]$ *has the property that* $\Omega_{Y/X}$ *is a free* $\mathcal{O}_X$*-module of rank* n *trivialized by* $dx_1, \dots, dx_n$ *where* $Y = \mathbb{A}_X^n$ *and* $X = Spec(R)$. *Write* $\delta x_1, \dots, \delta x_n$ *for the dual basis of* $\Theta_{Y/X}$ *and recall that* $\Omega_{Y/X}(X) = \oplus_{i=1}^n S dx_i$. *We can identify connections on* $\Omega_{Y/X}$ *with* $n \times n$ *matrices* $(\omega_i^k)_{i,k=1}^n$ *with coefficients in* $\oplus_{i=1}^n S dx_i$ *via the formula above acting* *on local sections V and $m = \sum_{i=1}^{n} f^i \delta x_i$ by*

$$\nabla_V(m) = \sum_{i,k=1}^{n} f^i \omega_i^k(V)\delta x_k + \sum_{k=1}^{n} V(f^k)\delta x_k.$$

Example 6. *Suppose that $S \subset T$ are fields where $\{x_i\}_{i=1}^{l}$ is a collection of elements. Assume that either char$(S) = 0$ and $\{x_i\}_{i=1}^{l}$ is a transcendence basis of T over S, or that char$(S) = p > 0$ and $\{x_i\}_{i=1}^{l}$ is a p-basis for T over S. Then the global sections of $\Omega_{T/S}$ form a vector space over T of dimension l with basis $\{dx_i\}_{i=1}^{l}$. As a result, the contangent sheaf a free $\mathcal{O}_T$-sheaf of rank l. Write $x^1, \ldots, x^l$ as the dual basis of $\{dx_i\}_{i=1}^{l}$. From remarks above, we can identify connections on $\Omega_{T/S}$ with $l \times l$ matrices $(\omega_i^k)_{i,k=1}^{l}$ with coefficients in $\oplus_{i=1}^{l} T dx_i$ via the formula above acting on local sections V and $m = \sum_{i=1}^{l} f^i x^i$ by*

$$\nabla_V(m) = \sum_{i,k=1}^{l} f^i \omega_i^k(V) x^k + \sum_{k=1}^{l} V(f^k) x^k.$$

Proposition 7. *(Correspondence for connections) Given the case where $\Omega_{X/S}$ is a free $\mathcal{O}_X$-module of finite rank, then there exists a one-to-one correspondence with $n \times n$ matrices with entries in $\Gamma(X, \Omega_{X/S})$ and connections on $\Theta_{X/S}$.*

Remark 4. *Suppose that on some open subset $U \subset X$, $\Theta_{X/S}(U)$ has a local basis $V_1, \ldots, V_n$, and let $\sigma^1, \ldots, \sigma^n$ be the dual local basis for $\Omega_{X/S}(U)$, so $\sigma^i(V_k) = \delta_k^i$ for $1 \leq i, k \leq n$. The connection forms here may be written as*

$$\omega_i^k = \sum_{j=1}^{n} \Gamma_{ij}^k \sigma^i$$

where each Γ_{ij}^k is a local section of $\mathcal{O}_X(U)$ where $1 \leq i, k \leq n$ and $1 \leq j \leq n$.

Definition 8. *Each Γ_{ij}^k are called the* ***local algebraic Christoffel symbols*** *of ∇ with respect to $b_1, \ldots, b_n$ (above) and $V_1, \ldots, V_n$, and they determine ∇ according to the expression*

$$\nabla(\sum_j g^j V_j, \sum_i f^i b_i) = \sum_k \big(\sum_{i,j} f^i g^j \Gamma_{ij}^k + \sum_i g^i V_i(f^k)\big) b^k.$$

Definition 9. *For a connection ∇ on $\Theta_{X/S}$, we say that a section W of $\Theta_{X/S}$ is* ***horizontal*** *if for all other local sections V of $\Theta_{X/S}$, $\nabla_V(W) = 0$.*

For an open set $U \subset X$, we write $\Theta_{X/S}^{\nabla}(U) \subset \Theta_{X/S}(U)$ as the $\mathcal{O}_X$-submodule of horizontal local sections. In the case where we restrict to an open set U where $\Omega_{X/S}(U)$ is a free $\mathcal{O}_X(U)$-module, let $b_1, \ldots, b_n$ be local sections for a basis of $\Theta_{X/S}(U)$ and $(\omega_i^k)_{i,k=1}^n$ be the local connection matrix of ∇ with respect to this basis. We see that for a local section $W \in \Gamma(U, \Theta_{X/S})$ written in the form $W = \sum_{i=1}^n f^i b_i$, it is horizontal if, and only if, the $f^1, \ldots, f^n$ are local sections of $\mathcal{O}_X$ that satisfy the following expression

$$\begin{pmatrix} V(f^1) \\ \vdots \\ V(f^n) \end{pmatrix} = -(w_i^k(V)) \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix}$$

for every $V \in \Gamma(U, \Theta_{X/S})$. Denote the associated local dual basis for $\Omega_{X/S}$ by $\sigma^1, \ldots, \sigma^n$. Write the local algebraic Christoffel symbols with respect to these local bases by Γ_{ij}^k with $1 \leq i, k \leq n$ and $1 \leq j \leq n$. We see that W is a local horizontal section of ∇ if, and only if, the following expression is satisfied

$$\begin{pmatrix} b_j(f^1) \\ \vdots \\ b_j(f^n) \end{pmatrix} = -(\Gamma_{ij}^k) \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix}$$

for each $1 \leq j \leq n$. We say that the connection ∇ has **sufficiently many solutions** if $\Theta_{X/S}^{\nabla}$ is an $\mathcal{O}_X$-subsheaf of $\Theta_{X/S}$ that generates $\Theta_{X/S}$ where $\Theta_{X/S}^{\nabla}$ is the sheaf associated to the presheaf $U \mapsto \Theta_{X/S}^{\nabla}(U)$. For local sections V, W of $\Theta_{X/S}(U)$ and unit ϵ of $\mathcal{O}_X^{\times}(U)$, we have the following **quotient rule**

$$\nabla_V(\epsilon^{-1}M) = \epsilon^{-1}\nabla_V(W) + V(\epsilon^{-1})W = \epsilon^{-2}(\epsilon\nabla_V(W) - V(\epsilon)W).$$

Fix some $\mathfrak{p} \in X$. We call the stalk of the connection ∇ at this point the **localization of ∇ at $\mathfrak{p}$**. Since $\Omega_{X/S}$ is finite locally free, we see that $\Omega_{X/S,\mathfrak{p}}$ is a finite free $\mathcal{O}_{X,\mathfrak{p}}$-module. The localization $\nabla_{\mathfrak{p}}$ induces a $\mathcal{O}_{X,\mathfrak{p}}$-bilinear map

$$\nabla(\mathfrak{p}) : \Theta_{X/S,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\Theta_{X/S,\mathfrak{p}} \times \Theta_{X/S,\mathfrak{p}} \to \Theta_{X/S}(\mathfrak{p})$$

where $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal of the local ring $\mathcal{O}_{X,\mathfrak{p}}$ and $\Theta_{X/S}(\mathfrak{p}) := \Theta_{X/S,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\Theta_{X/S,\mathfrak{p}}$. From this, we see that there exists a canonical isomorphism of $\mathcal{O}_{X/S,\mathfrak{p}}$-modules,

1. $\Theta_{X/S,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\Theta_{X/S,\mathfrak{p}}$

2. $\mathbf{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\Omega_{X/S,\mathfrak{p}}, \mathcal{O}_{X,\mathfrak{p}})/\mathfrak{m}_{\mathfrak{p}}\mathbf{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\Omega_{X/S,\mathfrak{p}}, \mathcal{O}_{X,\mathfrak{p}})$

3. $T_{X/S}(\mathfrak{p})$

where $T_{X/S}(\mathfrak{p}) := \Omega_{X/S,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\Omega_{X/S,\mathfrak{p}}$ (whose elements are called tangent vectors at $\mathfrak{p}$). We may regard $\nabla(\mathfrak{p})$ as an $\mathcal{O}_{X/S,\mathfrak{p}}$-bilinear map

$$\nabla(\mathfrak{p}) : T_{X/S}(\mathfrak{p}) \times \Theta_{X/S,\mathfrak{p}} \to \Theta_{X/S}(\mathfrak{p}).$$

Since this map is induced by the localization $\nabla_{\mathfrak{p}}$, it satisfies the following rule,

$$\nabla(\mathfrak{p})(V, fZ) = f(\mathfrak{p})\nabla(\mathfrak{p})(V, Z) + V(f)Z(\mathfrak{p})$$

for each tangent vector $V \in T_{X/Y}(\mathfrak{p})$, "germs at $\mathfrak{p}$" of elements $f \in \mathcal{O}_{X,\mathfrak{p}}$, and "germs at $\mathfrak{p}$" of $Z \in \Theta_{X/S,\mathfrak{p}}$. By "germs at $\mathfrak{p}$", we mean the images of the respective elements obtained from the Leibniz rule on $\nabla(\mathfrak{p})$ induced by $\nabla_{\mathfrak{p}}$. We write $\nabla(\mathfrak{p})(V, Z)$ as $\nabla(\mathfrak{p})_V(W)$ and called this the directional derivative of Z with respect to V (at $\mathfrak{p}$). Furthermore, it can be seen that

$$\nabla(\mathfrak{p})_{V(\mathfrak{p})}(W) = (\nabla_V(W))(\mathfrak{p})$$

for V, W sections of $\Theta_{X/S}$ about $\mathfrak{p}$. One could consider the connection ∇ as a measurement in the rate of change of W at $\mathfrak{p}$ in the direction of $V(\mathfrak{p})$.

Consider the map of abelian scheaves

$$\phi : \Theta_{X/S} \times \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S}$$

defined by $\phi(V, W, Z) = [\nabla_V, \nabla_W]Z - \nabla_{[V,W]}(W)$ for sections V, W, Z of $\Theta_{X/S}$ where

$$[\nabla_V, \nabla_W]Z = \nabla_V\nabla_W(Z) - \nabla_W\nabla_V(Z).$$

Clearly, ϕ is an $\mathcal{O}_X$-trilinear map on sections. Indeed, let f be a section of $\mathcal{O}_X$ and V, W, Z be sections of $\Theta_{X/S}$. We see linearity in V by the following computation,

$$\begin{aligned}
[\nabla_{fV}, \nabla_W](Z) - \nabla_{[fV,W]}(Z) &= \nabla_{fV}(\nabla_W(Z)) - \nabla_W(\nabla_{fV}(Z)) - \nabla_{(fV)\circ W - W\circ(fV)}(Z) \\
&= f\nabla_V(\nabla_W(Z)) - f\nabla_W(Z) - W(f)\nabla_V(Z) \\
&- \nabla_{f[V,W]}(Z) + W(f)\nabla_V(Z) \\
&= f\big([\nabla_V, \nabla_W](Z) - \nabla_{[V,W]}(Z)\big).
\end{aligned}$$

Since ϕ is alternating in V and W, the linearity in the second component follows as well. Lastly, we show the linearity of Z as follows:

$$\begin{aligned}
[\nabla_V, \nabla_W](Z) - \nabla_{[fV,W]}(Z) &= \nabla_V(f\nabla_W(Z) + W(f)Z) - \nabla_W(f\nabla_V(Z) + V(f)Z) \\
&- f\nabla_{[V,W]}(Z) - [V,W](fZ) \\
&= f\nabla_V(\nabla_W(Z)) + V(f)\nabla_W(Z) + W(f)\nabla_V(Z) \\
&+ V(W(f))Z - f\nabla_W(\nabla_V(Z)) - W(f)\nabla_V(Z) \\
&- V(f)\nabla_W(Z) \\
&- W(V(f))Z - f\nabla_{[V,W]}(Z) - [V,W](fZ) \\
&= f([\nabla_V, \nabla_W]Z - \nabla_{[V,W]}(Z)).
\end{aligned}$$

From this, we see that ϕ defines a map of abelian sheaves

$$\mathcal{R}(\nabla) : \Theta_{X/S} \to \mathbf{Mod}_{\mathcal{O}_X}(\bigwedge^2 \Theta_{X/S}, \Theta_{X/S})$$

given by

$$\mathcal{R}(\nabla)(Z)(V \wedge W) = [\nabla_V, \nabla_W]Z - \nabla_{[V,W]}(Z)$$

where V, W, Z are sections of $\Theta_{X/S}$. This map ϕ also gives a map of abelian sheaves

$$\mathcal{B}(\nabla) : \Theta_{X/S} \times \Theta_{X/S} \to \mathbf{Mod}_{\mathcal{O}_X}(\Theta_{X/S}, \Theta_{X/S})$$

which is given by $(V, W) \mapsto \phi(V, W, -)$ for sections V, W of $\Theta_{X/S}$.

Definition 10. *We that that $\mathcal{R}(\nabla)$ and $\mathcal{B}(\nabla)$ are respectively the* ***curvature*** *and* ***curvature form*** *of the connection ∇. The connection ∇ is* ***integrable*** *when $\mathcal{R}(\nabla) = 0$.*

There exists another map of abelian sheaves

$$\psi : \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S}$$

given by

$$(V, W) \mapsto \nabla_V(W) - \nabla_W(V) - [V, W]$$

where V, W are sections of $\Theta_{X/S}$. In fact, it is an alternating map and is $\mathcal{O}_X$-bilinear as for each section f of $\mathcal{O}_X$,

$$\begin{aligned}\nabla_{fV}(W) - \nabla_W(fV) - [fV, W]&\\ &= f(\nabla_V(W) - \nabla_W(V)) - W(f)V - fV \circ W + fW \circ V + W(f)V\\ &= f\psi(V, W)\end{aligned}$$

This map ϕ induces an $\mathcal{O}_X$-linear map

$$\mathcal{T}(\nabla) : \bigwedge^2 \Theta_{X/S} \to \Theta_{X/S}$$

given by for sections V, W of $\Theta_{X/S}$,

$$\mathcal{T}(\nabla)(V \wedge W) = \nabla_V(W) - \nabla_W(V) - [V, W].$$

Definition 11. *We say that $\mathcal{T}(\nabla)$ is called the **torsion** of the connection ∇. If $\mathcal{T}(\nabla) = 0$, then we say that ∇ is **symmetric**. In this case when $\mathcal{R}(\nabla) = \mathcal{T}(\nabla) = 0$, we call the connection **flat**.*

Proposition 8. *For any $\mathfrak{p} \in X$, the curvature $\mathcal{R}(\nabla_{\mathfrak{p}})$ is the image of $\mathcal{R}(\nabla)$ of the canonical map from*

$$\boldsymbol{Mod}_{\mathcal{O}_X(X)}(\Theta_{X/S}(X), \boldsymbol{Mod}_{\mathcal{O}_X(X)}(\bigwedge^2 \Theta_{X/S}(X), \Theta_{X/S}(X)))$$

to

$$\boldsymbol{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\Theta_{X/S,\mathfrak{p}}, \boldsymbol{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\bigwedge^2 \Theta_{X/S,\mathfrak{p}}, \Theta_{X/S,\mathfrak{p}})),$$

and the torsion $\mathcal{T}(\nabla_{\mathfrak{p}})$ is the canonical image of $\mathcal{T}(\nabla)$ by

$$\boldsymbol{Mod}_{\mathcal{O}_X(X)}(\bigwedge^2 \Theta_{X/S}(X), \Theta_{X/S}(X)) \to \boldsymbol{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\bigwedge^2 \Theta_{X/S,\mathfrak{p}}, \Theta_{X/S,\mathfrak{p}}).$$

Proof. The image of $[V, W]$ in $\Theta_{X/S,\mathfrak{p}}$ is the commutator of the images of V and W in $\Theta_{X/S,\mathfrak{p}}$. From this, the diagram

$$\begin{tikzcd}
\Theta_{X/S}(X) \times \Theta_{X/S}(X) \arrow[r, "\nabla"] \arrow[d] & \Theta_{X/S}(X) \arrow[d] \\
\Theta_{X/S,\mathfrak{p}} \times \Theta_{X/S,\mathfrak{p}} \arrow[r, "\nabla_{\mathfrak{p}}"] & \Theta_{X/S,\mathfrak{p}}
\end{tikzcd}$$

commutes, so we see that

$$[(\nabla_{\mathfrak{p}})_V, (\nabla_{\mathfrak{p}})_W]Z = \nabla_{\mathfrak{p}}(V, \nabla_{\mathfrak{p}}(W, Z))$$

is the canonical image of $[\nabla_V, \nabla_W]Z$ in $\Theta_{X/S,\mathfrak{p}}$. A similar thing could be stated about

$$(\nabla_{\mathfrak{p}})_{[V,W]}Z = \nabla_{\mathfrak{p}}([V, W], Z).$$

From this, the claims follow. $\square$

4 Algebraic Levi-Cività

Within this section, we only consider psuedo-Riemannian schemes (X, g) for which the number 2 is always a section (both globally and locally) of the sheaf of units $\mathcal{O}_X^\times$. For example, such could be the field of rational numbers $\mathbb{Q}$, any field extension k of $\mathbb{Q}$ (i.e. characteristic of k is 0), any field K of characteristic p not equal to 2, finite commutative rings of characteristic p not equal to 2, any non-zero algebra of finite type over any of the aforementioned commutative rings, etc. Furthermore, it will be shown that any such psuedo-Riemannian schemes (X, g) above have a canonical symmetric compatible connection that serves as an algebraic analog of the Levi-Cività.

Theorem 2. *(Local Existences of Algebraic Levi-Cività connections) Consider a psuedo-Riemannian scheme (X, g) over a scheme S for which the number 2 is always a section of the sheaf of units $\mathcal{O}_X^\times$. There exists a unique symmetric connection*

$$\nabla : \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S}$$

satisfying the following compatibility condition for any sections V, W, Z *of* $\Theta_{X/S}$,

$$Z(g(V, W)) = g(V, \nabla_Z(W)) + g(W, \nabla_Z(V)).$$

Proof. First, we will assume the existence of such an connection ∇. We see that for sections V, W, Z of $\Theta_{X/S}$,

$$\begin{aligned}
g(Z, \nabla_V(W)) &= g(Z, \nabla_W(V)) + g(Z, [V, W]) \\
&= W(g(V, Z)) - g(V, \nabla_W(Z)) + g(Z, [V, W]) \\
&= W(g(V, Z)) - g(V, \nabla_W(Z)) + g(Z, [V, W]) \\
&= W(g(V, Z)) - Z(G(V, W)) + g(W, \nabla_Z(V)) \\
&- g(V, [W, Z]) + g(Z, [V, W]) \\
&= W(g(V, Z)) - Z(g(V, W)) + g(W, \nabla_V(Z)) \\
&+ g(W, [Z, V]) - g(V, [W, Z]) + g(Z, [V, W]) \\
&= W(g(V, Z)) - Z(g(V, W)) + V(g(W, Z)) - \\
&g(Z, \nabla_V(W)) + g(W, [Z, V]) - g(V, [W, Z]) + g(Z, [V, W]).
\end{aligned}$$

From this expression, we obtain the following formula,

$$\begin{aligned}
2g(Z, \nabla_V(W)) = W(g(V, Z)) - Z(g(V, W)) + V(g(W, Z)) \\
+ g(W, [Z, V]) - g(V, [W, Z]) + g(Z, [V, W]).
\end{aligned}$$

Since 2 is always a section of $\mathcal{O}_X^\times$ and g is non-degenerate, the uniqueness of ∇ follows.

Next, we must show the existence of such an connection ∇. Denote the right side of the expression above by $F(V, W, Z)$. We show $\mathcal{O}_X$-linearity in each of the components Z and V of F to derive the desired properties of the connection. Let f a section of $\mathcal{O}_X$. Then

$$\begin{aligned}
F(V, W, fZ) &= W(fg(V, Z)) - fZ(g(V, W)) + V(fg(W, Z)) \\
&+ g(W, (fZ) \circ V - V \circ (fZ)) \\
&- g(V, W \circ (fZ) - (fZ) \circ W) + fg(Z, [V, W]) \\
&= fF(V, W, Z) + W(f)g(V, Z) + V(f)g(W, Z) \\
&- g(W, V(f)Z) - g(V, W(f)Z) \\
&= fF(V, W, Z).
\end{aligned}$$

For fixed local sections V, W of $\Theta_{X/S}$, the function F defines an element of $\Omega_{X/S}$ (identified with its canonical isomorphism). Hence, there exists a unique section $\nabla_V(W)$ depending only on the sections V and W with the property that

$$F(V, W, Z) = 2g(\nabla_V(W), Z)$$

for every section of $\Theta_{X/S}$. This shows $\mathcal{O}_X$-linearity in Z of F. Next, we show

$\mathcal{O}_X$-linearity in V,

$$F(fV,W,Z) = W(f)g(V,Z) - Z(f)g(V,W) + fV(g(W,Z))$$

$$+ g(W, Z \circ (fV) - (fV) \circ Z) -$$

$$fg(V,[W,Z]) + g(Z,(fV) \circ W - W \circ (fV))$$

$$= fF(V,W,Z).$$

Hence, for every section Z of $\Theta_{X/S}$, we have that

$$g(\nabla_{fV}(W), Z) = fg(\nabla_V(W), Z) = g(f\nabla_V(W), Z),$$

and this ensures that $\nabla_{fV}(W)) = f\nabla_V(W)$ on sections. We make the following observation in the W component of F,

$$F(V,fW,Z) = fW(g(V,Z)) - Z(fg(V,W)) + V(fg(W,Z)) + fg(W,[Z,V])$$

$$- g(V,(fW) \circ Z - Z \circ (fW))$$

$$- g(Z, V \circ (fW) - (fW) \circ V)$$

$$= fF(V,W,Z) - Z(f)g(V,W) + V(F)g(W,Z) + F(f)g(W,V)$$

$$+ V(f)g(W,Z)$$

$$= fF(V,W,Z) + 2g(Z,V(f)W).$$

This implies that for all sections Z of $\Theta_{X/S}$,

$$g(\nabla_V(fW), Z) = g(f\nabla_V(W) + V(f)W, Z).$$

Hence,

$$\nabla_V(fW) = f\nabla_V(W) + V(f)W,$$

so ∇ is an connection on $\Theta_{X/S}$. Furthermore, we see that for sections Z of $\Theta_{X/S}$, symmetry of ∇ follows from

$$g(Z.\nabla_V(W)) - g(Z, \nabla_W(V)) = g(Z, [V, W]),$$

and compatibility follows from the fact that

$$F(Z, W, V) + F(Z, V, W) = 2Z(g(V, W))$$

implies that $g(V, \nabla_Z(W)) + g(W, \nabla_Z(V)) = Z(g(V, W))$. This completes the proof. $\square$

Definition 12. *The unique connection ∇ above is called the **algebraic Levi-Civitá connection** on X over S.*

Consider a psuedo-Riemannian scheme (X, g) with algebraic Levi-Civitá connection ∇. For a section f of $\mathcal{O}_X$, we write Δf to be the section $\mathrm{div}^{\nabla}(\mathrm{Grad} f)$ of $\mathcal{O}_X$ and call it the **Laplacian of** f, and the map $\Delta : \mathcal{O}_X \to \mathcal{O}_X$ the **Laplace operator**. We all define the **Hessian** of f as $\mathrm{Hess} f := \Delta(-, \mathrm{Grad} f)$. The $\mathcal{O}_X$-bilinear map

$$\mathrm{hess} f : \Theta_{X/S} \times \Theta_{X/S} \to \Theta_{X/S}$$

with $(\mathrm{hess} f)(V, W) = g(\nabla_V(\mathrm{Grad} f), W)$ for V, W sections of $\Theta_{X/S}$ is called the

Hesse form of f, and the tensor in $\Omega_{X/S} \otimes_{\mathcal{O}_X} \Omega_{X/S}$ corresponding to it is called the **Hesse tensor**.

Proposition 9. *For each section f, the Hesse tensor f is a symmetric tensor.*

Proof.

$$
\begin{aligned}
(\mathrm{hess} f)(V, W) &= g(\nabla_V(\mathrm{Grad} f), W) \\
&= g(W, \nabla_V(\mathrm{Grad} f) \\
&= -g(\mathrm{Grad} f, \nabla_V(W)) + V(g(\mathrm{Grad} f, W)) \\
&= -g(\mathrm{Grad} f, \nabla_V(W)) - g(\mathrm{Grad} f, [V, W]) + V(g(\mathrm{Grad} f, W)) \\
&= g(V, \nabla_W(\mathrm{Grad} f)) - W(g(\mathrm{Grad} f, W)) - g(\mathrm{Grad} f, [V, W]) \\
&+ V(g(\mathrm{Grad} f, W)) \\
&= (\mathrm{hess} f)(W, V) - W(V(f)) - [V, W](f) + V(W(f)) \\
&= (\mathrm{hess} f)(W, V)
\end{aligned}
$$

□

Now suppose that $\Omega_{X/S}$ has a local basis $\sigma^1, \ldots, \sigma^n$ and $V_1, \ldots, V_n$ is its local dual basis. Let $(\omega_i^j)_{i,j=1}^n$ be the **local connection matrix** of ∇ with respect to $\sigma^1, \ldots, \sigma^n$. That is,

$$\nabla_V(V_i) = \sum_{j=1}^{n} \omega_i^j(V) W_j$$

for $1 \leq i \leq n$ and local section V of $\Theta_{X/S}$. The entries ω_i^j are called the **local connection forms** of ∇ with respect to $\sigma^1, \ldots, \sigma^n$. If $(g_{ik})_{i,k=1}^n$ is the local fundamental matrix of (X, g) with respect to $\sigma^1, \ldots, \sigma^n$, then

$$g(\nabla_V(V_i), -) = \delta_g(\nabla_V(V_i)) = \sum_{j=1}^{n} \omega_i^j(V)\sigma_j = \sum_{j,k} \omega_i^j(V) g_{kj}\sigma^k = \sum_k \omega_{ki}\sigma^k$$

where $\omega_{ki} := \sum_j g_{kj}\omega_i^j$. The ω_{ki} are called **local connection forms of the first kind**, whereas given the inverse $(g^{ik})_{i,k=1}^n$ of the fundamental matrix, the forms $\omega_i^k = \sum_k g^{jk}\omega_{ki}$ are called **local connection forms of the second kind**. By identifying $\Omega_{X/S}$ with its double dual $\Omega_{X/S}^{**}$ of $\mathcal{O}_X$-modules by its canonical isomorphism, we can write these forms as

$$\omega_{ki} = \sum_{j=1}^{n} \Gamma_{kij}\sigma^j, \omega_i^j = \sum_{k=1}^{n} \Gamma_{ik}^j \sigma^k.$$

We respectively call Γ_{kij} and Γ_{ik}^j the local Christoffel symbols of the **first kind** and **second kind** of the psuedo-Riemannian scheme (X, g).

Proposition 10. *For each $1 \leq i, j, k \leq n$ with notation above, we have that one may write:*

1. $\omega_{ik} + \omega_{ki} = dg_{ik}$ *(where d is the universal derivation);*
2. $\Gamma_{kij} = \frac{1}{2}\big(V_i(g_{jk}) - V_k(g_{ij}) + V_j(g_{ik}) + g(V_i, [V_k, V_j]) - g(V_j, [V_i, V_k]) + g(V_k, [V_j, V_i])\big)$*;*
3. $d\sigma^i = \sum_{k=1}^n \sigma^k \wedge \omega_k^i$.

Suppose that $x^1, \ldots, x^n$ is a local differential basis of $\Omega_{X/S}$ (i.e. $dx^1, \ldots, dx^n$ generated $\Omega_{X/S}$ as a finite free $\mathcal{O}_X$-module on some open subset). We may take σ^i and V_i above respectively as dx^i and $\frac{\partial}{\partial x^i}$ for $1 \leq i \leq n$. We have that for

$1 \leq i, j \leq n$, $[V_i, V_j] = 0$ and the local Christoffel symbols of the first kind simplify to

$$\Gamma_{kij} = \frac{1}{2}(\frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j}),$$

and it may be deduced that $\Gamma_{kij} = \Gamma_{kji}$. If for $1 \leq i, k \leq n$ it happens that $g_{ik} = \delta_{ik}$, we say that $V_1, \ldots, V_n$ form a **local orthonormal basis** of $\Theta_{X/S}$. In such a situation $(\omega_{ik})_{i,k=1}^n$ is a skew-symmetric matrix (i.e $\omega_{ik} = -\omega_{ki}$ for all $1 \leq i, k \leq n$). Thus, we have the following equations

$$\Gamma_{kij} = \frac{1}{2}\big(g(V_i, [V_k, V_j]) - g(V_j, [V_i, V_k]) + g(V_k, [V_j, V_i])\big),$$

and also that for $1 \leq i, j, k \leq n$, one has $\Gamma_{kij} = -\Gamma_{jik}$.

Fix a psuedo-Riemannian schemes (X, g) with algebraic Levi-Cività connection ∇. Let $\mathcal{I}$ be a sheaf of ideals on $\mathcal{O}_X$. Write $\mathcal{O}_X/\mathcal{I}$ for its quotient sheaf of $\mathcal{O}_X$-modules. There exists a split exact sequence of $\mathcal{O}_X/\mathcal{I}$-modules,

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S} \to \Omega_{X/S}/\mathcal{O}_X d(\mathcal{I}) \to 0.$$

Writing $\Omega_{X/S}/\mathcal{O}_X d(\mathcal{I})$ as $\Theta_{\mathcal{I}/S}$ and dualizing with respect to $\mathcal{O}_X/\mathcal{I}$ yields the following split-exact sequence,

$$0 \to \Theta_{\mathcal{I}/S} \to \Theta_{X/S}/\mathcal{I}\Theta_{X/S} \to (\mathcal{I}/\mathcal{I})^2 \to 0.$$

The algebraic metric

$$\overline{g} : \Theta_{X/S}/\mathcal{I}\Theta_{X/S} \times \Theta_{X/S}/\mathcal{I}\Theta_{X/S} \to \mathcal{O}_X/\mathcal{I}$$

induced by g is non-degenerate. We call the triple $(X, \overline{g}, \mathcal{I})$ a **psuedo-Riemannian residual** of (X, g) (over S). For V a section of $\Theta_{X/S}$, denote $\overline{V}$ as the image of V in $\Theta_{X/S}/\mathcal{I}\Theta_{X/S}$.

Proposition 11. $\Theta_{\mathcal{I}/S} = \langle \{\overline{Gradf}\}_{f \in \mathcal{I}} \rangle^{\perp}$.

Proof. We know that $\Theta_{\mathcal{I}/S}$ is equivalent to $\mathbf{Mod}_{\mathcal{O}_X}(\Omega_{X/S}/\mathcal{I}\Omega_{X/S}, \mathcal{O}_X/\mathcal{I})$, and so sections of $\Theta_{\mathcal{I}/S}$ can be identified with those linear forms

$$\bar{l} : \Omega_{X/S}/\mathcal{I}\Omega_{X/S} \to \mathcal{O}_X/\mathcal{I}$$

which vanish on the image of $\mathcal{I}/\mathcal{I}^2$ in $\Omega_{X/S}/\mathcal{I}\Omega_{X/S}$, i.e. on the residue classes of the *df* for f sections of $\mathcal{I}$. Furthermore, for any section V of $\Theta_{X/S}$, we have that $\overline{V}$ is a section of $\Theta_{\mathcal{I}/S}$ if, and only if, for every section f of $\mathcal{I}$, one has $V(f)$ is also a section of $\mathcal{I}$. However, $V(f) = g(\mathrm{Grad} f, V)$, so this condition is equivalent to $\overline{g}(\overline{\mathrm{Grad} f}, \overline{V}) = 0$. □

Corollary 1. *The following are equivalent:*

1. $\overline{g} : \Theta_{X/S}/\mathcal{I}\Theta_{X/S} \times \Theta_{X/S}/\mathcal{I}\Theta_{X/S} \to \mathcal{O}_X/\mathcal{I}$ *is non-degenerate;*

2. *the restriction of* $\overline{g}$ *to* $\langle \{\overline{Gradf}\}_{f \in \mathcal{I}(U)} \rangle^{\perp}$ *on any open* U *of* $\Theta_{X/S}/\mathcal{I}\Theta_{X/S}$ *is non-degenerate;*

3. *for any open* U *of* $\Theta_{X/S}/\mathcal{I}\Theta_{X/S}$,

$$\Theta_{X/S}/\mathcal{I}\Theta_{X/S}(U) = \Theta_{\mathcal{I}/S}(U) \times \langle \{\overline{Gradf}\}_{f \in \mathcal{I}(U)} \rangle^{\perp}.$$

Fix a psuedo-Riemannian residual $(X, \overline{g}, \mathcal{I})$, so $\overline{g}$ is non-degenerate. Let $\overline{\nabla}$ be its algebraic Levi-Cività connection. Notice that we can write

$$\Theta_{X/S}/\mathcal{I}\Theta_{X/S} = \Theta_{\mathcal{I}/S} \times \Theta_{\mathcal{I}/S}^{\perp}.$$

Write $\overline{\Theta_{X/S}}$ for $\Theta_{X/S}/\mathcal{I}\Theta_{X/S}$. For each section $\overline{V}$ of $\overline{\Theta_{X/S}}$, we have a unique decomposition $V = V^{\top} + V^{\perp}$ where $V^{\top}$ (called the **tangential component**) is a section of $\Theta_{\mathcal{I}/S}$ and $V^{\perp}$ (called the **normal component**) is a section of $\Theta_{\mathcal{I}/S}^{\perp}$. The connection ∇ induces a well-defined map

$$\overline{\nabla} : \Theta_{\mathcal{I}/S} \times \overline{\Theta_{X/S}} \to \overline{\Theta_{X/S}}$$

as follows. For $\overline{V}$ a section of $\Theta_{\mathcal{I}/S}$ and $\overline{W}$ a section of $\overline{\Theta_{X/S}}$ (with representative sections V, W of $\Theta_{X/S}$), we have that $\overline{\nabla}_{\overline{V}}(\overline{W}) = \overline{\nabla_V(W))}$. Since ∇ is $\mathcal{O}_X$-linear in V, $\overline{\nabla_V(W))}$ does not depend on the choice of V. Furthermore, if W' were another representative section of $\overline{W}$ of $\Theta_{X/S}$, then $W - W' = \sum_{k=1}^{m} a_k W_k$ for some sections a_k and W_k respectively for $\mathcal{I}$ and $\Theta_{X/S}$. Then

$$\nabla_V(W) - \nabla_V(W') = \nabla_V(W - W') = \sum_{k=1}^{m} V(\alpha_k)W_k + \sum_{k=1}^{m} a_k \nabla_V(W_k).$$

Since $\overline{V}$ is a section of $\Theta_{\mathcal{I}/S}$ and the a_k are sections of $\mathcal{I}$, we have that $V(a_k)$ is also a section of $\mathcal{I}$, so the right side of the expression above is zero. Therefore, $\nabla_V(W) = \nabla_V(W')$. Clearly, $\overline{\nabla}$ forms an connection on $\overline{\Theta_{X/S}}$.

Theorem 3. *Relative to the notation above, if $\overline{V}, \overline{W}$ are sections of $\overline{\Theta_{X/S}}$ with* *representative sections V, W of $\Theta_{X/S}$, then*

$$\overline{\nabla}_{\overline{V}}(\overline{W}) = \overline{\nabla_V W}^{\perp}.$$

Definition 13. *The maps*

$$\Pi : \Theta_{\mathcal{I}/S} \times \Theta^{\perp}_{\mathcal{I}/S} \to \Theta_{\mathcal{I}/S}$$

given by $(\overline{V}, \overline{N}) \mapsto -(\overline{\nabla}_{\overline{V}}(\overline{N}))^{\perp}$ is called the ***second fundamental forms of*** *$(X, \overline{g}, \mathcal{I})$. Fixing some section $\overline{N}$, the map*

$$\mathcal{S}_{\overline{N}} : \Theta_{\mathcal{I}/S} \to \Theta_{\mathcal{I}/S}$$

given by $\overline{V} \mapsto \Pi(\overline{V}, \overline{N})$ is called the ***shape operator with respect to*** *$\overline{N}$.*

The maps $\mathcal{S}_{\overline{N}}$ and Π are respectively $\mathcal{O}_X/\mathcal{I}$-linear and $\mathcal{O}_X/\mathcal{I}$-bilinear. In particular, the shape operator with respect to a section $\overline{N}$ defines an $\mathcal{O}_X/\mathcal{I}$-bilinear map

$$l_{\overline{N}} : \Theta_{\mathcal{I}/S} \otimes_{\mathcal{O}_X/\mathcal{I}} \Theta_{\mathcal{I}/S} \to \mathcal{O}_X/\mathcal{I}$$

given by $(\overline{V}, \overline{W}) \mapsto \overline{g}(\mathcal{S}_{\overline{N}}(\overline{V}), \overline{W})$, which uniquely determines $\mathcal{S}_{\overline{N}}$ as $\overline{g}$ is non-degenerate.

Proposition 12. *If N is a representative of $\overline{N}$, then the following conditions hold:*

1. *$l_{\overline{N}}$ is symmetric and $l_{\overline{N}}(\overline{V}, \overline{W}) = \overline{g}(\overline{\nabla}_{\overline{V}}(\overline{W}), \overline{N})$*

2. *For V,W representative sections of $\Theta_{X/S}$ of sections $\overline{V}, \overline{W}$ of $\Theta_{\mathcal{I}/S}$ and ϕ a section of $\mathcal{I}$, for each $\overline{N} := \overline{Gradf}$,*

$$l_{\overline{N}}(\overline{V}, \overline{W}) = -\overline{(hessf)(V, W)}.$$

The symmetric tensor $\Pi_{\overline{N}}$ of $\Omega_{\mathcal{I}/S} \otimes_{\mathcal{O}_X/\mathcal{I}} \Omega_{\mathcal{I}/S}$ corresponding to $l_{\overline{N}}$ is called the **second fundamental tensor of** $(X, \overline{g}, \mathcal{I})$. Suppose that $\Omega_{X/S}$ has a basis $\sigma^1, \ldots, \sigma^n$ such that for the dual basis $V_1, \ldots, V_n$ of $\Theta_{X/S}$, the residues $\overline{V_1}, \ldots, \overline{V_m}$ form a basis of $\Theta_{\mathcal{I}/S}$ and $\overline{V_{m+1}}, \ldots, \overline{V_n}$ form a basis for $\Theta^{\perp}_{\mathcal{I},S}$. Writing $\overline{N} = \sum_{k=m+1}^{n} v^k \overline{V_k}$ for v^k a section of $\mathcal{O}_X/\mathcal{I}$. Let $(\omega_i^j)_{1 \leq i \leq m}^{1 \leq j \leq n}$ be the connection matrix of $\overline{\nabla}$ with respect to $\overline{V_1}, \ldots, \overline{V_m}$ and

$$\overline{g} = \sum_{k,l=1}^{m} \overline{g_{kl}} \overline{\sigma^k} \otimes \overline{\sigma^l},$$

for $\overline{g_{kl}}$ sections of $\mathcal{O}_X/\mathcal{I}$, be the fundamental tensor of this psuedo-Riemannian residual with $\overline{\sigma^l}$ the image of σ^l as a section of $\Omega_{\mathcal{I}/S}$.

Proposition 13. *With respect to the notation above,*

1. $\Pi_{\overline{N}} = \sum_{i,j=1}^{m} b_{ij}(\overline{N}) \overline{\sigma^i} \otimes \overline{\sigma^j}$ *with* $b_{ij}(\overline{N}) = -\sum_{s=m+1}^{n} v^s \omega_{js}(\overline{V_i})$;

2. *the matrix of $\mathcal{S}_{\overline{N}}$ with respect to the basis $\overline{V_1}, \ldots, \overline{V_m}$ is $(\overline{g}^{ij})(b_{ij}(\overline{N}))$.*

5 Curvature of Algebraic Levi-Cività connections

Fix a pseudo-Riemannian scheme (X, g) where 2 is always a section of $\mathcal{O}_X^{\times}$ and let ∇ be its algebraic Levi-Cività connection. Recall that the curvature $\mathcal{R}$ of ∇

is map of abelian sheaves

$$\mathcal{R} : \Theta_{X/S} \to \mathbf{Mod}_{\mathcal{O}_X}(\bigwedge^2 \Theta_{X/S}, \Theta_{X/S})$$

with the property that

$$\mathcal{R}(V)(W \wedge Z) = [\nabla_W, \nabla_Z]V - \nabla_{[W,Z]}(V)$$

for sections V, W, Z of $\Theta_{X/S}$.

Proposition 14. *Assume that (X, g) has zero torsion. For any sections V, W, Z of $\Theta_{X/S}$,*

$$\mathcal{R}(V)(W \wedge Z) + \mathcal{R}(W)(Z \wedge V) + \mathcal{R}(Z)(V \wedge W) = 0.$$

Proof.

$$
\begin{aligned}
\mathcal{R}(V)(W \wedge Z) + \mathcal{R}(W)(Z \wedge V) + \mathcal{R}(Z)(V \wedge W) \\
&= [\nabla_W, \nabla_Z]V - \nabla_{[W,Z]}(V) + [\nabla_Z, \nabla_V]W - \nabla_{[Z,V]}(W) \\
&+ [\nabla_V, \nabla_w]Z - \nabla_{[V,W]}(Z) \\
&= \nabla_W(\nabla_Z(V)) - \nabla_Z(\nabla_W(V)) - \nabla_{[W,Z]}(V) + \nabla_Z(\nabla_V(W)) \\
&- \nabla_V(\nabla_Z(W)) - \nabla_{[Z,Y]}(W) + \nabla_V({}_W(Z))) \\
&- \nabla_W(\nabla_V(Z)) - \nabla_{[V,W]}(Z) \\
&= \nabla_W(\nabla_Z(V)) - \nabla_Z(\nabla_V(W)) - \nabla_Z([W,V]) - \nabla_{[W,Z]}(V) \\
&+ \nabla_Z({}_V(W)) - \nabla_V(\nabla_W(Z)) \\
&- \nabla_V([Z,W]) - \nabla_{[Z,V]}(W) + \nabla_V(\nabla_W(Z)) \\
&- \nabla_W(\nabla_Z(V)) - \nabla_W([V,Z]) - \nabla_{[V,Z]}(Z) \\
&= \nabla_Z([V,W]) - \nabla_{[W,Z]}(V) + \nabla_V([W,Z]) - \nabla_{[Z,V]}(W) \\
&+ \nabla_W([Z,V]) - \nabla_{[V,W]}(Z) \\
&= [Z,[V,W]] + [V,[W,Z]] + [W,[Z,V]] \\
&= 0.
\end{aligned}
$$

$\square$

The map of abelian sheaves

$$\mathcal{K} : \Theta_{X/S} \times \Theta_{X/S} \times \Theta_{X/S} \times_{\Theta\ X/S} \to \mathcal{O}_X$$

given by

$$(U, V, W, Z) \mapsto g(U, \mathcal{R}(V)(W \wedge Z))$$

is $\mathcal{O}_X$-linear in each of its components. We may think of $\mathcal{K}$ as a 4-tensor of $\Omega_{X/S}$,

Definition 14. *We call $\mathcal{K}$ the **curvature tensor** of (X, g) with connection ∇.*

Proposition 15. *(Symmetric For Curvature Tensor) Recall that 2 is always a section of $\mathcal{O}_X^\times$. For sections U, V, W, Z of $\Theta_{X/S}$, we have the following expressions holding:*

1. $\mathcal{K}(U, V, W, Z) + \mathcal{K}(U, W, Z, V) + \mathcal{K}(U, Z, V, W) = 0$;
2. $\mathcal{K}(U, V, W, Z) = -\mathcal{K}(V, U, W, Z) = -\mathcal{K}(U, V, Z, W)$;
3. $\mathcal{K}(W, Z, U, V) = \mathcal{K}(U, V, W, Z)$.

When 3 is not a zerodivisor section of $\mathcal{O}_X$, for any 4-tensor $\mathcal{K}'$ above satisfying the properties above and if for any sections U, V of $\Theta_{X/S}$ it happens that

$$\mathcal{K}'(U, V, U, V) = \mathcal{K}(U, V, U, V),$$

then $\mathcal{K}' = \mathcal{K}$.

Proof. The first item follows from proposition above. By the definition of $\mathcal{K}$, we see that $\mathcal{K}(U,V,W,Z) = -\mathcal{K}(U,V,Z,W)$. Then

$$\begin{aligned}
\mathcal{K}(V,V,W,Z) &= g(V,\mathcal{R}(V)(W\wedge Z)) \\
&= g(V,\nabla_W(\nabla_Z(V))) - g(V,\nabla_Z(\nabla_W(V))) - Z(g(V,\nabla_W(V)))+ \\
& g(\nabla_W(V),\nabla_Z(V)) - \frac{1}{2}[W,Z](g(V,V)) \\
&= \frac{1}{2}\big((W\circ Z)g(V,V) - (Z\circ W)(g(V,V)) - [W,Z](g(V,V))\big) \\
&= 0,
\end{aligned}$$

so we see that $\mathcal{K}$ is alternating with respect to the sections U and V, and this proves the second item. The last item and assertion follow from the similarly. $\square$

Suppose that $\Omega_{X/S}$ has a local basis $\sigma^1,\ldots,\sigma^n$ and $V_1,\ldots,V_n$ is the local dual basis for $\Theta_{X/S}$. We may regard $\mathcal{R}$ as a local section of

$$\mathbf{Mod}_{O_X}(\Theta_{X/S},\Omega^2_{X/S}\otimes_{\mathcal{O}_X}\Theta_{X/S})$$

and write for $1\leq j\leq n$,

$$\mathcal{R}(V_j) = \sum_{k=1}^{n}\Omega_j^k\otimes V_k$$

where each Ω_j^k is a local section of $\Omega^2_{X/S}$. Then we have that

$$\mathcal{K}(V_i,V_j,V_r,V_s) = g(V_i,\mathcal{R}(V_j)(V_r\wedge V_s)) = \sum_{k=1}^{n} g_{ij}\Omega_j^k(V_r\wedge V_2).$$

Writing each Ω_{ij} as $\sum_{k=1}^n g_{ik}\Omega_j^k$ for $1\leq i,j\leq n$, we obtain that

$$\mathcal{K}(V_i,V_j,V_r,V_s) = \Omega_{ij}(V_r\wedge V_s)$$

for $1\leq i,j,r,s\leq n$. This proves the following statement.

Corollary 2. *The matrix $(\Omega_{ij})_{i,j=1}^n$ is skew-symmetric, i.e. for any $1 \leq i, j \leq n$ we have that $\Omega_{ij} = -\Omega_{ji}$.*

Proposition 16. *For any $1 \leq i, j \leq n$, let ω_{ij} and ω_j^i respectively be the local connection forms of the first and second kind for ∇ with respect to the basis $\sigma^1, \dots, \sigma^n$.*

1. $\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^n w_{li} \wedge \omega_j^l$;

2. $d\Omega_{ij} = \sum_{l=1}^n (\Omega_{il} \wedge \omega_j^l - \omega_i^l \wedge \Omega_{lj})$.

Proof. There exists the following expressions

1. $\Omega_j^k = d\omega_j^k + \sum_{l=1}^n \omega_l^k \wedge \omega_j^l$,

2. $dg_{ik} = \omega_{ik} + \omega_{ki}$.

On one hand,

$$
\begin{aligned}
\Omega_{ij} &= \sum_{k=1}^n g_{ik} \Omega_j^k \\
&= \sum_{k=1}^n g_{ik} d\omega_j^k + \sum_{k=1}^n g_{ik} \sum_{l=1}^n \omega_l^k \wedge \omega_j^l \\
&= \sum_{k=1}^n d(g_{ij}\omega_j^k) - \sum_{k=1}^n dg_{ik} \wedge \omega_j^k + \sum_{l=1}^n \omega_{ik} \wedge \omega_j^k \\
&= d\omega_{ij} - \sum_{k=1}^n \omega_{ik} \wedge \omega_j^k - \sum_{k=1}^n \omega_{ki} \wedge \omega_j^k + \sum_{k=1}^n \omega_{ik} \wedge \omega_j^k \\
&= d\omega_{ij} - \sum_{k=1}^n \omega_{ik} \wedge \omega_j^k - \sum_{k=1}^n \omega_{ki} \wedge \omega_j^k + \sum_{k=1}^n \omega_{ik} \wedge \omega_j^k \\
&= d\omega_{ij} \sum_{l=1}^n \omega_{li} \wedge \omega_j^l,
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
d\Omega_{ij} &= -\sum_{l=1}^{n} d\omega_{li} \wedge \omega_j^l + \sum_{l=1}^{n} \omega_{li} \wedge d\omega_j^l \\
&= -\sum_{l=1}^{n} \Omega_{li} \wedge \omega_j^l + \sum_{l,t=1}^{n} \omega_{ti} \wedge \omega_l^t \wedge \omega_j^l + \sum_{l=1}^{n} \omega_{li} \wedge \Omega_j^l - \sum_{l,t=1}^{n} \omega_{li} \wedge \omega_t^l \wedge \omega_j^t \\
&= \sum_{l=1}^{n} (\Omega_{il} \wedge \omega_j^l - \omega_i^l \wedge \Omega_{jl}).
\end{aligned}
$$

□

Proposition 17. *If $\mathfrak{p}$ is an element of X, then the curvature of $\mathcal{R}_\mathfrak{p}$ is the image of the curvature $\mathcal{R}$ by the canonical map from*

$$\boldsymbol{Mod}_{\mathcal{O}_X(X)}(\Theta_{X/S}(X), \boldsymbol{Mod}_{\mathcal{O}_X(X)}(\bigwedge^2 \Theta_{X/S}(X), \Theta_{X/S}(X)))$$

to

$$\boldsymbol{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\Theta_{X/S,\mathfrak{p}}, \boldsymbol{Mod}_{\mathcal{O}_{X,\mathfrak{p}}}(\bigwedge^2 \Theta_{X/S,\mathfrak{p}}, \Theta_{X/S,\mathfrak{p}})).$$

The curvature $\mathcal{K}_\mathfrak{p}$ is the canonical image of $\mathcal{K}$ induced from the map $\Omega_{X/S} \to \Omega_{X/S,\mathfrak{p}}$.

Proof. In order to prove the statement for $\mathcal{R}_\mathfrak{p}$, it is enough to show that for sections V, W, Z of $\Theta_{X/S}$ with images $V_\mathfrak{p}, W_\mathfrak{p}, Z_\mathfrak{p}$ in $\Theta_{X/S,\mathfrak{p}}$, the derivation $\mathcal{R}_\mathfrak{p}(V_\mathfrak{p})(W_\mathfrak{p} \wedge Z_\mathfrak{p})$ is the image of $\mathcal{R}(V)(W \wedge Z)$ in $\Theta_{X/S,\mathfrak{p}}$. But this follows from the fact that the diagram

$$
\begin{array}{ccc}
\Theta_{X/S}(X)_{\mathcal{O}_X(X)}\Theta_{X/S}(X) & \xrightarrow{\nabla} & \Theta_{X/S}(X) \\
\downarrow & & \downarrow \\
\Theta_{X/S,\mathfrak{p}\mathcal{O}_{X,\mathfrak{p}}}\Theta_{X/S,\mathfrak{p}} & \xrightarrow{\nabla_\mathfrak{p}} & \Theta_{X/S,\mathfrak{p}}
\end{array}
$$

commutes and that $[V,W]_{\mathfrak{p}}$ is the image of $[V,W]$ in $\Theta_{X,\mathfrak{p}}$. The statement for $\mathcal{K}_{\mathfrak{p}}$ then follows. $\square$

For sections V, W of $\Theta_{X/S}$, define

$$\mathrm{Ric}(V,W) = \mathrm{Trace}(U \mapsto \mathcal{R}(W)(U \wedge V))$$

where U is a section of $\Theta_{X/S}$. We see that Ric is $\mathcal{O}_X$-linear in both V and W, so it defines a tensor Ric of $\Omega_{X/S} \otimes_{\mathcal{O}_X} \Omega_{X/S}$, which will be called the **Ricci tensor** of (X, g).

Proposition 18. *The Ricci tensor is symmetic. If $(\sigma^1, \ldots, \sigma^n)$ is a basis for $\Omega_{X/S}$ and $(V_1, \ldots, V_n)$ is its dual basis, then*

$$Ric = \sum_{r,s=1}^{n} (\sum_{i,j=1}^{n} g^{ij} \mathcal{K}(V_i, V_r, V_j, V_s)) \sigma^r \otimes \sigma^s.$$

Proof. We write $\mathcal{R}(V_r)(V_j \wedge V_s) = \sum_k \alpha^k_{rjs} V_k$ with α^k_{rjs} a section of $\mathcal{O}_X$. It follows that

$$\mathcal{K}(V_i, V_r, V_j, V_s) = g(V_i, \mathcal{R}(V_r)(V_j \wedge V_s)) = \sum_k \alpha^k_{rjs} g_{ik},$$

and that

$$\alpha^k_{rjs} = \sum_i g^{ik} \mathcal{K}(V_i, V_r, V_j, V_s).$$

The definition of Ric then yields the expression

$$\mathrm{Ric}(V_r, V_s) = \sum_j \alpha^j_{rjs} = \sum_{i,j} g^{ij} \mathcal{K}(V_i, V_r, V_j, V_s).$$

These results show that Ric is a symmetric tensor and completes the proof. $\square$

The composed map

$$\mathrm{Trace}_g : \Omega_{X/S} \otimes_{\mathcal{O}_X} \Omega_{X/S} \xrightarrow{\mathrm{id} \otimes \delta_g^{-1}} \Omega_{X/S} \otimes_{\mathcal{O}_X} \Theta_{X/S} \to \mathcal{O}_X$$

that is given by

$$\omega_1 \otimes \omega_2 \mapsto \omega_1 \otimes \delta_g(\omega_2) \mapsto \delta_g(\omega_2)(\omega_1)$$

is called the **g-trace** of 2-tensors.

Definition 15. *The map ρ defined by $Trace_g(Ric)$ is called the* ***scalar curvature of*** *(X, g).*

Given a basis $V_1, \ldots, V_n$ as above, we may write

$$\rho = \sum_{i,j,r,s=1}^{n} g^{ij} g^{rs} \mathcal{K}(V_i, V_r, V_j, V_s).$$

Definition 16. *We call (X, g) an* ***Einstein scheme*** *and g an* ***Einstein metric*** *if for some global section γ of $\mathcal{O}_X$, it happens that $Ric = \gamma g$.*

Proposition 19. *Suppose that $\Omega_{X/S}$ has constant rank n. If (X, g) is an Einstein scheme and ρ is its scalar curvature, then $\rho = n\gamma$ for γ a section of $\mathcal{O}_X$ such that $Ric = \gamma g$.*

Proof.

$$\rho = \mathrm{Trace}_g(\mathrm{Ric}) = \mathrm{Trace}_g(\gamma g) = \gamma \mathrm{Trace}_g(g) = n\gamma.$$

□

Proposition 20. *Suppose that $\Omega_{X/S}$ has rank 2. Assume that 2 is always a section of $\mathcal{O}_X$, Then (X, g) is an Einstein scheme. If σ^1 and σ^2 form a basis for $\Omega_{X/S}$, V_1, V_2 is its dual, and $(g_{ij})_{i,j=1}^2$ is the fundamental matrix of (X, g) with respect to σ^1, σ^2, then it happens that $Ric = \gamma g$ with*

$$\gamma = \frac{\mathcal{K}(V_1, V_2, V_1, V_2)}{det(g_{ij})_{i,j=1}^2}.$$

Also $\rho = 2\gamma$ is independent of the choice of the basis σ^1 and σ^2 for $\Omega_{X/S}$.

Proof. Notice that it is enough to prove the second assertion of the proposition. We see that

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{\det(g_{ij}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix},$$

so the symmetries of $\mathcal{K}$ yield the following expression, thus completing the proof,

$$\begin{aligned} Ric &= \sum_{r,s=1}^{2} \sum_{i,j=1}^{2} g^{ij} \mathcal{K}(V_i, V_r, V_j, V_s) \sigma^r \otimes \sigma^s \\ &= \mathcal{K}(V_1, V_2, V_1, V_2)\big(g^{22}\sigma^1 \otimes \sigma^1 - g^{21}\sigma^1 \otimes \sigma^2 - g^{12}\sigma^2 \otimes \sigma^1 + g^{11}\sigma^2 \otimes \sigma^2\big) \\ &= \frac{\mathcal{K}(V_1, V_2, V_1, V_2)}{\det(g_{ij})} g. \end{aligned}$$

$\square$

In this situation above, we call γ the **Guassian curvature** of (X, g) and it is denoted K. We will generalize as follows. Let $\mathcal{G}$ be a sheaf $\Omega_{X/S}$-submodules such that it is of rank 2 and its dual sheaf $\mathcal{G}^*$ also has rank 2 as well as a sheaf $\Theta_{X/S}$-submodules. The results above regarding Einstein schemes applies to the sheaf $\mathcal{G}$. This allows one to consider the **Gaussian curvature of** (X, g) **relative to** $\mathcal{G}$, which is denoted by $K_{\mathcal{G}}$ and clearly constructed from above. Fix such a sheaf $\mathcal{G}$.

Corollary 3. *Supose that V_1, V_2 are an orthonormal basis of $\mathcal{G}$ and write Ω_{12} as $K\sigma^1 \wedge \sigma^2$ for some K a section of $\mathcal{O}_X$. Then K is the Guassian curvature $K_{\mathcal{G}}$ of $\mathcal{G}$.*

Theorem 4. *(Residuals Relation To Global Curvature) Let $(X, \overline{g}, \mathcal{I})$ be a psuedo-Riemannian residual of (X, g). For $\overline{U}, \overline{V}, \overline{W}, \overline{Z}$ sections of $\Theta_{\mathcal{I}/S}$ with respective representative sections U, V, W, Z of $\Theta_{X/S}$, the following expression holds,*

$$\mathcal{K}(\overline{U}, \overline{V}, \overline{W}, \overline{Z}) = \overline{\mathcal{K}(U, V, W, Z)} + \overline{g}((\overline{\nabla_W(U)})^{\perp}, (\overline{\nabla_V(Z)})^{\perp}) - \overline{g}((\overline{\nabla_V(U)})^{\perp}, (\overline{\nabla_W(Z)})^{\perp}).$$

Proof. We know that the following expressions hold,

1. $\overline{\nabla}_{\overline{V}}(\overline{\nabla}_{\overline{W}}(\overline{V}) = (\overline{\nabla}(\overline{\nabla_W(X)})^{\top})^{\top} = (\nabla_V(\overline{\nabla_W(Z)}))^{\top} - \overline{\nabla}_{\overline{V}}((\overline{\nabla_W(Z)})^{\perp})^{\top}$,
2. $\overline{\nabla}_{\overline{W}}(\overline{\nabla}_{\overline{V}}(\overline{Z}) = (\nabla_W(\overline{\nabla_V(Z)}))^{\top} - \overline{\nabla}_{\overline{W}}((\overline{\nabla_V(Z)})^{\perp})^{\top}$,
3. $\overline{\nabla}_{[\overline{V}, \overline{W}]}(\overline{Z}) = (\overline{\nabla_{[V,W]}(Z)})^{\top}$.

From this, it follows that

$$\mathcal{R}(\overline{V} \wedge \overline{W})(\overline{Z}) = (\overline{\mathcal{R}(V \wedge W)(Z)})^{\top} + (\overline{\nabla}_{\overline{W}}(\overline{\nabla_V(Z)})^{\perp})^{\top} - (\overline{\nabla}_{\overline{V}}((\overline{\nabla_W(Z)})^{\perp}))^{\top}.$$

Therefore, we obtain

$$\begin{aligned}\mathcal{K}(\overline{U},\overline{V},\overline{W},\overline{Z}) &= \overline{g}(\overline{U},\mathcal{R}(\overline{V}\wedge\overline{W})(\overline{Z}))\\ &= \overline{g}(\overline{U},\overline{\mathcal{R}(V\wedge W)(Z)}) + \overline{g}(\overline{U},\overline{\nabla}_{\overline{W}}(\overline{\nabla_V(Z)})^{\perp})\\ &- \overline{g}(\overline{U},\overline{\nabla}_{\overline{V}}((\overline{\nabla_W(Z)})^{\perp})),\end{aligned}$$

which implies the desired result,

$$\mathcal{K}(\overline{U},\overline{V},\overline{W},\overline{Z}) = \overline{\mathcal{K}(U,V,W,Z)} + \overline{g}((\overline{\nabla_W(U)})^{\perp},(\overline{\nabla_V(Z)})^{\perp}) -$$
$$\overline{g}((\overline{\nabla_V(U)})^{\perp},(\overline{\nabla_W(Z)})^{\perp}).$$

□

6 Applications & Computations

Example 7. *Let k be a field of characteristic not equal to two, and $A = k[x,y]/(y^2 - x^3)$. We have that*

$$\Omega_{A/k} \cong (Adx \oplus Ady)/(2ydy - 3x^2dx).$$

At the origin, this module is free of rank two, but outside of th origin, it is of rank one, so the sheaf associated to the module $\Omega_{A/k}$ is not locally free.

Theorem 5. *If X is an S-scheme, then there exists a metric on $\Omega_{X/S}$ if, and only if, there exists a symmetric isomorphism of $\Omega_{X/S}$ and $\Theta_{X/S}$ as $\mathcal{O}_X$-modules in the following sense:*

- *There exists an $\mathcal{O}_X$-module isomorphism $u : \Theta_{X/S} \to \Omega_{X/S}$ that factors as*

$$\Theta_{X/S} \to \Omega^*_{X/S} \xrightarrow{u^*} \Theta^*_{X/S} \to \Omega_{X/S}$$

  where the left and right arrows are the canonical isomorphisms, and u^* is the dual map of u.

Proof. This is an implication from previous work. □

Proposition 21. *Consider $(B, \mathfrak{m})$ a regular local ring which contains its residue field $B/\mathfrak{m}$ (up to an isomorphism). If $B/\mathfrak{m}$ is a perfect field where B is a location of a finitely generated k-algebra, then there exists a metric on $X = Spec(B)$.*

Proof. It is enough to show that $\Omega_{X/k}$ is a free module of finite rank, but this naturally follows from that fact the rank $\Omega_{X/k}$ is a free B-module of rank being equal to the Krull dimension of B. □

Corollary 4. *In the notation above, any chose of a regular system of parameters generate a differential basis for $\Omega_{X/k}$, and hence, a metric on X over k.*

Example 8. *If Y is any scheme and $X = \mathbb{A}^n_Y$ is affine n-space over Y, then $\Omega_{X/Y}$ is a free $\mathcal{O}_X$-module of rank n and basis $(d_{X/Y}(x_1), \cdots, d_{X/Y}(x_n))$, which generate a metric on X.*

Proposition 22. *If X is a smooth variety of dimension n over a perfect field k, then each point $p \in X$ has an open neighborhood U for which $\Omega_{U/k}$ is finite free as an $\mathcal{O}_X|_U$-module and there exists a metric on $\Omega_{U/k}$.*

Proof. The existence of such an open neighborhood follows from the fact that $\Omega_{X/k}$ is locally free of rank n. By the Correspondence of Metrics Proposition, there exists a one-to-one correspondence between symmetric non-degenerate $\mathcal{O}_X$-bilinear morphisms $\Theta_{X/S} \times \Theta_{S/S} \to \mathcal{O}_X$ of abelian sheaves and symmetric matrices $(g_{ij}) \in \mathrm{Mat}_n(\mathcal{O}_X(X))$ with the property that $\det(g_{ij}) \in \mathcal{O}_X(X)^\times$. $\square$

Proposition 23. *If G is a group scheme over a field k, then there exists a metric on $\Omega_{G/k}$.*

Proof. This follows from the fact that there exists some $r \in \mathbb{N}_0$ such that $\Omega_{G/k} \cong \mathcal{O}_G^{\oplus r}$ as O_G-modules (Bhatt, Math 731). In this case, we have that $\Theta_{G/k} \cong \Omega_{G/k}$ as O_G-modules, and completes the proof. $\square$

Example 9. *Let k be a field. Consider an elliptic curve X over k. The identification of the canonical line bundle K_X with the cotangent sheaf $\Omega_{X/k}$ implies that it is trivial. In fact, this comes from that fact that X is a group scheme over k.*

Corollary 5. *Any abelian variety over a field k is a pseudo-Riemannian scheme.*

Proposition 24. *Consider the affine coordinate ring $k[x_1, \cdots, x_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ where k is a field of characteristic zero. There exists an*

$$f \in k[x_1, \cdots, x_n]/\mathfrak{p}$$

such that the affine scheme

$$Spec((k[x_1, \cdots, x_n]/\mathfrak{p})[f^{-1}])$$

has a metric.

Proof. Let $R = k[x_1, \cdots, x_n]/\mathfrak{p}$. It is enough to show that the cotangent module $\Omega_{R[f^{-1}]/k}$ is free of rank d for some $f \in R$ and $d \in \mathbb{N}$. We know that $\Omega_{R/k}$ is finitely generated as an R-module. Denote $K(R)$ as the field of fractions for R, and note that

$$\Omega_{R/k} \otimes_R K(R) = \Omega_{K(R)/k}$$

is a finite-dimensional $K(R)$-vector space with dimension tr.deg$(K(R)/k)$. Clearly, this is equal to some d. Choose elements $x_1, \cdots, x_d \in \Omega_{R/k}$ which form a basis for $\Omega_{K(R)/k}$. This gives a map $R^d \to \Omega_{R/k}$ which becomes an isomorphism after one localizes at the zero ideal (o). Hence, there exists an $f \in R$ such that the map above is an isomorphism after localization at f. We find that $\Omega_{R[f^{-1}]/k}$ is a free of rank d as desired. $\square$

Example 10. *Let $\mathbb{F}_p$ denote the field of p elements. Consider the ring $R = \mathbb{F}_p[x]/(x^p - 1)$ and the group scheme Spec(R) over $\mathbb{F}_p$. The contangent sheaf $\Omega_{R/\mathbb{F}_p}$ is Rdx, which is free of rank 1. Hence, there exists a metric on R.*

Proposition 25. *Let k be an algebraically closed field and R be a quotient ring $k[x_1, \cdots, x_n]$ such that $\Omega_{R/k}$ is a projective R-module of rank dim(R). For any closed point $\mathfrak{p} \in$ Spec(R), there exists a metric on Spec$(R_{\mathfrak{p}})$.*

Proof. This comes from the fact that $\Omega_{R/k}$ is locally free and projective of rank $\dim(R)$. Hence, $\Omega_{R_{\mathfrak{p}}}$ is free of dimension $\dim(R_{\mathfrak{p}})$. $\square$

Example 11. *Let k be an imperfect field of characteristic p nonzero. We can choose an $a \in k$ such that $a^{1/p} \notin k$. Let $k' = k(a^{1/p})$, and as a ring, this isomorphic to $k[t]/(t^p - \alpha)$. One can see that $\Omega_{k'/k} = k'$ as it is the cokernel of the map $k' \to k'$ given by multiplication $\frac{d}{dt}|_{a^{1/p}}(t^p - a)$. Hence, it is possible to see that a generator for $\Omega_{k'/k}$ is t where t is a p-th root of a, and so $\Omega_{k'/k} = k'$.*

Proposition 26. *Let k be an imperfect field of characteristic p nonzero. If $k' = k(a_1, \cdots, a_n)$ where each a_i has the property that $a_i^{1/p} \notin k$, then there exists a metric on Spec(k') and is determined by the generators da_i.*

Proof. This follows from that fact the $\Omega_{k'/k}$ is free of rank n over k' by adapting the argument in the example above. □

REFERENCES

[1] Kunz, Enrst (2002). Algebraic Differential Calculus. Unpublished manuscript.

[2] Jaoui, R. (2019). Generic planar algebraic vector fields are disintegrated. arXiv preprint arXiv:1905.09429.

[3] Jaoui, R. (2019). Differential fields and geodesic flows II: Geodesic flows of pseudo-Riemannian algebraic varieties. Israel Journal of Mathematics, 230(2), 527-561.

[4] Kähler, Erich. "Der innere differentialkalkül." Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. Vol. 25. No. 3-4. Springer-Verlag, 1962.

[5] Schmidt, Friedrich Karl. "Zur arithmetischen Theorie der algebraischen Funktionen. II. Allgemeine Theorie der Weierstraßpunkte." Mathematische Zeitschrift 45.1 (1939): 75-96

[6] N.M. Katz, "On the differential equations satisfied by period matrices" Publ. Math. IHES , 35 (1968) pp. 71–106

[7] Katz, Nicholas M. "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin." Publications mathématiques de l'IHES 39 (1970): 175-232.

[8] Yu. Manin, "Algebraic curves over fields with differentiation" Transl. Amer. Math. Soc. , 37 (1964) pp. 59–78 Izv. Akad. Nauk. SSSR Ser. Mat. , 22 (1958) pp. 737–756