Near-to-Far Field Signal Propagation for the Wave and Maxwell Equations

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B.S., Physics with Mathematics, University of Ghana, 2016

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Dedication

To all Mathematicians and Physicists, Humility is key:

The universe, full of intricate phenomena, yet such awe-inspiring beauty and elegance, is something humans have vested interest to describe and understand as long as we have been aware of our presence within it. Curious by nature, we have tried relentlessly to find a way to connect the dots between events, to determine the striking relationships between the interactions we perceive, as well as those that we do not. We have made impressive technological advancements from the smallest particle to the grand scheme of the universe that have given us at least a glimpse at the truth of the inner workings of the universe and all of the accomplishments through unique logical languages known as Mathematics and Physics. Of all of our stupendous progress and transcending breakthroughs that has stemmed from our ability to think logically and become well-versed about our universe, we still encounter problems and continue to stumble on conundrums which cannot be resolved within the spectrum of our rational and intellectual resources. We are therefore, forced to capitulate to the fact that, there is a super-natural power that functions independently and operates outside the scope of our perceptual and intellectual range. Holding such a tenet that, there is super-natural power which functions in a mystical way beyond the human comprehension is something which is innately grafted in the conscience of all sentient beings.

Acknowledgments

"Whosoever does not acknowledge and appreciate people, can never be appreciative of God" ∼ Prophet of Islam.

It is against this background that I stand to extend my profound gratitude to my adviser Prof. Stephen Lau for his staunch support, exemplary patience and tutelage that has brought me this far, in addition to all remaining UNM Department of Mathematics and Statistics faculty and staff, especially Ana Parra Lombard, for their overwhelming support throughout my stay here in the Department. I will finally thank my family and friends for their moral support.

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Abstract

The Maxwell equations may be viewed as evolution equations which develop an initial state of the electromagnetic field forward in time. Such evolution can be simulated numerically, that is modeled on a computer, in which case the domain of simulation is typically finite in extent. Nonetheless, one is often interested in the electromagnetic waves which reach infinity (of course which is outside of the simulation domain). Thus we are interested in near-to-far field signal propagation, that is a mathematical process where a signal or solution recorded at a finite radius $r = r_1$ can be converted to a signal at $r = r_2 > r_1$. We achieve such a conversion via application of convolution kernels in the time-domain, although the derivation of the appropriate kernels relies on Laplace transform arguments. Decomposing the wave and Maxwell equations using scalar and vector spherical harmonics respectively, we have solved the equations on the assumption that the source and initial data are compactly supported. We further assume that we work at a large distance outside of the supports. We develop from a theoretical standpoint signal-conversion formulas for the $3d$ wave and Maxwell equations and these generalize the simple time delay

associated with the propagation between two radii of a solution to the $1d$ wave equation.

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Chapter 1

Introduction

The primary objective of this work is to explore outgoing solutions to the 3-dimensional wave equation and Maxwell equations. The Maxwell equations remain the corner-stone as far as the concept of electromagnetism is concerned. The equations describe how electric and magnetic fields interact and propagate. The static Maxwell equations describe the structure of an electric field due to an electric charge. The equations imply the non-existence of magnetic monopoles and that the fundamental magnetic object is a dipole with north and south pole. The Maxwell equations further show how a change in magnetic field through a loop gives rise to an induced current. The Faraday Law, which is one of the four equations, explains how a circulating electric field gives rise to a magnetic field changing in time. The last Maxwell equation generally known as Ampere's Law tells us how a flowing electric current gives rise to a magnetic field that encircles that current. We will describe the mathematical representation of these equations in detail in the subsequent chapters (Chapter 2).

It is impractical to treat or solve a hyperbolic equation in a numerical computation with an infinite domain. The Maxwell equations may be viewed as evolution equations which develop an initial state of the electromagnetic field forward in time. Such evolution can be simulated numerically, that is, modeled on a computer, in which case the domain of simulation is typically finite in extent. Nonetheless, one is often interested in the electromagnetic waves which reach infinity (of course which is outside of the simulation domain). Thus we are interested in near-to-far field propagation, that is a mathematical process where a signal or solution recorded at a finite radius $r = r_1$ can be converted to a signal at $r = r_2$ and $r_2 > r_1$. We shall achieve such a conversion via application of convolution kernels in the time-domain, although the derivation of the appropriate kernels relies on Laplace transform arguments.

From a theoretical standpoint, another interest of this work (which is, in fact, related to near-to-far field propagation) is the form of outgoing solutions to wave and Maxwell equations. These generalize the simple right-advecting solutions to the wave equation on a 1-dimensional string $\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2}$. We will first decompose the 3-dimensional scalar wave equation using scalar spherical harmonics. This reduces the wave equation to the "radial form". By invoking the Laplace transformation, we rewrite the "radial form" as a special ODE known as the Modified Bessel Equation. Exploiting McDonald's function which is a solution to the Modified Bessel Equation, we will write the general form of the outgoing multipole solution to the wave equation. Unlike the wave equation, we will decompose the Maxwell equations using vector spherical harmonics. This reduces the equations into radial equations in time-radius. After we apply Laplace transformation, we end up with differential equations in r for the transverse components of the electric and magnetic fields. Our aim then is to solve these differential equation by writing them in terms of the Modified Bessel Equation. We will then write an explicit general form for the outgoing multipole solutions to the source free Maxwell equations.

We will develop the *teleportation kernel* for both the wave and the Maxwell equations. In particular, we will consider an explicit propagation formula for the wave equation for $\ell = 2$, and also $\ell = 2$ and $\ell = 3$ for the Maxwell equation. The propagation in both cases will be carried out at a time delay of $(r_2 - r_1)/c$.

Chapter 2

Wave Equation

The wave equation is a second-order, linear, hyperbolic partial differential equation. It describes the propagation of a variety of waves, such as sound waves, light waves and water waves. It arises in such fields as acoustics, electromagnetics, and fluid dynamics. For one time t and three space variables x,y,z the wave equation is

$$\frac{\partial^2 \psi(t,x,y,z)}{\partial t^2} = c^2 \nabla^2 \psi, \tag{1}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ and c are the Laplacian and speed of propagation [1]. In this work we will set the value of the speed of propagation to unity, i.e $c = 1$. The speed can be recovered by sending $t \to ct$.

2.1 Multipole Solutions

2.1.1 Separation of Variables

Consider a 3-dimensional wave equation given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial t^2} = S(t, x, y, z), \tag{2}$$

where S is a source term. Throughout this analysis, we assume S is compactly supported and that we work at a large distance from the source. Therefore, we set $S \equiv 0$. In spherical polar coordinates, the Laplacian takes the following form:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}, \tag{3}$$

where θ is the polar and ϕ is the azimuthal angle. We also write

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\nabla_S^2\psi, \tag{4}$$

with

$$\nabla_S^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{5}$$

as the Laplacian on the unit-radius sphere. We assume a "multipole expansion" for the wave field of the form

$$\psi(t, x, y, z) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r}\Psi_{\ell m}(t, r)Y_{\ell m}(\theta, \phi). \tag{6}$$

Here $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi} \tag{7}$$

is a standard spherical harmonic with $P_\ell^m(\cos\theta)$ as the associated Legendre function. The $Y_{\ell m}(\theta, \phi)$ eigenfunctions obey the identity

$$\nabla_S^2 Y_{\ell m}(\theta, \phi) = -\ell(\ell+1)Y_{\ell m}(\theta, \phi). \tag{8}$$

For our purposes, it is sufficient to analyze a single mode:

$$\psi = \frac{\Psi(t,r)}{r} Y_{\ell m}(\theta,\phi). \tag{9}$$

Here, we have suppressed the ℓm on the Ψ in (6) and we will maintain this trend throughout the analysis, although sometimes we keep just the ℓ subscript as you will see later. Then eqn(1) becomes

$$\frac{Y_{\ell m}(\theta,\phi)}{r}\frac{\partial^2 \Psi}{\partial t^2} = Y_{\ell m}(\theta,\phi)\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{\Psi}{r}\right) + \frac{\Psi}{r^3}\nabla_S^2 Y_{\ell m}(\theta,\phi) \tag{10}$$

and upon the use of (8), we have

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{\Psi}{r}\right) - \frac{\ell(\ell+1)}{r^2}\Psi.$$

Consequently, we also have that,

$$\frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{\Psi}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\Psi' - \Psi\right) = \Psi'' + \frac{\Psi'}{r} - \frac{\Psi'}{r} = \Psi'',$$

and so

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}\Psi. \tag{11}$$

This is known as the "radial wave equation" or *Euler-Poisson-Darboux* equation and it features an "effective potential", i.e.

$$V(r) = \frac{\ell(\ell+1)}{r^2}. \tag{12}$$

Our goal is to find solutions to (11) which are anologous to the simple rightward propagating solutions $f(t-r)$ to the 1 dimensional wave equation

$$\Psi_{tt} = \Psi_{rr}. \tag{13}$$

We proceed via the technique of the Laplace transform in time. The Laplace transform of Ψ is defined as

$$\mathcal{L}[\Psi(\cdot, r)](s) = \lim_{a\to\infty} \int_0^a e^{-st}\Psi(t,r)dt. \tag{14}$$

We assume that this transformation exists and proceed formally. This assumption rests on the fact that at a fixed radius the solutions to the radial wave equation tend to decay in time, not grow. We can through integration by parts establish that

$$\begin{aligned}\mathcal{L}[\dot{\Psi}(\cdot, r)](s) &= \lim_{a\to\infty} \left[e^{-sa}\Psi(a,r) - \Psi(0,r) + s\int_0^a e^{-st}\Psi(t,r)dt \right] \\ &= s\mathcal{L}[\Psi(\cdot, r)](s) - \Psi(0,r).\end{aligned} \tag{15}$$

Here $\dot{\Psi} = \partial\Psi/\partial t$. Clearly, it follows that

$$\mathcal{L}[\ddot{\Psi}(\cdot, r)](s) = s^2\mathcal{L}[\Psi(\cdot, r)](s) - s\Psi(0,r) - \dot{\Psi}(0,r). \tag{16}$$

With these results, the Laplace transform of the radial wave equation (11) is given by

$$s^2\hat{\Psi}(s,r) = \frac{d^2\hat{\Psi}(s,r)}{dr^2} - \frac{\hat{\Psi}(s,r)}{r^2}\ell(\ell+1) + \frac{\partial\Psi(0,r)}{\partial t} + s\Psi(0,r), \tag{17}$$

where $\hat{\Psi}(s,r) = \mathcal{L}[\Psi(\cdot, r)](s)$. For the initial condition, we assume a radial location r large enough that the initial data vanishes. That is,

$$\frac{\partial\Psi(0,r)}{\partial t} = s\Psi(0,r) = 0.$$

The equation now becomes

$$s^2\hat{\Psi}(s,r) = \frac{d^2\hat{\Psi}(s,r)}{dr^2} - \frac{\hat{\Psi}}{r^2}\ell(\ell+1). \tag{18}$$

To solve this equation, we substitute the equation

$$\hat{\Psi} = \sqrt{r}y(s,r) \tag{19}$$

into (18) and find that y obeys the following

$$-r^2y'' - ry' + y\left[\frac{1}{4} + \ell(\ell+1) + s^2r^2\right] = 0.$$

With $z = sr$, the above equation becomes

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - y\left[(\ell + \frac{1}{2})^2 + z^2\right] = 0. \tag{20}$$

Equation (20) is known as the Modified Bessel Equation. Here the equation is of half-integer order[2].

2.1.2 MacDonald Function

The solutions to equation (20) are $I_\nu(z)$ and $K_\nu(z)$, with $\nu = \ell + 1/2$. We shall exclusively be concerned with $K_\nu(z)$ which is known as the MacDonald function of order ν. This is because, along the positive z-axis, we have that

$$\lim_{z\to\infty} K_\nu(z) = 0.$$

They are determined by [2]

$$\sqrt{\frac{\pi}{2z}}K_{\ell+\frac{1}{2}}(z) = \frac{\pi}{2z}e^{-z}\sum_{k=0}^{\ell}\left(\ell + \frac{1}{2}, k\right)\frac{1}{(2z)^k}, \tag{21}$$

where

$$\left(\ell + \frac{1}{2}, k\right) = \frac{(\ell+k)!}{k!\Gamma(\ell - k + 1)}. \tag{22}$$

Thus, we can write following equations:

$$K_{\ell+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}W_\ell(z), \qquad W_\ell(z) = \sum_{k=0}^{\ell}\frac{c_{\ell k}}{z^k}, \qquad c_{\ell k} = \frac{1}{2^k k!}\frac{(\ell+k)!}{(\ell-k)!}. \tag{23}$$

2.1.3 Some Examples of MacDonald Functions

The first five MacDonald functions are the following.

$$
\begin{aligned}
K_{1/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}(\text{this has no root}) \\
{\color{red}+}\ K_{3/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{1}{z}\right) \\
{\color{blue}\diamond}\ K_{5/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{3}{z}+\frac{3}{z^2}\right) \\
\circ\ K_{7/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{6}{z}+\frac{15}{z^2}+\frac{15}{z^3}\right) \\
{\color{magenta}*}\ K_{9/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{10}{z}+\frac{45}{z^2}+\frac{105}{z^3}+\frac{105}{z^4}\right).
\end{aligned} \tag{24}
$$

The symbols used in Figure 1 below which include $+$ $\diamond$ $\circ$ and $*$ respectively correspond to $\ell = 1, 2, 3, 4$. Watson [4] shows that each $b_{\ell j}$ lies in the left-half plane and is simple. Watson's analysis is presented in the Appendix of [3]. When one watches closely to the curve with the naked eye, the scaled roots appear to be lying on the curve even for small ℓ. The scaled roots actually tend to lie on the curve when $\ell \to \infty$. Nonetheless, the fact that roots appear to be lying on the curve even for small ℓ illustrates that often asymptotics are good even when the parameter is not large.

2.1.4 Time Domain Expressions

Recall that we have the equation

$$\sqrt{\frac{2z}{\pi}}K_{\ell+\frac{1}{2}}(z) = e^{-z}W_\ell(z). \tag{25}$$

Therefore, the preceding analysis shows that we can from (19) have

$$\hat{\Psi}_\ell(s,r) = \alpha(s)e^{-z}W_\ell(sr) \tag{26}$$

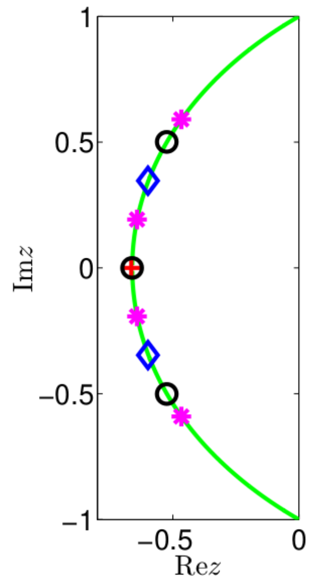

Figure 2.1: Scaled zeros $\dfrac{b_{\ell j}}{(\ell+1/2)}$ of $K_{\ell+1/2}(z)$ and $W_\ell(z)$.

as a solution to (18), despite multiplying the solution with a factor $\alpha(s)$. We now choose $\alpha(s) = a(s)s^\ell$ for the later convenience. Here $a(s)$ is an analytic function of s which encodes the nature of the wave in the time-domain. Explicitly,

$$\hat{\Psi}_\ell(s,r) = a(s)e^{-sr}\sum_{k=0}^{\ell} c_{\ell k}\frac{s^{\ell-k}}{r^k}. \tag{27}$$

We claim that the inverse Laplace transform of eqn(27) is

$$\Psi_\ell(t,r) = \sum_{k=0}^{\ell}\frac{1}{r^k}c_{\ell k}f^{(\ell-k)}(t-r), \qquad c_{\ell k} = \frac{1}{2^k k!}\frac{(\ell+k)!}{(\ell-k)!}. \tag{28}$$

Where the profile function f is determined by $a(s)$. To establish this result, we need the following two lemmas.

Lemma 2.1.1. *Let $f \in C^\infty(\mathbb{R})$, with $f(u) = 0$ for $u \notin [-B,-A]$, where $u = t - r$*

is retarded time, and $r > B$. Then

$$\int_0^\infty e^{-st} f(t-r)dt = e^{-sr} a(s). \tag{29}$$

Proof. Letting $u = t - r$; via change of variable, we have that

$$\int_0^\infty e^{-st} f(t-r)dt = e^{-sr} \int_{-r}^\infty e^{-su} f(u)du = e^{-sr} a(s),$$

where

$$a(s) \equiv e^{-sr} \int_{-B}^{-A} e^{-su} f(u)du. \tag{30}$$

Clearly $a(s)$ is independent of r because the function is only supported on the interval $[-B, -A]$ which excludes $-r$ from the region where the function is nonzero. $a(s)$ is an entire function, because the integral above is convergent for any $s \in \mathbb{C}$. □

Lemma 2.1.2. *From lemma 1, when we have a "smooth profile function " supported on the interval $[-B, -A]$, for a fixed $r > B$ then,*

$$\int_0^\infty e^{-st} f^p(t-r)dt = e^{-sr} s^p a(s). \tag{31}$$

Proof. We prove this lemma using the method of induction. The base case of $p = 0$ was shown in the last lemma. We know that

$$I = \int_0^\infty e^{-st} f^p(t-r)dt = e^{-sr} \int_{-B}^{-A} e^{-su} \frac{d}{du} f^{(p-1)}(u)du \tag{32}$$

$$I = se^{-sr} \int_{-B}^{-A} e^{-su} f^{(p-1)}(u)du + e^{-sr} \int_{-B}^{-A} \frac{d}{du}\Big(e^{-su} f^{(p-1)}(u)\Big)du. \tag{33}$$

The above integral

$$e^{-sr} \int_{-B}^{-A} \frac{d}{du}\Big(e^{-su} f^{(p-1)}(u)\Big)du = 0,$$

since the function f is only supported on $[-B,-A]$ and is $C^\infty(\mathbb{R})$ so the boundary terms vanishes. We can write our integral as

$$I = s\int_0^\infty e^{-st} f^{(p-1)}(t-r)dt. \tag{34}$$

Clearly, by induction

$$\int_0^\infty e^{-st} f^p(t-r)dt = e^{-sr}s^p a(s).$$

$\square$

As an example of this type of solution, we consider the case $\ell = 2$, with $\hat{\Psi}_2(s,r)$ given by

$$\hat{\Psi}_2(s,r) = a(s)s^2 e^{-sr} W_2(sr), \qquad W_2(z) = 1 + \frac{3}{z} + \frac{3}{z^2}$$

with corresponding quadrupole expansion in the time domain as

$$\Psi_2(t,r) = f''(t-r) + \frac{3}{r}f'(t-r) + \frac{3}{r^2}f(t-r). \tag{35}$$

With the above two lemmas, it is seen that Laplace transform of (35) yields the preceding equation.

2.2 Teleportation for Flatspace Multipoles

2.2.1 Formulas for General ℓ

Teleportation refers to the process whereby a signal $\Psi(t,r)$ recorded at $r = r_1$ is converted to the one recorded at $r = r_2$ and $r_2 > r_1$. We shall achieve such a conversion via application of a convolution kernel in the time-domain, although the derivation of the appropriate kernel relies on Laplace transform arguments. From (26) of section 2.1.4 we have previously seen that

$$\hat{\Psi}_\ell(s,r_1) = a(s)s^\ell e^{-z_1} W_\ell(z_1), \qquad \hat{\Psi}_\ell(s,r_2) = a(s)s^\ell e^{-z_2} W_\ell(z_2) \tag{36}$$

where $z_1 = sr_1$, $z_2 = sr_2$, and we assume that $r_2 > r_1 >$ support of the initial data. The relationship between the solutions at different radii is given by

$$\hat{\Psi}_\ell(s,r_2) = e^{-s(r_2-r_1)} \frac{W_\ell(sr_2)}{W_\ell(sr_1)} \hat{\Psi}(s,r_1), \tag{37}$$

or upon rearranging terms, we have

$$e^{(z_2-z_1)} \hat{\Psi}_\ell(s,r_2) = \left[\frac{W_\ell(z_2)}{W_\ell(z_1)} - 1\right] \hat{\Psi}_\ell(s,r_1) + \hat{\Psi}_\ell(s,r_1). \tag{38}$$

We will write the last equation as

$$e^{s(r_2-r_1)} \hat{\Psi}_\ell(s,r_2) = \hat{\Phi}_\ell(s,r_1,r_2) \hat{\Psi}_\ell(s,r_1) + \hat{\Psi}_\ell(s,r_1) \tag{39}$$

where

$$\hat{\Phi}_\ell(s,r_1,r_2) = -1 + \frac{W_\ell(sr_2)}{W_\ell(sr_1)}. \tag{40}$$

Remark: The -1 here ensures that the *frequency domain kernel* $\hat{\Phi}(s,r_1,r_2)$ decays for large s and this guarantees that the inverse Laplace transform exists. For the case $r_2 = \infty$, we see that $W_\ell(sr_2) = 1$ so the frequency domain kernel becomes

$$\hat{\Phi}_\ell(s,r_1,\infty) = \frac{-W_\ell(sr_1) + 1}{W_\ell(sr_1)}. \tag{41}$$

As shown below, the frequency domain teleportation kernel is given as sum of poles which is [5, 6]

$$\hat{\Phi}_\ell(s,r_1,r_2) = \sum_{j=1}^{\ell} \frac{a_{\ell j}(r_1,r_2)}{s - b_{\ell j}/r_1}, \tag{42}$$

whereas the time-domain teleportation kernel is a corresponding sum of exponentials,

$$\Phi_\ell(t,r_1,r_2) = \sum_{k=1}^{\ell} a_{\ell k}(r_1,r_2)\exp\left(\frac{b_{\ell k}t}{r_1}\right). \tag{43}$$

To establish the teleportation formula, we take the inverse Laplace transform of eqn(39), and this gives

$$\Psi_\ell(t+(r_2-r_1),r_2) = \int_0^t \Phi_\ell(t-t',r_1,r_2)\Psi_\ell(t',r_1)dt' + \Psi_\ell(t,r_1). \tag{44}$$

2.2.2 Explicit Formulas for $\ell = 2$

In terms of its zeros, $W_\ell(z)$ has the form

$$W_\ell(z) = \frac{1}{z^\ell}\prod_{j=1}^{\ell}(z-b_{\ell j}) \tag{45}$$

where $b_{\ell j}$ are roots. This is demonstrative of the fact that the zeros of $W_\ell(z)$ are the same as those of the MacDonald function $K_{\ell+1/2}(z)$. Since

$$(z_1 - b_{\ell j}) = r_1(s - b_{\ell j}/r_1),$$

we can therefore proceed to calculate the residue as

$$a_{\ell k}(r_1,r_2) = \lim_{s\to\ b_{\ell k}/r_1}(s-b_{\ell k}/r_1)\hat{\Phi}_\ell(s,r_1,r_2) \tag{46}$$

since the poles are simple. This follows because the roots of the $K_{\ell+1/2}(z)$ are known to be simple. We will therefore from eqn(45), define the residue as [5]

$$a_{\ell j}(r_1,r_2) = \frac{W_\ell(b_{\ell j}r_2r_1^{-1})}{r_1W_\ell'(b_{\ell j})}. \tag{47}$$

Let us consider $\ell = 2$ as an example. We have already seen in the previous analysis that

$$W_2(z) = 1 + \frac{3}{z} + \frac{3}{z^2}.$$

When we set $W_2(z) = 0$, the roots are

$$b_{21} = -\frac{3}{2} + i\frac{\sqrt{3}}{2}, \qquad b_{22} = -\frac{3}{2} - i\frac{\sqrt{3}}{2}.$$

Expanding eqn(43) for the $\ell = 2$ case gives

$$\Phi_2(t, r_1, r_2) = a_{21}(r_1, r_2) \exp\left(\frac{b_{21}}{r_1}t\right) + a_{22}(r_1, r_2) \exp\left(\frac{b_{22}}{r_1}t\right).$$

When we substitute the values of b_{21} and b_{22} in the previous equation, we have

$$\Phi_2(t, r_1, r_2) = \exp(-\frac{3}{2r_1}t)\left[(a_{21}+a_{22})\cos\left(\frac{\sqrt{3}}{2r_1}t\right) + i(a_{21}-a_{22})\sin\left(\frac{\sqrt{3}}{2r_1}t\right)\right]. \quad (48)$$

Here, our purpose is to calculate a_{21} and a_{22}. When we set $r = r_2 r_1^{-1}$ and also choose $r_1 = 1$ for convenience sake, then eqn(47) reduces to

$$a_{\ell k}(1, r) = \frac{W_\ell(b_{\ell k} r)}{W'_\ell(b_{\ell k})}. \quad (49)$$

We can recover the actual residue equation as

$$a_{\ell k}(r_1, r_2) = a_{\ell k}(1, r_2 r_1^{-1})/r_1$$

. We can also in an alternative way rewrite

$$W_2(z) = \frac{1}{z^2}\Big(z - b_{21}\Big)\Big(z - b_{22}\Big). \quad (50)$$

With this expression we find, for example, that

$$a_{21}(1, r) = \frac{W_\ell(b_{21} r)}{W'_\ell(b_{21})} = \frac{r-1}{r^2}\frac{b_{21}^2 r - b_{21}b_{22}}{b_{21} - b_{22}}.$$

From here

$$a_{21}(1, r) = \frac{(r-1)(rb_{21}^2 - b_{21}b_{22})}{ir^2\sqrt{3}}. \quad (51)$$

With this result, we then compute the values of $a_{21}(r_1, r_2)$ and $a_{22}(r_1, r_2)$ by feeding into (51) the values of b_{21} and b_{22}. The computation finally yields

$$a_{21}(r_1, r_2) = -\frac{r_2 - r_1}{2r_1 r_2^2}[3r_2 + i\sqrt{3}(r_2 - 2r_1)]$$

and

$$a_{22}(r_1, r_2) = -\frac{r_2 - r_1}{2r_1 r_2^2}[3r_2 - i\sqrt{3}(r_2 - 2r_1)], \tag{52}$$

since a_{22} is the complex conjugate of a_{21}. This implies that

$$\Phi_2(t, r_1, r_2) = \exp\left(-\frac{3}{2r_1}t\right)\left[(a^R)\cos\left(\frac{\sqrt{3}}{2r_1}t\right) + i(a^I)\sin\left(\frac{\sqrt{3}}{2r_1}t\right)\right], \tag{53}$$

where

$$a^R = \frac{-3(r_2 - r_1)}{r_1 r_2} \qquad a^I = -\frac{\sqrt{3}}{r_1 r_2^2}[(r_2 - r_1)(r_2 - 2r_1)]. \tag{54}$$

Chapter 3

Maxwell Equations

The central equations that govern electromagnetic theory are the Maxwell equations. Fundamentally, the equations can be written in the following microscopic Gaussian unit form [7]:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{55a}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{55b}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \tag{55c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{55d}$$

These equations are respectively known as Gauss' Law, Faraday's Law, Generalized Ampere's Law and the Magnetic Law. Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is volume charge density, $\mathbf{J}$ is the current charge density. These equations are typically expressed in the time dependent circumstance. In the "static" case, the equations are independent of time and therefore the time dependence of the previous equations vanishes.

In the source free case, we set $\rho = 0$ and $\mathbf{J} = 0$. We will, as in the case of wave equation, take $c = 1$. The speed can be recovered by sending $t \to ct$. Our previous equations then reduce to the following:

$$\nabla \cdot \mathbf{E} = 0 \tag{56a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{56b}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \tag{56c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{56d}$$

This shows that $\mathbf{E}$ and $\mathbf{B}$ are *coupled.* This implies that variations in $\mathbf{E}$ act as a source for $\mathbf{B}$, which in turn act as a source for $\mathbf{E}$. We can expand both the $\mathbf{E}$ and $\mathbf{B}$ in terms of the orthogonal basis of vector spherical harmonics.

3.1 Vector Spherical Harmonics (VSH)

3.1.1 Derivation and Properties of VSH

We will succinctly go through some analysis in constructing the VSH. Several techniques have been used in constructing the VSH. There is no universally agreed upon methodology for constructing VSH. Mostly, the choice of the technique or method has to do with convenience of analysis. In our case, we will be using the Barrera *et al* approach [8] of constructing the VSH, an attempt to construct the VSH, which is analogous to the scalar spherical harmonics, would be to treat each of the three components of a vector as a separate scalar field. Naively, one might attempt to first express a given vector field

$$\mathbf{V}(r,\theta,\phi) = \mathbf{e}_r V^r(r,\theta,\phi) + \mathbf{e}_\theta V^\theta(r,\theta,\phi) + \mathbf{e}_\phi V^\phi(r,\theta,\phi) \tag{57}$$

in terms of the standard spherical polar frame, and then subsequently expand each component in a scalar spherical harmonic expansion. That is, for a given field $\mathbf{V}$, we might write

$$\begin{aligned}\mathbf{V}(r,\theta,\phi) = \mathbf{e}_r \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} V^r_{\ell m}(r) Y_{\ell m} \\ + \mathbf{e}_\theta \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} V^\theta_{\ell m}(r) Y_{\ell m} + \mathbf{e}_\phi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} V^\phi_{\ell m}(r) Y_{\ell m}.\end{aligned} \tag{58}$$

The representation of eqn(58) is certainly valid, since the scalar spherical harmonics form a complete set. Consider an equation of the form

$$\nabla \cdot \mathbf{V} = f. \tag{59}$$

We might expect that eqn(59) and the scalar spherical harmonic decomposition of the scalar function function f:

$$f = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(r) Y_{\ell m}(\theta,\phi) \tag{60}$$

would eventually relate the coefficients, $f_{\ell m}$, $V^\theta_{\ell m}$, $V^\phi_{\ell m}$ and finally $V^r_{\ell m}$ in a useful way. Unfortunately, this is not the case. To show why, we first find the divergence of $\mathbf{V}$ in eqn(59) in spherical polar coordinates,

$$\begin{aligned}\nabla \cdot \mathbf{V} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \Bigg[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V^r_{\ell m}(r) Y_{\ell m}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V^\theta_{\ell m}(r) Y_{\ell m}) \\ + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(V^\phi_{\ell m}(r) Y_{\ell m})\Bigg].\end{aligned} \tag{61}$$

Considering eqn(61), we see the presence of terms like $Y_{\ell m}/\sin\theta$, which shows that, the angular dependence for ℓm is not simply $Y_{\ell m}$. Therefore, our aim to sweep away all the angular dependence, in order to simply relate the r-dependent coefficients, cannot be achieved. Since we cannot cancel the $Y_{\ell m}$, we will not go through this futile exercise of constructing the vector spherical harmonics by taking the divergence of $\mathbf{V}$. Rather, we consider a scalar field

$$f(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(r) Y_{\ell m}(\theta,\phi), \tag{62}$$

and obtain a vector field by taking its gradient

$$\nabla f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{d}{dr} f_{\ell m}(r) Y_{\ell m} \mathbf{e}_r + f_{\ell m} \nabla Y_{\ell m} \right). \tag{63}$$

We have succeeded in expressing the radial part in terms $Y_{\ell m}$. We now have to expand the new mathematical object, $\nabla Y_{\ell m}$, in terms of $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. This motivates the following notation: $\mathbf{\Phi}_{\ell m}$, $\mathbf{\Psi}_{\ell m}$ and $\mathbf{Y}_{\ell m}$, with $\mathbf{Y}_{\ell m} = \mathbf{e}_r Y_{\ell m}$. We will for convenience sake define the new notations as

$$\mathbf{\Psi}_{\ell m} = r \nabla Y_{\ell m}(\theta, \phi) \tag{64}$$

and

$$\mathbf{\Phi}_{\ell m} = \mathbf{e}_r \times \mathbf{\Psi}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}. \tag{65}$$

Where $\mathbf{r} = r\mathbf{e}_r$. This guarantees that for any given field $\mathbf{V}(r, \theta, \phi)$, we can write the field as

$$\mathbf{V}(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} V_{\ell m}^{r} \mathbf{Y}_{\ell m} + V_{\ell m}^{(1)} \mathbf{\Psi}_{\ell m} + V_{\ell m}^{(2)} \mathbf{\Phi}_{\ell m}. \tag{66}$$

As collected in [8], some identities regarding the scalar and vector spherical harmonics include the following:

1. Divergence:

$$\nabla \cdot (F(r) \mathbf{Y}_{\ell m}) = \left(\frac{1}{r^2} \frac{d}{dr} r^2 F(r) \right) Y_{\ell m},$$

$$\nabla \cdot (F(r) \mathbf{\Psi}_{\ell m}) = -\frac{\ell(\ell+1)}{r} F(r) Y_{\ell m},$$

$$\nabla \cdot (F(r) \mathbf{\Phi}_{\ell m}) = 0.$$

2. Curl:

$$\nabla \times (F(r)\mathbf{Y}_{\ell m}) = -\frac{F(r)}{r}\mathbf{\Phi}_{\ell m},$$

$$\nabla \times (F(r)\mathbf{\Psi}_{\ell m}) = \left(\frac{1}{r}\frac{d}{dr}rF(r)\right)\mathbf{\Phi}_{\ell m},$$

$$\nabla \times (F(r)\mathbf{\Phi}_{\ell m}) = -\left(\frac{\ell(\ell+1)}{r}F(r)\right)\mathbf{Y}_{\ell m} - \left(\frac{1}{r}\frac{d}{dr}rF(r)\right)\mathbf{\Psi}_{\ell m}.$$

3. Gradient:

$$\nabla(F(r)Y_{\ell m}) = \left(\frac{d}{dr}F(r)\right)\mathbf{Y}_{\ell m} + \frac{F(r)}{r}\mathbf{\Psi}_{\ell m}.$$

Now, it is evident from the previous analysis that, when we consider the equation,

$$\nabla \cdot \mathbf{V} = f$$

in terms of the vector spherical harmonics, we are able to relate the coefficients in a useful way.

3.1.2 Maxwell Equations Expressed in Terms of VSH

With the previous analysis, we now expand both the electric **E** and magnetic **B** fields in terms of the vector spherical harmonics as

$$\mathbf{E}(r,\theta,\phi) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} E^r_{\ell m}\mathbf{Y}_{\ell m} + E^{(1)}_{\ell m}\mathbf{\Psi}_{\ell m} + E^{(2)}_{\ell m}\mathbf{\Phi}_{\ell m}, \tag{67}$$

$$\mathbf{B}(r,\theta,\phi) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} B^r_{\ell m}\mathbf{Y}_{\ell m} + B^{(1)}_{\ell m}\mathbf{\Psi}_{\ell m} + B^{(2)}_{\ell m}\mathbf{\Phi}_{\ell m}. \tag{68}$$

It is crystal clear that our expansion for both $\mathbf{E}$ and $\mathbf{B}$ field ignores terms involving $\ell = 0$. We have ignored that because $\mathbf{\Psi}_{00}$ and $\mathbf{\Phi}_{00}$ vanish identically, and also $E^r_{00}\mathbf{Y}_{00}$ is Coulomb term which is easy to handle. The presence of $B^r_{00}\mathbf{Y}_{00}$ is indicative of a magnetic monopole (magnetic charge) which we do not want to have in the expansion. Here $E^r_{\ell m}$ is the radial component of the vector field, while $E^{(1)}_{\ell m}$ and $E^{(2)}_{\ell m}$ are the transverse components with

$$\mathbf{Y}_{\ell m} = Y_{\ell m}\mathbf{e}_r. \tag{69}$$

In this paragraph, we write an explicit expansion for both $\mathbf{\Psi}_{\ell m}$ and $\mathbf{\Phi}_{\ell m}$, in terms of $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. Then we will describe how to relate the VSH components of $\mathbf{B}$ and $\mathbf{E}$ to the Cartesian components. We proceed as follows, we know the gradient of a function f in the spherical polar coordinates is given by

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{70}$$

In our case $f = Y_{\ell m}(\theta,\phi)$, and clearly

$$\nabla Y_{\ell m}(\theta,\phi) = \frac{1}{r}\frac{\partial Y_{\ell m}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\mathbf{e}_\phi. \tag{71}$$

Since we have previously defined $\mathbf{\Psi}_{\ell m} = r\nabla Y_{\ell m}(\theta,\phi)$, then

$$\mathbf{\Psi}_{\ell m} = \frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\mathbf{e}_\phi + \frac{\partial Y_{\ell m}}{\partial \theta}\mathbf{e}_\theta. \tag{72}$$

Similarly, given that $\mathbf{\Phi}_{\ell m} = \mathbf{e}_r \times \mathbf{\Psi}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}$, then we have

$$\mathbf{\Phi}_{\ell m} = -\frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\mathbf{e}_\theta + \frac{\partial Y_{\ell m}}{\partial \theta}\mathbf{e}_\phi, \tag{73}$$

where, of course $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$ are the standard unit basis vectors in the spherical co-ordinate system.

Substituting $\mathbf{\Psi}_{\ell m}$ and $\mathbf{\Phi}_{\ell m}$ into eqn(67) and eqn(68), and rearranging the terms gives (isolating single ℓm-terms in each):

$$\mathbf{E} = E^r_{\ell m}Y_{\ell m}\mathbf{e}_r + \left[E^{(1)}_{\ell m}\frac{\partial Y_{\ell m}}{\partial \theta} - \frac{E^{(2)}_{\ell m}}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\right]\mathbf{e}_\theta + \left[\frac{E^{(1)}_{\ell m}}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi} + E^{(2)}_{\ell m}\frac{\partial Y_{\ell m}}{\partial \theta}\right]\mathbf{e}_\phi, \tag{74}$$

$$\mathbf{B} = B^r_{\ell m} Y_{\ell m} \mathbf{e}_r + \left[B^{(1)}_{\ell m} \frac{\partial Y_{\ell m}}{\partial \theta} - \frac{B^{(2)}_{\ell m}}{\sin\theta} \frac{\partial Y_{\ell m}}{\partial \phi} \right] \mathbf{e}_\theta + \left[\frac{B^{(1)}_{\ell m}}{\sin\theta} \frac{\partial Y_{\ell m}}{\partial \phi} + B^{(2)}_{\ell m} \frac{\partial Y_{\ell m}}{\partial \theta} \right] \mathbf{e}_\phi. \quad (75)$$

From these expressions we read off E^r, E^θ and E^ϕ which we can use as formula in determining the Cartesian components of $\mathbf{E}$ and $\mathbf{B}$. Next, we want to decompose the Maxwell equations (56a)-(56d) using the properties of the vector spherical harmonics. We can now write the curl of both the electric and magnetic field in terms of the VSH as

$$\nabla \times \mathbf{E} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left[\zeta_E \mathbf{Y}_{\ell m} - \eta_E \mathbf{\Psi}_{\ell m} + \chi_E \mathbf{\Phi}_{\ell m} \right], \quad (76)$$

where

$$\zeta_E = -\frac{\ell(\ell+1)}{r} E^{(2)}_{\ell m},$$

$$\eta_E = \left(\frac{\partial E^{(2)}_{\ell m}}{\partial r} + \frac{1}{r} E^{(2)}_{\ell m} \right),$$

and

$$\chi_E = \left(-\frac{1}{r} E^r_{\ell m} + \frac{\partial E^{(1)}_{\ell m}}{\partial r} + \frac{1}{r} E^{(1)}_{\ell m} \right).$$

Similarly,

$$\nabla \times \mathbf{B} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left[\zeta_B \mathbf{Y}_{\ell m} - \eta_B \mathbf{\Psi}_{\ell m} + \chi_B \mathbf{\Phi}_{\ell m} \right]. \quad (77)$$

$$\zeta_B = -\frac{\ell(\ell+1)}{r} B^{(2)}_{\ell m},$$

$$\eta_B = \left(\frac{\partial B^{(2)}_{\ell m}}{\partial r} + \frac{1}{r} B^{(2)}_{\ell m} \right),$$

$$\chi_B = \left(-\frac{1}{r} B^r_{\ell m} + \frac{\partial B^{(1)}_{\ell m}}{\partial r} + \frac{1}{r} B^{(1)}_{\ell m} \right)$$

3.2 Multipole Solutions

3.2.1 Laplace Transform

The goal of this subsection is to show via Laplace transformation, that the Maxwell equations are equivalent to a denumerable set of ODE for all ℓ, m (see (92) below). We now want to take the Laplace transform of both eqn(56b) and eqn(56c). We will begin with the partial time derivative of both equations. We have already in the previous analysis defined the the Laplace transform as

$$\mathcal{L}[\mathbf{E}(\cdot, r)](s) = \lim_{a\to\infty} \int_0^a e^{-st}\mathbf{E}(t,r)dt \tag{78}$$

$$\begin{aligned}\mathcal{L}[\mathbf{E}_t(\cdot, r)](s) &= \lim_{a\to\infty} \left[e^{-sa}\mathbf{E}(a,r) - \mathbf{E}(0,r) + s\int_0^a e^{-st}\mathbf{E}(t,r)dt\right] \\ &= s\mathcal{L}[\mathbf{E}(\cdot, r)](s) - \mathbf{E}(0,r).\end{aligned} \tag{79}$$

Similarly, when one takes the Laplace transform of time dependent $\mathbf{B}$ field, we have

$$\begin{aligned}\mathcal{L}[\mathbf{B}_t(\cdot, r)](s) &= \lim_{a\to\infty} \left[e^{-sa}\mathbf{B}(a,r) - \mathbf{B}(0,r) + s\int_0^a e^{-st}\mathbf{B}(t,r)dt\right] \\ &= s\mathcal{L}[\mathbf{B}(\cdot, r)](s) - \mathbf{B}(0,r).\end{aligned} \tag{80}$$

For the initial condition, we will assume that the data is compactly supported, and that $r >$ support of the initial data. Therefore,

$$\mathbf{B}(0,r) = \mathbf{E}(0,r) = 0,$$

and the Laplace transforms of eqn(56b) and eqn(56c) are as follows:

$$\nabla \times \hat{\mathbf{E}} = -s\hat{\mathbf{B}}, \tag{81}$$

and

$$\nabla \times \hat{\mathbf{B}} = s\hat{\mathbf{E}}. \tag{82}$$

Since the expansions (76) and (77) involve no time derivatives, we may immediately take their Laplace transforms by simply "hatting" all the r-dependent expansion coefficients. When we expand the previous equations, (81) and (82) in terms of $\mathbf{Y}_{\ell m}$ $\mathbf{\Psi}_{\ell m}$ and $\mathbf{\Phi}_{\ell m}$, and compare both sides of the equations, then clearly we will have the following equations;

$$s\hat{E}^r_{\ell m} = -\frac{\ell(\ell+1)}{r}\hat{B}^{(2)}_{\ell m}, \tag{83}$$

$$s\hat{E}^{(1)}_{\ell m} = -\Bigg(\frac{d\hat{B}^{(2)}_{\ell m}}{dr} + \frac{1}{r}\hat{B}^{(2)}_{\ell m}\Bigg), \tag{84}$$

$$s\hat{E}^{(2)}_{\ell m} = \Bigg(-\frac{1}{r}\hat{B}^r_{\ell m} + \frac{d}{dr}\hat{B}^{(1)}_{\ell m} + \frac{1}{r}\hat{B}^{(1)}_{\ell m}\Bigg). \tag{85}$$

Similarly, we also have for the $\mathbf{B}$ field,

$$s\hat{B}^r_{\ell m} = \frac{\ell(\ell+1)}{r}\hat{E}^{(2)}_{\ell m}, \tag{86}$$

$$s\hat{B}^{(1)}_{\ell m} = \Bigg(\frac{d\hat{E}^{(2)}_{\ell m}}{dr} + \frac{1}{r}\hat{E}^{(2)}_{\ell m}\Bigg), \tag{87}$$

$$s\hat{B}^{(2)}_{\ell m} = \Bigg(\frac{1}{r}\hat{E}^r_{\ell m} - \frac{d}{dr}\hat{E}^{(1)}_{\ell m} - \frac{1}{r}\hat{E}^{(1)}_{\ell m}\Bigg). \tag{88}$$

Now, we want to eliminate $\hat{B}^r_{\ell m}$ from (85) so that the equation reduces to terms involving only $\hat{E}^{(2)}_{\ell m}$ and $\hat{B}^{(1)}_{\ell m}$. To achieve this, we will substitute (86) into (85). This gives

$$s\hat{E}^{(2)}_{\ell m} = \Bigg(-\frac{1}{r}\frac{\ell(\ell+1)}{sr}\hat{E}^{(2)}_{\ell m} + \frac{d}{dr}\hat{B}^{(1)}_{\ell m} + \frac{1}{r}\hat{B}^{(1)}_{\ell m}\Bigg). \tag{89}$$

Simplifying and rearranging the terms will give

$$\hat{E}^{(2)}_{\ell m}\Bigg(s + \frac{\ell(\ell+1)}{sr^2}\Bigg) = \Bigg(\frac{d}{dr} + \frac{1}{r}\Bigg)\hat{B}^{(1)}_{\ell m}. \tag{90}$$

In the same vein, we want to eliminate $\hat{E}^r_{\ell m}$ from (88), so that our equation will now contain only $\hat{B}^{(2)}_{\ell m}$ and $\hat{E}^{(1)}_{\ell m}$ terms. We will do this by substituting (83) into (88), resulting in

$$\hat{B}^{(2)}_{\ell m}\left(-s-\frac{\ell(\ell+1)}{sr^2}\right)=\left(\frac{d}{dr}+\frac{1}{r}\right)\hat{E}^{(1)}_{\ell m}. \tag{91}$$

Collection of (84), (87), (90) and (91) in matrix form yields

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\begin{pmatrix}\hat{E}^{(1)}_{\ell m}\\ \hat{E}^{(2)}_{\ell m}\\ \hat{B}^{(1)}_{\ell m}\\ \hat{B}^{(2)}_{\ell m}\end{pmatrix}=s\begin{pmatrix}0 & 0 & 0 & -1-\frac{\ell(\ell+1)}{s^2r^2}\\ 0 & 0 & 1 & 0\\ 0 & 1+\frac{\ell(\ell+1)}{s^2r^2} & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\hat{E}^{(1)}_{\ell m}\\ \hat{E}^{(2)}_{\ell m}\\ \hat{B}^{(1)}_{\ell m}\\ \hat{B}^{(2)}_{\ell m}\end{pmatrix} \tag{92}$$

We now turn to solution of this matrix system.

3.2.2 Frequency Domain Solution

In this phase of our analysis, we will further simplify the differential equations and find their respective solution in the frequency domain. To do this, we will we proceed as follows. We know from the previous calculation that

$$\hat{E}^{(2)}_{\ell m}\left(s+\frac{\ell(\ell+1)}{sr^2}\right)=\left(\frac{d}{dr}+\frac{1}{r}\right)\hat{B}^{(1)}_{\ell m}, \tag{93}$$

but we know from (87) that we can express $\hat{B}^{(1)}_{\ell m}$ in terms of $\hat{E}^{(2)}_{\ell m}$:

$$s\hat{B}^{(1)}_{\ell m}=\left(\frac{d\hat{E}^{(2)}_{\ell m}}{dr}+\frac{1}{r}\hat{E}^{(2)}_{\ell m}\right). \tag{94}$$

Therefore, substituting (87) into (93) yields,

$$\hat{E}^{(2)}_{\ell m}\left(s+\frac{\ell(\ell+1)}{sr^2}\right)=\frac{1}{s}\left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{d\hat{E}^{(2)}_{\ell m}}{dr}+\frac{1}{r}\hat{E}^{(2)}_{\ell m}\right). \tag{95}$$

However, performing the differentiation on the right hand side of the equation (95) and simplifying the terms, then we will have

$$s\hat{E}^{(2)}_{\ell m}\left(s+\frac{\ell(\ell+1)}{sr^2}\right)=\frac{d^2\hat{E}^{(2)}_{\ell m}}{d^2r}-\frac{1}{r^2}\hat{E}^{(2)}_{\ell m}+\frac{1}{r^2}\hat{E}^{(2)}_{\ell m}+\frac{2}{r}\frac{d\hat{E}^{(2)}_{\ell m}}{dr}. \tag{96}$$

This subsequently reduces to

$$r^2\frac{d^2\hat{E}^{(2)}_{\ell m}}{d^2r}+2r\frac{d\hat{E}^{(2)}_{\ell m}}{dr}-\left(s^2r^2+\ell(\ell+1)\right)\hat{E}^{(2)}_{\ell m}=0, \tag{97}$$

that is the Modified Spherical Bessel Equation. In the same vein, we know that $\hat{B}^{(2)}_{\ell m}$ and $\hat{E}^{(1)}_{\ell m}$ are related by

$$\hat{B}^{(2)}_{\ell m}\left(-s-\frac{\ell(\ell+1)}{sr^2}\right)=\left(\frac{d}{dr}+\frac{1}{r}\right)\hat{E}^{(1)}_{\ell m}, \tag{98}$$

where

$$s\hat{E}^{(1)}_{\ell m}=-\left(\frac{d\hat{B}^{(2)}_{\ell m}}{dr}+\frac{1}{r}\hat{B}^{(2)}_{\ell m}\right).$$

Following the same procedure without necessarily showing the nitty-gritty, we obtain

$$r^2\frac{d^2\hat{B}^{(2)}_{\ell m}}{d^2r}+2r\frac{d\hat{B}^{(2)}_{\ell m}}{dr}-\left(s^2r^2+\ell(\ell+1)\right)\hat{B}^{(2)}_{\ell m}=0. \tag{99}$$

Here we take the solution to these differential equations as

$$\begin{pmatrix}\hat{E}^{(1)}_{\ell m}\\ \hat{E}^{(2)}_{\ell m}\\ \hat{B}^{(1)}_{\ell m}\\ \hat{B}^{(2)}_{\ell m}\end{pmatrix}=a_{\ell m}(s)s^{\ell+2}\begin{pmatrix}k'_\ell(sr)+\frac{k_\ell(sr)}{sr}\\ 0\\ 0\\ -k_\ell(sr)\end{pmatrix}+b_{\ell m}(s)s^{\ell+2}\begin{pmatrix}0\\ k_\ell(sr)\\ k'_\ell(sr)+\frac{k_\ell(sr)}{sr}\\ 0\end{pmatrix}. \tag{100}$$

In (100) the factors of $s^{\ell+2}$ are included for later convenience. Here, k_ℓ is the Modified Spherical Bessel Function which is expressed as

$$k_\ell(z)=\sqrt{\frac{\pi}{2z}}K_{\ell+\frac{1}{2}}(z)=\frac{\pi}{2z}e^{-z}\sum_{k=0}^{\ell}\frac{c_{\ell k}}{z^k}. \tag{101}$$

where $K_{\ell+\frac{1}{2}}(z)$ is the MacDonald function which was discussed in the previous section. This Bessel function decays for a large z, $Re(z) > 0$

$$k_\ell(z) \sim \frac{\pi}{2z} e^{-z}. \tag{102}$$

In principle, we should be getting four different independent solutions to (100). We could write down analogous solutions involving [2]

$$i_\ell(z) = \sqrt{\frac{\pi}{2z}} I_{\ell+\frac{1}{2}}(z) = z^\ell \Big(\frac{1}{z}\frac{d}{dz}\Big)^\ell \frac{\sinh z}{z}. \tag{103}$$

We neglected the latter form of the solution because $i_\ell(z)$ grows as $z \to \infty$, which is incompatible with the outgoing wave propagation. Our focus is for the outgoing case. This is the reason why we only focus on the solution with form k_ℓ.

3.2.3 Time Domain Solutions

We have in the previous analysis written the solution to the Laplace-transform, VSH-decomposed Maxwell equations which corresponds to outgoing boundary conditions. We now proceed to compute the inverse Laplace transform. We know that

$$\hat{B}^{(2)}_{\ell m}(s,r) = -a_{\ell m}(s) s^{\ell+2} k_\ell(sr). \tag{104}$$

Substitution of $k_\ell(sr)$ from (101) into the previous equation gives

$$\begin{aligned} \hat{B}^{(2)}_{\ell m}(s,r) &= -a_{\ell m}(s) s^{\ell+2} \frac{\pi}{2z} e^{-z} \sum_{k=0}^{\ell} \frac{c_{\ell k}}{z^k} \\ &= -a_{\ell m}(s) s^{\ell-k+1} \frac{\pi}{2} e^{-sr} \sum_{k=0}^{\ell} \frac{c_{\ell k}}{r^{k+1}}. \end{aligned} \tag{105}$$

By lemmas 2.1.1 and 2.1.2 the inverse Laplace transform of eqn(105) is given by

$$B^{(2)}_{\ell m}(t,r) = -\frac{\pi}{2} \sum_{k=0}^{\ell} \frac{c_{\ell k}}{r^{k+1}} f^{(\ell-k+1)}(t-r). \tag{106}$$

Considering the radial component from (83), we have that

$$\begin{aligned}\hat{E}^r_{\ell m}(s,r) &= -\frac{\ell(\ell+1)}{z}\hat{B}^{(2)}_{\ell m}(s,r) \\ &= \ell(\ell+1)e^{-sr}a_{\ell m}(s)s^{\ell-k}\frac{\pi}{2}e^{-z}\sum_{k=0}^{\ell}\frac{c_{\ell k}}{r^{k+2}}.\end{aligned} \tag{107}$$

Upon inverse Laplace transform of (107), we also have

$$E^r_{\ell m}(t,r) = \ell(\ell+1)\frac{\pi}{2}\sum_{k=0}^{\ell}\frac{c_{\ell k}}{r^{k+2}}f^{(\ell-k)}(t-r). \tag{108}$$

By (101), it naturally follows that

$$\begin{aligned}\hat{E}^{(1)}_{\ell m}(s,r) &= a_{\ell m}(s)s^{\ell+2}\left(k'_\ell(z)+\frac{k_\ell(z)}{z}\right) \\ &= -a_{\ell m}(s)s^{\ell+2}\frac{\pi}{2}e^{-z}\sum_{k=0}^{\ell}c_{\ell k}\left(\frac{1}{z^{k+1}}+\frac{k}{z^{k+2}}\right). \\ &= -a_{\ell m}(s)s^{\ell-k+1}\frac{\pi}{2}e^{-sr}\sum_{k=0}^{\ell}c_{\ell k}r^{-k-1} - a_{\ell m}(s)s^{\ell-k}\frac{\pi}{2}e^{-sr}\sum_{k=0}^{\ell}kc_{\ell k}r^{-k-2}.\end{aligned} \tag{109}$$

Thus, one can write

$$E^{(1)}_{\ell m}(t,r) = -\frac{\pi}{2}\sum_{k=0}^{\ell}c_{\ell k}\left[\frac{1}{r^{k+1}}f^{(\ell-k+1)}(t-r)+\frac{k}{r^{k+2}}f^{(\ell-k)}(t-r)\right]. \tag{110}$$

We have so far recovered the time domain VSH components corresponding to the coefficient $a_{\ell m}(s)$ in (100). Similarly, without showing detailed calculations,

$$B^{(1)}_{\ell m}(t,r) = -\frac{\pi}{2}\sum_{k=0}^{\ell}c_{\ell k}\left[\frac{1}{r^{k+1}}g^{(\ell-k+1)}(t-r)+\frac{k}{r^{k+2}}g^{(\ell-k)}(t-r)\right].$$

$$E^{(2)}_{\ell m}(t,r) = -\frac{\pi}{2}\sum_{k=0}^{\ell}\frac{c_{\ell k}}{r^{k+1}}g^{(\ell-k+1)}(t-r),$$

and finally

$$B^r_{\ell m}(t,r) = \ell(\ell+1)\frac{\pi}{2}\sum_{k=0}^{\ell}\frac{c_{\ell k}}{r^{k+2}}g^{(\ell-k)}(t-r).$$

Here a different underlying profile function $g(t-r)$ appears since these components refer to the sector in (100) corresponding to $b_{\ell m}(s)$.

3.3 Near-to-Far Field Propagation

3.3.1 Teleportation Kernels

To derive the *teleportation kernels* for the Maxwell equations, we will use similar arguments to those of the wave equation. We know from previous analysis that

$$\begin{aligned}\hat{B}^{(2)}_{\ell m}(s,r_2) &= -\frac{\pi}{2r_2}a_{\ell m}(s)s^{\ell+1}e^{-z_2}W_\ell(z_2),\\ \hat{B}^{(2)}_{\ell m}(s,r_1) &= -\frac{\pi}{2r_1}a_{\ell m}(s)s^{\ell+1}e^{-z_1}W_\ell(z_1),\end{aligned} \tag{111}$$

where $z_1 = sr_1$, $z_2 = sr_2$, and we assume that $r_2 > r_1 >$ support of the initial data. Evidently, the relationship between the two solutions is given by

$$e^{s(r_2-r_1)}\hat{B}^{(2)}_{\ell m}(s,r_2) = \frac{r_1}{r_2}\left[\frac{W_\ell(sr_2)}{W_\ell(sr_1)} - 1\right]\hat{B}^{(2)}_{\ell m}(s,r_1) + \frac{r_1}{r_2}\hat{B}^{(2)}_{\ell m}(s,r_1). \tag{112}$$

We will define it as in the case of the wave equation,

$$\hat{\Phi}_\ell(s,r_1,r_2) = -1 + \frac{W_\ell(sr_2)}{W_\ell(sr_1)}.$$

Taking the inverse Laplace transform of eqn(112) we see that

$$B^{(2)}_{\ell m}(t+(r_2-r_1),r_2) = \frac{r_1}{r_2}\int_0^t \Phi_\ell(t-t',r_1,r_2)B^{(2)}_{\ell m}(t',r_1)dt' + \frac{r_1}{r_2}B^{(2)}_{\ell m}(t,r_1). \tag{113}$$

Obviously, the kernel for $B^{(2)}_{\ell m}(t,r)$ is the same as that in the case of the wave equation. We will now proceed to calculate the *teleportation kernel* between the expression

$$(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s,r_1) = a_{\ell m}(s)s^{\ell+1}\frac{\pi}{2r_1}e^{-sr_1}W'_\ell(sr_1),$$

$$(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_2) = a_{\ell m}(s) s^{\ell+1} \frac{\pi}{2r_2} e^{-sr_2} W'_\ell(sr_2).$$

They are related by the identity

$$\begin{aligned}
e^{s(r_2-r_1)}(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_2) &= \frac{r_1}{r_2}\frac{W'_\ell(sr_2)}{W'_\ell(sr_1)}(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1) \\
&= \left(\frac{r_1}{r_2}\right)^3 \left[\left(\frac{r_2}{r_1}\right)^2 \frac{W'_\ell(sr_2)}{W'_\ell(sr_1)}\right](\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1) \\
&= \left(\frac{r_1}{r_2}\right)^3 \left[\left(\frac{r_2}{r_1}\right)^2 \frac{W'_\ell(sr_2)}{W'_\ell(sr_1)} - 1\right](\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1) \\
&+ \left(\frac{r_1}{r_2}\right)^3 (\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1).
\end{aligned} \tag{114}$$

We will before defining our new kernel as Υ_ℓ consider the following analysis. We know that

$$W'_\ell(sr) = -\sum_{k=1}^{\ell} \frac{kc_{\ell k}}{(sr)^{k+1}} = -\frac{c_{\ell 1}}{r^2} s^{-(\ell+1)} \sum_{k=1}^{\ell} \frac{kc_{\ell k}}{c_{\ell 1}} s^{\ell-k} r^{1-k}. \tag{115}$$

The factor

$$\sum_{k=1}^{\ell} \frac{kc_{\ell k}}{c_{\ell 1}} s^{\ell-k} = O(s^{\ell-1}) \tag{116}$$

is a monic polynomial of degree $(\ell-1)$ in s, and we *assume* that its roots $\{d_{\ell j} : j = 1, \ldots \ell - 1\}$ are simple. Whence

$$\sum_{k=1}^{\ell} \frac{kc_{\ell k}}{c_{\ell 1}} s^{\ell-k} = \prod_{j=1}^{\ell-1}(s - d_{\ell j}). \tag{117}$$

The roots $d_{\ell j}$ are also the roots of $W'_\ell(z)$. When we consider

$$\begin{aligned}
\sum_{k=1}^{\ell} \frac{kc_{\ell k}}{c_{\ell 1}} s^{\ell-k} r^{1-k} &= r^{1-\ell} \sum_{k=1}^{\ell} \frac{kc_{\ell k}}{c_{\ell 1}} (sr)^{\ell-k} \\
&= r^{1-\ell} \prod_{j=1}^{\ell-1}(sr - d_{\ell j}) \\
&= \prod_{j=1}^{\ell-1}(s - \frac{d_{\ell j}}{r}).
\end{aligned} \tag{118}$$

Combination of eqn(115) and (118), gives

$$W_\ell'(sr) = -\frac{c_{\ell 1}}{r^2} s^{-(\ell+1)} \prod_{j=1}^{\ell-1} (s - \frac{d_{\ell j}}{r}). \tag{119}$$

One can clearly see that as $s \to \infty$

$$\frac{W_\ell'(sr_2)}{W_\ell'(sr_1)} \to \left(\frac{r_1}{r_2}\right)^2.$$

In order to ensure that our kernel decays, we define it as

$$\hat{\Upsilon}_\ell(s, r_2, r_1) = \left(\frac{r_2}{r_1}\right)^2 \frac{W_\ell'(sr_2)}{W_\ell'(sr_1)} - 1. \tag{120}$$

When we feed into (120) the expression

$$\frac{W_\ell'(sr_2)}{W_\ell'(sr_1)}$$

as it appeared in (119), we have

$$\hat{\Upsilon}_\ell(s, r_2, r_1) = \left[\frac{\prod_{j=1}^{\ell-1}(s - \frac{d_{\ell j}}{r_2})}{\prod_{j=1}^{\ell-1}(s - \frac{d_{\ell j}}{r_1})}\right] - 1. \tag{121}$$

Returning to (114) to rewrite the equation in terms of Υ_ℓ, we find

$$\begin{aligned} \exp(s(r_2 - r_1))(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_2) &= \left(\frac{r_1}{r_2}\right)^3 (\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1) \\ &\quad + \left(\frac{r_1}{r_2}\right)^3 \hat{\Upsilon}_\ell(s, r_1, r_2)(\hat{E}^{(1)}_{\ell m} - \hat{B}^{(2)}_{\ell m})(s, r_1). \end{aligned} \tag{122}$$

The inverse Laplace transform of (122) is given by:

$$\begin{aligned} &(E^{(1)}_{\ell m} - B^{(2)}_{\ell m})(t + (r_2 - r_1), r_2) \\ &\qquad = \left(\frac{r_1}{r_2}\right)^3 \int_0^t \Upsilon_\ell(t - t', r_1, r_2)(E^{(1)}_{\ell m} - B^{(2)}_{\ell m})(t', r_1)dt' \\ &\qquad + \left(\frac{r_1}{r_2}\right)^3 (E^{(1)}_{\ell m} - B^{(2)}_{\ell m})(t, r_1). \end{aligned} \tag{123}$$

The residues $g_{\ell k}(r_1, r_2)$ in the pole expansion

$$\hat{\Upsilon}_\ell(s, r_2, r_1) = \sum_{k=1}^{\ell-1} \frac{g_{\ell k}(r_1, r_2)}{s - d_{\ell k}/r_1} \tag{124}$$

are given by

$$\begin{aligned} g_{\ell k}(r_1, r_2) &= \lim_{s \to d_{\ell k}/r_1} (s - d_{\ell k}/r_1) \left[\frac{\prod_{j=1}^{\ell-1}(s - \frac{d_{\ell j}}{r_2})}{\prod_{j=1}^{\ell-1}(s - \frac{d_{\ell j}}{r_1})} - 1 \right] \\ &= \left[\frac{\prod_{j=1}^{\ell-1} \Big(d_{\ell k}/r_1 - d_{\ell j}/r_2\Big)}{\prod_{j=1, j\neq k}^{\ell-1} \Big(d_{\ell k}/r_1 - d_{\ell j}/r_1\Big)} \right]. \end{aligned} \tag{125}$$

Notice also that, the derivative $W_\ell'(z)$, where $z = d_{\ell k} r_2/r_1$ is given by

$$W_\ell'(d_{\ell k} r_2/r_1) = -\frac{c_{\ell 1}}{r_2^2}(d_{\ell k}/r_1)^{-(\ell+1)} \prod_{j=1}^{\ell-1}(d_{\ell k}/r_1 - d_{\ell j}/r_2). \tag{126}$$

Finally, from the fact that

$$W_\ell'(sr) = -c_{\ell 1}(sr)^{-(\ell+1)} \prod_{j=1}^{\ell-1}(sr - d_{\ell j}), \tag{127}$$

it guarantees that the second derivative $W_\ell''(d_{\ell k})$ can computed as follows

$$\begin{aligned} W_\ell''(d_{\ell k}) &= -c_{\ell 1}(d_{\ell k})^{-(\ell+1)} \prod_{j=1, j\neq k}^{\ell-1} (d_{\ell k} - d_{\ell j}) \\ &= -c_{\ell 1}(d_{\ell k})^{-(\ell+1)} r_1^{(\ell-2)} \prod_{j=1, j\neq k}^{\ell-1} (d_{\ell k} - d_{\ell j})/r_1 \\ &= -(c_{\ell 1}/r_1^3)(d_{\ell k}/r_1)^{-(\ell+1)} \prod_{j=1, j\neq k}^{\ell-1} (d_{\ell k} - d_{\ell j})/r_1. \end{aligned} \tag{128}$$

Therefore, the residue is found to be

$$g_{\ell k}(r_1, r_2) = \Big(\frac{r_2}{r_1}\Big)^2 \frac{W_\ell'(d_{\ell k} r_2/r_1)}{r_1 W_\ell''(d_{\ell k})}. \tag{129}$$

3.3.2 Explicit Formulas for $\ell = 2$ and $\ell = 3$

The frequency domain teleportation kernel is given as sum poles which is

$$\hat{\Upsilon}_\ell(s, r_1, r_2) = \sum_{j=1}^{\ell-1} \frac{g_{\ell j}(r_1, r_2)}{s - d_{\ell j}/r_1} \tag{130}$$

whereas the time-domain teleportation kernel is sum of exponentials, which is given as

$$\Upsilon_\ell(t, r_1, r_2) = \sum_{k=1}^{\ell-1} g_{\ell k}(r_1, r_2) \exp(\frac{d_{\ell k} t}{r_1}). \tag{131}$$

For the $\ell = 2$ case, we have

$$\Upsilon_2(t, r_1, r_2) = g_{21}(r_1, r_2) \exp(\frac{d_{21}}{r_1} t). \tag{132}$$

We know from previous analysis that,

$$W_2(z) = 1 + \frac{3}{z} + \frac{3}{z^2},$$

therefore

$$W_2'(z) = -\frac{3}{z^2} - \frac{6}{z^3}, \quad W_2''(z) = \frac{6}{z^3} + \frac{18}{z^4}.$$

Clearly the root of $W_2'(z)$ is

$$d_{21} = -2.$$

Our goal here is to calculate g_{21}. Setting $r = r_2 r_1^{-1}$ and choosing $r_1 = 1$, eqn(129) reduces to

$$g_{21}(1, r) = r^2 \frac{W_2'(r d_{21})}{W_2''(d_{21})}, \tag{133}$$

we can recover the actual equation later by substituting $r = r_2 r_1^{-1}$ and dividing by r_1. When we substitute the values of d_{21} and $r d_{21}$ in the previous equation (133), we have

$$r^2 \frac{W_2'(-2r)}{W_2''(-2)} = \frac{2}{r}(1 - r),$$

therefore

$$g_{21}(r_1, r_2) = \frac{2}{r_1 r_2}\Big(r_1 - r_2\Big). \tag{134}$$

Here we have

$$\Upsilon_2(t, r_1, r_2) = \frac{2}{r_1 r_2}\Big(r_1 - r_2\Big)\exp(\frac{-2t}{r_1}). \tag{135}$$

For $\ell = 3$, we have

$$\Upsilon_3(t, r_1, r_2) = g_{31}(r_1, r_2)\exp(\frac{d_{31}}{r_1}t) + g_{32}(r_1, r_2)\exp(\frac{d_{32}}{r_1}t),$$

Similarly we know that

$$W_3(z) = 1 + \frac{6}{z} + \frac{15}{z^2} + \frac{15}{z^3}.$$

Therefore

$$W_3'(z) = -\frac{6}{z^2} - \frac{30}{z^3} - \frac{45}{z^4} = -\frac{6}{z^4}\Big(z^2 + 5z + 15/2\Big),$$

and

$$W_3''(z) = \frac{12}{z^3} + \frac{90}{z^4} + \frac{180}{z^5} = \frac{6}{z^5}\Big(2z^2 + 15z + 30\Big)$$

The roots of $W_3'(z)$ are given by

$$d_{31} = -\frac{5}{2} + i\frac{\sqrt{5}}{2}, \qquad d_{32} = -\frac{5}{2} - i\frac{\sqrt{5}}{2}.$$

When we substitute the values of d_{31} and d_{32} in the previous equation, we have

$$\Upsilon_3(t, r_1, r_2) = \exp(-\frac{5}{2r_1}t)\left[(g_{31}+g_{32})\cos\left(\frac{\sqrt{5}}{2r_1}t\right) + i(g_{31}-g_{32})\sin\left(\frac{\sqrt{3}}{2r_1}t\right)\right] \tag{136}$$

Our goal here is to calculate g_{31} and g_{32}. When we set $r = r_2 r_1^{-1}$ and $r_1 = 1$ as we did in the previous calculation, we have

$$g_{31}(1, r) = r^2\frac{W_3'(rd_{31})}{W_3''(d_{31})} = -\frac{d_{31}}{r^2}\Big[\frac{(rd_{31})^2 + 5rd_{31} + 15/2}{2d_{31}^2 + 15d_{31} + 30}\Big]. \tag{137}$$

We can after substituting the respective values of d_{31} and d_{32} arrive at

$$g_{31}(1,r) = -\frac{(r-1)}{2r^2}\Big(5r + i\sqrt{5}(2r-3)\Big) \tag{138}$$

since g_{32} is the complex conjugate of g_{31}, it implies that

$$\Upsilon_3(t,r_1,r_2) = \exp\left(-\frac{5}{2r_1}t\right)\left[(g^R)\cos\left(\frac{\sqrt{5}}{2r_1}t\right) + i(g^I)\sin\left(\frac{\sqrt{5}}{2r_1}t\right)\right] \tag{139}$$

where

$$g^R(1,r) = \frac{5(1-r)}{r} \implies g^R(r_1,r_2) = \frac{5(r_1-r_2)}{r_1r_2} \tag{140}$$

and

$$g^I(1,r) = \frac{\sqrt{5}(1-r)(2r-3)}{r^2} \implies g^I(r_1,r_2) = \frac{\sqrt{5}(r_1-r_2)(2r_2-3r_1)}{r_1r_2^2}. \tag{141}$$

Chapter 4

Conclusion

We will conclude by highlighting some of the important aspects of this analysis. In chapter 1, we solved the 3-dimensional scalar wave equation based on the assumption that our source S and initial data are compactly supported and that we work at a large distance from the source. By lemmas 2.1.1 and 2.1.2, we derived the time-domain outgoing multipole solution $\Psi_\ell(t,r)$ to the wave equation. We found the kernel for the wave equation and its corresponding residue to be

$$\Phi_\ell(t,r_1,r_2) = \sum_{k=1}^{\ell} a_{\ell k}(r_1,r_2)\exp\left(\frac{b_{\ell k}t}{r_1}\right),$$

$$a_{\ell j}(r_1,r_2) = \frac{W_\ell(b_{\ell j}r_2r_1^{-1})}{r_1W'_\ell(b_{\ell j})},$$

respectively. In the final phase of this analysis, we expressed the Maxwell equations in terms of the Vector Spherical Harmonics (VSH). By invoking the properties of the VSH, we solved the Maxwell equations. Using the same lemmas 2.1.1 and 2.1.2, we found the time-domain representation of outgoing solution to the Maxwell equations. In developing the *teleportation kernel* for $B^{(2)}_{\ell m}(t,r)$ we found that, it is the same kernel as in the case of the wave equation. On the contrary, the *teleportation* for

$$E^{(2)}_{\ell m}(t,r) - B^{(2)}_{\ell m}(t,r)$$

yields a different *kernel* and a residue. We found the kernel and the residue in this case respectively to be

$$\Upsilon_\ell(t,r_1,r_2) = \sum_{k=1}^{\ell-1} g_{\ell k}(r_1,r_2)\exp\left(\frac{d_{\ell k}t}{r_1}\right)$$

and

$$g_{\ell k}(r_1,r_1) = \left(\frac{r_2}{r_1}\right)^2 \frac{W'_\ell(d_{\ell k}r_2/r_1)}{r_1 W''_\ell(d_{\ell k})}.$$

The teleportation kernels Φ_ℓ and Υ_ℓ as well the as the residues $a_{\ell k}(r_1,r_2)$ and $g_{\ell k}(r_1,r_2)$ obtained in this work, were used in developing the general formula for propagating the multipole solutions to the wave and Maxwell equations for different values of ℓ.

Greengard, Hagstrom, Jiang [9] have examined the asymptotic behavior of the residue

$$a_{\ell j} = \frac{W_\ell(b_{\ell j}r_2 r_1^{-1})}{r_1 W'_\ell(b_{\ell j})}.$$

They showed in their work that as $\ell \to \infty$ for fixed large r_1, r_2, the coefficient of $a_{\ell j}$ grows exponentially. That is,

$$\max_k |a_{\ell k}| \sim \frac{\ell^\ell}{\ell!} \sim e^\ell.$$

One might also investigate the asymptotic behavior of $g_{\ell k}$ from (129) as

$$\ell \to \infty$$

for fixed large value of r. Such an investigation would also involve the larger-order asymptotics of Hankel functions (as [9]), but also of their derivatives. We have in our approach for solving Maxwell equations used Vector Spherical Harmonics. As an alternative, one could have reduced the Maxwell equations to the scalar wave equation by first considering the following calculation. We know that

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{142}$$

$$\nabla \times \mathbf{B} = -\frac{\partial \mathbf{E}}{\partial t}. \tag{143}$$

When we take the curl of (142) equation, we have

$$\nabla \times (\nabla \times \mathbf{E}) = \frac{\partial(\nabla \times \mathbf{B})}{\partial t}. \tag{144}$$

Substituting (143) into (144), we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{145}$$

For the source free case $\nabla \cdot \mathbf{E} = 0$, therefore our previous equation reduces to

$$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{146}$$

A similar wave equation can be derived for the Cartesian components of $\mathbf{B}$. Once this resolution is reached, the *teleportation kernel* for the propagation of both $\mathbf{E}$ and $\mathbf{B}$ field can be found using only the kernel for the wave equation. We did not use this approach in solving Maxwell equations because it would have led us in getting six (6) convolutions which is tedious to handle. We preferred the use of the Vector Spherical Harmonics in solving the Maxwell equations to the approach discussed above because the latter (VSH approach) is cheaper and relatively simpler. We only had four (4) convolutions as apposed six convolution in former approach. The VSH approach gave us two sectors (100). One of the sectors comprises of $B^{(2)}_{\ell m}(t,r)$ and $(E^{(1)}_{\ell m} - B^{(2)}_{\ell m})(t,r)$ which correspond to the coefficient $a_{\ell m}(s)$. The second sector comprises of $E^{(2)}_{\ell m}(t,r)$ and $(B^{(1)}_{\ell m} - E^{(2)}_{\ell m})(t,r)$ which corresponds to the coefficient $b_{\ell m}(s)$. Our analysis indicates that $B^{(2)}_{\ell m}(t,r)$ is similar to $E^{(2)}_{\ell m}(t,r)$ as $(E^{(1)}_{\ell m} - B^{(2)}_{\ell m})(t,r)$ is to $(B^{(1)}_{\ell m} - E^{(2)}_{\ell m})(t,r)$ in terms of their teleportation kernels. The four convolutions agree with two transverse degrees of freedom for radiating waves in Maxwell's theory.

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