Summer 7-28-2020

Lattice of Maximal-Primary Ideals in Quadratic Orders

Ryan Bridges
University of new mexico

Ryan Lewayne Bridges

Candidate

Mathematics

Department

This dissertation is approved, and it is acceptable in quality and form for publication:

Approved by the Thesis Committee:

Dr. Janet Vassilev, Chair

Dr. Alexandru Buium, Member

Dr. Hongnian Huang, Member

Lattice of Maximal-Primary Ideals in Quadratic Orders

Authored by Ryan Bridges, Advised by Doctor Janet Vassilev

BA in Mathematics and Philosophy, Austin College, 2017

THESIS

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Masters of Science

Mathematics

The University of New Mexico

Albuquerque, New Mexico

July, 2020

Acknowledgements

Firstly, I would like to acknowledge Doctor Janet Vassilev. Doctor Vassilev taught me Algebra from my first graduate Abstract Algebra course to Commutative Algebra and Algebraic Topology. My interests and knowledge of Algebra has been developed by Doctor Vassilev more than anyone I can name. Moreover, when I needed mathematical direction, Doctor Vassilev gave it to me by giving me the project that turned into this thesis. I would not be the mathematician I am today without them.

I would like to thank Doctor Alexandru Buium for providing his excellent algebraic expertise to this thesis on short notice.

Likewise, I would like to thank my other Committee Member Doctor Hongnian Huang, who has instructed me in every geometry course I've had the pleasure to take, and whom it is a great pleasure to have see my mathematical journey at UNM off.

Additionally, I'd like to thank some of the Professors I've had that have greatly aided my mathematical development; Doctors J'Lee Bumpous, Andrea Overbay, Rod Stewart, David Baker, and Terry Loring.

Lastly, I would like to thank Doctor Kerry Brock, who taught me how to write a proof, introduced me to Lagranges Theorem, and was the first person to spark my love of Algebra and Abstract Mathematics in general.

Lattice of Maximal-Primary Ideals in Quadratic Orders

Authored by Ryan Bridges

B.A., Mathematics, Austin College, 2017

Abstract

An order is a subring of the ring of integers of an algebraic extension, Peruginelli and Zanardo classified the lattices of orders with prime index inside te ring of integers of quadratic extensions of the rational numbers. The lattices are quite striking and have different layered structure depending on whether the prime is inert, split, or ramified. This thesis considers the orders which have prime power index inside the Gaussian integers. This is a nice generalization of the work of Peruginelli and Zanardo, and succeeds in a few classifications of specific instances of orders derived from inert primes.

Contents

1 Introduction

Our primary objects of interests in algebraic number theory are **algebraic number fields** which commonly occur as some finite, algebraic extension of $\mathbb{Q}$. A quadratic number field is an algebraic extension of $\mathbb{Q}$ of degree 2, for example $\mathbb{Q}[i]$. For each algebraic number field $\mathbb{Q}[\alpha]$, there is an associated **ring of integers** given by $\mathbb{Z}[\alpha]$. Algebraic number theory is largely concerned with the extension of the properties of the integers to algebraic extensions of the integers, i.e., rings of integers. A quadratic ring of integers is defined similarily as a degree 2 algebraic extension of $\mathbb{Z}$, one such example is the Gaussian integers $\mathbb{Z}[i]$. The primary object whose sub-lattice we are investigating is an **order**. At its most basic, an order is a subring O of the ring of integers of a field that is a $\mathbb{Z}$-module. For an order O of a number ring O_K, its conductor is given by $\mathfrak{F} = \{\alpha \in K | \alpha K \subseteq O\}$, the largest ideal of O_K contained in O.

In this paper, we begin with the general theory and full characterizations of the lattice of $\mathfrak{F}$-primary ideals of the number ring $\mathbb{Z}[\alpha f]$, for quadratic algebraic integer α and prime f, formed by **[PZ]**. We then attempt to extend the general theory to the case $\mathbb{Z}[\alpha f^\sigma]$, α and f the same, but now with f raised to a power. Further, we characterize specific cases of these lattices, and show the existence of a canonical sub-lattice associated to the conductor $\mathfrak{F}$. At the end of the Peruginelli and Zanardo paper, we come across in the remarks that the Hasse diagram of a number ring $\mathbb{Z}[\alpha n]$ for quadratic α and integer n will be given by the disjoint union of the lattices we are investigating above. This work is a small extension towards a general characterization of all such lattices, and eventually, a characterization of the lattices of all such integers themselves.

2 Preliminary Algebraic Number Theoretic Results

Now, to establish some of the preliminary algebra necessary for our constructions and discussion. As stated, the objects of primary interest are the lattices of particular algebraic sub-objects of a common algebraic number theory setting, a number field. A **number field** is some finite, algebraic extension of the rationals, i.e., an algebraic number field is of the form $\mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_1, ..., \alpha_n$ are algebraic over $\mathbb{Q}$. [**RS**]

Remark 2.1 *If K is an algebraic number field, then K is an algebra over $\mathbb{Q}$.*

An **algebraic integer** of an algebraic number field K is any element $\beta \in K$ whose monic minimal polynomial in $\mathbb{Q}$ has only integer coefficients instead of rational coefficients. The collection of algebraic integers of K form a sub-ring called the **ring of numbers** (colloquially, number ring), commonly denoted as O_K. [**Mat**] (Theorems 9.1 and 9.4 from Matsumara applied to number fields yield the above)

In this paper, we focus on an algebraic integers α of quadratic, i.e., algebraic integers whose monic, irreducible minimal polynomial is of degree 2 (both rational and integral by Gauss' Lemma). So we have that there exists an irreducible polynomial $h(x) = x^2 + ax + b$ such that $f(\alpha) = 0$ for $a, b \in \mathbb{Z}$. This polynomial is unique. So for f^σ, this implies that there exists a monic, irreducible polynomial with coefficients in $\mathbb{Z}$ such that $g(x) = x^2 + Ax + B$, and $g(\alpha f^\sigma) = \alpha^2 f^{2\sigma} + Af^\sigma \alpha + B = 0$. From this we see that $A = f^\sigma a$ and $B = f^{2\sigma} b$. We will refer to these as the **algebraic equations**, and use them further down the line to elucidate the structure of particular ideals as modules.

These rings O_K are no longer fields, but inherit a variety of nice structure from K [**S**], as follows:

Remark 2.2 *If K is an algebraic number field then O_K*

i) is an integral domain

ii) is integrally closed in K

iii) is a Noetherian ring

iv) has Krull dimension 1

v) is a Dedekind domain

So, for all algebraic number fields K, its ring of integers O_K is an integrally closed Noetherian integral domain of Krull dimension 1, and thus a Dedekind domain. While not possessing the full structure of a field, it is clear that our rings of integers inherit a lot of 'nice' algebraic structure from the field that they are derived from.

For a number field K, and it's ring of integers O_K, an **order** of O_K is a subring of O_K, denoted O, that is also a $\mathbb{Z}$-module. As we make explicit the sructures we desire to investigate, we will put conditions on our order to make it unique amongst the possibilities in a number ring.

In every order O of the ring of integers O_K of number field K there is an ideal called the **conductor**, denoted $\mathfrak{F}$. The conductor is given to us as the the largest ideal of O contained in O_K. For number ring and order O_K, O, $O \subseteq O_K$, the conductor of S in R is given as $\mathfrak{F} = \{s \in S | sR \subseteq S\}$. $\mathfrak{F}$, as mentioned is an ideal of O, as shown below.

Lemma 2.1 *For order O of ring of integers O_K, the conductor of O is an ideal of both rings.*

Proof: Obviously $O \subseteq O_K \subseteq K$ are commutative so the conductor of O is well defined. First to check that $\mathfrak{F}$ is a subring of O. Let α, $\beta \in O$, then $(\alpha + \beta)O_K = \alpha O_K + \beta O_K$. αO_K and βO_K are both sub rings of O, which is closed under addition, so $\alpha O_K + \beta O_K$ is certainly contained in O.

To show $\mathfrak{F}$ is an ideal of O, let $\alpha \in \mathfrak{F}$, $\beta \in O$, then $\alpha\beta O_K = \alpha(\beta O_K) = \alpha O_K \subseteq O$, so $\mathfrak{F}$ defines an ideal.

Alternatively, for commutative rings $S \subseteq R$, the conductor can be characterized as $\mathfrak{F} = (S :_K R)$, the colon ideal of R and S over K.

Remark 2.3 *If $\mathfrak{F}$ is the conductor of O, then $\mathfrak{F} = (O_K :_K O)$.*

The conductor under non vacuous conditions is always a proper ideal, as shown below.

Lemma 2.2 *Let $S \subseteq R$ be commutative rings. $\mathfrak{F} = S \iff S = R$.*

Proof: ($\leftarrow$) let $S = R$. Then $RS = R \subseteq S$, which implies that $\mathfrak{F} = S$.
($\rightarrow$) let $\mathfrak{F} = S$, then $1 \in \mathfrak{F}$, so $R\mathfrak{F} = R \subseteq S$. By definition, $S \subseteq R$, so we have that $S = R$.

One notion we need from abstract algebra comes to us in a formulation of Cox. [**C**] When we are considering an order O of a ring of integers O_K of quadratic number field K, he calls an ideal I of O proper if
$\{\omega \in O_K | \omega I \subset I\} = O$. In other words, the set of elements multiplying I into itself are all in O.

Theorem 2.3 *Let O be an order of quadratic number field K. If I is a finitely generated, non-zero ideal of O, then I is a proper ideal if and only if I is an invertible ideal.*

Given our common use of proper to refer to a set inclusion property, this will be the last time we use the term in Cox's sense. However, it should be noted that any ideal that is a module of an order of O_K containing O, or O_K itself, then it is NOT proper in the sense of Cox. So, the above essentially tells us that any O ideal that is NOT a module of any 'upper' order or O_K is invertible.

To explicitly represent the algebraic structure of ideals contained in the conductor of an order, we use a **lattice**. A lattice in its most general form is a categorial object that we can build in the required algebraic categories (Grp, Mon, etc.) to capture the notions we need. For a category $\mathfrak{C}$, $\mathfrak{C}^{\to}$ is the collection of arrows of $\mathfrak{C}$, and $\mathfrak{C}^{obj}$ is the collection of objects in $\mathfrak{C}$.. A **pre-order** is a category $\mathfrak{C}$ in which there is at most one arrow $f : C \to D$ for $f \in \mathfrak{C}^{\to}$, $C, D \in \mathfrak{C}^{Obj}$. Note, this allows for two $\mathfrak{C}$-objects C, D to have 2 arrows between them, with opposite domains/codomains ($f : C \to D, g : D \to C$). **[Mac]** A **partial ordering** is a preorder in which for any two $\mathfrak{C}$-objects have at most one arrow between them, i.e. for $C, D \in \mathfrak{C}^{Ob}$ there is either an arrow $f : C \to D$ or $g : D \to C$ or no arrows. The meaning of the arrows is specific to the preorder, of which category the lattice is a subcategory of. Partial orders can have a notion of supremum and infimum, both globally and locally, called the **join** and **meet** respectively. If for any two objects C, D in some partial order such that there exists object E such that there exists $f : C \to E$, $g : D \to E$, and for any object W that also satisfies this property (an arrow from C, D to W), there exists an arrow from E to W, then E is called the **join** of C and D. This captures our notion of least upper bound in a partial order. If for any two objects C, D in a partial order such that there exists E such that there exists $f : E \to C$, $g : E \to D$, and for any object W that also satisfies this property (an arrow from W to C, D), there is an arrow from W to E, then E is called the **meet** of C and D. A global maximum of a partial order is an object in which every other (comparable) object has an arrow to it, and a global minimum is an object in which it has an arrow to every other (comparable) object. A **lattice** is a partial order in which any two elements have a **join** and a **meet**. **[G]**

The **radical** of an ideal J of ring R is the set

$$\sqrt{J} = \{x \in R | x^n \in J \text{for some n} \in \mathbb{N}\}$$

which will be a prime ideal. We call J P-primary if J is a primary ideal and prime ideal P is the radical of J [**H**]. In this thesis, we attempt to extend the previous general theory of the lattice of $\mathfrak{F}$-primary ideals in the quadratic algebraic order $\mathbb{Z}[\alpha f]$ for prime f, to the setting of $\mathbb{Z}[\alpha f^\sigma]$, and to provide a characterization of the structure for inert primes. For order O, the $\mathfrak{F}$-primary lattice will be the lattice of ideals of O who are $\mathfrak{F}$-primary ordered by inclusion. For the interests of our paper, this will either be the lattice of subideals of $(f, \alpha f^\sigma)$, whcih is the conductor in the $\sigma = 1$ case.

Definition 2.1 *For an order $\mathbb{Z}[\alpha f^\sigma]$ the maximal primary ideals are the ideals primary to $(f, \alpha f^\sigma)$. This will be the set of ideals whose radical is $(f, \alpha f^\sigma)$. For order $\mathbb{Z}[\alpha f]$, the maximal primary ideals are those whose radical is the conductor.*

Theorem 2.4 *If O is an order of ring of integers K, then the lattice of maximal primary ideals ordered by inclusion forms a lattice bounded on top (i.e. lattice of $\mathfrak{F}$-primary ideals).*

Having established the basic algebraic objects underpinning our investigation, we narrow our focus on the primes themselves. While an ideal P may be prime when considered in $\mathbb{Z}$, it's extension in higher order rings containing $\mathbb{Z}$ may no longer be prime, and have its own decomposition into prime factors. Generalizing further, for prime ideals of some number ring O_K we can classify their behavior in order O_F of extension field of K, F, though it is dependent on which extension field F of K we consider.

For prime ideal of O_K P, PO_F is an ideal in O_F, so we can look at its decomposition into prime ideals; $PO_F = \Pi P_i^j$ If we have $j > 1$ for any of the products components, we declare P to be **ramified** in this extension. By the nature of multiplicativity, we can say $[F : K] = \Sigma j_i a_i$, some integers a_i. If all j_i, a_i are equivalent to one, then we say P splits completely, or is **split** in our extension. If P remains prime in our extension, we say it is **inert** in our extension (in which case $PO_F = \Pi P = P$). We give formulas that can determine the splitting type of our ideals later on. As stated, the case we are most interested in is prime ideals of

$\mathbb{Z}$ in field extensions of $\mathbb{Q}$. In other words, we are generally looking at the splitting type of ideals generated by prime integers in quadratic algebraic extensions of $\mathbb{Q}$.

This 'splitting type' of primes was shown to determine the $\mathfrak{F}$-primary lattice structure of the conductor of a quadratic order $\mathbb{Z}[\alpha f]$ **[PZ]**. While splitting type is not entirely sufficient for our generalization considered here, it is an essential piece of information. In this thesis, we only specifically delve into the structure of inert primes, however the general theory developed intially applies to all three cases. Moreover, we present results from Peruginelli and Zanardo on the other types of primes to fully illuminate the setting this machinery is being developed to operate in.

This ends our account of the basic algebraic objects under consideration. To reiterate, we are looking at the underlying theory determining the structure of the maximal primary lattice of the quadratic order $\mathbb{Z}[\alpha f^\sigma]$.

3 Preliminary Results of Orders and $\mathfrak{F}$-Primary Ideals

Here we go over the theory determining the actual structure of $\mathfrak{F}$ and the ideals associated to its f-primary lattice. Let d be a square-free integer. The ring of integers of $K = \mathbb{Q}(\sqrt{d})$ is equal to $D = \mathbb{Z}[\omega]$, where either $\omega = \sqrt{d}$, when $d = 2, 3$ modulo 4, or $\omega = (1 + \sqrt{d})/2$, when $d = 1$ modulo 4. In the latter case, we get $\omega^2 = \theta - (1 - d)/4$. Let now f be a positive prime integer and $O = Z[f\omega]$ be the unique quadratic order in K such that $[D : O] = f$.

An important definition of this preliminary theory is the notion of $\mathfrak{F}$-basic ideals, or the $\mathfrak{F}$-primary ideals who are not contained by $\mathfrak{F}^2$. A large part of this theory is that we can rely on the $\mathfrak{F}$-basic layer to characterize our lattices. Additionally we can characterize elements of our order as basic and ideal-primary as well. For $t \in O$, ideal $I \subset O, t$ is I-primary if $\sqrt{tO} = I$. For our purposes, we concentrate on the case of elements of the order who are primary to the conductor. We call an element $t \in O$ $\mathfrak{F}$-basic if tO is an $\mathfrak{F}$-basic ideal. For our single power of f case, we get a number of simple results from the norms and above definitions. **[PZ]**

Remark 3.1 *$\mathfrak{F}$ is a prime ideal of O if and only if f is a prime number.*

Remark 3.2 *There are no intermediate ideals between $\mathfrak{F}$ and fO since $|\mathfrak{F} : fO| = f$.*

Remark 3.3 *An ideal I of O is $\mathfrak{F}$-primary if and only if $N(I) = f^a$, some $a \geq 0$.*

Lemma 3.1 *(Lemma 2.1 of **[PZ]**) Let $\omega \in \mathfrak{F}/fO$. Then $\mathfrak{F} = (f, \omega)$.*

Theorem 3.2 *(Lemma 2.5 of **[PZ]**) Let Q be a $\mathfrak{F}$-primary ideal and let $k = max\{n \in \mathbb{N} | Q \subset \mathfrak{F}^n\}$. Then;*
i) $Q = f^{k-1}Q'$, where Q' is a $\mathfrak{F}$-basic ideal.
ii) If Q/f^m is $\mathfrak{F}$-basic for some $m > 0$, then m coincides with $k - 1$.

Lemma 3.3 *(Proposition 2.6 of **[PZ]**) Let $t = fx + f\omega y \in \mathfrak{F}$ be $\mathfrak{F}$-primary, $x, y \in \mathbb{Z}$. Then $g.c.d(x, y) = f^a$, for some $a \geq 0$. Moreover, t is $\mathfrak{F}$-basic if and only if $g.c.d(x, y) = 1$. If the latter conditions hold, then t is an irreducible element of O which is not prime.*

Proof: Let $t = fx + f\omega y \in \mathfrak{F}$ be $\mathfrak{F}$-primary, so $\sqrt{tO} = \mathfrak{F}$. Then by the above lemma, $N(tO) = |O/tO| = |(1)/f(x + wy)| = f^a, a > 0$. But $|O/tO| = fgcd(x, y) \Rightarrow gcd(x, y) = f^{a-1}, a > 0$. For any $\mathfrak{F}$-basic ideal $J, N(J) = |O/J| = |N : (fO)| = f$ This immediately implies tO is an $\mathfrak{F}$-basic ideal if and only if $\gcd(x, y)$ by our previous work. To show the last condition, as given by **[PZ]**, let us use a proof by contradiction, and assume that $t = rs$, where $r, s \in O$, and both r, s are not units in O. Since the norm is multiplicative on O, r, s are $\mathfrak{F}$-primary elements. In particular, $r, s \in \mathfrak{F}$. But then $t = rs \in \mathfrak{F}^2$, a contradiction. Moreover, tO is not a prime ideal, since it is strictly contained in the conductor $\mathfrak{F}$ (the only prime ideal containing t), which is not principal.

We further characterize the basic ideals found between $\mathfrak{F}$ and $\mathfrak{F}^2$ by the basic elements $t \in O$ such that $\mathfrak{F}^2 \subset tO \subset \mathfrak{F}$.

Lemma 3.4 *(Lemma 3.4 of **[PZ]**) A principal ideal tO lies properly between $\mathfrak{F}$ and $\mathfrak{F}^2$, i.e. t is $\mathfrak{F}$-basic if and only if $t = fk$, for a suitable unit of $k \in D$. Moreover, $fkO = fk'O$ if and only if $k/k' \in O$.*

Here, we give a characterization of the most important layer of these structures for all lattices, the first conductor layer composed of the conductor and the conductor primary ideals where our number field is given by $\mathbb{Q}(\omega)$, with ring of integers $D = \mathbb{Z}[\omega]$. The order's this theorem is concerned with are $O = \mathbb{Z}[f\omega]$.**[PZ]**

Theorem 3.5 *(Theorem 3.3 of **[PZ]**) Let $Q = (f^k, f\alpha)$ be an $\mathfrak{F}$-basic ideal different from fO. Then;*
i) f^k is the minimum power of $\mathfrak{F}$ contained in Q
ii) If Q is a D-module, then there are exactly $f+1$ ideals of O lying properly between Q and fQ, namely the pairwise distinct ideals

$$\boldsymbol{J} = \{J = (f^k, f^2\alpha), J_a = (f^{k+1}, af^k + f^k\alpha)\}$$

iii) If $Q \neq QD, \exists!$ ideal of O lying properly between Q & fQ of the form

$$J = (f^k, f^2\alpha)$$

This theorem is meant to characterize the local lattice structure of any quadratic order, where our order is given by $O = \mathbb{Z}[f\alpha]$. A key part of our work will be extending this theorem to the case $\mathbb{Z}[\alpha f^\sigma]$. This theorem is central in our research into these objects, allowing us to build an unknown structure up piece by piece to be investigated for its global properties. Moreover, it will be seen to be a key in many proofs regarding the actual characterizations of our lattice's of interest. As an example;

Lemma 3.6 *For number field $D = \mathbb{Q}[i]$, ring of integers $\mathbb{Z}[i]$, and order $O = \mathbb{Z}[if]$, the conductor $F = (f, if)$ is a D-module.*

Proof: $\mathbb{Z}[i]F = (1, i)(f, if) = (f) + (if) + (if) - (f) = (f, if) = F$, which shows the action of D on the generators of F preserve all of F, and satisfy F being a D-module.

If f is an inert prime, this lemma combined with the above theorem actually fully characterizes the f-primary lattice of the conductor in $\mathbb{Z}[\alpha f]$, as we will come to see. This just illustrates the power and utility of the previous theorem.

Now, to determine which of the ideals of $\mathbf{J}$ is a D-module requires a little more work and knowledge of whether or not f is inert/ramified/split. Let D^*, O^* denote the multiplicative group of the two rings. The following gives us an easy method to calculate the splitting type of f relative to the algebraic structures we already have. [**P&Z**]

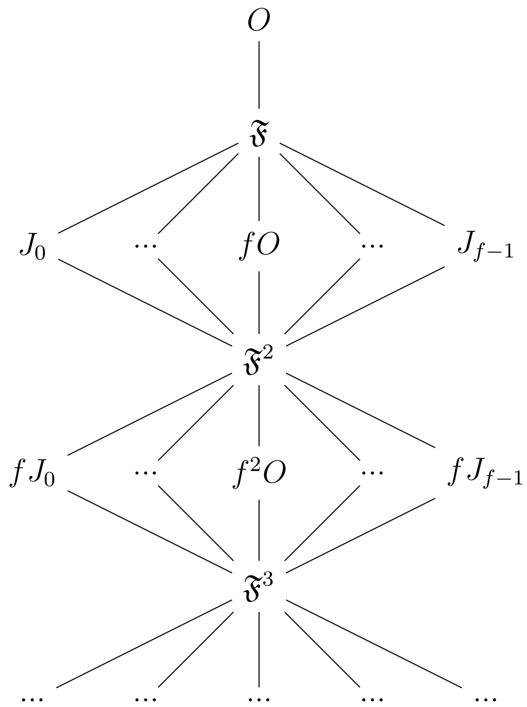
Lemma 3.7 *(Proposition 3.5 of [**PZ**]) Let $\tau = |D^*/O^*|$. Then we have;*
i) if f is inert in D, then $\tau | f + 1$.
ii) if f is split in D, then $\tau | f - 1$.
iii) if f is ramified in D, the $\tau | f$.

For the case which we will work in, where $O = \mathbb{Z}[\alpha f^\sigma]$, this lemma can still help us determine the splitting type of f, we will just jave to replace O in the above lemma with the single power case ($\sigma = 1$).

4 Preliminary Results of Lattice Structures

Drawing primarily from [**PZ**], we restate the results characterizing the lattices of $\mathfrak{F}$-primary ideals of orders $\mathbb{Z}[f\omega]$, of the ring of integers $D = \mathbb{Z}[\omega]$, where $f \in \mathbb{Z}$ is a prime. As stated previously, the lattice structure of O is dependent upon the splitting type of f, so we get three cases, each corresponding to when f is inert/split/ramified (see previous section).

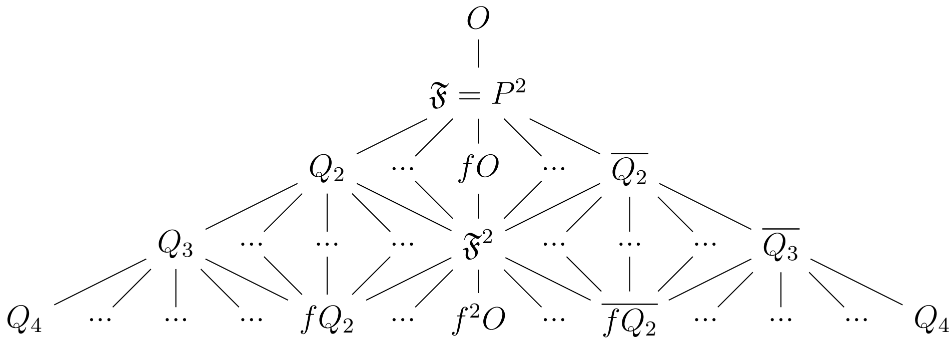
4.1 Inert Case



Consider the case for $O = \mathbb{Z}[f\omega]$, where f is a prime element of $\mathbb{Z}$

Theorem 4.1 *(Theorem 4.1 of [**PZ**]) Suppose $\mathfrak{F} = fD$ is a prime ideal of D. Then every basic $\mathfrak{F}-$primary ideal of O containing $\mathfrak{F}^2$, and lies in the following set of pairwise distinct ideals $\boldsymbol{J} = \{J = (f, f^2\omega), J_a = (f^2, f(a+\omega))\ for\ 0 \leq a \leq f-1\}$.*

4.2 Split Case



Consider the case for $O = \mathbb{Z}[f\omega]$, where f is a prime element of $D = \mathbb{Z}[\omega]$ as above, but $\mathfrak{F} = fD$ splits as an ideal of D, i.e. $\exists$ ideals $P, \overline{P}$ such that $\mathfrak{F} = fD = PQ$, where $P \neq \overline{P}$. $P, \overline{P}$ principal if and only if f is not irreducible in D, which by definition of f implies that $P, \overline{P}$ are not principal. However, since the class group of D is finite, $\exists$ integers n such that P^n is principal.

Lemma 4.2 *(Lemma 4.2 of **[PZ]**) For ideals $P, \overline{P}$ such that $\mathfrak{F} = P\overline{P}$, β a fixed generator of P. $\forall n \in \mathbb{Z}, \beta^n \notin O$.*

This next theorem characterizes all the $\mathfrak{F}$-basic elements of O. Unlike the above inert case, there are basic elements of arbitrarily large norms, and thus infinitely many.

Theorem 4.3 *(Theorem 4.3 of **[PZ]**) For each $n \in \mathbb{N}$, let $t_n = f\beta^n$. An element $t \in O$ is basic if and only if t is associated in D either to t_n or its conjugate, for some $n \in \mathbb{N}$. Moreover, the principal ideals $t_n wO, t_n{}^- w'O$, for $n > 0$ and $w, w' \in D^*, w, w' \notin O$, are pariwise incomparable and do not contain $\mathfrak{F}^2$.*

Note a Special Principal Ideal Ring (Special PIR) R is a principal ideal ring with a unique prime ideal M, such that M is nilpotent. Special PIR's are chained rings i.e, have linearly ordered ideals.

Lemma 4.4 *(Lemma 4.4 of **[PZ]**) The quotient ring O/t_nO is a Special PIR for all $n \in \mathbb{N}$. In particular, the ideals (necessarily $\mathfrak{F}$-primary) that contain t_nO are equal to (f^i, t_n), for $i = 1, ..., mn+2$, and their norm of (f^i, t_n) is f^i.*

Lemma 4.5 *(Proposition 4.5 of* ***[PZ]****) Let* $t \in O$ *be a basic* $\mathfrak{F}$*-primary element of norm* f^m*, and let* $i \in \mathbb{N}$ *be such that* i, m*. Then the ideal* $I = (f^i, t)$ *of* O *is a* D*-module, equal either to* P^iQ *or* PQ^i*. In particular we get* $(f^i, t_i) = (f^i, t_n)$*, for every* $n \geq i$*.*

The next theorem gives a description of the ideals of O that contain a basic element.

For every $k \geq 1$, let $Q_k = (f^k, t_k) = P^k\overline{P}$, and $Q_1 = \mathfrak{F}$.

Theorem 4.6 *(Theorem 4.6 of* ***[PZ]****)*

i) Let Q *be a* $\mathfrak{F}$*-basic ideal, then exists* $k \geq 1$ *such that*

$fQ_k \subset Q \subseteq Q_k$.

ii) The ideals $Q_k = (f^k, t_k)$*, for* $k \in \mathbb{N}$*, are pairwise distinct.*

iii) An ideal Q *of* O *contains* $Q_k \iff Q \in \{Q_i | i = 0, ..., k\}$*.*

iv) If Q *contains a basic element and it is not principal, then either* $Q = Q_k$ *or* $Q = \overline{Q}_k$ *for some* $k \in \mathbb{N}$*.*

Now we characterize the ideals of O that do not conatin a F-basic element.

Theorem 4.7 *(Theorem 4.7 of* ***[PZ]****) Let* Q *be a basic* $\mathfrak{F}$*-primary ideal not containing any basic element. Then;*

i) Q *lies properly between* Q_k *and* fQ_k*, for some* $k > 0$*.*

ii) $Q = (f^{k+1}, af^k + t_k)$ *for some* $1 \leq a \leq f - 1$*.*

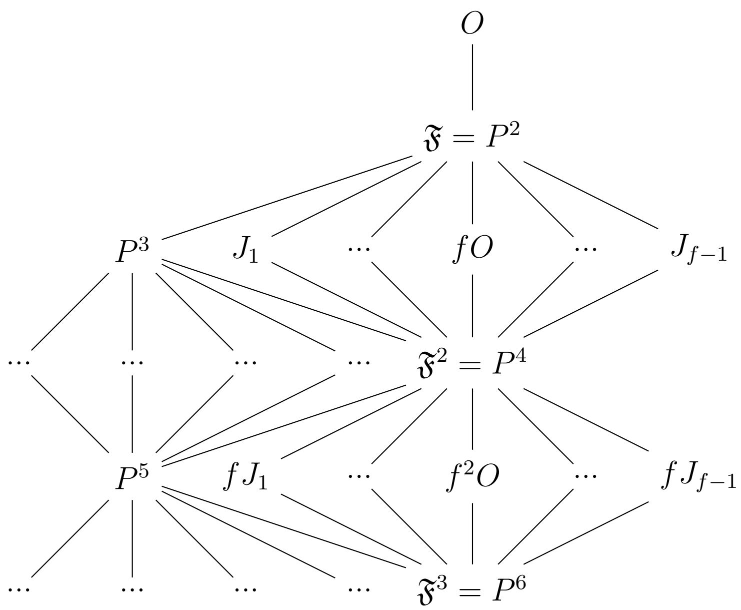
iii) Q *does not contain any other basic* $\mathfrak{F}$*-primary ideal.*

iv) Q *is an invertible ideal of* O*.*

v) Q *is not a* D*-module.*

Combined the previous two theorems characterize the lattice of $\mathfrak{F}$-primary ideals for a splitting prime.

4.3 Ramified



Consider the case where $O = \mathbb{Z}[f\omega]$, where f is a prime element of D, but $\mathfrak{F} = fD = P^2$,P a prime ideal of D. I.e. the conductor splits into two prime ideals. Following results from [**PZ**].

Theorem 4.8 *(Theorem 4.9 of [**PZ**])*

i) If $d \equiv 1, 2 \pmod 4$ or $d \equiv 3 \pmod 4$ and $f \neq 2$, then we have $P = fD + \sqrt(2)dD$. If $d \equiv 3 \pmod 4$ and $f = 2$, then $P = 2D + (1 + \sqrt(2)d)D$.

ii) Let $Q \subseteq \mathfrak{F}$ be a basic $\mathfrak{F}$-primary ideal. Then either $P^4 \subset Q \subseteq P^2$ or $P^5 \subset Q \subseteq P^3$.

iii) If $\mathfrak{F}^2 \subset Q \subset \mathfrak{F}^2$, then either $Q = J_a = (f^2, f(a + \sqrt{2d}))$, for some $a = 0, 1, ..., f-1$, or $Q = J = (f, f^2\sqrt{2d}) = fO$.

iv) if $fP^3 = P^5 \subset Q \subset P^3$, then $Q = H_a = (f^3, af^2 + f\sqrt{2d})$, for some $a = 0, 1, ..., f-1$, or $Q = (f^2, f^2\sqrt(2)d) = f\mathfrak{F} = P^4$, except when $f = 2$ and $d \equiv 3 \pmod 4$; in this latter case, we either get $Q = (8, 2(1 + \sqrt(2)d))$, or $Q = (4, 4(1 + \sqrt(2)d)) = P^4$.

This next result shows that besides the basic elements of $t \in \mathfrak{F}$ such that $F^2 \subset tO \subset \mathfrak{F}$, we have other basic elements dependent on whether or not P is a principal ideal of D.

Lemma 4.9 *(Proposition 4.10 of **[PZ]**) There exists a basic element $t \in O$ such that $P^5 \subset tO \subset P^3$ if and ony if P is a principal ideal of D. If this condition holds, say $P = \beta D$, for some $\beta \in D$, then every basic element is associated to $f\beta$ by a unit of D.*

So by the previous two results, we can characterize the lattice of $\mathfrak{F}$-primary ideals of a ramified prime.

5 Extension of General Results to the Orders $\mathbb{Z}[\alpha f^\sigma]$

Now, to begin our study of the case of orders $\mathbb{Z}[\alpha f^\sigma]$ of the quadratic number ring $\mathbb{Z}[\alpha]$. We show how our general theorems from above carry over into this new setting, characterize the ideals our lattice will have, and note how the changed structure differs in general from our prime case. We show that the original primary theorem used to characterize an ideals maximal properly contained ideals by whether or not they were a module holds, if in a slightly different manner. We will see our global maximum (bound in a lattice) changes, causing separation in powers between αf and αf^σ, and as we will see, many of our new properties discovered can be characterized in terms of this 'separation'.

Theorem 5.1 *Let V be an ideal in $\mathbb{Z}[\alpha f^\sigma]$, of the form $(f^k, \sum c_j f^j + \alpha f^\sigma)$, where $\mathfrak{F} = (f^\sigma, \alpha f^\sigma)$ and $\mathfrak{m} = (f, \alpha f^\sigma)$. Then;*
i) if V is a $\mathbb{Z}[\alpha f^\beta]$ module, where $\beta \in \{0, ..., k-1\}$, then there uniquely exists $f+1$ pairwise distinct ideals between V and fV, of the form

$$\boldsymbol{J} = \{J = (f^k, \sum c_j f^{j+1} + \alpha f^{l+1}),$$
$$J_a = (f^{k+1}, af^k + \sum b_j f^j + \alpha f^l) for 0 \leq a \leq f-1\}$$

ii) if V is NOT a $\mathbb{Z}[\alpha f^{\sigma}]$ module, then there is only one proper ideal between V and fV.

Proof: i) $V/fV =\cong Z/fZ \oplus Z/fZ$ as abelian groups. $Z/fZ \oplus Z/fZ$ has exactly $f+1$ proper, non-zero subgroups, so it suffices to show that J, J_a are pairwise distinct, as their containment and structure as ideals is obvious. To verify distinctness:

Suppose that $J_a = J_b$. Then for proper x_0, x_1, y_0, and $y_1 \in \mathbb{Z}$, we have that

$$f^{\delta}(af^{k-\delta} + \sum c_j f^{j-\delta} + \alpha f^{l-\delta}) =$$
$$f^{\delta}((x_0 + x_1 f^{\sigma}\alpha)f^{k-\delta} + (y_0 + y_1 f^{\sigma}\alpha)(bf^{k-\delta} + \sum c_j f^{j-\delta} + \alpha f^{l-\delta})$$

Which implies that

$$af^{k-\delta} + \sum c_j f^{j-\delta} + \alpha f^{l-\delta} - x_0 f^k - y_0(bf^{k-\delta} + \sum c_j f^j + \alpha f^{l-\delta}) =$$
$$x_1 f^{\sigma} f^k \alpha + y_1 f^{\sigma}\alpha(bf^{k-\delta} + \sum c_j f^{j-\delta} + \alpha f^{l-1})$$

which is an element of $f^{\sigma}\alpha V \subset V$ where δ is the smallest power of k, l, or j for a non-zero c_j term. So we get three cases depending on what δ is:

Case $\delta = k$ Then we have that $(1-y_0)f^{l-k}\alpha \in O$ where $\sigma \leq l < \sigma + k$, so $(1-y_0) \in f\mathbb{Z}$

Case $\delta = l$ Then we have that $(1-y_0)\alpha \in O$, so $(1-y_0) \in f\mathbb{Z}$.

Case $\delta = j$ Then for some c_j, we have that $(1-y_0)c_j \in O$ where $c_j \in \{1, 2, ..., f-1\}$, so we have that $(1-y_0) \in f\mathbb{Z}$.

So in all cases, we get that $(1-y_0) \in f\mathbb{Z}$. By minimality of k $af^{k-\delta} - y_0 bf^{k-1} \in V$, so we can conclude that $1 \equiv y_0, a \equiv y_0 b \mod f$, so $a \equiv b \mod f$, and so $a = b$ since they are both in $\{0, 1, ..., f-1\}$. So we have shown that $J_a = J_b$ if and only if $a = b$.

We have $J \neq J_a$ for all a, as J contains f^k whereas J_a does not.

ii) Suppose V is NOT a module of $\mathbb{Z}[\alpha f^{\beta}]$ for any $0 \leq \beta \leq \alpha - 1$. So V is O-proper and thus invertible. Let $D + \mathbb{Z}[\alpha f^{\sigma-1}]$. So $fV \subset fDV \subset V$ since V is O-proper. Let J be any proper ideal between I and fI. Let $Q = JI^{-1}$, so Q is f-primary, as it is the product of two f-primary ideals. So $J = QI \subset I\mathfrak{m} = IfD$, but $[V : fV] = f^2$ so we have that $J = IfD = I\mathfrak{m}$.

It should be immediately noted that this is the generalized version of our Thereom 3.8, the thoerem **[PZ]** used to locally characterize the lattice structure, and is similarily one of the central pieces of this work. Both in this fact it constituted significant work in its own right, and is relied upon by many of the results to be given. Our Theorem 5.1 differs from **[PZ]** in two key ways. First, now an ideal I will contain more than one proper ideal between itself and fI if it is a module of any order between our $O = \mathbb{Z}[\alpha f^{\sigma}]$ and $D = \mathbb{Z}[\alpha f]$. This largely increases the complexity in determining if an ideal is a module in a relevant sense, as it vastly expands the possible relevant orders to be considered as σ increases. Secondly, there is no part of this theorem showing every ideal is contained by some f multiple of $\mathfrak{F}$. As the conductor is no longer maximal, not all ideals satisfy this property nymore, and much more of the structure is determined by ideals not contained by $\mathfrak{F}$. However, it should be noted for all ideals in our lattice I, there exists some postive integer n such that $f^n I$ is containe by some f multiple of $\mathfrak{F}$, $f^m\mathfrak{F}$. This has not been examined in any depth in this work, and may be relevant to future undertakings.

As the above theorem is the only relevant context modules will come up again in this paper, from here on out, whenever we are discussing whether or not an f-primary ideal of $\mathbb{Z}[\alpha f^{\sigma}]$ is a module, we are referring to whether or not it is a module of $\mathbb{Z}[\alpha f^{\beta}]$, for any $0 \leq \beta < \alpha$. We will only refer to ideals being modules with respect to orders between ours and the base ring of integers.

Note that the above theorems give two possibilities for the number of proper sub-ideals containing the f-multiple and whether or not an ideal is a module, also a Boolean possibility. Because of this, we get a new way to determine if an ideal is a module in our orders as follows;

Theorem 5.2 *Consider the f-primary lattice of ideals of $\mathbb{Z}[\alpha f^{\sigma}]$. If I is an ideal of our order, then;*
i) I is a module of $\mathbb{Z}[\alpha f^{\beta}]$, $0 \leq \beta < \alpha$ if and only if there exists $f+1$ proper sub-ideals of I properly containing fI
ii) I is NOT a module if and only if there uniquely exists 1 proper sub-ideal of I properly containing fI

Proof: Theorem 5.1 above fulfills the forward directions of both i) and ii). For the backwards direction of i), suppose I is *not* a module. Then Theorem 5.1 implies we have a contradiction. Similarily, we have for the backwards direction of ii), that assuming I *is* a module results in a contradiction by Theorem 5.1. So we have the above.

To start characterizing these lattices, we must know which ideals will be in these lattices in the first place. In the prime case described previously, there were relatively few ideals. In our higher power case, we have far more ideals by several orders of magnitude, and they may be quite similar but not arranged closely. As such, it will be very useful to give a way to list all the ideals one will have, and have a general notion of there location in the lattice, if a specific location cannot always be given canonically. As such, while this wasn't a particularly useful result to give for our original case as it was fully characterized, a general treatment of the theory will require it until a fully algorithmic characterization of all powers can be given, so one can check they have obtained a correct characterization of a constructed lattice. As such:

Theorem 5.3 *Consider the f-primary lattice of $\mathbb{Z}[\alpha f^\sigma]$. All ideals whose integer components are generated by f^k, $k \geq 1$, are given by $(f^k, \sum_{k/2}^{k-1} a_i f^i + \alpha f^\delta)$ if k is even, or $(f^k, \sum_{(k+1)/2}^{k-1} a_i f^i + \alpha f^\delta)$ if k is odd, where a_i is 0 for $i = 0$, and $a_i \in \{0, 1, ..., f-1\}$ for $i \geq 0$, max$(\alpha, k/2$ or $(k+1)/2) \leq \sigma \leq k+\alpha$, and $\delta \geq \sigma$, except for any ideal where $(f^k, \alpha f^\delta)$ could generate any ideal of the form $f^\gamma \mathfrak{m}$. The only such ideals of that form are $f^\gamma \mathfrak{m}$, $\gamma \in \mathbb{Z}_+$.*

Proof: Consider the lowest term with the lowest power of f times our algebraic element. Then any other term will have a totally rational residual when compared to this lowest algebraic power element. So the residual will be divisible by the lowest power rational power, so the whole ideal can always be generated by the lowest rational power and term with lowest algebraic power.
Now consider an ideal of the form $(f^k, \beta + \alpha f^{\sigma+k})$, where β is a linear combination of the lower powers of f and some element of $\mathbb{Z}/f\mathbb{Z}$ as a constant as allowed above. Then we see this ideal contains $\alpha f^{\sigma+k}$ as a lone element, and thus has β also as a lone element, so we have $\gcd(f^k, \beta)$, which will be some lower power of f, and thus not actually of the supposed form if written in the manner above.

A key structural difference between the prime case and all other higher power cases is that the conductor $\mathfrak{F}$ is no longer the maximal ideal of our lattice. Our lattice is still bounded, but our maximal lattice is now $(f, \alpha f^\sigma)$ for number ring $\mathbb{Z}[\alpha f^\sigma]$. We denote our maximal ideal of $\mathbb{Z}[\alpha f^\sigma]$ by $\mathfrak{m}$, which will be of particular interest to us, as in the previous case of $\mathbb{Z}[\alpha f]$, $\mathfrak{m}$ and $\mathfrak{F}$ were one and the same. We denote ideals I found between $\mathfrak{m}$ and $\mathfrak{F}$ to be called $\mathfrak{m}$-basic.As it turns out, like in our prime case the $\mathfrak{F}$-basic ideals characterized our lattice structure, it is now the $\mathfrak{m}$-basic and $\mathfrak{F}$-basic ideals that induce our structure. A full characterization of our lattices of interest will likely require a more articulated understanding of the interactions between these two layers.

Theorem 5.4 *Consider the f-primary lattice of ideals of $\mathbb{Z}[\alpha f^\sigma]$. The maximal ideal $\mathfrak{m}$ is given by $(f, \alpha f^\sigma)$ and the conductor $\mathfrak{F}$ is given by $(f^\sigma, \alpha f^\sigma)$.*

Proof: Consider the above lattice. We want to show that the above ideal is our maximal ideal, in other words, for all ideals I in our lattice, $I \subset (f, \alpha f^\sigma)$. If I is in the lattice, it's f-primary, i.e., all element of I are generated by multiples and powers of f and αf^σ, which by definition is our $\mathfrak{m}$, so $I \subset \mathfrak{m}$ for all I in our lattice. Now we want to show that $\mathfrak{F} = (f^\sigma, \alpha f^\sigma)$ is the largest ideal of $\mathbb{Z}[\alpha]$ that is still an $\mathbb{Z}[\alpha f^\sigma]$ ideal. As αf^σ is the lowest power of f times α in our structure, and any algebraic component $\beta + \alpha f^\sigma$ where β is a linear combination of lower powers of f, or an integer component of f^k, $k < \sigma$, would no longer be the same ideal under multiplication by α. I.e, $(f^k, \beta + \alpha f^\sigma)$ would contain αf^k, which would certainly not be in $\mathbb{Z}[\alpha f^\sigma]$ much less our ideal, and it would also contain $\alpha\beta + \alpha^2 f^\sigma$, which would result in far lower powers of f in the algebraic or integer component than is possible. So we see that the largest ideal in both our number ring and order is $(f^\sigma, \alpha f^\sigma)$.

Recall in the inert/ramified/split prime cases given above, the particular importance of the conductor $\mathfrak{F}$ and fO, and their various powers $f^k\mathfrak{F}$ and f^kO. If we were to take the sub-lattices given only by those ideals, they graphically represent the 'center line' of our diagrams. We want a formal definition of this structure, and in our higher power case for it to include not only the conductor $\mathfrak{F}$ and fO, but also the maximal ideal $\mathfrak{m}$ and the ideals between $\mathfrak{m}$ and $\mathfrak{F}$.

Definition 5.1 *Consider the order $\mathbb{Z}[\alpha f^\sigma]$. The* ***spine*** *of this order is the set of ideals*

$$\{(f, \alpha f^\sigma), (f^k, \alpha f^\sigma) \forall 1 \leq k \leq \sigma, f^n(f^\sigma, \alpha f^\sigma), f^n(f^{\sigma+1}, \alpha f^\sigma) \forall n\}$$

We now show that for the case $\sigma \geq 2$, all spinal ideals are modules.

Theorem 5.5 *Consider the f-primary lattice of ideals of $\mathbb{Z}[\alpha f^\sigma]$, $\sigma \geq 2$. All ideals of the spine of the f-primary lattice are modules of $\mathbb{Z}[\alpha f^\beta]$, some $\beta \in \{0, 1, ..., \alpha - 1\}$.*

Proof: Preliminarily, α is a quadratic algebraic integer, i.e, there exists $a, b, c \in \mathbb{Z}$, such that $ax^2 + bx + c = 0$ minimally in $\mathbb{Z}[\alpha]$. So we get that αf^σ is an algebraic integer solving $Ax^2+Bx+C = 0$ minimally, where $A = a, B = f^\sigma b$, and $C = 2f^\sigma c$. We know that we have three cases to consider along the spine. Ideals of the form;

i) $(f^k, \alpha f^\sigma)$, where $1 \leq k < \sigma$

ii) $(f^\delta, \alpha f^\delta)$ where $\sigma \leq \delta$

iii) $(f^\delta, \alpha f^{\delta+1})$ where $\sigma \leq \delta$

i) Assume I is of the form in *i*). We want to show that it contains more than one distinct proper sub-ideal that properly contains fI. We know if I is a module what its proper sub-modules of the above form will be. Suppose I is not a module, then all of its proper sub-ideals properly containing fI should be equivalent, so we consider $(f^{k+1}, \alpha f^\sigma)$ to be equal to $(f^{k+1}, f^k + \alpha f^\sigma)$. We have by a generalized Bezout's Identity, there exists x_0, y_0, x_1, and $y_1 \in \mathbb{Z}$ such that

$(x_0 + \alpha f^\sigma y_0)f^{k+1} + (x_1 + \alpha f^\sigma y_1)\alpha f^\sigma = f^k + \alpha f^\sigma$, which implies

$x_0 f^{k+1} + y_0\alpha f^{\sigma+k+1} + x_1\alpha f^\sigma + y_1\alpha^2 f^{2\sigma} = f^k + \alpha f^\sigma$ and

$x_0 f^{k+1} + y_0\alpha f^{\sigma+k+1} + x_1\alpha f^\sigma - y_1 A\alpha f^\delta - y_1 B = f^k + \alpha f^\delta$, matching up components we see

$x_0 f^{k+1} - y_1 B = f^k$, so $y_1 B = f^k$, but f^σ divides B, which is a contradiction. So we have the above ideals are not distinct.

Remark about the next two cases, they all follow the same argumentative structure.

ii) Let I be of the form in *ii*). Set the two sub-ideals to be equal;

$$(f^{\delta+1}, \alpha f^\sigma) = (f^{\delta+1}, af^\delta + \alpha f^\delta)$$

So we suppose there exists x_0, y_0, x_1, and y_2 such that

$$(x_0 + \alpha f^\delta y_0)f^{\delta+1} + (x_1 + \alpha f^\delta y_1)\alpha f^\delta = f^\delta + \alpha f^\delta$$

so similiar to the above we have, $x_0 f^{\delta+1} - y_1 B = f^\delta$, but $f^{\delta+1}$ divides B, so by Bezout's identity we once again get a contradiction, and have that the two supposed sub-ideals are distinct.

iii) Lastly, let I be of the form in iii). Suppose the sub-ideals $(f^{\delta+1}, \alpha f^{\delta+1})$ and $(f^{\delta+1}, f^\delta + \alpha f^{\delta+1})$ are equivalent. So there exists x_0, y_0, x_1, and y_1 such that

$$(x_0 + y_0 \alpha f^\sigma)f^{\delta+1} + (x_1 + y_1 \alpha f^\sigma)\alpha f^{\delta+1} = f^\delta + \alpha f^{\delta+1}$$

so as above, we get that
$x_0 f^{\delta+1} - y_1 B = f^\delta$, which is again a contradiction by Bezout's Identity.
All of the sub-ideals used are both proper and contain the ideal fI properly as inspection of their powers readily shows. As any ideal that has more than one proper sub-ideal properly containing itself times f is a module, we have finished showing all spinal ideals of the f-primary lattice of $\mathbb{Z}[\alpha f^\sigma]$ are modules for $\sigma \geq 2$.

6 Bar Notation for Ideals

Our lattice structures quickly become too large to easily represent and work with at a global level. However, we can take advantage of the two-generator structure and the common structure of containmnet given to us by Theorem 5.1 and base some notation off of these properties to more easily represent our lattices.

$$
\begin{tikzcd}
\Big|_{a}^{b}(O,\sigma) \arrow[d, dash] \\
F
\end{tikzcd}
$$

$$
\begin{tikzcd}
(O,\sigma) \arrow[drr, dash] & (O,\sigma+a) \arrow[dr, dash] & \cdots \arrow[d, dash] & (O,\sigma+(b-1)a) \arrow[dl, dash] \\
 & & F &
\end{tikzcd}
$$

Definition 6.1 *Let the first structure above be called a **grouping**, and to be equivalent to the below set of ideals containing F. When we write a grouping with more n-bars after the generators, we call it an **n-grouping**. The figure immediately below is a two grouping, and is equivalent to the collection of groupings below it. We apply multiple bars recursively as the further two equivalent figures below shows.*

$$
\begin{tikzcd}
(O,\sigma)\Big|_{a}^{b}\Big|_{c}^{d} \arrow[d, dash] \\
F
\end{tikzcd}
$$

$$
\begin{tikzcd}
(O,\sigma)\Big|_{c}^{d} \arrow[drr, dash] & (O,\sigma+a)\Big|_{c}^{d} \arrow[dr, dash] & \cdots \arrow[d, dash] & (O,\sigma+(b-1)a)\Big|_{c}^{d} \arrow[dl, dash] \\
 & & F &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\alpha,\beta)\Big|_{a1}^{b1}\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[d, dash] \\
F
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\alpha,\beta)\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[drr, dash] & (\alpha,\beta+a1)\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[dr, dash] & \cdots\cdots \arrow[d, dash] & (\alpha,\beta+a1(b1-1))\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[dl, dash] \\
& & F &
\end{tikzcd}
$$

Note we start with the innermost bar, and work outwards from there. As Theorem 5.1 shows, each ideal that is a module contains f ideals who differ by an addition of some power of f. Now we give rules for groupings containing other groupings, with the caveat that the the i-th grouping bar of both groupings must have the same top number for $1 \leq i \leq \min(n,m)$. For an n-grouping that contains an m-grouping, we consider the j-th (n-1)-grouping to contain the j-th (m-1) grouping for $1 \leq j \leq n$ as such;

$$
\begin{tikzcd}
(\alpha,\beta)\Big|_{a1}^{b1}\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[d, dash] \\
(\eta,\gamma)\Big|_{c1}^{d1}\Big|_{c2}^{d2}\cdots\Big|_{cm}^{dm}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\alpha,\beta)\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[d, dash] & (\alpha,\beta+a1)\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[d, dash] & \cdots \arrow[d, dash] & (\alpha,\beta+(b1-1)a1)\Big|_{a2}^{b2}\cdots\Big|_{an}^{bn} \arrow[d, dash] \\
(\eta,\gamma)\Big|_{c2}^{d2}\cdots\Big|_{cm}^{dm} & (\eta,\gamma+c1)\Big|_{c2}^{d2}\cdots\Big|_{cm}^{dm} & \cdots & (\eta,\gamma+(d1-1)c1)\Big|_{c2}^{d2}\cdots\Big|_{cm}^{dm}
\end{tikzcd}
$$

These definitions allow us to work with this notation recursively, and to collapse data in our lattices into an easier to interpret format. Essentially, when we have two ideals with multiple of our expansion bars, we delete the innermost bars on top and bottom, and connect the lattices of like indice. This does mean that for two expanded sets of lattices to be connected, they must have the same number of multiples.

For any n-grouping $(\alpha, \beta)\Big|_{a1}^{b1}\Big|_{a2}^{b2}...\Big|_{an}^{bn}$ we refer to its **base** as the ideal (α, β).

7 The Canonical Conductive Sub-lattice

We have not fully characterized the structure of every single f-primary lattice of $\mathbb{Z}[\alpha f^{\sigma}]$, however, there is one sub-lattice we were able to prove the existence of and fully characterize for all number rings of the above form. This sub-lattice is derived from the sub-lattice having the conductor $\mathfrak{F}$ as the maximal ideal, and ignoring that the ideals in between multiples of the conductor are modules. In other words, we get this sub-lattice by picking out the structural features most similar to our original prime case, where the multiples of fO were not modules. Note, that the following sub-lattice is characterized solely from applications of Theorem 5.1, so is not dependent on the prime in question being inert/ramified/split. So, while we did not manage to characterize any of the higher power lattices of ramified or split primes, we are not at a complete loss of the structure of these lattices. The characterization of this sub-lattice once again shows the importance of the conductor, and how much structure it determines even when not the maximal ideal of our lattice.

As stated, we are interested in the sub-lattice where the conductor $\mathfrak{F}$ is the maximal ideal, and all other sub-ideals of the conductor *except* the non-spinal ideals that ideals of the form $(f^{\beta}, \alpha f^{\beta+1})$ contain. In other words, we ignore that $(f^{\beta}, \alpha f^{\beta+1})$ is a $\mathbb{Z}[\alpha f^{\delta}]$-module, $\delta < \alpha$, and choose the structure most similar to the spine of our prime case. To define this sublattice, we need the spine, and two other different sets of ideals

Definition 7.1 *Consider the order $\mathbb{Z}[\alpha f^\sigma]$. The* ***co-spine*** *is given to us by the set of ideals;*

$$\mathfrak{C}_{\mathfrak{S}} = \{(f^{\sigma+1}, \alpha f^\sigma)\Big|_{f^\sigma}^{f}, (f^{\sigma+2}, \alpha f^\sigma)\Big|_{f^\sigma}^{f}\Big|_{f^{\sigma+2}}^{f}, ..., (f^{2\sigma}, \alpha f^\sigma)\Big|_{f^\sigma}^{f}...\Big|_{f^{2\sigma-1}}^{f},$$
$$(f^{2\sigma+k}, \alpha f^{\sigma+k})\Big|_{f^{\sigma+k}}^{f}...\Big|_{f^{2\sigma+k-1}}^{f}, (f^{2\sigma+k}, \alpha f^{\sigma+k+1})\},$$
$$\textit{all } k \geq 0$$

Furthermore, we need to define the **in between** ideals of the conductive sublattice, in other words, the ideals between the cospine and spine;

Definition 7.2 *Consider the order $\mathbb{Z}[\alpha f^\sigma]$. An* ***in between*** *ideal is any ideal at the k-th layer whose integral/algebraic powers are both between the integral/algebraic power of the k-th layers spinal ideal and co-spinal base.*

Now we can define the conductive sublattice;

Definition 7.3 *Consider the order $\mathbb{Z}[\alpha f^\sigma]$. The* ***conductive sublattice*** *of this order is the sublattice containing the spine, cospine, and in between ideals.*

This sublattice will be shown to exist and be fully characterized for al lattices under our investigation regardless of power or the splitting type of our prime. This gives a universal starting point to aid research focused on full characterizations of such lattices. We now go about proving said existence and characterizing. As the spines structure is already well known (and we are ignoring the module structure of the ideals in between f multiples of the conductor), we begin by characterizing the co-spine;

Theorem 7.1 *Consider the conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$. The first σ layers of the cospine are given by $(f^{\sigma+k}, \alpha f^\sigma)\Big|_{f^\sigma}^{f}...\Big|_{f^{\sigma+k-1}}^{f}$, all of which are modules except the σ-th grouping itself (will either contain the next layer of spinal ideal or in between ideals besides the next layer of cospinal ideals)*

The k-th cospinal grouping after the σ-th layer is;

a) if k is even, the cospinal grouping is, for $v = k/2$,
$(f^{2\sigma+v}, \alpha f^{\sigma+v})\Big|_{f^v f^\sigma}^{f} \cdots \Big|_{f^v f^{2\sigma-1}}^{f}$,
which are NOT modules

b) if k is odd, the cospinal grouping is, for $n = ((k-1)/2$ and $m = (k+1)/2$,
$(f^{2\sigma+n}, \alpha f^{\sigma+m})\Big|_{f^m f^{\sigma+1}}^{f} \cdots \Big|_{f^m f^{2\sigma-1}}^{f}$
which are modules

Proof: We know the first co-spinal grouping $(f^{2\sigma+1}, \alpha f^\sigma)\Big|_{f^\sigma}^{f}$, is given to us by Theorem 5.1 on $\mathfrak{F}$. The rest are given to us by induction on the k-th layer by Theorem 5.1, and collecting all the ideals resulting form increasing the rational power for the next cospinal layer, up until $k = \sigma$. We show they are modules below.

To show the first σ co-spinal groupings are modules, consider the ideal $(f^{\sigma+k}, \sum_{i=0}^{k-1} a_i f^{\sigma+i} + \alpha f^\sigma)$ for $0 < k < \alpha$. We see that the leading rational power is not σ higher than the algebraic power, so the sub-ideal with a leading generator one power higher is proper. Consider the ideals

$$(f^{\sigma+k+1}, vf^{\sigma+k} + \sum_{i=0}^{k-1} a_i f^{\sigma+i} + \alpha f^\sigma) \text{ and}$$
$$(f^{\sigma+k+1}, jf^{\sigma+k} + \sum_{i=0}^{k-1} a_i f^{\sigma+i} + \alpha f^\sigma)$$

for distinct $v, j \in \mathbb{Z} \bmod f$. If they are equal, then there exists $x_0, y_0, x_1, y_1 \in \mathbb{Z}$ such that

$$(x_0 + \alpha f^\sigma y_0)f^{\sigma+k+1} + (x_1 + \alpha f^\sigma y_1)(vf^{\sigma+k} + \sum_{i=0}^{k-1} a_i f^{\sigma+i} + \alpha f^\sigma)$$
$$= jf^{\sigma+k} + \sum_{i=0}^{k-1} a_i f^{\sigma+i} + \alpha f^\sigma$$

We see this implies that, along with our algebraic equations,

$$x_0 f^{\sigma+k+1} + x_1 \sum a_i f^{\sigma+i} + x_1 v f^{\sigma+k} - y_1 f^{2\sigma} b$$
$$= \sum a_i f^{\sigma+i} + j f^{\sigma+k}$$

By subtraction, we see that

$$x_0 f^{\sigma+k+1} + (x_1 - 1) \sum a_i f^{\sigma+i} + (x_1 v - j) f^{\sigma+k} - y_1 f^{2\sigma} b$$
$$= 0 \text{ if and only if } x_1 = 1 \text{ and} x_1 v = j$$

But $v \neq j$, a contradiction.

Proof by Induction: Consider the k-th ideal after the σ-th layer. Let k=1 (base case). We know that the σ-th layer grouping was NOT a module. So Theorem 5.1 tells us it contains $(f^{2\sigma}, \alpha f^{\sigma+1})\Big|_{f^{\sigma+1}}^{f} \cdots \Big|_{f^{2\sigma-1}}^{f}$. By the above we have that this IS a module. Applying Theorem 5.1 to this first cospinal ideal and collecting all ideals whose rationaly power increased shows shows the base case for when k is even. The above once again tells us this 'even' layered grouping is *not* a module. Let k be odd, and assume this statement holds for all layers up to and including $k+1$-st layer. We will show this implies the $k+2$-nd and $k+3$-rd layers are as hypothesized. The $k+1$-st layer is *not* a module, so Theorem 5.1 gives us the desired form. Recognizing the $k+2$-nd layer as f times the k-th layer, we see it is a module. A further application of Theorem 5.1 and collection of ideals constituting the ones with an increased rational power yields the desired form. This completes the induction.

We want to characterize all the groupings found between the spine and co-spine in this sub-lattice. We call a layer of the conductive sub-lattice **conductive** if the k-th layer spinal ideal is of the form $f^n \mathfrak{F}$ for some integer n. A layer which has a spinal ideal that is not such an f multiple of the conductor is **sub-conductive**. With this notion in mind, we prove that at the k-th layer of the spinal conductive lattice, if β and δ are the rationals powers of the k-th spinal ideal and the k-th co-spinal grouping, then there are as a many groupings between the spine and co-spine as $\delta - \beta - 1$, and whose forms are depndent upon whether or not we are at a conductive or subcondctive layer. Note, we will refer to a function sep_k for each conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$ as the difference in powers of the rational components of the k-th layer spinal and co-spinal ideals, minus one. I.e., if we are at the k-th layer of the conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$, and the spinal and co-spinal ideals have base rational component as f^β, f^γ respectively, then $sep_k = \gamma - \beta - 1$. So we claim that we have as many in between ideals at the k-th level as sep_k, a claim we show in the next theorem. One last notational consideration, we denote the k-th layers t-th in-between ideals by ${S_t}^k$, and denote the whole set of in-between ideals at the k-th layer by S^k.

Theorem 7.2 *Consider $\mathbb{Z}[\alpha f^\sigma]$ and it's spinal conductive lattice. All in-between ideals are modules. The first in-between ideal exists at the third layer $\iff$ $\sigma \geq 3$. There are as many in between ideals at the k-th layer as sep_k, of the form (assuming $sep_k > 0$);*
if the k-th layer is sub-conductive, then ${S_b}^k$ is contained by/contains
${S_{b-1}}^{k-1}/{S_{b-1}}^{k+1}$ and ${S_b}^{k-1}/{S_b}^{k+1}$. Moreover, ${S_b}^k$ is a $2b-1$ grouping, or;
if the k-th layer is conductive, then then ${S_b}^k$ is contained by/contains
${S_b}^{k-1}/{S_b}^{k+1}$ and ${S_{b+1}}^{k-1}/{S_{b+1}}^{k+1}$. Moreover, ${S_b}^k$ is a $2b$ grouping.

Proof: We wish to show that all of these in-between ideals are in fact modules. Consider the (conductive layer) ideal
${S_t}^k = (f^{\sigma+v+t}, \sum_{i=0}^{2t-1} a_i f^{\sigma+v-t+i} + \alpha f^{\sigma+v-t})$ for $k = 2v \leq \sigma$, $t < v$
Since the power of the rational term is not a full σ higher, we know that the sub-ideal with leading generator $f^{\sigma+v+t+1}$ is proper. As before, we will show the

ideals

$$(f^{\sigma+v+t+1}, jf^{\sigma+v+t} + \sum_{i=0}^{2t-1} a_i f^{\sigma+v-t+i} + \alpha f^{\sigma+v-t}) \text{ and}$$
$$(f^{\sigma+v+t+1}, gf^{\sigma+v+t} + \sum_{i=0}^{2t-1} a_i f^{\sigma+v-t+i} + \alpha f^{\sigma+v-t}) \text{ for distinct } j, g \in \mathbb{Z} \bmod f$$

are distinct. If they contain each other, then there exists $x_0, y_0, x_1, y_1 \in \mathbb{Z}$ such that

$$(x_0 + \alpha f^{\sigma} y_0) f^{\sigma+v+t+1} + (x_1 + \alpha f^{\sigma} y_1)(jf^{\sigma+v+t} + \sum_{i=0}^{2t-1} a_i f^{\sigma+v-t+i} + \alpha f^{\sigma+v-t})$$
$$= gf^{\sigma+v+t} + \sum_{i=0}^{2t-1} a_i f^{\sigma+v-t+i} + \alpha f^{\sigma+v-t})$$

Using the algebraic equations referenced in the first section on preliminary algebra, we see this implies that

$x_0 f^{\sigma+v+t-1} + x_1 \sum a_i f^{\sigma+v-t+i} - x_1 f^{2\sigma+v-t} b + x_1 j f^{\sigma+v+t}$

$= \sum a_i f^{\sigma+v-t+i} + gf^{\sigma+v+t}$

Subtracting the terms, we get that

$x_0 f^{\sigma+v+t-1} + (x_1 - 1) \sum a_i f^{\sigma+v-t+i} - x_1 f^{2\sigma+v-t} b + (x_1 j - g) f^{\sigma+v+t} = 0$

which occurs if and only if $x_1 = 1$ and $x_1 j = g$. But $j \neq g$, thus a contradiction. So these ideals cannot be equal, so by previous theorems we have that our in-between ideal is a module.

To show this for the sub-conductive case, we repeat the same argument above with the algebraic power one higher. Similarily, we end with the equation

$$x_0 f^{\sigma+v+t-1} + x_1 \sum a_i f^{\sigma+v-t+i+1} - x_1 f^{2\sigma+v-t+1} b + x_1 j f^{\sigma+v+t+1}$$
$$= \sum a_i f^{\sigma+v-t+i+1} + gf^{\sigma+v+t+1}$$

which results in the same argument under subtraction. So we conclude that all in-between ideals are themselves modules.

Let the first in-between ideal exist at the third layer. Supose $\sigma < 3$, Then the first three cospinal base's are of the form $(f^{\sigma+1}, \alpha f^{\sigma})$, $(f^{\sigma+2}, \alpha f^{\sigma})$, and $(f^{\sigma+2}, \alpha f^{\sigma+1})$, with the difference in cases being the rank as a grouping. The first three spinal ideal's after the conductor are $(f^{\sigma}, \alpha f^{\sigma+1}), (f^{\sigma+1}, \alpha f^{\sigma+1})$ and $(f^{\sigma+1}, \alpha f^{\sigma+2})$. Applying Theorem 5.1 to these ideals (not the spinal subconductive ideals) shows that only spinal and copsinal ideals are contained. Which implies no in between ideal exists at the third level, a contradiction. Suppose $\sigma \geq 3$. Then the first three cospinal base's are given by $(f^{\sigma+1}, \alpha f^{\sigma}), (f^{\sigma+2}, \alpha f^{\sigma})$, and $(f^{\sigma+3}, \alpha f^{\sigma})$, applying Theorem 5.1 shows that the first cospinal base nly contains cospinal and spinal ideals, but that the second cospinal base contains ideals not in the spine or cospine, hence in between ideals (one in particular). Note, that the third spinal ideal after the conductor will never be of the form of an f multiple of $\mathfrak{F}$, so the first in between grouping will always occur on a sub-conductive layer.

Base Case(s): By the work immediately above, we know the first in-between ideal occurs at the third layer on a subconductive level. We know $\sigma \geq 3$, so the spinal ideal and co-spinal grouping at the third layer are $(f^{\sigma+1}, \alpha f^{\sigma+2})$ and $(f^{\sigma+3}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} \ldots \Big|_{f^{\sigma+2}}^{f}$. By Theorem 5.1 applied to the previous layers spinal ideal $(f^{\sigma+1}, \alpha f^{\sigma+1})$, we get that it contains the grouping $(f^{\sigma+2}, \alpha f^{\sigma+1})\Big|_{f^{\sigma+1}}^{f}$. Observation shows these are the collection all the mximally proper sub-ideals resulting from increasing the algebraic power of the previous cospinal grouping $(f^{\sigma+2}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} \ldots \Big|_{f^{\sigma+1}}^{f}$ when applying Theorem 5.1. So ${S_1}^3$ is contained by ${S_0}^2$ and ${S_1}^2$ as desired. Applying Theorem 5.1 (will show these in-between ideals are modules at the end) shows this first in-between grouping contains $(f^{\sigma+3}, \alpha f^{\sigma+1})\Big|_{f^{\sigma+1}}^{f} \Big|_{f^{\sigma+2}}^{f}$ (either the first in between grouping for a conductive layer, or the next co-spinal grouping for $\sigma = 3$) and $(f^{\sigma+2}, \alpha f^{\sigma+2})$. This shows the next layers in-between ideal, our base case for the conductive layers, exists. Moreover this shows that ${S_1}^3$ contains ${S_0}^4$ and ${S_1}^4$, completing our base case for the sub-conductive layer. Consider the first conductive in between grouping at the 4th layer (assuming $\sigma > 3$) $(f^{\sigma+3}, \alpha f^{\sigma+1})\Big|_{f^{\sigma+1}}^{f}\Big|_{f^{\sigma+2}}^{f}$. We know the previous in-between grouping is containted by ${S_1}^4$ by previous work and the third cospinal grouping $(f^{\sigma+3}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} \cdots\Big|_{f^{\sigma+2}}^{f}$ by Theorem 5.1 (the collection of ideals resulting from raising the algebraic power). So ${S_1}^4$ is contained by ${S_1}^3$ and ${S_2}^3$. Applying Theorem 5.1 to ${S_1}^4$ yields the desired containments. This finishes the base case.

Next we prove existence of the bases of the groupings, then prove their complete structure. Assume a strong induction on k. Now assuming our theorem holds for all layers up to and for $k-1$. Let β_k, δ_k are the respective rational powers of the k-th spinal ideal and co-spinal grouping. Let $j_k = max(\delta_k - \beta_k, 0)$, and assume j_k is greater than 0. We can list the ideals of interest, assume the k-th spinal ideal is of the form $(f^{\beta}, \alpha f^{\iota})$ and the co-spinal base grouping is of the form $(f^{\eta}, \alpha f^{\zeta})$. By our inductive hypothesis that the bases groupings between the spine and co-spine at the k-th layer are $J_k = \{(f^{\beta+1}, \alpha f^{\iota-1}), ..., (f^{\eta-1}, \alpha f^{\zeta+1})\}$

We get four cases dependent on the spinal ideal and co-spinal base grouping of the k-1-th layer.

Case 1: k-1-th layer $(f^{\beta-1}, \alpha f^{\beta})/(f^{\eta-1}, \alpha f^{\zeta})$ The k-1-th layer implies that $\iota = \beta$, so

$S_{k-1} = \{(f^{\beta}, \alpha f^{\beta-1}), ..., (f^{\eta-1}, \alpha f^{\zeta})\}$

and

$S_k = \{(f^{\beta+1}, \alpha f^{\beta-1}), ..., (f^{\eta-1}, \alpha f^{\zeta+1}\}$

and so we clearly see the base ideals of S_k are contained in the base ideals of S_{k-1}, the co-spine and the spine.

Case 2: $(k-1)$-th layer $(f^{\beta}, \alpha f^{\beta})/(f^{\eta-1}, \alpha f^{\zeta})$ The $(k-1)$-th layer implies that $\iota = \beta + 1$, so

$S_{k-1} = \{(f^{\beta+1}, \alpha f^{\beta-1}), ..., (f^{\eta-1}, \alpha f^{\zeta+1})\}$

and

$S_k = \{(f^{\beta+1}, \alpha f^{\beta}), ..., (f^{\eta-1}, \alpha f^{\zeta+1})\}$

and so we clearly see the base ideals of S_k are contained in the ideals of S_{k-1}, the co-spine and the spine.

Case 3: $(k-1)$-th layer $(f^{\beta-1}, \alpha f^{\beta})/(f^{\eta}, \alpha f^{\zeta-1})$ The $(k-1)$-th layer implies that $\iota = \beta$, so

$S_{k-1} = \{(f^{\beta}, \alpha f^{\beta-1}, ..., (f^{\eta-1}, \alpha f^{\zeta})\}$

and

$S_k = \{(f^{\beta+1}, \alpha f^{\beta-1}), ..., (f^{\eta-1}, \alpha f^{\zeta+1}\}$

and so we clearly see the base ideals of S_k are contained in the ideals of S_{k-1}, the co-spine and the spine.

Case 4: $(k-1)$-th layer $(f^{\beta}, \alpha f^{\beta})/(f^{\eta}, \alpha f^{\zeta-1})$ The k-1-th layer implies that $\iota = \beta + 1$, so

$S_{k-1} = \{(f^{\beta+1}, \alpha f^{\beta-1}), ..., (f^{\eta-1}, \alpha f^{\zeta+1})\}$

and

$S_k = \{(f^{\beta+1}, \alpha f^{\beta}), ..., (f^{\eta-1}, \alpha f^{\zeta+1})\}$

and so we clearly see the base ideals of S_k are contained in the ideals of S_{k-1}, the co-spine and the spine. This completes our induction on the number of distinct bases of groupings we have between layers, i.e., the number of distinct groupings we have at each layer of our sub-lattice.

Now to use induction to fully characterize the structure for our in-between ideals. Let this theorem hold for all layers up to k, $k \leq \sigma$. Want to establish an induction on the k-th layer's collection of in-between ideals itself.

Supose k is a subconductive layer, so $k = 2v + 1, v \in \mathbb{Z}$.

Base Case: Consider ${S_1}^k$. The k-th and $(k-1)$-st spinal ideals and cospinal groupings are given by

$(f^{\sigma+v}, \alpha f^{\sigma+v+1}), (f^{\sigma+v}, \alpha f^{\sigma+v})$ and

$(f^{\sigma+k}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} ... \Big|_{f^{\sigma+k-1}}^{f}, (f^{\sigma+k-1}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} ... \Big|_{f^{\sigma+k-2}}^{f}$ respectively.

${S_1}^k$ is contained by the $k-1$-th spinal ideal, whose only other maximal proper ideal is the k-th spinal ideal. Applying Theorem 5.1 and the previous inductive hypothesis, this characterizes ${S_1}^k$ as a 1-grouping contained by ${S_0}^{k-1}$ and ${S_1}^{k-1}$. ${S_1}^k = (f^{\sigma+v+1}, \alpha f^{\sigma+v})\Big|_{f^{\sigma+v}}^{f}$. An application of Theorem 5.1 shows this grouping contains the $k+1$-th spinal ideal (an f multiple of $\mathfrak{F}$, is the raised algebraic power) and $(f^{\sigma+v+2}, \alpha f^{\sigma+v})\Big|_{f^{\sigma+v}}^{f}\Big|_{f^{\sigma+1}}^{f}$. This completes the base case for our induction on the subconductive collection of in between ideals S^k.

Assume our theorem holds for the first $t-1$ in between groupings of layer k. Consider ${S_t}^k$. By our inductive hypothesis, we have this grouping is contained by ${S_{t-1}}^{k-1}$ and ${S_t}^{k-1}$ a $2t-2$ and $2t$ grouping respectively, and that ${S_{t-1}}^k$ is a $2t-3$ grouping. Knowing these are all modules, by counting we have that ${S_t}^k$ is a $2t-1$-grouping. We have the forms of the t and $t-1$-th layers spinal ideals, so we know ${S_t}^k = (f^{\sigma+v+t}, \alpha f^{\sigma+v+1-t})\Big|_{f^{\sigma+v+1-t}}^{f} ... \Big|_{f^{\sigma+v+t-1}}^{f}$. By Theorem 5.1, this means ${S_t}^k$ contains

$(f^{\sigma+v+t+1}, \alpha f^{\sigma+v+1-t})\Big|_{f^{\sigma+v+1-t}}^{f} ... \Big|_{f^{\sigma+v+t}}^{f}$ and

$(f^{\sigma+v+t}, \alpha f^{\sigma+v+2-t})\Big|_{f^{\sigma+v+2-t}}^{f} ... \Big|_{f^{\sigma+v+t-1}}^{f}$

In other words, contains ${S_{t-1}}^{k+1}$ and ${S_t}^{k+1}$.

Suppose layer k is a conductive layer (so k is even, $k = 2v$). Assume a second induction as above. Base Case: Consider ${S_1}^k$, contained by ${S_1}^{k-1}$ and ${S_2}^{k-1}$. ${S_1}^{k-1}$ also contains the k-th layers spinal ideal, and is a 1-grouping. As such, an application of Theorem 5.1 shows us ${S_1}^k$ is a 2-grouping. Similarily, applying Theorem 5.1 to ${S_1}^k$ itself yields the desired containments. ${S_1}^k = (f^{\sigma+v+1}, \alpha f^{\sigma+v-1})\Big|_{f^{\sigma+v-1}}^{f}\Big|_{f^{\sigma+v}}^{f}$, which Theorem 5.1 implies contains

${S_1}^{k+1} = (f^{\sigma+v+1}, \alpha f^{\sigma+v})\Big|_{f^{\sigma+v}}^{f}$ and

${S_2}^{k+1} = (f^{\sigma+v+2}, \alpha f^{\sigma+v-1})\Big|_{f^{\sigma+v-1}}^{f}\Big|_{f^{\sigma+v}}^{f}\Big|_{f^{\sigma+v+1}}^{f}$. This completes the base case of the conductive layer.

Assume the same strong induction on t as above. Consider ${S_t}^k =$
$(f^{\sigma+v+t}, \alpha f^{\sigma+v-t})\Big|_{f^{\sigma+v-t}}^{f} \cdots \Big|_{f^{\sigma+v+t-1}}^{f}$
which is contained by ${S_t}^{k-1}$, a $2t$-grouping. Since ${S_t}^{k-1}$ also contains the $2t-2$ grouping ${S_{t-1}}^k$, an application of Theorem 5.1 shows ${S_t}^k$ is a $2t$-grouping. Likewise, Theorem 5.1 shows it is also contained by ${S_{t+1}}^{k-1}$. Theorem 5.1 applied to ${S_t}^k$ itself shows it to contain
$(f^{\sigma+v+t}, \alpha f^{\sigma+v-t+1})\Big|_{f^{\sigma+v-t+1}}^{f} \cdots \Big|_{f^{\sigma+v+t-1}}^{f}$ and
$(f^{\sigma+v+t+1}, \alpha f^{\sigma+v-t})\Big|_{f^{\sigma+v-t}}^{f} \cdots \Big|_{f^{\sigma+v+t}}^{f}$
This completes our induction.

This theorem allows us to compute the number of groupings between the spine and co-spine of each layer, given that we know the structure of the lattice up until that layer. However, we can go further, and using the first σ layers of the conductive sub-lattice to characterize the full substructure. Much like the basic layer of ideals in our original case characterized the structure fully, analogously the first alpha layers of groupings determine the structure of the conductive sub-lattice. In particular, we show that the $\sigma - 1$ and σ layer can be used to calculate the rest of the sub-structure once the sub-lattice has been built up to that point. Note, we will refer to a function sep_k for each conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$ as the difference in powers of the rational components of the k-th layer spinal and co-spinal ideals, minus one. In other words, if we are at the k-th layer of the conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$, and the spinal and co-spinal ideals have base rational component as f^β, f^γ respectively, then $sep_k = \gamma - \beta - 1$. Now we characterize the 'in between' groupings of the conductive sub-lattice for the pre-σ layers of the conductive sub-lattice.

Theorem 7.3 *Consider the conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$. If the k-th layer has δ groupings in between the spine and co-spine, and the k-1-th layer has $\delta - 1$ groupings, $1 \leq k < \sigma$, then the k+1-th layer will have δ groupings in between the* *spine and co-spine.*

Proof: Let the above hold. Since each pre-σ cospinal layer has its rational power increase by one level to level, and the *sep* function increases by 1 between the k and $k+1$-th index, we have that the k-1-th and k-th layer of spinal ideals are given by $(f^\beta, \alpha f^\beta)$ and $(f^\beta, \alpha f^{\beta+1})$ respectively. So $sep_{k-1} = k-1-\beta-1 = \delta$ and $sep_k = k-\beta-1 = \delta$. If the rational power of the spinal ideal increased instead, the separation would remain constant. So the k+1-th layer spinal ideal is given by $(f^{\beta+1}, \alpha f^{\beta+1})$. So $sep_{k+1} = k+1-(\beta+1)-1 = (k-\beta)-1 = \delta$.

Theorem 7.4 *Consider the conductive sub-lattice of* $\mathbb{Z}[\alpha f^\sigma]$*. If the k-th and k-1-th layer has* σ *groupings in between the spine and cospine,* $1 \leq k < \sigma$*, then the k+1-th layer will have* $\delta + 1$ *groupings in between the spine and co-spine.*

Proof: Let the above hold. So $sep_{k-1} = sep_k = \sigma + 1$. We know the rational power of the cospinal groupings increase by one each successive layer prior to the α-th layer, so the rational power of the spinal ideals must also have increased by one from the $(k-1)$-th layer to the k-th layer. In other words, the $(k-1)$-th spinal ideal is of the form $(f^\beta, \alpha f^{\beta+1})$ and the k-th spinal ideal is of the form $(f^{\beta+1}, \alpha f^{\beta+1})$. So the k+1-th spinal ideal will be of the form $(f^{\beta+1}, \alpha f^{\beta+2})$.
So $sep_k = k-(\beta+1)-1 = k-\beta-2 = \delta$ which implies
$sep_{k+1} = k+1-(\beta+1)-1 = k-\beta-1 = \delta+1$.

Theorem 7.5 *Consider the conductive sub-lattice of* $\mathbb{Z}[\alpha f^\sigma]$*. If the* σ*-layer and previous layer have* δ *in between groupings, then all subsequent layers will have* δ *in between groupings.*

Proof: Let the above hold. So $sep_\sigma = sep_{\sigma-1} = \sigma + 1$. So the σ and $\sigma - 1$ layer spinal ideal are given by $(f^\beta, \alpha f^\beta)$ and $(f^{\beta-1}, \alpha f^\beta)$ respectively. So $sep_{\sigma-1} =$

$(2\sigma - 1) - (\beta - 1) - 1 = 2\sigma - \beta - 1 = sep_\sigma = \delta$. Let i be odd. Then the $\sigma + i$-th layer has spinal and co-spinal ideals of the form $(f^{\beta+(i-1)/2}, \alpha f^{\beta+(i+1)/2})$ and $(f^{2\sigma+(i-1)/2}, \alpha f^{\sigma+(i+1)/2})$ respectively. So

$sep_{\sigma+i} = 2\sigma + (i-1)/2 - \beta - (i-1)/2 - 1 = 2\sigma - \beta - 1 = \delta$

Let i be even. Then the $\sigma + i$-th layer has spinal and co-spinal ideals of the form $(f^{\beta+i/2}, \alpha f^{\beta+i/2})$ and $(f^{2\beta+i/2}, \alpha f^{\beta+i/2})$ respectively. So $sep_{\sigma+i} = 2\beta + i/2 - \beta - i/2 - 1 = 2\sigma - \beta - 1 = \delta$.

Theorem 7.6 *Consider the conductive sub-lattice of $\mathbb{Z}[\alpha f^\sigma]$. If the σ-th layer has δ in between groupings, and the previous layer has $\delta - 1$ in between groupings, then the i-th layer after the σ-th layer has σ groupings if i is even, and $\sigma - 1$ groupings if i is odd.*

Proof: Let the above hold. So $sep_\sigma = \delta$ and $sep_{\sigma-1} = \delta$- 1. So the σ and $\sigma - 1$ layer spinal ideal are given by $(f^\beta, \alpha f^\beta)$ and $(f^\beta, \alpha f^{\beta-1})$ respectively. Let i be odd. Then the $\sigma + i$-th layer has spinal and co-spinal ideals of the form $(f^{\beta+(i+1)/2}, \alpha f^{\beta+(i-1)/2})$ and $(f^{2\sigma+(i-1)/2}, \alpha f^{\sigma+(i+1)/2})$ respectively. So $sep_{\sigma+i} = 2\sigma + (i-1)/2 - \beta - (i+1)/2 - 1 = 2\sigma - \beta - 2 = \sigma - 1$ Let i be even. Then the $\sigma + i$-th layer has spinal and co-spinal ideals of the form $(f^{\beta+i/2}, \alpha f^{\beta+i/2})$ and $(f^{2\sigma+(i/2)}, \alpha f^{\sigma+i/2})$ respectively. Then

$sep_{\sigma+i} = 2\sigma + i/2 - \beta - i/2 - 1 = 2\sigma - \beta - 1 = sep_\sigma = \sigma$

These theorems allow us to know how many groupings we will have in between the spine and co-spine by just being able to construct the first layer of the sub-structure. They also show that the number of groupings in between the spine and co-spine is either fixed or alternating after the first σ layers.

We are now prepared to give our master theorem fully characterizing the conductive sublattice for all of our f-primary, quadratic number rings.

Theorem 7.7 *Consider the f-primary lattice of $\mathbb{Z}[\alpha f^{\sigma}]$. The sub-lattice given by all sub-ideals of the conductor $\mathfrak{F}$ except those who are maximally proper to the spinal ideals $(f^k, \alpha f^{k+1})$, $k \geq \sigma$. This sub-lattice exists and is characterized by;*

i) The first σ layers of the cospine are given by $(f^{\sigma+k}, \alpha f^{\sigma})\Big|_{f^{\sigma}}^{f} \cdots \Big|_{f^{\sigma+k-1}}^{f}$, all of which are modules except the σ-th grouping itself (will either contain the next layer of spinal ideal or inbetween ideals besides the next layer of cospinal ideals)

ii) The k-th cospinal grouping after the σ-th layer is;

a) if k is even, the cospinal grouping is, for $v = k/2$,
$(f^{2\sigma+v}, \alpha f^{\sigma+v})\Big|_{f^{v} f^{\sigma}}^{f} \cdots \Big|_{f^{v} f^{2\sigma-1}}^{f}$
which are NOT modules

b) if k is odd, the cospinal grouping is, for $n = ((k-1)/2$ and $m = (k+1)/2$,
$(f^{2\sigma+n}, \alpha f^{\sigma+m})\Big|_{f^{m} f^{\sigma+1}}^{f} \cdots \Big|_{f^{m} f^{2\sigma-1}}^{f}$
which are modules

iii) All in-between ideals are modules. There are as many in between ideals at the k-th layer as sep_k. The first in-between ideal exists at the third layer $\iff$ $\sigma \geq 3$. If the first in-between ideal exists, the spinal ideal at that level is NOT an f multiple of $\mathfrak{F}$.

iv) If $sep_k > 0$, and the k-th spinal ideal is an f multiple of $\mathfrak{F}$, then ${S_b}^k$ is contained by/contains ${S_b}^{k-1}/{S_b}^{k+1}$ and ${S_{b+1}}^{k-1}/{S_{b+1}}^{k+1}$. Moreover, ${S_b}^k$ is a 2b grouping.

v) If $sep_k > 0$, and the k-th spinal ideal is of the form $(f^{\beta}, \alpha f^{\beta+1})$, then ${S_b}^k$

is contained by/contains ${S_{b-1}}^{k-1}/{S_{b-1}}^{k+1}$ *and* ${S_b}^{k-1}/{S_b}^{k+1}$*. Moreover,* ${S_b}^k$ *is a* $2b-1$ *grouping.*

vi) If the $\sigma - 1$ *and* σ *layer have;*

a) the same number of in -between groupings, all subsequent layers will have the same number of in-between groupings.

b) $\beta - 1$ *and* β *in-between groupings respectively, all subsequent layers will alternately have* $\beta - 1$ *or* β *groupings.*

Proof: First, to address existence. By nature of the objects considered, $\mathfrak{F}$ certainly exists. By succesive application of Theorem 5.1 (and recalling all spinal ideals are modules) we easily know that the spine exists. Defining our conductive sub-lattice as above, containing all sub-ideals of $\mathfrak{F}$, besides those maximally proper to the $'fO'$ ideals, we see that any two ideals would have a smallest ideal that would contain them both, a largest ideal that both would contain, and can only be maximally primary to one ideal. In other words, the structure has joins, meets, and only one 'arrow' between objects, so along with $\mathfrak{F}$ obviously being the maximal ideal, this sturcture is a sub-lattice. Addressing i), ii), and iii) show existence of the rest of the non-spinal structures.

i) We know this by Theorem 6.1

ii) We know this by Theorem 6.1.

For iii), iv), and v) Theorem 6.2 shows all these statements.

vi) Theorem's 6.5 and 6.6 provide the proof for these cases.

Q.E.D

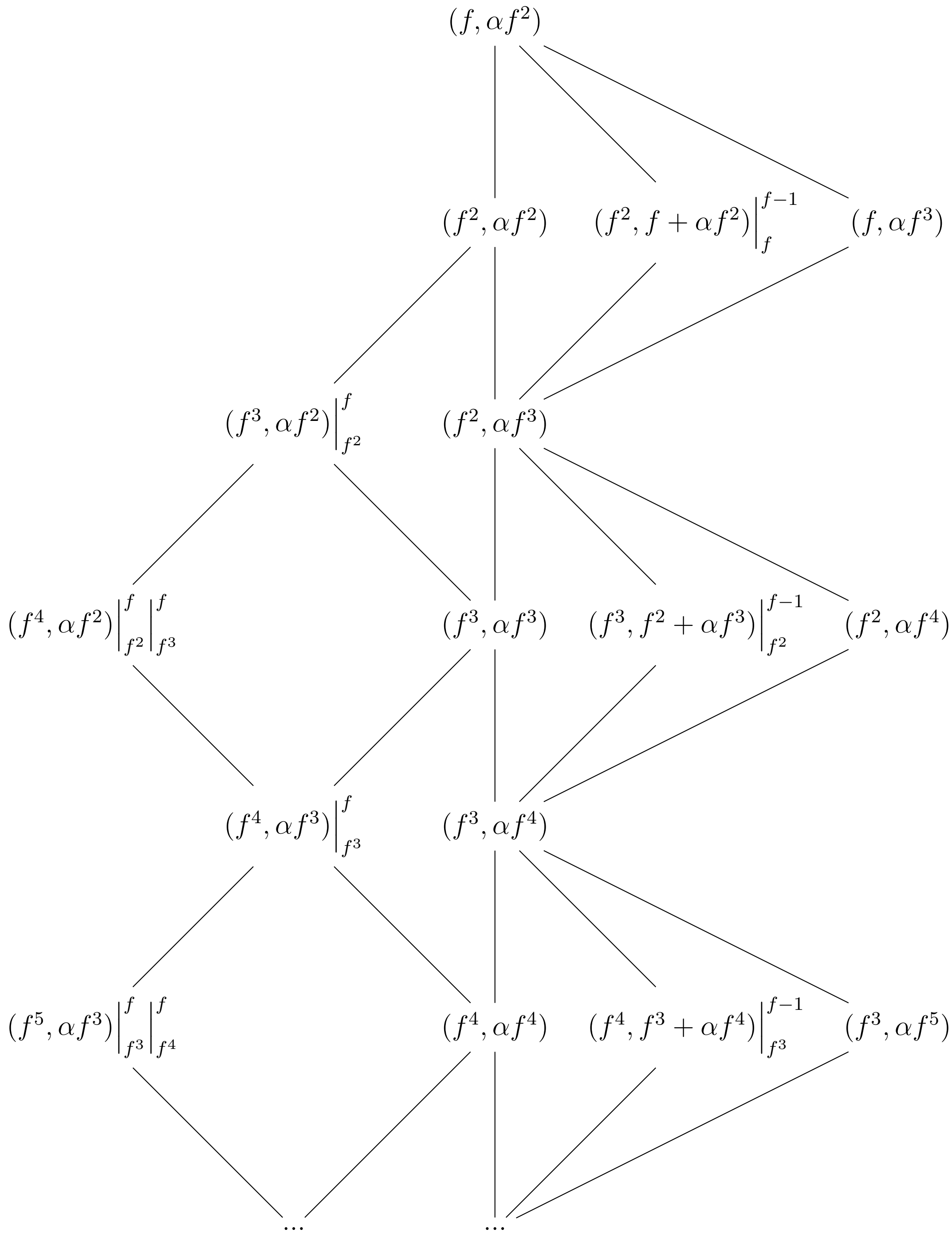
It should be noted that the conductive sub-lattice for the prime case fully and completely characterizes the inert case, and forms the vast majority of the second power. While we will see that in the higher power cases the conductive sub-lattice constitutes far less of the lattice, a significant portion of the lattice is still quite similar to the conductive sub-lattice in terms of structure.

8 Lattice of f-primary Ideals of $\mathbb{Z}[\alpha f^\sigma]$ for f Inert

Let f be some prime element of our quadratic number field $D = \mathbb{Z}[\alpha]$. Here, $\mathfrak{m} = (f, \alpha f^\sigma)$, and $\mathfrak{F} = (f^\alpha, \alpha f^\sigma)$. Let an ideal found between $\mathfrak{m}^n$ and $\mathfrak{m}^{n+1}$ be called n-th $\mathfrak{m}$-layer ideals, and ideals found particulary between $\mathfrak{m}$ and $\mathfrak{F}$, $\mathfrak{m}$-basic ideals. Let an ideal found between $f^n\mathfrak{F}$ and $f^{n+1}\mathfrak{F}$ be called n-th $\mathfrak{F}$-layer ideals. We will not delve into these notions deeply, or make use of the idea of basic to a conductor as previous work has done. The relation between 'maximal basic' and 'conductor basic' is not fully elucidated. Instead, we take all the maximal and conductor basics as a whole and attempt to derive a few lattices of particular orders by examining the initial layers with the above theory. We will once again find initial layers of ideals determine the rest of the structure. However, in contrast to our initial prime case, the lattice structures of our higher power orders are determined by the layers between the maximal ideal $\mathfrak{m}$ and the conductor $\mathfrak{F}$, and the layer between the conductor $\mathfrak{F}$ and $f^2\mathfrak{F}$, not just the $\mathfrak{F}$-basic layer as in our initial case.

8.1 $\mathbb{Z}[\alpha f^2]$ Case for Inert Prime

Here, we characterize the lattice of f-primary ideals of $\mathbb{Z}[\alpha f^2]$ for prime f inert. Much like our other cases, the conductor $\mathfrak{F}$ no longer being the maximal ideal of our lattice induces a larger, more expansive structure than our original case of $\mathbb{Z}[\alpha f]$. However, unlike all other cases of $\mathbb{Z}[\alpha f^\sigma]$, our second power case is the only other than our original case that has no proper ideals between the conductor and maximal ideal. Because of this, while the structure of $\mathbb{Z}[\alpha f^2]$ is quite different from our original case, it is also quit different than most of our other cases when compared with each other. This illustrates how radically the changes to the structure of our lattices occur when the maximal ideal and conductor are 'separated' by proper ideals between them along the spine. We will end up seeing that the lattice of $\mathbb{Z}[\alpha f^2]$ is essentially the conductive sub-lattice given above, and a disjoint structure descending from the maximal ideal resembling our original case, joining only along the spine. This separation of structure allows $\mathbb{Z}[\alpha f^2]$ to be an excellent example of how the structure of our lattices initially changes when the conductor is no longer the maximal ideal, and the changes that occur from the maximal and conductor ideals being separated by proper sub-ideals of the maximal ideal.


$(f,\alpha f^2)$
$(f^2,\alpha f^2)$
$(f^2, f+\alpha f^2)\Big|_{f}^{f-1}$
$(f,\alpha f^3)$
$(f^3,\alpha f^2)\Big|_{f^2}^{f}$
$(f^2,\alpha f^3)$
$(f^4,\alpha f^2)\Big|_{f^2}^{f}\Big|_{f^3}^{f}$
$(f^3,\alpha f^3)$
$(f^3, f^2+\alpha f^3)\Big|_{f^2}^{f-1}$
$(f^2,\alpha f^4)$
$(f^4,\alpha f^3)\Big|_{f^3}^{f}$
$(f^3,\alpha f^4)$
$(f^5,\alpha f^3)\Big|_{f^3}^{f}\Big|_{f^4}^{f}$
$(f^4,\alpha f^4)$
$(f^4, f^3+\alpha f^4)\Big|_{f^3}^{f-1}$
$(f^3,\alpha f^5)$
...
...

Theorem 8.1 *Consider the lattice of f-primary ideals of $\mathbb{Z}[\alpha f^2]$. If ideal I is of the form;*

i) $(f^k, \alpha f^{k+1})$, it is a module which contains $(f^{k+1}, \alpha f^{k+1})$, $(f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}$, and $(f^k, \alpha f^{k+2})$. All of these ideals are NOT modules except the initial module, and all of whom contain $(f^{k+1}, \alpha f^{k+2})$.

ii) $(f^k, \alpha f^k)$, it is a module containing $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ and $(f^k, \alpha f^{k+1})$. $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ is a module containing the next layers spinal ideal and $(f^{k+2}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}$. This grouping is not a module, and just contains $f((f^{k+1}, \alpha f^k)\Big|_{f^k}^{f})$.

Proof: Let the above hold. We know all spinal ideals are modules whose existence is easily shown by succesive applications of Theorem 5.1.

i)Ideals of the form $(f^k, \alpha f^{k+1})$ are spinal ideals and thus modules. An application of Theorem 5.1 shows that it contains the ideals $(f^{k+1}, \alpha f^{k+1})$, $(f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}$, and $(f^k, \alpha f^{k+2})$. We will show these latter ideals are not modules, which will show they contain $(f^{k+1}, \alpha f^{k+2})$ by Theorem 5.1.

Consider $(f^k, \alpha f^{k+2})$

$(f^k, \alpha f^{k+2})(1, \alpha) = (f^k, \alpha f^{k+2}, \alpha f^k, \alpha^2 f^{k+2}) = (f^k, \alpha f^k)$, so the ideal in question is not a $\mathbb{Z}[\alpha]$ module.

$(f^k, \alpha f^{k+2})(1, \alpha f) = (f^k, \alpha f^{k+2}, \alpha f^{k+1}, \alpha^2 f^{k+3}) = (f^k, \alpha f^{k+1})$, so the ideal in question is not a $\mathbb{Z}[\alpha f]$.

So $(f^k, \alpha f^{k+2})$ is not a module of any relevant orders.

Consider $(f^{k+1}, af^k + \alpha f^{k+1})$.

$$\begin{aligned}&(f^{k+1}, af^k + \alpha f^{k+1})(1, \alpha)\\ &= (f^{k+1}, af^k + \alpha f^{k+1}, \alpha f^{k+1}, \alpha^2 f^{k+1} + a\alpha f^k)\\ &= (f^k, \alpha f^{k+1})\end{aligned}$$

So $(f^{k+1}, af^k + \alpha f^{k+1})$ is not a $\mathbb{Z}[\alpha]$ module.

$$(f^{k+1}, af^k + \alpha f^{k+1})(f, \alpha f) = (f^{k+2}, af^{k+1} + \alpha f^{k+2}, \alpha f^{k+2}, \alpha^2 f^{k+2} + a\alpha f^{k+2})$$

Note the first three generators of the last ideal are certainly contained by our ideal in question. However, by our algebraic equations:

$$\alpha^2 = -\alpha a - b \text{ which implies}$$
$$\alpha^2 f^{k+2} + a\alpha f^{k+2}$$
$$= -\alpha a f^{k+2} - bf^{k+2} + a\alpha f^{k+1}$$

Which would imply that our ideal in question contains $a\alpha f^{k+1}$, a contradiction.

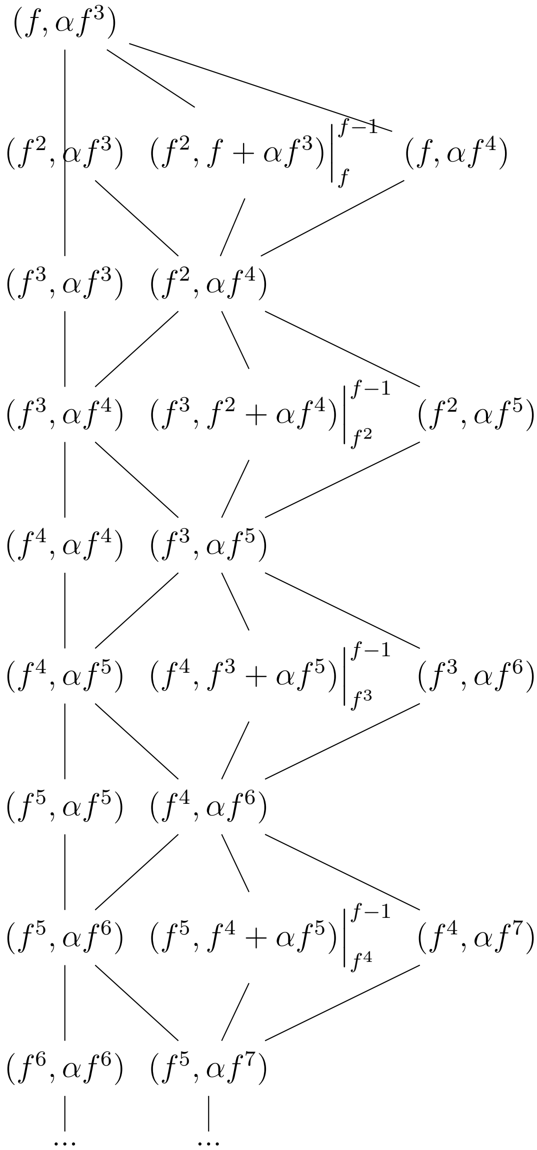
ii) Ideals of the form $(f^k, \alpha f^k)$ are the f multiples of the conductor, so there containments are given by the conductive sublattice. We see that our ideal contains $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ and $(f^k, \alpha f^{k+1})$. The latter ideal is characterized above and is part of the spine. The former grouping constitutes the cospine of the conductive sublattice. As such, since $(f^{k+1}, \alpha f^k)\Big|_{p^k}^{p}$ is an $\sigma - 1$ grouping on the cospine, by Theorem 5.1 we know that it is a module containing the next spinal ideal and $(f^{k+2}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}$. As the latter is a σ grouping on the cospine, we know that it isnt a module, and just contains the next cospinal layer.

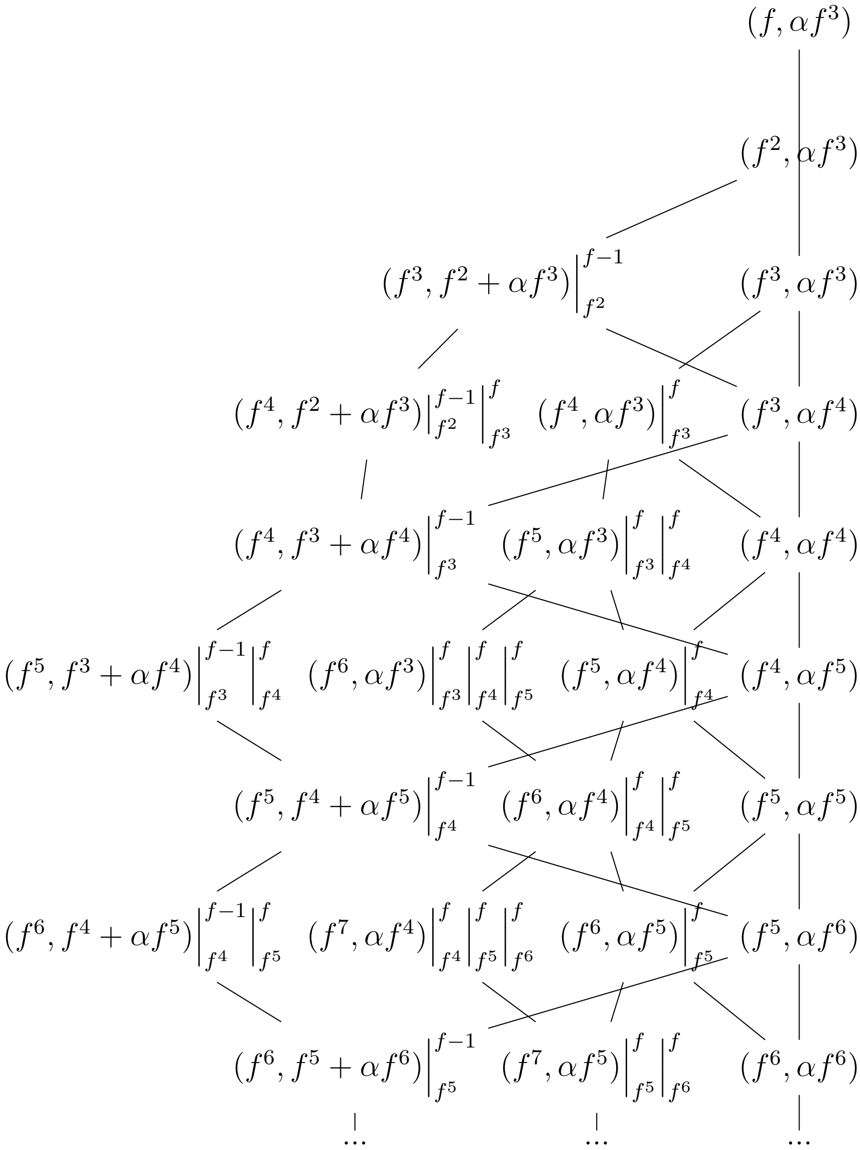
This finsishes the characterization of the f-primary lattice of $\mathbb{Z}[\alpha f^2]$.

8.2 $\mathbb{Z}[\alpha f^3]$ Case for Inert Prime

Here, we characterize the lattice of f-primary ideals of $\mathbb{Z}[\alpha f^3]$ for prime f inert. This is our first lattice structure in which the maximal ideal $(f, \alpha f^3)$ does not contain the conductor $(f^3, \alpha f^3)$ with no proper ideals separating them. As we shall see, this causes a distinct departure in structural characteristics of our lattice from our two earlier cases. Whereas our $\mathbb{Z}[\alpha f^2]$ could be viewed as two separate sub-lattices with only the spine in common, in our $\mathbb{Z}[\alpha f^3]$ case we find the ideals between our powers of the conductor are the meeting points of the sub-lattice induced by both the maximal ideal and the ideal between the maximal and conductor. While the structures branching off of the spine are still mostly disjoint, we see that the sub-lattice induced by $(f^2, \alpha f^3)$, the ideal between $\mathfrak{m}$ and $\mathfrak{F}$, seems reminiscent of our conductive sub-lattice. As our first structure in which separation between $\mathfrak{m}$ and $\mathfrak{F}$ occur, this structure is a prime example of the class of structures given by $\mathbb{Z}[\alpha f^\sigma]$.

The figure for the lattice is given by the disjoint union of the following two figures. Essentially, one can have the whole figure by identifying the spines of the two figures together. (Was split up for formatting requirements)


$(f, \alpha f^3)$
$(f^2, \alpha f^3)$
$(f^2, f + \alpha f^3)\Big|_{f}^{f-1}$
$(f, \alpha f^4)$
$(f^3, \alpha f^3)$
$(f^2, \alpha f^4)$
$(f^3, \alpha f^4)$
$(f^3, f^2 + \alpha f^4)\Big|_{f^2}^{f-1}$
$(f^2, \alpha f^5)$
$(f^4, \alpha f^4)$
$(f^3, \alpha f^5)$
$(f^4, \alpha f^5)$
$(f^4, f^3 + \alpha f^5)\Big|_{f^3}^{f-1}$
$(f^3, \alpha f^6)$
$(f^5, \alpha f^5)$
$(f^4, \alpha f^6)$
$(f^5, \alpha f^6)$
$(f^5, f^4 + \alpha f^5)\Big|_{f^4}^{f-1}$
$(f^4, \alpha f^7)$
$(f^6, \alpha f^6)$
$(f^5, \alpha f^7)$
...
...


$(f, \alpha f^3)$
$(f^2, \alpha f^3)$
$(f^3, f^2 + \alpha f^3)\Big|_{f^2}^{f-1}$
$(f^3, \alpha f^3)$
$(f^4, f^2 + \alpha f^3)\Big|_{f^2}^{f-1}\Big|_{f^3}^{f}$
$(f^4, \alpha f^3)\Big|_{f^3}^{f}$
$(f^3, \alpha f^4)$
$(f^4, f^3 + \alpha f^4)\Big|_{f^3}^{f-1}$
$(f^5, \alpha f^3)\Big|_{f^3}^{f}\Big|_{f^4}^{f}$
$(f^4, \alpha f^4)$
$(f^5, f^3 + \alpha f^4)\Big|_{f^3}^{f-1}\Big|_{f^4}^{f}$
$(f^6, \alpha f^3)\Big|_{f^3}^{f}\Big|_{f^4}^{f}\Big|_{f^5}^{f}$
$(f^5, \alpha f^4)\Big|_{f^4}^{f}$
$(f^4, \alpha f^5)$
$(f^5, f^4 + \alpha f^5)\Big|_{f^4}^{f-1}$
$(f^6, \alpha f^4)\Big|_{f^4}^{f}\Big|_{f^5}^{f}$
$(f^5, \alpha f^5)$
$(f^6, f^4 + \alpha f^5)\Big|_{f^4}^{f-1}\Big|_{f^5}^{f}$
$(f^7, \alpha f^4)\Big|_{f^4}^{f}\Big|_{f^5}^{f}\Big|_{f^6}^{f}$
$(f^6, \alpha f^5)\Big|_{f^5}^{f}$
$(f^5, \alpha f^6)$
$(f^6, f^5 + \alpha f^6)\Big|_{f^5}^{f-1}$
$(f^7, \alpha f^5)\Big|_{f^5}^{f}\Big|_{f^6}^{f}$
$(f^6, \alpha f^6)$
$\cdots$ $\cdots$ $\cdots$

Theorem 8.2 *Consider the f-primary lattice of* $\mathbb{Z}[\alpha f^3]$*. If ideal I is of the form;*
i) of our maximal ideal $(f, \alpha f^3)$*, it is a module containing*

$$(f^2, \alpha f^3), (f^2, f + \alpha f^3)\Big|_{f}^{f-1} \text{ and } (f, \alpha f^4)$$

ii) $(f^k, f^{k-1}+\alpha f^{k+1})\Big|_{f^{k-1}}^{f-1}$ *or* $(f^{k-1}, \alpha f^{k+2})$ *are NOT modules and contain* $(f^k, \alpha f^{k+2})$*, a module containing*

$$(f^{k+1}, \alpha f^{k+2}), (f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}, \text{ and} (f^k, \alpha f^{k+3})$$

iii) $(f^k, \alpha f^k)$ *is a module containing* $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ *and* $(f^k, \alpha f^{k+1})$*. For k=3,* $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ *is the first cospinal grouping containing* $(f^{k+1}, \alpha f^{k+1})$ *and the first iteration of* $(f^{k+2}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}$*, a module containing* $(f^{k+3}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}\Big|_{f^3}^{f}$ *and* $(f^{k+2}, \alpha f^{k+1})\Big|_{f^{k+1}}^{f}$*. The former grouping are not module and contain* $(f^{k+3}, \alpha f^{k+1})\Big|_{f^{k+1}}^{f}\Big|_{f^{k+2}}^{f}$*. The latter constitute the rest of the iterations of* $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$*, in between modules containing* $(f^{k+3}, \alpha f^{k+2})\Big|_{f^{k+1}}^{f}\Big|_{f^{k+2}}^{f}$ *and* $(f^{k+2}, \alpha f^{k+2})$*.*

iv) $(f^k, \alpha f^{k+1})$*, for* $k \geq 2$*, then it is a module containing*

$$(f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}, (f^{k+1}, \alpha f^{k+1}), \text{ and } (f^k, \alpha f^{k+2})$$

The latter two have been characterized above, $(f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}$ *are modules containg*

$$(f^{k+2}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}\Big|_{f^{k+1}}^{f} \text{ and } (f^{k+1}, \alpha f^{k+2})$$

$(f^{k+2}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}\Big|_{f^{k+1}}^{f}$ are NOT modules and contain $(f^{k+2}, f^{k+1} + \alpha f^{k+2})\Big|_{f^{k+1}}^{f-1}$. This fully characterizes the f-primary lattice of $\mathbb{Z}[\alpha f^3]$.

Proof: Let the above assumptions hold and consider the f-primary lattice of $\mathbb{Z}[\alpha f^3]$.

i) $(f, \alpha f^3)$ is a spinal ideal, and thus a module. Theorem 5.1 tells us it contains the supposed ideals. To show these containments modularity (or lack theoreof);

ii)Consider $(f^k, vf^{k-1} + \alpha f^{k+1})$ v a non-zero integer mod f.

$(f^k, vf^{k-1} + \alpha f^{k+1})(1, \alpha) = (f^k, vf^{k-1} + \alpha f^{k+1}, \alpha f^k, \alpha^2 f^{k+1} + \alpha v f^{k-1})$. Obviously this ideal does not contain αf^k, so is not a $\mathbb{Z}[\alpha]$ module.

$(f^k, vf^{k-1} + \alpha f^{k+1})(1, \alpha f) = (f^k, vf^{k-1} + \alpha f^{k+1}, \alpha f^{k+1}, \alpha^2 f^{k+2} + \alpha v f^k)$. Obviously the ideal does not contain αf^{k+1}, so is not an $\mathbb{Z}[\alpha f]$.

$(f^k, vf^{k-1} + \alpha f^{k+1})(1, \alpha f^2) = (f^k, vf^{k-1} + \alpha f^{k+1}, \alpha f^{k+2}, \alpha^2 f^{k+3} + \alpha v f^{k+1})$. By our algebraic equations;

$\alpha^2 f^{k+3} + \alpha v f^{k+1} = -af^{k+3}\alpha - bf^{k+3} + vf^{k+1}\alpha$ which implies the above contains $vf^{k+1}\alpha$ a contradiction. So our ideal is not a $\mathbb{Z}[\alpha f^2]$.

So we conclude $\alpha^2 f^{k+3} + \alpha v f^{k+1}$ is not a module, and contains $(f^k, \alpha f^{k+2})$.

Consider $(f^{k-1}, \alpha f^{k+2})$.

$(f^{k-1}, \alpha f^{k+2})(1, \alpha) = (f^{k-1}, \alpha f^{k+2}, \alpha f^{k-1}, \alpha^2 f^{k+2})$. Which implies our ideal contains αf^{k-1} a clear contradiciton. So our ideal is not a $\mathbb{Z}[\alpha]$ module.

$(f^{k-1}, \alpha f^{k+2})(1, \alpha f) = (f^{k-1}, \alpha f^{k+2}, \alpha f^k, \alpha^2 f^{k+3})$. which implies our ideal contains αf^k a clear contradiction. So our ideal is not a $\mathbb{Z}[\alpha f]$ module.

$(f^{k-1}, \alpha f^{k+2})(1, \alpha f^2) = (f^{k-1}, \alpha f^{k+2}, \alpha f^{k+1}, \alpha^2 f^{k+4})$. Which implies our ideal contains αf^{k+1} a clear contradiction. So our ideal is not a $\mathbb{Z}[\alpha f^2]$.

So we see $(f^{k-1}, \alpha f^{k+2})$ is not a module and contains $(f^k, \alpha f^{k+2})$.

Consider $(f^k, \alpha f^{k+2})$.

$(f^k, \alpha f^{k+2})(1, \alpha f^2) = (f^k, \alpha f^{k+2}, \alpha f^{k+2}, \alpha^2 f^{k+4})$. By our algebraic equations;

$\alpha^2 f^{k+4} = -af^{k+4}\alpha - bf^{k+4}$ which is in our ideal. Thus we see our ideal is a $\mathbb{Z}[\alpha f^2]$ module, and by Theorem 5.1, has the supposed containments.

iii) $(f^k, \alpha f^k)$ are the f muliples of our conductor and thus spinal. So by Theorem 5.1 $(f^k, \alpha f^k)$ contains $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$ and $(f^k, \alpha f^{k+1})$.
Consider $(f^{k+1}, \alpha f^k)\Big|_{f^k}^{f}$. For its first existing grouping, $k = 3$, this is the first cospinal ideal. So we know its a module, and by Theorem 5.1 it contains the next cospinal grouping, the first iteration of $(f^{k+2}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}$ and $(f^{k+1}, \alpha f^{k+1})$. The first set of ideals are also cospinal and 2-groupings, so we see that they are modules containing $(f^{k+3}, \alpha f^k)\Big|_{f^k}^{f}\Big|_{f^{k+1}}^{f}\Big|_{f^{k+2}}^{f}$, a cospinal 3 grouping, and thus not a module (and who contains the next cospinal layer $(f^{k+3}, \alpha f^{k+1})\Big|_{f^{k+1}}^{f}\Big|_{f^{k+2}}^{f}$), and $(f^{k+2}, \alpha f^{k+1})\Big|_{f^{k+1}}^{f}$. This latter grouping for
$k > 3$ constitutes the rest of the ideals of the form we first considered here, and constitutes the in between ideals of our conductive sub-lattice. As such, this is a grouping of modules containing $(f^{k+3}, \alpha f^{k+1})\Big|_{f^{k+1}}^{f}\Big|_{f^{k+2}}^{f}$ and $(f^{k+2}, \alpha f^{k+2})$.

iv) Consider $(f^k, \alpha f^{k+1})$. It is spinal, so a module containing

$$(f^{k+1}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}, (f^{k+1}, \alpha f^{k+1}), \text{ and } (f^k, \alpha f^{k+2})$$

The latter two are given by a above work. Let v be a non-zero integer modulo f. Consider $(f^{k+1}, vf^{k-1} + \alpha f^{k+1})$.
$(f^{k+1}, vf^{k-1} + \alpha f^{k+1})(1, \alpha f^2) = (f^{k+1}, vf^{k-1} + \alpha f^{k+1}, \alpha f^{k+3}, \alpha^2 f^{k+3} + \alpha v f^{k+2})$.
By our algebraic equations;
$\alpha^2 f^{k+3} + \alpha v f^{k+2} = -af^{k+3}\alpha = bf^{k+3} + vf^{k+2}\alpha$, which ends up implying our ideal contains $vf^{k+2}\alpha$, which it does. So we have that $(f^{k+1}, vf^{k-1} + \alpha f^{k+1})$ is a $\mathbb{Z}[\alpha f^2]$ module. Theorem 5.1 implies it contains $(f^{k+2}, f^k + \alpha f^{k+1})\Big|_{f^k}^{f-1}\Big|_{f^{k+1}}^{f}$ and $(f^{k+1}, \alpha f^{k+2})$.

Let v, w be non-zero integers modulo f. Consider:

$$(f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1})$$
$$(f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1})(1, \alpha)$$
$$= (f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1}, \alpha f^{k+2}, \alpha^2 f^{k+1} + \alpha w f^{k+1} + \alpha v f^k)$$

implying our ideal contains αf^{k+2}, a clear contradiction. So our ideal is not a $\mathbb{Z}[\alpha]$ module.

$(f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1})(1, \alpha f) =$

$(f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1}, \alpha f^{k+3}, \alpha^2 f^{k+2} + \alpha w f^{k+2} + \alpha v f^{k+1})$

By our algebraic equations, we get:

$$\alpha^2 f^{k+2} + \alpha w f^{k+2} + \alpha v f^{k+1}$$
$$= -af^{k+2}\alpha - bf^{k+2} + wf^{k+2}\alpha + vf^{k+1}\alpha$$

which implies our ideal contains $wf^{k+2}\alpha + vf^{k+1}\alpha$, a clear contradiction. So our ideal is not a $\mathbb{Z}[\alpha f]$.

$$(f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1})(1, \alpha f^2)$$
$$= (f^{k+2}, vf^k + wf^{k+1} + f^k + \alpha f^{k+1}, \alpha f^{k+4}, \alpha^2 f^{k+3} + \alpha w f^{k+3} + \alpha v f^{k+2})$$

By our algebraic equations, we get:

$$\alpha^2 f^{k+3} + \alpha w f^{k+3} + \alpha v f^{k+2}$$
$$= -af^{k+3}\alpha - bf^{k+3} + \alpha w f^{k+3} + \alpha v f^{k+2}$$

which implies that our ideal contains $\alpha v f^{k+2}$, a clear contradiction. So our ideal is not a $\mathbb{Z}[\alpha f^3]$. So $\alpha^2 f^{k+3} + \alpha w f^{k+3} + \alpha v f^{k+2})$ is not a modules, and contain $(f^{k+2}, f^{k+1} + \alpha f^{k+2})\Big|_{f^{k+1}}^{f-1}$. This completes our characterization.

9 Conclusion

To review. We began by listing out previously known results of the lattice structure of the conductor ideal in the order of an algebraic number ring of a number field (ring of integers of $\mathbb{Q}$ with algebraic solutions adjoined). Specifically, we were interested in the cases of our adjoined elements to be quadratic algebraic integers multiplied by a known prime. The lattice structures corresponded with the splitting type of the prime itself, with ideals generated by our integer component and f-primary multiples of our algebraic integer, our algebraic component, in accordance with our orders being Dedekind domains. Locally, the structure around a particular ideal in the lattice corresponded with whether or not that ideal was a module of the underlying number ring, $\mathbb{Z}[\alpha]$. It was found that the structure as a whole depended on the first layer of f-basic, f-primary ideals. For inert and ramified primes, our lattice structures were globally iterations of this f-basic layer, and for the split case, a local recursion of this f-basic layer. Having found full characterizations of the lattice structures of these objects, we turned our attention to extending this theory to the lattice structures of the f-primary ideals of a f-primary quadratic number ring, $\mathbb{Z}[\alpha f^\sigma]$.

The first major differences in the higher power case were twofold. Firstly, when generalizing Theorem 3.8, for an ideal I to contain more than one proper ideal between itself and fI, it could be a module of any order between the number ring $\mathbb{Z}[\alpha]$ and our order under study $\mathbb{Z}[\alpha f^\sigma]$. This resulted in lattice structures above the prime case being significantly larger than the original case. While no characterizations of a ramified or split prime are presented in this paper, in pre-liminary inestigations of these objects, they, like the inert case, also seemed far large in the higher power cases. Given that the split case itself already lends itself to large lattics for the single power case, this points to the higher order lattices of the split case to be very large indeed. It should be noted that the proof of the generalized Theorem 5.1 we gave here was certainly guided and inspired by Peruginelli and Zanardo. It's safe to say the proof of the second statement was almost wholly the same except for one minor alteration, and followed the same strategy. The statement of the first proof did follow a similar stratagem, but had to be modified quiet heavily. It is a theme throughout this research that arithmetic analysis of these ideals increases in complexity and difficulty when looking at the higher power cases, but generally is inspired by and follows the same principles as the single power case. Secondly, the maximal ideal of our lattices is no longer the conductor. This contributes to our higher power cases resulting in much larger lattices, but also has more subtle ramifications (and not just for the ramified case). A major effect of this larger bound , is the $\mathfrak{F}$-basic layer can no longer be used to characterize the lattice entirely, as ideals not contained by the conductor are present and adding the structure of their own proper sub-ideals to the lattice. A distinct mathematical effect of this is that our generalized Theorem 5.1 no longer has the third statement that was present in [**PZ**].

We were not able to distinctly characterize all effects the 'seperation' between the conductor and maximal ideal caused, or distinclty clarify the relationship between the maximal basic and conductor basic layers. However, attempts to explicate the nature of this seperation led us to proving the existence of, and fully characterizing the conductive sub-lattice. This sub-structure is present in all lattices of our orders of $\mathbb{Z}[\alpha f^\sigma]$, regardless of the splitting type of the prime or power of f. In fact, this sub-structure can be used to fully characterize the single power case with an inert prime, and provides an alternate proof to the one found in [**PZ**]. Arguably, Theorem 5.1 and Theorem 7.7 are the two most important theorems in this paper, as they both allow the local analyses of our lattices in question, but also a pre-constructed reference point to start an investigation into a lattice with.

In investigating the nature of ideals around the spine post-conductor, we realized the number of 'in between' groupings corresponded to the difference in powers of the spine and cospine. This property is directly measurable with and induced by the seperation in powers between $\mathfrak{m}$ and $\mathfrak{F}$. Another easy measure of how much bigger the higher power cases become, is the increased number of groupings associated to the conductive sub-lattice.

We were not able to provide a full characterization of any of the splitting types of a prime. However, we were able to characterie the orders $\mathbb{Z}[\alpha f^2]$ and $\mathbb{Z}[\alpha f^3]$. These orders provide important examples for the basis of further study of these objects. $\mathbb{Z}[\alpha f^2]$ is the only order where the conductor is not the maximal ideal, but there are no proper ideals between $\mathfrak{m}$ and $\mathfrak{F}$, so provides an example of what effects the presence of $\mathfrak{m}$ has without the added structure of any other ideals containing $\mathfrak{F}$. $\mathbb{Z}[\alpha f^3]$ is our first order with a proper ideal between the maximal ideal and the conductor. As such, it is an excellent example to compare to $\mathbb{Z}[\alpha f^2]$. However, to get a full characterization of the inert primes, we will have to characterize a few more powers beyond $\mathbb{Z}[\alpha f^3]$ to have a full understanding of the effects of seperation on these lattices.

For future research endeavors, several goals must be met to have fully extended the work of Peruginelli and Zanardo to the higher power cases. Namely, a far greater explication of the maximal basic layers and conductor basic layers interactions are required. It should be further noted, that the idea of conductor basic itself may have to be extended. This could be a byproduct of the notation developed and used in this paper, but there are ideals generated in the in between not directly linked to the conductor that are not f-multiples of any ideal traditionally considered conductor basic. Given that the conductive sub-lattice does not 'stabilize' for σ layers, the notion of conductor basic may have to be extended out to the first σ layers. Like **[PZ]** characterized structures with respect to the basic layer, our lattices may require a characterization as the union (along the spine) of iterated layers from the maximal basic and condcutor basic layers. In the same vein, the use of basic elements was required by **[PZ]** to characterize the split case, which heavily points to a notion of maximal basic element being necessary for higher power characterizations. However, it is certain that a much deeper analysis of the difference between pre conductor ideals and post conductor ideals will be necessary.

10 Bibliography

References

[C] Cox, David. 'Primes of the Form $x^2 + ny^2$, Second Edition,'
Wiley Publishings, United States, 1989.

[G] Goldblatt, Robert. 'Topoi The Categorial Analysis of Logic,'
Dover Publications, Mineola, New York, 2006.

[H] Hungerford, Thomas(2003). 'Algebra,'
Graduate Texts in Mathematics, Springer-Verlag Publishing, 2003.

[Mac] Maclane, Saunders. 'Categories for the Working Mathematician,'
Graduate Texts in Mathematics, Springer-Verlag Publishing, New York City, New York, 1971.

[Mat] Matsumura, Hideyuki. 'Commutative Ring Theory, Translated by M. Reid, Second Edition,'
Cambridge Studies in Advanced Mathematics, 9. Cambridge University Press, Cambridge, 1989.

[PZ] Peruginelli, Giullio; Zanardo, Paolo(I-PADV), 'The lattice of primary ideals of orders in quadratic number fields,'
International Journal of Number Theory 12(2016) no. 7, 2025-2040

[RS] K. Rogers; H.P.F Swinnerton-Dyer. 'The Geometry of Numbers Over Algebraic Fields,'
Trans. Amer. Math. Soc. 88(1958), 227-242

[S] Samuel, Pierre. 'Algebraic Theory of Numbers,'
Dover Publications, United States, 2013.

[**?**]