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L^{\infty}-Estimates Of The Solution Of The Navier-Stokes Equations For Periodic Initial Data

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DISSERTATION

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Abstract

In this doctoral dissertation, we consider the Cauchy problem for the 3D incompressible Navier-Stokes equations. Here, we are interested in a smooth periodic solution of the problem which happens to be a special case of a paper by Otto Kreiss and Jens Lorenz. More precisely, we will look into a special case of their paper by two approaches. In the first approach, we will try to follow the similar techniques as in the original paper for smooth periodic solution. Because of the involvement of the Fourier expansion in the process, we encounter with some intriguing factors in the periodic case which are absolutely not a part in the original paper. While in the second approach, we decompose our solution space using the Helmholtz-Weyl decomposition and introduce a new tool "the Leray projector" to eliminate the pressure term from the Navier-Stokes equations and go from there. This approach is completely different than the technique of dealing with the pressure term in the paper by Otto Kreiss and Jens Lorenz.

Contents

1 Introduction

1.1 Function Spaces and Some Notations

In this section, we introduce some notations and some standard function spaces that we will require in our work. We are interested in a three dimensional flow, however we will introduce notations and spaces for $n \geq 3$. We focus on two domains without boundaries and one with boundary:

- the whole space $\mathbb{R}^n$.
- the n dimensional torus $\mathbb{T}^n$.
- the n dimensional half space $\mathbb{R}^n_+$.

Functions defined on the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ can be realized as periodic functions defined on all of $\mathbb{R}^n$, i.e

$$u(x + 2\pi k) = u(x) \quad \text{for all} \quad k \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n.$$

It is also convenient at times to identify u with its restriction to the fundamental domain $[0, 2\pi)^n$. If $u : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field then we can define the divergence of u as

$$\text{div} u = \nabla \cdot u = \sum_{i=1}^{n} D_i u_i, \quad D_i = \partial/\partial x_i.$$

For higher order derivatives, we will employ multi-index notation. We write

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \quad \alpha_i \geq 0 \quad \text{and} \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

For a vector $x = (x_1, x_2, \cdots, x_n)$ we define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and similarly, we set

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

For any simply connected domain with smooth boundary $\Omega \subseteq \mathbb{R}^n$, we use the following notation for function spaces.

- $C^\infty_{per}(\mathbb{R}^n)$ is the space of smooth 2π periodic functions.
- $C^\infty_c(\Omega)$ is the space of all smooth function with compact support in Ω.

- $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions on $\mathbb{R}^n$, that is the space of all smooth function such that

$$p_{k,\alpha} := \sup_{x\in\mathbb{R}^n} |x|^k |D^\alpha u(x)|$$

is finite for every choice of $k = 0, 1, 2\cdots$ and multi-index $\alpha \geq 0$.

- $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions on $\mathbb{R}^n$.

- By $L^p(\Omega)$ for $1 \leq p < \infty$, we denote the standard Lebesgue spaces of measurable p-integrable functions with the norm

$$||u||_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

- $L^\infty(\Omega)$ is space of bounded functions on Ω which is equipped with the standard norm

$$|u|_\infty := \sup_{x\in\Omega} |u(x)|.$$

- For any $j = 0, 1, 2, \cdots$, we set, for any multiindex α

$$|\mathcal{D}^j u(x)|_\infty = \max_{|\alpha|=j} |D^\alpha u(x)|_\infty$$

i.e $|\mathcal{D}^j u(x)|_\infty$ measures all space derivatives of order j in maximum norm.

- $L^p_{loc}(\Omega)$ for $1 \leq p \leq \infty$ consists of those functions that are contained in $L^p(K)$ for every compact subset K of Ω.

- For integer $k > 0$ and $1 \leq p < \infty$ the space $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^k u \in L^p(\Omega)\}$ where D^k is weak derivative of order k. The standard norm in $W^{k,p}(\Omega)$ is given by

$$||u||^p_{W^{k,p}} = \sum_{|\alpha|\leq k} ||D^\alpha u||^p_{L^p}.$$

- For any vector valued space X, we define $X^\sigma = \{u \in X : \nabla \cdot u = 0\}$.

- For $s \geq 0$ the Sobolev space $H^s(\mathbb{T}^n)$ consists of all function such that

$$||u||^2_{H^s} := (2\pi)^n \sum_{k\in\mathbb{Z}^n} (1 + |k|^{2s})|\widehat{u}(k)|^2 < \infty.$$

- BMO consists function of bounded mean oscillation. See [16] page: 29.
- We denote $[L^2(\Omega)]^n = \mathbb{L}^2(\Omega), \quad [H^s(\Omega)]^n = \mathbb{H}^s(\Omega)$.
- $\dot{\mathbb{L}}^2(\mathbb{T}^n) = \{u \in \mathbb{L}^2(\mathbb{T}^n) : \widehat{u}(0) = 0\}$.
- $\dot{\mathbb{Z}}^n = \mathbb{Z}^n \backslash \{(0, \cdots, 0)\}$
- For a function $u(x,t)$ where $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, we denote $u(t) = u(.,t)$ also $|u(t)|_\infty = \max_{x\in\mathbb{R}^n} |u(x,t)|$ for fixed t.

1.2 Motivation: A Brief History of the Navier-Stokes Equations

In this paper, we consider the three-dimensional incompressible Navier-Stokes equations. The three -dimensional incompressible Navier-Stokes equations form the fundamental mathematical model of fluid dynamics. Derived from basic physical principles under the assumption of a linear relationship between the stress and the rate-of-strain in the fluid, their applicability to real-life problems is undisputed. However, a rigorous mathematical theory for these equations is still far from complete: in particular there is no guarantee of the global existence of unique solution. The aim of this introductory section is to give an overview of the existing results in the literature for the existence and uniqueness of the solution of the Navier-Stokes equations.

The Navier-Stokes equations are a set of partial differential equations that describe fluid motion. Since we are interested in the 3D incompressible Navier-Stokes equations, therefore we begin with $x \in \mathbb{R}^3$ as the space variable and $t \in [0,\infty)$ as the time variable. We consider the fluid velocity field of a fluid with notation $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$ for the scalar pressure field. The conservation of linear momentum leads to

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \triangle u \tag{1.1}$$

while the conservation of mass yields the divergence-free (incompressible) condition

$$\nabla \cdot u = 0 \tag{1.2}$$

where the viscosity constant is ν, and

$$u \cdot \nabla = u_1 D_1 + u_2 D_2 + u_3 D_3, \qquad D_i = \frac{\partial}{\partial x_i}, \qquad 1 \leq i \leq 3.$$

Furthermore, we assume that

$$u(x,0) = f(x) \quad \text{with} \quad \nabla \cdot f = 0. \tag{1.3}$$

Throughout this paper when referring to the Navier-Stokes equations, we will normalize the constant ν so that $\nu = 1$. In this work, we will consider the problem on two types of domain:

i. the whole space $\mathbb{R}^3$ with a certain class of solution

ii. the torus $\mathbb{T}^3 = (\mathbb{R}\backslash 2\pi\mathbb{Z})^3$, sometimes described as the case of "periodic boundary conditions" ; for mathematical convenience, we will also impose the zero-average condition $\int_{\mathbb{T}^3} u = 0$.

These cases have the advantage that they do not involve any boundaries, which greatly simplifies the analysis in many instances: when we carry out an analysis restricted to these two cases, we will refer to the equation " in the absence of boundaries".

Equations that describe physical phenomena beg for solution, and the Navier-Stokes equations are no exception. There is a vast wealth of literature devoted to the solutions of Navier-Stokes equations (eg [4],[5],[6],[7],[8],[10],[11]). For example, in 1934 Leray [15] constructed a global in time weak solution, and a local strong solution in $\mathbb{R}^3$. So far, it is not achieved that whether such weak solution is unique or the unique strong solution exists globally in time. This paper, among others, reinforced the idea of specialized solutions. Among the different classes of solutions studied for the Navier-Stokes equations one can find classical, strong, mild, weak, very weak, uniform weak, and local Leray solutions. These different classes have themselves produced a variety of methods designed to explore the various types of solutions. Fourier analysis, statistical mechanics, distribution theory, and harmonic analysis have all played a part in attempting to analyze the equations for over two centuries. A large portion of this work focuses on the well-known fact that at its core, the Navier-Stokes equation (1.1) is basically a non-linear heat equation. Thus, it can be written using Duhamel's principle in an integral form with heavy dependence on the initial data $u(x,0) = f(x)$. Exploiting the integral form of the Navier-Stokes equations has been used to explore other aspects of the solutions, such as existence, uniqueness, and the dependence on initial data. For example, it is known that for initial data $u(x,0) = f(x) \in L^\infty(\mathbb{R}^n)$ the equations (1.1)- (1.3) admit a local in time

(regular) solution u with the pressure p is determined by

$$p = \sum_{i,j} R_i R_j (u_i u_j) \tag{1.4}$$

where R_i is the Riesz transform ([3], [8]). For the L^r case, where $3 \leq r < \infty$, the equations (1.1)-(1.3) admit a unique local in time solution u for some pressure p. If u decays at the space infinity, then (1.4) follows a posteriori for L^r ([11]). Kato [10] observed that for initial data $f \in L^\infty(\mathbb{R}^n)$, the constructed solution is bounded and may not decay at the space infinity. So even if u solves (1.1)- (1.3) the equation (1.4) may not follow. Kato further noted that in the most simple case, for $x \in \mathbb{R}^3$ and $t \in (0, \infty)$, one could construct a solution of the form $u(x,t) = g(t), p(x,t) = -g'(t) \cdot x$. This function pair (u, p) solves (1.1) and (1.2) no matter what the function $g(t)$ is. So if u has constant initial data, the solution is not unique without assuming (1.4). Not only does this demonstrate a non-uniqueness to the solution, it also implies a non-decaying pressure spatially. Kato further observed that one would need to impose some control on p to obtain uniqueness other than controlling u.

In [6], it was noted that uniqueness holds if u is bounded, and p is of the form

$$p(x,t) = \pi_0 + \sum_{i,j} R_i R_j \pi_{ij} \tag{1.5}$$

for bounded functions π_0 and π_{ij}. In particular, it was noted that for $t \in (0, T)$ for a maximal time T, then

$$\pi_0, \pi_{ij} \in L^\infty \cap L^1_{loc}$$

The paper by Kato [10] improved upon the result by simply assuming that $p \in L^1_{loc} \cap BMO$. A theorem of Uchiyama [21] indicated that if a function g was BMO, then it was of the form

$$g = \nu_0 + \sum_{i,j} R_i R_j \nu_{ij}$$

with some $\nu_{ij}, \nu_0 \in L^\infty(\mathbb{R}^n)$.

As the pressure term is a key aspect in the analysis of the Navier-Stokes equations, in chapter three, we will discuss the fact that the formal pressure term (and its modification) is the solution of the Poisson pressure equation; it is the convolution of the term $\sum_{i,j}(D_i u_j)(D_j u_i)$ with a Calderon-Zygmund kernel. It is of some interest to note the fact that if the velocity field has derivatives that are sufficiently smooth and small at infinity, it then turns out that the pressure is additionally a Riesz potential (see [18]). Interestingly, it is not obvious that such a pressure given by convolution is periodic in space even u is smooth periodic in space. We will focus on some detail in proving that the pressure given as a convolution is indeed smooth periodic, provided u is smooth periodic in space.

Furthermore, the papers by Giga specifically [7], [8] , took a different approach, and chose initial data in the space of bounded uniformly continuous functions (BUC) in $\mathbb{R}^n$, or in $L^\infty(\mathbb{R}^n)$. In the paper, it was shown that if the initial value function $u(x,0) = f(x)$ was BUC, then so was the unique solution u. In this case, however, the focus was on u as a solution to be integral (heat equation) version of the Navier-Stokes equations. In Giga's work, the set $C([0,T], BUC)$ was defined to be the set of all bounded, uniformly continuous on $[0,T]$, and the set C^α was representative of the set of Hölder continuous functions of order α. It was additionally shown in the same paper that

$$u \in C([\delta, T]; BUC) \quad \text{and} \quad t^{1/2}\nabla u \in C([0,T]; BUC) \tag{1.6}$$

and

$$\nabla u \in C^\alpha([\delta, T]; BUC)$$

for some α with $0 < \alpha < 1/2$ and δ such that $0 < \delta < T$. This showed that $t^{1/2}\nabla u$ was bounded in some sense. An additional result was that if $u(x,0) = f(x)$ was a BUC function with u satisfying the integral form of the solution to the Navier-Stokes equations, and if $\nabla u \in C^\alpha([\delta, T]; BUC)$, then by writing

$$p = \sum_{i,j} R_i R_j (u_i u_j)$$

we have (u,p) solving (1.1)-(1.2). However, again in [7], it was noted that if one replaced the space BUC with L^∞, the results were different. Expressions (1.6) are replaced by

$$u \in C_w([\delta, T]; L^\infty) \quad \text{and} \quad t^{1/2}\nabla u \in C_w([0,T]; L^\infty)$$

where C_w is the space of all L^∞ valued weakly continuous functions defined on $[0,T]$.

These results, among others, are similar to the results in the work by Kreiss-Lorenz ([12]). The Kreiss-Lorenz paper is the main focus and source for this doctoral thesis. The main difference between the KL paper and the works of Kato and Giga is the restriction on the solutions. The KL's paper concentrates on classical solutions in $C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^3)$, while Kato and Giga's ([8],[10]) developed solutions that existed in the distributional sense.(eg.weakly continuous), with some sort of control on the pressure term p. Additionally, Kato and Giga's papers assumed their space of contention to be $\mathbb{R}^n$, for $n \geq 2$. Their results obviously can be restricted to $\mathbb{R}^3$, where the Kreiss-Lorenz results are only particular to $\mathbb{R}^3$. It must be noted that the KL paper was able to construct similar results, but the only structure mentioned on the pressure term was that it was simply a BMO valued function- no breakdown of the pressure term into Riesz transforms is required or even alluded to.

2 The Kreiss-Lorenz Paper (KL)

In this section, we will discuss important aspects of the KL paper. Since our work is closely related to the work of the Kreiss and Lorenz, it is worth discussing the significant details of the paper to make it easier for the readers to understand this doctoral dissertation. The KL paper considers the Cauchy problem for the three-dimensional Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \triangle u, \qquad \nabla \cdot u = 0 \tag{2.1}$$

with initial condition

$$u(x,0) = f(x), \qquad x \in \mathbb{R}^3 \tag{2.2}$$

where $f \in L^\infty(\mathbb{R}^3)$ and also assumes $\nabla \cdot f = 0$ in the distributional sense for the sake of compatibility. The main goal of the KL paper is to prove the following theorem:

Theorem 2.1. Consider the Cauchy problem for the Navier-Stokes equations (2.1) and (2.2) where $f \in L^\infty, \nabla \cdot f = 0$. There is a constant c_0 and for every $j = 0, 1, \cdots$, there is a constant K_j so that

$$t^{j/2}|\mathcal{D}^j u(.,t)|_\infty \leq K_j |f|_\infty \quad \text{for} \quad 0 < t \leq c_0/|f|_\infty^2.$$

The constants c_0 and K_j are independent of t and f.

Let us observe the key ideas of the KL paper that are used to prove theorem 2.1. We will also outline the significant differences between the KL work and work in this doctoral dissertation.

It is not uncommon in the literature to consider the finite energy solution of the Navier-Stokes equations.
That is

$$\int |u(x,t)|^2 dx < \infty.$$

That means u exists in L^2. In contrast, the KL paper assumes only that $u \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$. Thus, the KL paper allows for an infinite energy. Their paper begins by discussing the parabolic system

$$u_t = \triangle u + D_i g(u(x,t)), \qquad x \in \mathbb{R}^3, \qquad t \geq 0, \qquad D_i = \partial/\partial x_i \tag{2.3}$$

with initial condition

$$u(x,0) = f(x), \qquad f \in L^\infty(\mathbb{R}^3) \tag{2.4}$$

on a maximum time interval $0 < t < T(f)$ where g was assumed to be quadratic in u. It was shown that under the assumptions given on f and g that there is a constant $c_0 > 0$ with

$$T(f) > c_0/|f|_\infty^2$$

and

$$|u(\cdot,t)|_\infty \leq 2|f|_\infty \quad \text{for} \quad 0 < t \leq c_0/|f|_\infty^2.$$

Additionally, the estimate of theorem 2.1 was shown for $j = 1, 2 \cdots$ for the solution of the parabolic system (2.3) and (2.4).

The result obtained for the parabolic system is as important as the method used in obtaining the main result of the original paper. The same method is used to analyze the Navier-Stokes equations and obtain the similar bounds on the velocity field u and its derivatives. Estimating u and its derivatives require the estimates on the pressure and its derivatives. In the KL paper, the pressure is determined from the Poisson equation and decomposed into the local and global part using the smooth cut-off function and u and derivatives of u are being used to obtain the bounds on local pressure and global pressure. To pursue the desire result, Kreiss and Lorenz rewrite the Navier-Stokes equations

$$u_t = \triangle u + Q, \qquad \nabla \cdot u = 0, \qquad u = f \quad \text{at} \quad t = 0$$

with

$$
\begin{aligned}
Q &= -\nabla p - u \cdot \nabla u \\
&= -\nabla p - \sum_j u_j D_j u.
\end{aligned}
$$

The pressure can be determined by the Poisson equation

$$
\begin{aligned}
-\triangle p &= \sum_{i,j} D_i D_j (u_i u_j) \\
&= \sum_{i,j} (D_i u_j)(D_j u_i).
\end{aligned}
$$

Dropping the t-dependence in the notation, one gets formally,

$$
p(x) = \frac{1}{4\pi} \sum_{i,j} \int \frac{1}{|x-y|} D_i D_j (u_i u_j)(y) dy.
$$

This was decomposed into local and global part, $p = p_{loc} + p_{glb}$, as follows : Choose a C^∞ cut-off function $\phi(r)$ with

$$
\phi(r) = 1 \quad \text{for} \quad 0 \le r \le 1, \qquad \phi(r) = 0 \quad \text{for} \quad r \ge 2.
$$

Then, for $\delta > 0$, define

$$
p_{loc}(x) = \frac{1}{4\pi} \sum_{i,j} \int |x-y|^{-1} D_i D_j (\phi(\delta^{-1}|x-y|) u_i(y) u_j(y)) dy. \tag{2.5}
$$

The global part, $p_{glb} = p - p_{loc}$, is determined correspondingly with ϕ replaced by $1-\phi$. For the application purpose in the later section, it will be chosen that $\delta = \sqrt{t}$.

The KL paper provides suitable bounds on p_{loc}, Dp_{loc} and Dp_{glb}. The problem with the integral for the pressure is that it may fail to exist because of the fact that $u \in L^\infty$. If the integral fails to exists the subsequent calculations and bounds are essentially incorrect. At the end of the paper, however a modification

$$
p^*(x,t) = PV \sum_{i,j} \frac{1}{4\pi} \int \big(G_{ij}(x-y) - G_{ij}(y)\big)(u_i u_j)(y,t) dy
$$

where $G(y) = |y|^{-1}$ and

$$
G_{ij}(y) = D_i D_j G(y)
$$

was given.

The modification was claimed to solve the Poisson pressure equation and has benefit of being bounded as long as $|x| < R$ for some $R > 0$ by an application of the Mean- Value Theorem. We will also prove that claim when the solution to the Navier-Stokes equations is considered to be smooth periodic in space. Thus the modified pressure integral exists even if the velocity field u may not have any decay at space infinity.

Consider a modified kernel

$$G_{ij}(x-y) - G_{ij}(y)$$

so the modified pressure term becomes $p^*(x,t) = p(x,t) + C(t)$ where $C(t)$ is a time dependent constant. Notice, in the Navier-Stokes equation only space derivatives of the pressure term appears, therefore adding or subtracting a suitably chosen time dependent constant would not matter.

It can be shown that the modified pressure solves the Poisson pressure equation and it also satisfies the estimates:

$$\begin{aligned}
|p_{lc}|_\infty &\leq C(|u|_\infty^2 + \delta|u|_\infty|\mathcal{D}u|_\infty);\\
|\mathcal{D}p_{lc}|_\infty &\leq c(\delta^{-1}|u|_\infty^2 + \delta|\mathcal{D}u|_\infty^2);\\
|\mathcal{D}p_{gl}|_\infty &\leq c\delta^{-1}|u|_\infty^2.
\end{aligned}$$

These estimates are required to obtain the main estimates in the original paper, however, they are not discussed rigorously in the KL paper for the modified pressure term and for its higher order derivatives.

Since we are interested in the smooth periodic solution of the Navier-Stokes equations, we are concerned whether we will be able to obtain similar estimates on the periodic pressure so that we could follow the same techniques as in the KL paper. Expecting the periodic pressure term of the same underlying structure as the modified pressure would not be surprising for the purpose of obtaining the similar estimates as in the KL paper. While dealing with the pressure in later section, we will keep that thing in our mind so that the pressure term possesses same underlying structure as the modified pressure.

On the other hand proving theorem 2.1 in the KL paper is identical to the results of the parabolic problem (2.3) and (2.4). This will prove that all derivatives of u are bounded in maximum norm by the max norm of initial function $f \in L^\infty(\mathbb{R}^3)$ in some maximum interval $(0, T(f))$ for $0 < T(f) \leq \infty$.

The next section of this doctoral dissertation will be focused on how to prove the KL paper when the initial function $f \in C^{\infty}_{per}(\mathbb{R}^3)$ which is a special case of the KL paper and our main work for this dissertation. The difficulties in doing so lie on the fact that whether we will be able to obtain estimates on the pressure and its derivatives as in the KL paper. However, for the parabolic system in the periodic case seems working exactly the same way as in the KL paper by the availability of the Poisson summation formula.

3 The KL Paper in Periodic Case

This section of this doctoral dissertation will be concentrated in details of obtaining the estimates of theorem 2.1 when the initial data is smooth periodic. It is well-known that the unique smooth periodic solution of (2.1), (2.2) for $f \in C^{\infty}_{per}(\mathbb{R}^n)$ exists in a maximum interval of time $0 \leq t < T(f)$ for some $T(f) \leq \infty$. Since our data are smooth periodic, we will seek the use of the Fourier expansion in the process of establishing many auxiliary results to prove theorem 2.1 whereas this was not the case in the KL paper since their solution is of class $L^{\infty}(\mathbb{R}^3) \cap C^{\infty}(\mathbb{R}^3)$. For that purpose, we will consider the Cauchy problem for 3D incompressible Navier-Stokes equations (2.1) and (2.2) where $f \in C^{\infty}_{per}(\mathbb{R}^3)$.

To follow the techniques of the KL paper, we are required to establish some auxiliary results of the heat equations in a torus $\mathbb{T}^n$ for $n \geq 3$.

3.1 Periodic Heat Flow in $\mathbb{R}^n$

Consider the Cauchy problem associated to the heat equation with periodic boundary conditions, that is

$$u_t(x,t) = \triangle u(x,t) \quad \forall \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{3.1}$$

$$u(x,0) = f(x) \qquad \forall \quad x \in \mathbb{R}^n. \tag{3.2}$$

Though we are considering smooth periodic initial function for the Navier-Stokes equations, for now we can relax the assumption of smoothness on f and consider $f \in C_{per}(\mathbb{R}^n)$. Taking the Fourier expansion of the PDE (3.1) and the initial condition in (3.2) we get, for $k \in \mathbb{Z}^n$,

$$\widehat{u}_t(k,t) = -|k|^2 \widehat{u}(k,t),$$
$$\widehat{u}(k,0) = \widehat{f}(k).$$

For each fixed k, this is an initial value problem for an ordinary differential equation. The unique solution is given by

$$\widehat{u}(k,t) = e^{-|k|^2 t}\widehat{f}(k). \tag{3.3}$$

To satisfy the initial condition, we use the superposition

$$u(x,t) = \sum_{k\in\mathbb{Z}^n} e^{-|k|^2 t} e^{ik\cdot x}\widehat{f}(k). \tag{3.4}$$

Formally, (3.4) can also be written as an integral operator

$$\begin{aligned} u(x,t) &= \sum_{k\in\mathbb{Z}^n}\left[\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-ik\cdot y} f(y)dy\right] e^{-|k|^2 t} e^{ik\cdot x} \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \theta(x-y,t) f(y)dy \end{aligned} \tag{3.5}$$

where

$$\theta(x,t) = \sum_{k\in\mathbb{Z}^n} e^{-|k|^2 t} e^{ik\cdot x}, \qquad t>0 \tag{3.6}$$

is the periodic heat kernel in $\mathbb{R}^n$.

Proposition 3.1. The series in (3.4) and (3.6) and their term by term derivatives of all orders, converge absolutely and uniformly on $\mathbb{R}^n \times [t,\infty)$ for all $t>0$. Both are solutions of the heat equation on $\mathbb{R}^n\times(0,\infty)$ and both belong to $C^\infty(\mathbb{R}^n\times(0,\infty))$. The initial condition is satisfied as $\lim_{t\to 0^+}\theta(.,t)*f = f$ in the maximum norm.

Proof. Left to the reader. □

There are other possible expansion for the periodic heat kernel. For example, in one dimension, it is easy to see that

$$\theta(x,t) = 1 + 2\sum_{k=1}^{\infty}\cos(kx)e^{-k^2 t}.$$

Thus the periodic heat kernel is a real valued function. But the fact that it is also a non-negative function is not so obvious. Analogous facts hold in higher dimension as well.

Next, we are going to state the Poisson summation formula and use it to prove that the heat kernel given by (3.6) is non-negative function.

(**The Poisson Summation Formula**) Suppose $f\in C(\mathbb{R}^n)$, satisfies $|f(x)|\le C(1+|x|)^{-n-\epsilon}$ for some $\epsilon>0, C>0$ and $|\widehat{f}(\xi)|\le C(1+|\xi|)^{-n-\epsilon}$ then

$$\sum_{k\in\mathbb{Z}^n} f(x+2\pi k) = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}^n}\widehat{f}(k)e^{ik\cdot x}. \tag{3.7}$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} dx$$

is the Fourier transform of f in $\mathbb{R}^n$.

Proof. The proof can be found in most standard Fourier Analysis books. □

With the use of the Poisson Summation Formula, we prove the following important proposition for this section.

Proposition 3.2. Let $\theta = \theta(x,t)$ be the function defined in (3.6). Then for all $t > 0$, $x \in \mathbb{R}^n$

$$\theta(x,t) = \sum_{k\in\mathbb{Z}^n} \left(\frac{\pi}{t}\right)^{n/2} \exp\left[\frac{-|x+2\pi k|^2}{4t}\right]. \tag{3.8}$$

Proof. Let us take the function $f(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}$ for $t > 0$, $x \in \mathbb{R}^n$. The Fourier transform of f with respect to the space variable is given by $\widehat{f}(\xi,t) = e^{-|\xi|^2 t}$. It is not difficult to see that f satisfies the requirements in the Poisson summation formula. Therefore, applying the Poisson summation formula, we get

$$\begin{aligned}
\frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \widehat{f}(k,t) e^{ik\cdot x} &= \sum_{k\in\mathbb{Z}^n} f(x+2\pi k, t) \\
\sum_{k\in\mathbb{Z}^n} e^{-|k|^2 t} e^{ik\cdot x} &= (2\pi)^n \sum_{k\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x+2\pi k|^2}{4t}} \\
\theta(x,t) &= \sum_{k\in\mathbb{Z}^n} \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{\frac{-|x+2\pi k|^2}{4t}}.
\end{aligned}$$

□

It is obvious from (3.8), the periodic heat kernel is non-negative, real valued function. Next, we are going to state properties of the heat kernel which will justify that the heat kernel is an approximation of identity. Later, we will be using these properties of the heat kernel to obtain some auxiliary results for the solution to the heat equation in periodic case.

Proposition 3.3. Let $\theta(x,t)$ be the function defined in (3.8). Then $\theta(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and

i. $\theta(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

ii. $\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \theta(x,t) dx = 1$ for all $t \in (0,\infty)$.

iii. $\lim_{t\to 0^+} \int_{\mathbb{T}^n \setminus B_\delta(0)} \theta(x,t)dx = 0$ for all $\delta \in (0,\pi)$.

Proof. Left to the reader. $\square$

With the heat kernel of the form (3.8) in hand and with proposition 3.3, we now are ready to establish some auxiliary results of the heat equation for periodic initial data. For that purpose, let us rewrite the Cauchy problem associated to the heat equation for $f \in C_{per}(\mathbb{R}^n)$. That is

$$u_t = \triangle u, \qquad u = f \qquad at \quad t = 0. \tag{3.9}$$

The solution of (3.9) is denoted by

$$u(t) := u(.,t) = e^{\triangle t} f = \theta * f$$

where $\theta(x,t) = \theta_t(x) = 1/(4\pi t)^{n/2} e^{\frac{-|x|^2}{4t}}, t > 0$. It is well-known that the solution of (3.9) satisfies the following estimates:

$$|e^{\triangle t} f|_\infty \le |f|_\infty, \qquad t \ge 0 \tag{3.10}$$

$$|\mathcal{D}^j e^{\triangle t} f|_\infty \le C_j t^{-j/2} |f|_\infty, \qquad t > 0, \qquad j = 1, 2, \cdots \tag{3.11}$$

Here, and in the following, C, C_j, etc. denote positive constants that are independent of t and f.

Proposition 3.4. Let $F \in C_{per}(\mathbb{R}^n \times [0,T])$. Then the solutions of

$$u_t = \triangle u + F(x,t), \qquad u = 0 \quad \text{at} \quad t = 0 \tag{3.12}$$

$$u_t = \triangle u + D_i F(x,t), \qquad u = 0 \quad \text{at} \quad t = 0 \tag{3.13}$$

respectively obey the estimates

$$|u(t)|_\infty \le 2t^{1/2} \max_{0\le s\le t} \{s^{1/2} |F(s)|_\infty\} \tag{3.14}$$

$$|u(t)|_\infty \le C \max_{0\le s\le t} \{s^{1/2} |F(s)|_\infty\}. \tag{3.15}$$

Proof. In the following, we will use $F(t) := F(.,t)$ to be consistent with the notations. To prove (3.14), let us write the solution of (3.12) using the *Duhamel's principle*

$$u(x,t) = \int_0^t e^{\triangle(t-s)} F(x,s) ds \qquad (x \in \mathbb{R}^n, t \ge 0)$$

We estimate $u(t)$ as

$$\begin{aligned}
|u(t)|_\infty &\le \int_0^t |F(s)|_\infty ds \\
&= \int_0^t s^{-1/2} s^{1/2} |F(s)|_\infty ds \\
&\le \max_{0\le s\le t}\{s^{1/2}|F(s)|_\infty\} \int_0^t s^{-1/2} ds \\
&\le 2t^{1/2} \max_{0\le s\le t}\{s^{1/2}|F(s)|_\infty\}.
\end{aligned}$$

To estimate the solution of (3.13), we note that D_i commutes with the heat semi-group. Again, using the *Duhamel's principle* we have

$$\begin{aligned}
u(t) &= \int_0^t e^{\triangle(t-s)} D_i F(s) ds \\
&= \int_0^t D_i e^{\triangle(t-s)} F(s) ds.
\end{aligned}$$

Using (3.11) for $j = 1$ we get

$$\begin{aligned}
|u(t)|_\infty &\le C \int_0^t (t-s)^{-1/2} |F(s)|_\infty ds \\
&= C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |F(s)|_\infty ds \\
|u(t)|_\infty &\le C \max_{0\le s\le t}\{s^{1/2}|F(s)|_\infty\} \int_0^t (t-s)^{-1/2} s^{-1/2} ds,
\end{aligned}$$

since $\int_0^t (t-s)^{-1/2} s^{-1/2} ds = C$ for some $C > 0$ independent of t. In particular, for $t = 1$, we have $C = \int_0^1 (1-s)^{-1/2} s^{-1/2} ds$. Therefore

$$|u(t)|_\infty \le C \max_{0\le s\le t}\{s^{1/2}|F(s)|_\infty\}.$$

□

3.2 Estimates for Parabolic System

In this section, we consider the system

$$u_t = \triangle u + D_i g(u), \qquad D_i = \partial/\partial x_i \tag{3.16}$$

$$u = f \quad \text{at} \quad t = 0 \tag{3.17}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be quadratic in u and $f \in C^\infty_{per}(\mathbb{R}^n)$. It is well known that the solution is C^∞_{per} in a maximal interval $0 \le t < T(f)$ where $0 < T(f) \le \infty$. Let us set

$$F(x,t) = g(u(x,t)) \quad \text{for} \quad x \in \mathbb{R}^n, \quad 0 \le t < T(f)$$

and consider u as the solution of the inhomogeneous heat equation $u_t = \triangle u + D_i F$. Since $g(u)$ is quadratic in u, therefore there exists a constant $C_g > 0$ with

$$|g(u)| \le C_g |u|^2, \quad |g_u(u)| \le C_g |u| \quad \text{for all} \quad u \in \mathbb{R}^n. \tag{3.18}$$

All second u- derivatives of g are constant. Next, we introduce the following important theorem.

Theorem 3.5. Under the assumptions on f and g as mentioned above, the solution of (3.16) and (3.17) satisfy the following:
(a) There is a constant $c_0 > 0$ with

$$T(f) > \frac{c_0}{|f|^2_\infty} \tag{3.19}$$

and

$$|u(t)|_\infty \le 2|f|_\infty \quad \text{for} \quad 0 \le t \le \frac{c_0}{|f|^2_\infty}. \tag{3.20}$$

(b) For every $j = 1, 2, \cdots$, there is a constant $K_j > 0$ with

$$t^{j/2} |\mathcal{D}^j u(.,t)|_\infty \le K_j |f|_\infty \quad \text{for} \quad 0 < t \le \frac{c_0}{|f|^2_\infty}. \tag{3.21}$$

The constants c_0 and K_j are independent of t and f.

Proof. To prove (a), let C_g denote the constant in (3.18) and C denote the constant in (3.15). Set $c_0 = \frac{1}{16 C_g^2 C^2}$. Using the *Duhamel's principle*, solution of (3.16) and (3.17) can be written as

$$u(t) = e^{\triangle t} f + \int_0^t e^{\triangle (t-s)} D_i F(s) ds.$$

Using (3.10) and (3.15), we get

$$|u(t)| \le |f|_\infty + C \max_{0 \le s \le t} \{ s^{1/2} |F(s)|_\infty \}.$$

Suppose (3.20) does not hold then we can find some t_0 with $0 < t_0 < c_0/|f|_\infty^2$ such that $|u(t_0)|_\infty = 2|f|_\infty$. Using (3.18), we have $|F(s)|_\infty \le C_g|u(s)|_\infty^2 \le C_g|u(t_0)|_\infty^2$. So all these give us

$$2|f|_\infty = |u(t_0)|_\infty \le |f|_\infty + CC_g t_0^{1/2} \max_{0\le s\le t_0} |u(s)|_\infty^2$$
$$2|f|_\infty \le |f|_\infty + CC_g t_0^{1/2} 4|f|_\infty^2.$$

This gives

$$1 \le 4CC_g t_0^{1/2}|f|_\infty,$$

thus $t_0 \ge 1/(16C^2C_g^2|f|_\infty^2) = c_0/|f|_\infty^2$. This contradiction proves (3.20). Moreover, if $T(f)$ is finite then we will have $\limsup_{t\to T(f)} |u(t)|_\infty = \infty$. This validates the estimate $T(f) > c_0/|f|_\infty^2$.

Next, we prove estimate (3.21) by induction on j. The base case $j = 0$ is proved by estimate (3.20). Suppose $j \ge 1$ and assume for $0 \le k \le j-1$

$$t^{k/2}|\mathcal{D}^k u(t)|_\infty \le K_k|f|_\infty \quad \text{for} \quad 0 \le t \le c_0/|f|_\infty^2. \tag{3.22}$$

Here c_0 is the same constant as defined in part (a). Now, we apply D^j to the equation $u_t = \triangle u + D_i g(u)$ to obtain

$$v_t = \triangle v + D^{j+1}g(u), \qquad v := D^j u.$$

Use the *Duhamel's principle* to obtain

$$v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^{j+1} g(u(s))ds.$$

Using (3.11) we get

$$t^{j/2}|v(t)|_\infty \le C|f|_\infty + t^{j/2}\left|\int_0^t e^{\triangle(t-s)} D^{j+1} g(u(s))ds\right|_\infty. \tag{3.23}$$

We split the integral into

$$\int_0^{t/2} + \int_{t/2}^t =: I_1 + I_2.$$

In what follows, we allow the constant C to change from line to line. Again, using the fact that D^{j+1} commutes with the heat semigroup we obtain

$$\begin{aligned}
|I_1(t)| &= \left|\int_0^{t/2} D^{j+1}e^{\triangle(t-s)}g(u(s))ds\right|_\infty \\
&\le \int_0^{t/2} |D^{j+1}e^{\triangle(t-s)}g(u(s))ds|_\infty ds \\
&\le C\int_0^{t/2}(t-s)^{-(j+1)/2}|g(u(s))|_\infty ds \\
&= C|f|_\infty^2 t^{(1-j)/2}.
\end{aligned}$$

The integrand in I_2 has singularity at $s = t$. Therefore, we can move only one derivative from $D^{j+1}g(u)$ to the heat semi-group.(If we move two or more derivatives then the singularity becomes non-integrable) Thus we have

$$\begin{aligned}
|I_2(t)|_\infty &= \left|\int_{t/2}^{t} De^{\triangle(t-s)}D^j g(u(s))ds\right|_\infty \\
&\le C\int_{t/2}^{t}(t-s)^{-1/2}|D^j g(u(s))|_\infty ds. \qquad (3.24)
\end{aligned}$$

Recall $g(u)$ is quadratic in u. Therefore

$$|D^j g(u)|_\infty \le C|u|_\infty|\mathcal{D}^j u|_\infty + C\sum_{k=1}^{j-1}|\mathcal{D}^k u|_\infty|\mathcal{D}^{j-k}u|_\infty.$$

We try to bound the sum in the above expression. By induction hypothesis (3.22) we get

$$\begin{aligned}
C\sum_{k=1}^{j-1}|\mathcal{D}^k u(s)|_\infty|\mathcal{D}^{j-k}u(s)|_\infty &\le C\sum_{k=1}^{j-1}s^{-k/2}|f|_\infty s^{(k-j)/2}|f|_\infty \\
&\le Cs^{-j/2}|f|_\infty^2
\end{aligned}$$

where C is independent of t and f but depends on j. Next, we obtain

$$|D^j g(u(s))|_\infty \le C|u(s)|_\infty|\mathcal{D}^j u(s)|_\infty + Cs^{-j/2}|f|_\infty^2.$$

Therefore

$$\begin{aligned}
|I_2(t)|_\infty &\le C\int_{t/2}^{t}(t-s)^{-1/2}(|u(s)|_\infty|\mathcal{D}^j u(s)|_\infty + s^{-1/2}|f|_\infty^2)ds \\
&\le C\int_{t/2}^{t}(t-s)^{-1/2}|u(s)|_\infty|\mathcal{D}^j u(s)|_\infty ds + C\int_{t/2}^{t}(t-s)^{-1/2}s^{-j/2}|f|_\infty^2 ds.
\end{aligned}$$

Second integral in the above expression can be bounded by $C|f|^2_\infty t^{(1-j)/2}$ because $\int_{t/2}^t (t-s)^{-1/2} s^{-j/2} ds = Ct^{(1-j)/2}$ and the first integral is bounded by

$$\int_{t/2}^t (t-s)^{-1/2}|u(s)|_\infty |\mathcal{D}^j u(s)|_\infty ds \le C|f|_\infty \int_{t/2}^t (t-s)^{-1/2} s^{-j/2} s^{j/2} |\mathcal{D}^j u(s)|_\infty ds.$$
$$\le C|f|_\infty t^{(1-j)/2} \max_{0\le s\le t}\{s^{j/2}|\mathcal{D}^j u(s)|_\infty\}.$$

We use these bounds to bound the integral in (3.23). We have $v = D^j u$. Then maximizing the resulting estimate for $t^{j/2}|D^j u(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) := t^{j/2}|\mathcal{D}^j u(t)|_\infty$$

we obtain the following estimate

$$\phi(t) \le C|f|_\infty + Ct^{1/2}|f|^2_\infty + C|f|_\infty t^{1/2} \max_{0\le s\le t}\phi(s) \quad \text{for} \quad 0 \le t \le c_0/|f|^2_\infty.$$

Since $t^{1/2}|f|_\infty \le \sqrt{c_0}$ then $Ct^{1/2}|f|^2_\infty \le C\sqrt{c_0}|f|_\infty$. Therefore

$$\phi(t) \le C_j|f|_\infty + C_j|f|_\infty t^{1/2} \max_{0\le s\le t}\phi(s) \quad \text{for} \quad 0 \le t \le c_0/|f|^2_\infty. \tag{3.25}$$

Let us fix C_j so that the above estimate holds and set

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}.$$

First, let us prove the following

$$\phi(t) < 2C_j|f|_\infty \quad \text{for} \quad 0 < t < c_j/|f|^2_\infty.$$

Suppose there is a smallest time t_0 such that $0 < t_0 < c_j/|f|^2_\infty$ with $\phi(t_0) = 2C_j|f|_\infty$.Then using (3.25), we obtain

$$2C_j|f|_\infty = \phi(t_0) \le C_j|f|_\infty + 2C_j^2|f|^2_\infty t_0^{1/2},$$

thus

$$1 \le 2C_j|f|_\infty t_0^{1/2} \quad \text{gives} \quad t_0 \ge c_j/|f|^2_\infty.$$

which contradicts the assertion. Therefore, we proved the estimate

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le 2C_j|f|_\infty \quad \text{for} \quad 0 \le t \le c_j/|f|^2_\infty. \tag{3.26}$$

If

$$T_j := \frac{c_j}{|f|_\infty^2} < t \le \frac{c_0}{|f|_\infty^2} =: T_0 \tag{3.27}$$

then we start the corresponding estimate at $t - T_j$. Using part (a) we have $|u(t - T_j)|_\infty \le 2|f|_\infty$ and obtain

$$T_j^{j/2} |\mathcal{D}^j u(t)|_\infty \le 4C_j |f|_\infty. \tag{3.28}$$

Finally, for any t satisfying (3.27),

$$t^{j/2} \le T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}$$

and (3.28) yield

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \le 4C_j \left(\frac{c_0}{c_j}\right)^{j/2} |f|_\infty.$$

This completes the proof of theorem 3.5. □

3.3 The pressure term in the Navier-Stokes Equations

This section of my work will be focused to an analysis of the pressure term in the Navier-Stokes equations. The pressure term can be determined from the Poisson equation which can be obtained by taking the divergence on the both sides of the momentum equations of the Navier-Stokes equations. The main focus of this section would be deriving a suitable smooth periodic pressure term that solves the Navier-Stokes equation along with the velocity field u. Before we get started determining such pressure and study its underlying structure, we start by establishing some known results of solutions of the Poisson equation in $\mathbb{R}^3$ to make readers familiar about them. In addition, we would like to focus that Proposition 3.6 and 3.7 of this sections are classical results, though I have provided very nice proofs with carefully added details. Let us consider the following Poisson equation

$$-\triangle p = g, \quad g \in C_0^2(\mathbb{R}^3). \tag{3.29}$$

A solution of (3.29) is given by

$$p(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{g(y)}{|x-y|} dy. \tag{3.30}$$

$$= \frac{1}{4\pi} G * g \tag{3.31}$$

where $G(x) = \frac{1}{|x|}$. Because the singularity at $y = x$ is integrable and $g \in C_0^2(\mathbb{R}^3)$, the integral given by (3.30) exists in the classical sense.

On the other hand, since $g \in C_0^\infty(\mathbb{R}^3)$, we can also implement the Fourier transform to obtain the solution of the Poisson equation as follows:

Let us assume apriori that $\widehat{p}$ is well-defined, and since $g \in C_0^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$ we have

$$
\begin{aligned}
|\xi|^2 \widehat{p}(\xi) &= \widehat{g}(\xi) \quad \xi \in \mathbb{R}^3, \\
\widehat{p}(\xi) &= \frac{1}{|\xi|^2}\widehat{g}(\xi) \quad \text{for} \quad \xi \neq 0.
\end{aligned}
$$

We know that the Fourier transform is an isomorphism between the spaces $L^2(\mathbb{R}^3)$, therefore $\widehat{p} \in L^2(\mathbb{R}^3)$ and we can write

$$
\begin{aligned}
p(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \widehat{p}(\xi) e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{\widehat{g}(\xi)}{|\xi|^2} e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{\widehat{g}(\xi)}{|\xi|^2} e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{1}{|\xi|^2} \left(\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} g(y) e^{-iy\cdot\xi} dy \right) e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i\xi\cdot(x-y)}}{|\xi|^2} g(y) dy d\xi \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left(PV \int_{\mathbb{R}^3} \frac{e^{i\xi\cdot(x-y)}}{|\xi|^2} d\xi \right) g(y) dy \\
&= \frac{1}{(2\pi)^3} (G * g)(x)
\end{aligned} \tag{3.32}
$$

where $G(x) = PV \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{|\xi|^2} d\xi$ is called the Poisson kernel in $\mathbb{R}^3$. Next, we prove the following:

Proposition 3.6. If (3.31) and (3.32) solve the same Poisson equation (3.29) then

$$
PV \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\xi\cdot x}}{|\xi|^2} d\xi = \frac{1}{4\pi|x|}. \tag{3.33}
$$

Proof. For $x = 0$, both sides of (3.33) become infinity. Therefore, let $x \neq 0$. In addition, the singularity at $\xi = 0$ in the integral of (3.33) is integrable whereas the slow decay of the integrand at space infinity not absolutely integrable. Therefore, the integral on the left of (3.33) needs to be treated as the limit of

$$
\lim_{R\to\infty} \frac{1}{(2\pi)^3} \int_{|\xi|<R} \frac{e^{i\xi\cdot x}}{|\xi|^2} d\xi.
$$

Therefore let us write

$$PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\xi\cdot x}}{|\xi|^2}d\xi = \lim_{R\to\infty}\frac{1}{(2\pi)^3}\int_{|\xi|<R}\frac{e^{i\xi\cdot x}}{|\xi|^2}d\xi \quad \text{for} \quad R>0.$$

First, let us prove this proposition for $x=(0,0,x_3), x_3>0$, therefore, we get

$$PV\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi = \lim_{R\to\infty}\int_{|\xi|<R}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi$$

Changing into spherical polar coordinates, we obtain

$$\begin{aligned} PV\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi &= \lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{e^{irx_3\cos\theta}}{r^2}r^2\sin\theta dr d\theta d\phi \\ &= (2\pi)\lim_{R\to\infty}\int_0^{R}\int_0^{\pi} e^{irx_3\cos\theta}\sin\theta d\theta dr \end{aligned}$$

Simple substitution yields

$$\begin{aligned} PV\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi &= (2\pi)\lim_{R\to\infty}\int_0^{R}\int_{-1}^{1} e^{irx_3 u}du dr \\ &= 2\pi\lim_{R\to\infty}\int_0^{R}\left.\frac{e^{irx_3 u}}{irx_3}\right]_{-1}^{1}dr \\ &= 4\pi\lim_{R\to\infty}\int_0^{R}\frac{\sin rx_3}{rx_3}dr \\ &= \frac{4\pi}{x_3}\lim_{R\to\infty}\int_0^{R}\frac{\sin u}{u}du. \end{aligned}$$

Since $\int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}$, we get

$$PV\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi = \frac{2\pi^2}{x_3}.$$

Therefore,

$$\begin{aligned} PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi &= \frac{1}{(2\pi)^3}\frac{2\pi^2}{x_3} \\ &= \frac{1}{4\pi x_3} \quad \text{for} \quad x_3>0. \end{aligned}$$

Proceeding exactly same way as above for $x_3<0$, we obtain

$$PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi = -\frac{1}{4\pi x_3}.$$

Finally for any $x_3 \neq 0$, we get

$$PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\xi_3 x_3}}{|\xi|^2}d\xi = \frac{1}{4\pi|x_3|}.$$

Thus, we have proved the proposition for $x = (0, 0, x_3)$. Moreover, we proved that the integral on (3.33) exists in the principal value sense for $x \neq 0$. Next, we rotate and scale the point x to prove the proposition in general case.

Let us rotate the point $x = (0, 0, x_3)$ by some orthonormal matrix A so that $y = Ax$ with $|x| = |y|$. Then

$$
\begin{aligned}
PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iy\cdot\xi}}{|\xi|^2}d\xi &= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iAx\cdot\xi}}{|\xi|^2}d\xi \\
&= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix\cdot A^T\xi}}{|\xi|^2}d\xi.
\end{aligned}
$$

Let $A^T\xi = \eta$. We also know $|\xi| = |\eta|$ and $|detA| = 1$. Therefore, with this change of variable, we obtain

$$
\begin{aligned}
PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iy\cdot\xi}}{|\xi|^2}d\xi &= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix\cdot\eta}}{|\eta|^2}|detA|d\eta \\
&= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix\cdot\eta}}{|\eta|^2}d\eta \\
&= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix_3\eta_3}}{|\eta|^2}d\eta \\
&= \frac{1}{4\pi|x_3|} \\
&= \frac{1}{4\pi|y|}.
\end{aligned}
$$

Finally, let us scale the point y obtained by rotating the point x as $z = \alpha y$ where $\alpha \neq 0$, $\alpha \in \mathbb{R}$. Clearly $|z| = |\alpha||y|$. Then,

$$
\begin{aligned}
PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iz\cdot\xi}}{|\xi|^2}d\xi &= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\alpha y.\xi}}{|\xi|^2}d\xi \\
&= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iy\cdot\alpha\xi}}{|\xi|^2}d\xi.
\end{aligned}
$$

Simple change of variable $\alpha\xi = \eta$ gives

$$
\begin{aligned}
PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iz\cdot\xi}}{|\xi|^2}d\xi &= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iy\cdot\eta}}{|\eta|^2}|\alpha|^2\frac{d\eta}{|\alpha|^3} \\
&= PV\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{iy\cdot\eta}}{|\alpha||\eta|^2}d\eta \\
&= \frac{1}{4\pi|y||\alpha|} \\
&= \frac{1}{4\pi|z|}.
\end{aligned}
$$

This completes the proof of proposition 3.6. $\square$

The Poisson equation with periodic boundary conditions is of considerable interest in this part of our work. First, we find a solution of the partial differential equation

$$-\triangle p(x,t) = g(x,t) \tag{3.34}$$

with periodic boundary condition: or in an alternative form, to compute the integral

$$p(x,t) = \int_{\mathbb{T}^3} G(x-y)g(y,t)dy \tag{3.35}$$

where $g \in C^{\infty}_{per}(\mathbb{R}^3)$ and $\mathbb{T}^3 = [0,2\pi)^3$. The $G(x)$ in (3.35) is the Green's function of the Poisson equation (3.34) and is given by

$$G(x) = \frac{1}{(2\pi)^3} \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \frac{e^{ik\cdot x}}{|k|^2} \tag{3.36}$$

Apparently, the solution of (3.35) exists and can be given by

$$p(x) = \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \widehat{p}(k)e^{ik\cdot x} \tag{3.37}$$

where $\widehat{p}(k) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{e^{ik\cdot y}}{|k|^2} g(y)dy$. Notice, we have suppressed the t-dependence in our notation.

The series in (3.37) does not converge absolutely; however, because of some oscillations in the summand, the series still manages to converge for $x \neq 0$. At the same time, we would like to modify the series given in (3.36) so that the convergence of such series is easy to observe. For that purpose, we proceed via the Fourier expansion on the Poisson equation (3.34)

$$\begin{aligned}
\widehat{p}(k) &= \frac{1}{|k|^2}\widehat{g}(k) \\
&= \widehat{g}(k)\int_0^{\infty} e^{-|k|^2\tau}d\tau \\
&= \widehat{g}(k)\int_0^{\alpha} e^{-|k|^2\tau}d\tau + \widehat{g}(k)\int_{\alpha}^{\infty} e^{-|k|^2\tau}d\tau, \qquad \text{for} \qquad \alpha > 0 \\
&= \widehat{g}(k)\int_0^{\alpha} e^{-|k|^2\tau}d\tau + \widehat{g}(k)\frac{e^{-|k|^2\alpha}}{|k|^2} \\
&= \widehat{p_1}(k) + \widehat{p_2}(k).
\end{aligned}$$

Let us rewrite (3.37)

$$
\begin{aligned}
p(x) &= \sum_{\substack{k\in\mathbb{Z}^3\\k\neq 0}} \widehat{p}(k)e^{ik\cdot x}\\
&= \sum_{\substack{k\in\mathbb{Z}^3\\k\neq 0}} \big(\widehat{p_1}(k)+\widehat{p_2}(k)\big)e^{ik\cdot x}\\
&= \sum_{\substack{k\in\mathbb{Z}^3\\k\neq 0}} \widehat{p_1}(k)e^{ik\cdot x} + \sum_{\substack{k\in\mathbb{Z}^3\\k\neq 0}} \widehat{p_2}(k)e^{ik\cdot x}\\
&= p_1(x)+p_2(x).
\end{aligned}
$$

Take

$$
\begin{aligned}
p_1(x) &= \sum_{k\in\mathbb{Z}^3} \widehat{p_1}(k)e^{ik\cdot x}\\
&= \sum_{k\in\mathbb{Z}^3} \widehat{g}(k)\int_0^\alpha e^{-|k|^2\tau}d\tau e^{ik\cdot x}.
\end{aligned}
$$

The Lebesgue dominated convergence theorem allows us to write

$$
\begin{aligned}
p_1(x) &= \int_0^\alpha \left(\sum_{k\in\mathbb{Z}^3} \widehat{g}(k)e^{-|k|^2\tau}e^{ik\cdot x}\right)d\tau\\
&= \int_0^\alpha \sum_{k\in\mathbb{Z}^3} \widehat{(g*\theta_\tau)}(k)e^{ik\cdot x}d\tau.
\end{aligned}
$$

Note that θ_t is the n-dimensional heat kernel. Since $\mathcal{F}^{-1}(e^{-|\xi|^2\tau})(x) = \frac{e^{-|x|^2/4\tau}}{(4\pi\tau)^{3/2}}$ for $\tau > 0$, here $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^3$, and using the Poisson summation formula we obtain

$$
\begin{aligned}
p_1(x) &= \sum_{k\in\mathbb{Z}^3}\int_0^\alpha \left(\int_{\mathbb{T}^3} \frac{e^{-|x-y+2\pi k|^2/4\tau}}{(4\pi\tau)^{3/2}}g(y)dy\right)d\tau\\
&= \int_{\mathbb{T}^3}\sum_{k\in Z^3}\int_0^\alpha \frac{e^{-|x-y+2\pi k|^2/4\tau}}{(4\pi\tau)^{3/2}}d\tau g(y)dy.
\end{aligned}
$$

Also

$$
\begin{aligned}
p_2(x) &= \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \widehat{p_2}(k)e^{ik\cdot x}\\
&= \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \widehat{g}(k)\frac{e^{-|k|^2\tau}}{|k|^2}e^{ik\cdot x}\\
&= \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{-|k|^2\tau}}{|k|^2}\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} g(y)e^{-ik\cdot y}dye^{ik\cdot x}\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{ik\cdot(x-y)}}{|k|^2}e^{-|k|^2\tau}g(y)dy.
\end{aligned}
$$

Finally, we arrive at the following expression

$$
\begin{aligned}
p(x) &= p_1(x) + p_2(x)\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\left(\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{ik\cdot(x-y)}}{|k|^2}e^{-|k|^2\alpha} + \frac{1}{8\pi\sqrt{\pi}}\sum_{k\in\mathbb{Z}^3}\int_0^\alpha e^{-|x-y+2\pi k|^2/4\tau}\tau^{-3/2}d\tau\right)g(y)dy.
\end{aligned}
\tag{3.38}
$$

Hence, we have proved the following proposition.

Proposition 3.7. The expression in (3.38) solves the Poisson equation (3.34) with the Green's function of the form

$$
G(x) = \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{ik\cdot x}}{|k|^2}e^{-|k|^2\alpha} + \frac{1}{8\pi\sqrt{\pi}}\sum_{k\in\mathbb{Z}^3}\int_0^\alpha e^{-|x+2\pi k|^2/4\tau}\tau^{-3/2}d\tau \quad \text{for} \quad \alpha > 0.
$$

Next, we will connect the Poisson equation to the Navier Stokes equations. To do this, we apply the divergence operator $\nabla\cdot$ to both sides of the momentum equation of the Navier-Stokes equations along with the divergence free condition $\nabla\cdot u = 0$ and obtain

$$
-\triangle p = g \tag{3.39}
$$

where

$$
g(x,t) = \sum_{i,j=1}^{3} D_iD_j(u_ju_i)(x,t).
$$

The pressure term in the Navier Stokes equations can be determined by the Poisson equation (3.39). Dropping the t-dependence in our notation we formally have

$$p(x) = \frac{1}{4\pi} \sum_{i,j} \int G(x-y) D_i D_j (u_i u_j)(y) dy \tag{3.40}$$

where $G(x) = \frac{1}{|x|}$ is called the Poisson kernel in $\mathbb{R}^3$.

There are couple of concerns in the integral (3.40). First, the singularity at $y = x$, and second slow decay of the integrand for large y. However, singularity at $y = x$ is harmless because it is integrable. On the other hand the slow decay may cause the non-existence of the integral. The integral (3.40) may not exist in the classical sense but the Calderon- Zygmund theory of singular integrals guarantees that p is in the space of bounded mean oscillation(BMO). Our main goal while dealing with the pressure is to obtain a smooth periodic pressure such that (u, p) solves the Navier-Stokes equations with appropritate initial condition. The pressure given by (3.40) is nowhere close to the type that we are looking for because either non-existence of the integral or the requirements of the periodic pressure. Since we are still interested to follow the techniques of the paper by Kreiss and Lorenz, we motivate ourselves to produce a smooth periodic pressure with the same underlying structure as the pressure in (3.40). To proceed in that direction, we next start with the following definition.

Definition 3.8. Let (u, p) be a solution to the Navier-Stokes equations (2.1) and (2.2) with $f \in C^\infty_{per}(\mathbb{R}^3)$ and suppose that $u \in C^\infty_{per}(\mathbb{R}^3)$ for $0 \leq t < T$ for some $T < \infty$. The ***modified Poisson pressure*** is given by

$$p^*(x,t) = \frac{1}{4\pi} \sum_{i,j} \int \left[G_{ij}(x-y) - G_{ij}(y)\right] (u_i u_j)(y,t) dy \tag{3.41}$$

where for $i, j \in \{1, 2, 3\}$,

$$G_{ij}(x) = D_i D_j (|x|^{-1}) \tag{3.42}$$

where $G(x) = |x|^{-1}$.

In the following, we would be focused to show the existence of the integral (3.41) in the principal value sense and p^* is a smooth periodic pressure so that (u, p^*) solves the Navier-Stokes equations (2.1),(2.2) with $f \in C^\infty_{Per}(\mathbb{R}^3)$. Before we do so we introduce the following elementary results for the kernel given by (3.42) and (3.43) whose proofs are left for reader since they are standard exercise problems of multivariable calculus.

Proposition 3.9. Let $S^2_{x,a}$ denote a sphere of radius a and centered at $x \in \mathbb{R}^3$ and dS denote the element of surface measure of sphere in $\mathbb{R}^3$, then for $i, j \in \{1, 2, 3\}$

$$\int_{S^2_{x,a}} (x_i - y_i)(x_j - y_j) dS_y = \begin{cases} 0 & \text{if} \quad i \neq j \\ \frac{4\pi a^4}{3} & \text{if} \quad i = j. \end{cases}$$

Proposition 3.10. For $i, j \in \{1, 2, 3\}$

$$G_{ij}(x - y) = \frac{3(x_i - y_i)(x_j - y_j)}{|x - y|^5}$$

and

$$G_{jj}(x - y) = \frac{3(x_j - y_j)^2 - |x - y|^2}{|x - y|^5}$$

then

$$\int_{S^2_{x,a}} G_{ij}(x - y) dS_y = 0.$$

Proposition 3.11. Let $\Omega = \{y \in \mathbb{R}^3 : \epsilon < |x - y| < \delta, x \in \mathbb{R}^3\}$ for any $\epsilon, \delta > 0$. Then

$$\int_{\Omega} G_{ij}(x - y) dS_y = 0 \quad \text{for any} \quad i, j \in \{1, 2, 3\}.$$

Lemma 3.12. Let G_{ij} be the kernel given by (3.42) and (3.43). Then

$$|G_{ij}(x - y) - G_{ij}(y)| \leq \frac{C|x|}{|y|^4} \quad \text{for} \quad |y| > 2|x|.$$

Proof. Let us define

$$\phi(t) = G_{ij}(y - tx), \quad 0 \leq t \leq 1.$$

For $|y| > 2|x|$, fundamental theorem of calculus applies and gives

$$G_{ij}(y - x) - G_{ij}(y) = \phi(1) - \phi(0) = \int_0^1 \phi'(t) dt$$

and

$$|G_{ij}(y - x) - G_{ij}(y)| = |\phi(1) - \phi(0)| \leq \max_{0 \leq t \leq 1} |\phi'(t)|.$$

Obtain

$$\phi'(t) = -x \cdot \nabla G_{ij}(y - tx)$$

and by Cauchy-Schwarz inequality

$$|\phi'(t)| \leq |x||\nabla G_{ij}(y - tx)| \leq \frac{C|x|}{|y - tx|^4}.$$

For $0 \leq t \leq 1$ and $|y| > 2|x|$ we have

$$|y - tx| \geq |y| - t|x| \geq |y| - |x| \geq |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Thus,

$$\max_{0 \leq t \leq 1} |\phi'(t)| \leq \frac{C|x|}{|y|^4}.$$

This proves lemma 3.12. □

Lemma 3.13. Suppose that $G_{ij}(y)$ is the kernel defined by (3.42) and (3.43). Then for any $\delta > 0$ and fixed $x \in \mathbb{R}^3$,

$$PV \int_{|x-y|<\delta} G_{ij}(x - y)g(y)dy < \infty \tag{3.43}$$

where $g(y) = (u_i u_j)(y)$, for any $i, j \in \{1, 2, 3\}$. However, a Lipschitz continuous g would suffice for the result of this lemma to be valid.

Proof. For any fixed $x \in \mathbb{R}^3$ and $0 < \epsilon < \delta$, denote

$$I(x) := \int_{\epsilon<|x-y|<\delta} G_{ij}(x - y)g(y)dy.$$

Since $g \in C^\infty_{per}(\mathbb{R}^3)$ and using the proposition (3.11) for any $x \in \mathbb{R}^3$, we can get

$$\int_{\epsilon<|x-y|<\delta} G_{ij}(x - y)g(x)dy = 0.$$

Therefore, we can also write

$$I(x) = \int_{\epsilon<|x-y|<\delta} G_{ij}(x - y)[g(y) - g(x)]dy.$$

Using $|G_{ij}(x - y)| \leq \frac{C}{|x-y|^3}$ for some $C > 0$ yields

$$|I(x)| \leq C \int_{\epsilon<|x-y|<\delta} \frac{1}{|x - y|^3} |g(y) - g(x)|dy.$$

Next, we determine the suitable bound for $|g(y)-g(x)|$. Let us start with the following

$$\begin{aligned} g(y) - g(x) &= (u_i u_j)(y) - (u_i u_j)(x) \\ &= u_i(y)u_j(y) - u_i(y)u_j(x) + u_i(y)u_j(x) - u_i(x)u_j(x) \end{aligned}$$

and

$$
\begin{aligned}
|g(y) - g(x)| &\leq |u_i(y)||u_j(y) - u_j(x)| + |u_j(x)||u_i(y) - u_j(x)| \\
&\leq |u|_\infty |u(y) - u(x)| + |u|_\infty |u(y) - u(x)| \\
&= 2|u|_\infty |u(y) - u(x)|.
\end{aligned}
$$

Let

$$
\phi(t) = u(x + (y - x)t), \qquad 0 \leq t \leq 1.
$$

Then

$$
\begin{aligned}
\phi(1) - \phi(0) &= \int_0^1 \phi'(t) dt \\
&= \int_0^1 \nabla u(x + (y - x)t) \cdot (y - x) dt.
\end{aligned}
$$

Therefore

$$
|u(y) - u(x)| = |\phi(1) - \phi(0)| \leq \max_{0 \leq t \leq 1} |\nabla u(x + (y - x)t)||y - x|
$$

$$
|u(y) - u(x)| \leq |\nabla u|_\infty ||y - x|.
$$

Since $u \in C^\infty_{per}(\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
|g(y) - g(x)| &\leq 2|u|_\infty |\nabla u|_\infty |y - x| \\
&\leq C|y - x|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|I(x)| &\leq C \int_{\epsilon < |x - y| < \delta} \frac{1}{|x - y|^3} |g(y) - g(x)| dy \\
&\leq C \int_{\epsilon < |x - y| < \delta} \frac{1}{|x - y|^2} dy.
\end{aligned}
$$

Changing into polar coordinates gives

$$
|I(x)| \leq C \int_\epsilon^\delta r^2 / r^2 dr = C(\delta - \epsilon).
$$

Finally

$$
\begin{aligned}
PV \int_{|x - y| < \delta} G_{ij}(x - y) g(y) dy &= \lim_{\epsilon \to 0} \int_{\epsilon < |x - y| < \delta} G_{ij}(x - y) g(y) dy \\
&= \lim_{\epsilon \to 0} I(x) \\
&\leq \lim_{\epsilon \to 0} C(\delta - \epsilon) \\
&= C\delta < \infty.
\end{aligned}
$$

Finally, we have proved that the integral given by (3.44) exists. Hence the Lemma 3.14 is proved. □

Next, we are ready to state and prove an important theorem of this section of our work.

Theorem 3.14. Suppose that (u, p) is a solution to the Navier-Stokes equations (2.1) and (2.2) with $f \in C^{\infty}_{per}(\mathbb{R}^3)$ and suppose that $u \in C^{\infty}_{per}(\mathbb{R}^3)$, $t \in [0, T)$ for some $T \leq \infty$. Then for any fixed $x \in \mathbb{R}^3$ the ***modified Poisson pressure*** given by (3.41) exists and does not grow faster than a *logarithmic* function of $|x|$ as $x \to \infty$.

Proof. In what follows, we suppress the t-dependence in our notations and denote $g(y) = (u_i u_j)(y)$. In addition, we allow the constant to change line to line. First, if $x = 0$ then $p^*(x) = 0$, therefore there is nothing to prove. Let $x \neq 0$ be fixed and write the integral given by (3.41) as

$$
\begin{aligned}
p^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y) dy \\
&\quad + PV \sum_{i,j} C \int_{|y|>2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y) dy \\
&= I_1 + I_2
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y) dy \\
&= PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(x-y) g(y) dy - PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(y) g(y) dy \\
&= J_1 + J_2.
\end{aligned}
$$

From the lemma 3.13 for fixed x, we obtain

$$
\begin{aligned}
|J_2(x)| &\leq C|x| \\
&\leq C.
\end{aligned}
$$

Denote the ball of radius r and centered at $x \in \mathbb{R}^3$ by $B_r(x)$. For some $\epsilon > 0$, write

$$
\begin{aligned}
J_1(x) &= PV \sum_{i,j} \int_{|y|<2|x|} G_{ij}(x-y) g(y) dy \\
&= \sum_{i,j} C \int_{|y|<2|x| \setminus B_\epsilon(x)} G_{ij}(x-y) g(y) dy + PV \sum_{i,j} C \int_{B_\epsilon(x)} G_{ij}(x-y) g(y) dy \\
&= J_1^* + J_1^{**}.
\end{aligned}
$$

Again, from lemma 3.13, we immediately get

$$|J_1^{**}(x)| \leq C\epsilon.$$

Let us notice that $\{y \in \mathbb{R}^3 : |y| < 2|x| \backslash B_\epsilon(x)\} \subset \{y \in \mathbb{R}^3 : \epsilon < |x-y| < 3|x|\}$ to get

$$|J_1^*(x)| \leq \sum_{i,j} C \int_{\epsilon<|x-y|<3|x|} |G_{ij}(x-y)||g(y)|dy.$$

Using lemma 3.13 again we get

$$\begin{aligned}|J_1^*(x)| &\leq \sum_{i,j} C \int_{\epsilon<|x-y|<3|x|} |G_{ij}(x-y)||g(y)|dy \\ &\leq C(3|x| - \epsilon) \\ &\leq C.\end{aligned}$$

Therefore, we have the following estimate for J_1:

$$\begin{aligned}|J_1(x)| &\leq |J_1^*(x)| + |J_2^{**}(x)| \\ &\leq C + C\epsilon \\ &\leq C.\end{aligned}$$

We obtain

$$\begin{aligned}|I_1(x)| &\leq |J_1(x)| + |J_2(x)| \\ &\leq C.\end{aligned}$$

Next, we will find bound for I_2 which is given by

$$I_2(x) = \lim_{R\to\infty} \sum_{i,j} C \int_{2|x|<|y|<R} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)dy.$$

Use lemma 3.12 to obtain

$$\begin{aligned}|I_2(x)| &\leq \lim_{R\to\infty} \sum_{i,j} C \int_{2|x|<|y|<R} \frac{|x|}{|y|^4} dy \\ &= \lim_{R\to\infty} \sum_{i,j} C|x| \int_{2|x|<|y|<R} \frac{1}{|y|^4} dy.\end{aligned}$$

Evaluating the integral using the polar coordinates gives us

$$\begin{aligned}|I_2(x)| &\leq \lim_{R\to\infty} \sum_{i,j} C|x| \left[\frac{1}{2|x|} - \frac{1}{R}\right] \\ &\leq \lim_{R\to\infty} \sum_{i,j} C \left[\frac{1}{2} - \frac{|x|}{R}\right] \\ &\leq C, \quad \text{for any } x \in \mathbb{R}^3.\end{aligned}$$

Therefore

$$\begin{aligned}|I(x)| &\leq |I_1(x)| + |I_2(x)| \\ &\leq C.\end{aligned}$$

Hence, we proved that the modified Poisson pressure p^* given by (3.41) is bounded for fixed x. This proves that the integral equation given by (3.41) exists in the principal value sense. Now, it remains to prove the p^* does not grow faster than the *logarithmic* function of $|x|$ for large x. For that, choose $|x| > 1$. Let us write p^* as

$$\begin{aligned}p^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)dy \\ &+ PV \sum_{i,j} C \int_{|y|>2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)dy \\ &= p_1^* + p_2^*.\end{aligned}$$

Notice, the estimate for p_2^* is same as that for I_2 in the previous case. Therefore

$$|p_2^*(x)| \leq C.$$

Here

$$\begin{aligned}p_1^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)dy \\ &= PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(x-y)g(y)dy - PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(y)g(y)dy\end{aligned}$$

We can write the above integral as

$$\begin{aligned}p_1^*(x) &= \sum_{i,j} C \int_{|y|<2|x| \setminus B_1(x)} G_{ij}(x-y)g(y)dy + PV \sum_{i,j} C \int_{B_1(x)} G_{ij}(x-y)g(y)dy \\ &- PV \sum_{i,j} C \int_{|y|<1} G_{ij}(y)g(y)dy - \sum_{i,j} C \int_{1<|y|<2|x|} G_{ij}(y)g(y)dy \\ &= T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Applying lemma 3.13, we immediately get

$$|T_2(x)| \leq C \quad \text{and} \quad |T_3(x)| \leq C.$$

To estimate T_1, we observe that $\{y \in \mathbb{R}^3 : |y| < 2|x| \setminus B_1(x)\} \subset \{y \in \mathbb{R}^3 : 1 < |x-y| < 3|x|\}$ and $|G_{ij}(x-y)| \leq \frac{C}{|x-y|^3}$, for some $C > 0$. Therefore

$$\begin{aligned}|T_1(x)| &\leq \sum_{i,j} C \int_{1<|x-y|<3|x|} |G_{ij}(x-y)|dy \\ &\leq \sum_{i,j} C \int_{1<|x-y|<3|x|} \frac{1}{|x-y|^3} dy.\end{aligned}$$

Changing into polar form gives

$$\begin{aligned}|T_1(x)| &\leq \sum_{i,j} C \int_1^{3|x|} \frac{1}{r} dr \\ &\leq C \ln 3|x|.\end{aligned}$$

Similarly,

$$\begin{aligned}|T_4(x)| &\leq C \ln 2|x| \\ &\leq C \ln 3|x|.\end{aligned}$$

Therefore, we obtain the following bound for p_1^*

$$\begin{aligned}|p_1^*(x)| &\leq |T_1(x)| + |T_2(x)| + |T_3(x)| + |T_4(x)| \\ &\leq C(1 + \ln 3|x|).\end{aligned}$$

Combine the results from above, we get a bound for

$$\begin{aligned}|p^*(x)| &\leq |p_1^*(x)| + |p_2^*(x)| \\ &\leq C(1 + \ln 3|x|).\end{aligned}$$

Finally, we have proved that the modified Poisson pressure exists and does not grow faster than *logarithmic* function of $|x|$ for large x. $\square$

Theorem 3.15. Let

$$p^*(x,t) = PV \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^3} [G_{ij}(x-y) - G_{ij}(y)](u_i u_j)(y,t)dy \tag{3.44}$$

Then p^* is a solution to the Poisson equation

$$-\triangle p(x,t) = \sum_{i,j} (D_i u_j)(D_j u_i)(x,t). \tag{3.45}$$

Proof. Once again, we will suppress the t-dependence in our notations. Suppose, there is an $R > 0$ such that $2|x| < R$. Let us introduce a C^∞ cut-off function $\phi(r)$

with $\phi(r) = 1$ if $0 \leq r \leq R$ and 0 for $r \geq R + 1$. We write p^* using this cut-off function as follows:

$$\begin{aligned}
p^*(x) &= PV \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^3} [G_{ij}(x-y) - G_{ij}(y)](\phi(|y|) - (1 - \phi(|y|)))(u_i u_j)(y) dy \\
&= PV \sum_{i,j} \frac{1}{4\pi} \int_{|y|<R+1} [G_{ij}(x-y) - G_{ij}(y)]\phi(|y|)(u_i u_j)(y) dy \\
&+ \sum_{i,j} \frac{1}{4\pi} \int_{|y|>R} [G_{ij}(x-y) - G_{ij}(y)](1 - \phi(|y|))(u_i u_j)(y) dy \\
&= p^*_{loc}(x) + p^*_{glb}(x).
\end{aligned}$$

Using integration by parts and the fact that the ϕ vanishes on the boundary, we obtain

$$\begin{aligned}
p^*_{loc}(x) &= \sum_{i,j} \frac{1}{4\pi} \int_{|y|<R+1} [G(x-y) - G(y)] D_i D_j (\phi(|y|) u_i u_j)(y) dy \\
&= \sum_{i,j} \frac{1}{4\pi} \int_{|y|<R} [G(x-y) - G(y)] D_i D_j (u_i u_j)(y) dy \\
&+ \sum_{i,j} \frac{1}{4\pi} \int_{R<|y|<R+1} [G(x-y) - G(y)] D_i D_j (\phi(|y|) u_i u_j)(y) dy \\
&= I_1(x) + I_2(x).
\end{aligned}$$

Then

$$\triangle_x p^*_{loc} = \triangle_x I_1(x) + \triangle_x I_2(x).$$

It is known that for values of y different from x, we have

$$\triangle_x G(x-y) = 0 \quad \text{and} \quad \triangle_x G_{ij}(x-y) = 0$$

This clearly implies $\triangle_x p^*_{glb}(x) = 0$ and $\triangle_x I_2(x) = 0$. Therefore, we arrive at

$$\begin{aligned}
\triangle_x p^*(x) &= \triangle_x I_1(x) \\
&= PV \sum_{i,j} \frac{1}{4\pi} \int_{|y|<R} \triangle_x G(x-y) D_i D_j (u_i u_j)(y) dy \\
&= PV \frac{1}{4\pi} \int_{|y|<R} \triangle_x G(x-y) \sum_{i,j} D_i D_j (u_i u_j)(y) dy \\
&= -\sum_{i,j} D_i D_j (u_i u_j)(x) \\
&= -\sum_{i,j} (D_i u_j)(D_j u_i)(x).
\end{aligned}$$

This completes the proof of theorem 3.15. $\square$

As per our quest of a smooth periodic pressure that has same underlying structure as the BMO valued pressure in (3.40), we have just been able to construct a modified pressure (3.45) which solves the Poisson equation (3.46) and exists in the principal value sense. However, we are still left to show that such modified pressure is smooth periodic for a smooth periodic velocity field.

Recall, we used the Fourier expansion on the Poisson equation (3.34) and formally obtained a smooth periodic pressure (3.38) . However, the problem persist with the fact that such pressure in (3.38) does not obviously have same underlying structure as the pressure used in the Kreiss and Lorenz paper. Therefore, we prove the following theorem which is an essential part of this paper.

Theorem 3.16. If p given by (3.38) and p^* given by (3.45) solve the same Poisson pressure equation (3.46) then $p(x,t) = p^*(x,t) + C(t)$ for some constant C that depends only on t. This concludes, p^* is also a smooth periodic solution of the Poisson pressure equation (3.46).

Proof. Let $p_1 = p - p^*$. Then p_1 is harmonic in $\mathbb{R}^3$. Suppose for any $x_1, x_2 \in \mathbb{R}^3$ with $x_1 \neq x_2$, we apply volume version of the mean value property of harmonic function to obtain

$$p_1(x_1) = \frac{1}{vol(B_r(x_1))} \int_{B_r(x_1)} p_1(y)dy$$

and

$$p_1(x_2) = \frac{1}{vol(B_r(x_2))} \int_{B_r(x_2)} p_1(y)dy$$

where dy is element of volume measure of sphere and $r > 0$ be arbitrarily large.

Let $A = B_r(x_1)\backslash B_r(x_2) \cup B_r(x_2)\backslash B_r(x_1)$.Then

$$\begin{aligned}|p_1(x_1) - p_1(x_2)| &\leq \frac{1}{vol(B_r(0))} \int_A |p_1(y)|dy \\ &\leq \frac{1}{vol(B_r(0))} \int_A |p(y) - p^*(y)|dy \\ &\leq \frac{1}{vol(B_r(0))} \int_A \big(|p(y)| + |p^*(y)|\big)dy.\end{aligned}$$

Since $p(y,t)$ in (3.38) is bounded in $\mathbb{R}^3 \times [0,T]$ for some $T > 0$ and from theorem 3.14, we have $|p^*(y,t)| \leq C(1 + \ln 3|y|)$ where C depends on t which we have suppressed in our notations. Therefore, we get

$$
\begin{aligned}
|p_1(x_1) - p_1(x_2)| &\leq \frac{1}{vol(B_r(0))} \int_A C(1 + \ln 3|y|)dy \\
&\leq \frac{C}{vol(B_r(0))}(1 + \ln 3r)vol(A).
\end{aligned}
$$

Notice if $B = B_{r+|x_1-x_2|}(x_1)\backslash B_{r-|x_1-x_2|}(x_2)$ then $vol(A) \leq vol(B)$. Also it is not difficult to see the $vol(B) = O(r^2)$. To this end, we have

$$
|p_1(x_1) - p_1(x_2)| \leq \frac{C(1 + \ln 3r)O(r^2)}{r^3}
$$

Therefore as $r \to \infty$, we obtain $p_1(x_1) = p_1(x_2)$ for any x_1, x_2. Hence we proved that p_1 is a constant function of x which proves $p(x,t) = p^*(x,t) + C(t)$. Finally, we proved p^* is also a smooth periodic solution of the Poisson pressure equation (3.46). $\square$

3.4 Bounds on the Pressure Derivatives

To prove the result analogous to the main theorem of the KL paper, we require the estimates on the pressure derivatives. Since we are in the periodic case, it is natural to expect to use the periodic pressure given by (3.38). However, this periodic pressure is not in the convenient form to obtain the estimates as we wish to follow the techniques of KL's paper. In theorem 3.16, we proved that the modified pressure coincides with periodic pressure (3.38) in torus $\mathbb{T}^3$. This fact influences us to use modified pressure to obtain necessary estimates using the similar techniques as in the KL paper.

Lemma 3.17. Let u be a solution to the Navier-Stokes equations (2.1) and (2.2) with $f \in C^\infty_{per}(\mathbb{R}^3)$ and $\nabla \cdot f = 0$ where pressure $p(x,t) = p(x)$ is given by, for some constant $C_0 > 0$,

$$
p(x) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} [G_{ij}(x-y) - G_{ij}(y)](u_i u_j)(y)dy.
$$

Then there is a constant $C > 0$, independent on t, δ and f so that the following estimates hold:

$$
|p_{loc}|_\infty \leq C(|u|^2_\infty + \delta |u|_\infty |\mathcal{D}u|_\infty) \tag{3.46}
$$

$$
|\mathcal{D}p_{loc}|_\infty \leq C(\delta^{-1}|u|^2_\infty + \delta|\mathcal{D}u|^2_\infty) \tag{3.47}
$$

$$
|\mathcal{D}p_{glb}|_\infty \leq C\delta^{-1}|u|^2_\infty. \tag{3.48}
$$

where p_{loc} and p_{glb} are defined in page 9 equation (2.5).

Proof. Follow Lemma 4.1 of [12]. □

In the KL paper, estimates on the higher order pressure derivatives were not achieved despite their need while obtaining the estimates on the higher order derivatives on the velocity field. In addition, the KL paper leaves the proof of the main theorem for the readers by giving an illustrative proof for the parabolic system (3.16) and (3.17). To fullfil the gap left by the KL paper, we start with the following theorem which provides the estimates on the higher order derivatives of the pressure.

Theorem 3.18. Consider the Navier-Stokes equations

$$u_t = \triangle u + Q, \quad \nabla \cdot u = 0 \quad u = f \quad \text{at} \quad t = 0$$

where

$$Q = -\nabla p - u \cdot \nabla u.$$

Let $j \geq 1$ and assume that for $0 \leq k \leq j-1$, there are constants c_0 and K_k independent of t, f such that

$$t^{k/2}|\mathcal{D}^k u(t)|_\infty \leq K_k |f|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{|f|_\infty^2}. \tag{3.49}$$

Then there exists a constant C independent of t and f and for $1 \leq l \leq 3$ such that

$$|\mathcal{D}^j p_{loc}(x)|_\infty \leq C(|f|_\infty |\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2) \tag{3.50}$$

and

$$|\mathcal{D}^j p_{glb}(x)|_\infty \leq C|f|_\infty^2 t^{-j/2}. \tag{3.51}$$

Proof. Let us begin by applying D^{j-1} to $u_t = \triangle u + Q$ to get

$$v_t = \triangle v + D^{j-1}Q, \quad v := D^{j-1}u$$

By taking the divergence on both side of the above equations we have the Poisson equation

$$\triangle q = -\sum_{i,k} D_i D_k (D^{j-1}(u_i u_k)) \quad q := D^{j-1}p$$

A solution of this Poisson equation is given by $q = q_{loc} + q_{glb}$ where

$$q_{loc}(x) = \sum_{i,k} C_0 \int |x-y|^{-1} D_i D_k (\phi(\delta^{-1}|x-y|) D^{j-1}(u_i u_k)(y)) dy \tag{3.52}$$

and

$$q_{glb}(x) = \sum_{i,j} C_0 \int |x-y|^{-1} D_i D_k((1-\phi(\delta^{-1}|x-y|))D^{j-1}(u_i u_k)(y))dy \tag{3.53}$$

where $\phi(r) = 1$ for $0 \le r \le 1$ and 0 if $r \ge 2$ is C^∞ cut-off function whose argument is always $\delta^{-1}|x-y|$ in this proof which we will suppress in our notations.

To estimate $|\mathcal{D}q_{loc}|_\infty$, we first apply $D_{l,x} = \partial/\partial x_l$ under the intergral sign of (3.53) and obtain

$$\begin{aligned}
D_{l,x}q_{loc}(x) &= \sum_{i,k} C_0 \int_{B(x,2\delta)} D_{l,x}(|x-y|^{-1}) D_i D_k(\phi D^{j-1}(u_i u_k)(y))dy \\
&+ \sum_{i,k} C_0 \int_{B(x,2\delta)} |x-y|^{-1}(D_i D_k(D_{l,x}(\phi))D^{j-1}(u_i u_k))(y)dy \\
&= I_1 + I_2
\end{aligned}$$

To estimate I_1, we first estimate $\sum_{i,k} D_i D_k(\phi D^{j-1}(u_i u_k))$ in maximum norm. In the following it is also important to use $\sum_{i,k} D_i D_k(u_i u_k) = \sum_{i,k}(D_i u_k)(D_k u_i)$.

$$\begin{aligned}
\sum_{i,k} D_i D_k(\phi D^{j-1}(u_i u_k)) &= \sum_{i,k}\{D_i D_k \phi D^{j-1}(u_i u_k) + D_k \phi D_i D^{j-1}(u_i u_k) \\
&+ D_i \phi D_k D^{j-1}(u_i u_k) + \phi D^{j-1}(D_i u_k)(D_k u_i)\} \\
&= \sum_{i,k}(J_1 + J_2 + J_3 + J_4)(i,k)
\end{aligned}$$

Let us first estimate the following:

$$\begin{aligned}
|D^l(u_i u_k)|_\infty &= \left| \sum_{m=0}^{l} C(l,m) D^m u_i D^{l-m} u_k \right|_\infty \\
&\le C \sum_{m=0}^{l} |D^m u|_\infty |D^{l-m} u|_\infty \\
&\le C(|u|_\infty |\mathcal{D}^l u|_\infty + \sum_{m=1}^{l-1} |\mathcal{D}^m u|_\infty |\mathcal{D}^{l-m} u|_\infty \\
&\le C(|u|_\infty |\mathcal{D}^l u|_\infty + t^{-m/2}|f|_\infty t^{-(l-m)/2}|f|_\infty) \\
&\le C(|u|_\infty |\mathcal{D}^j u|_\infty + t^{-l/2}|f|_\infty^2)
\end{aligned} \tag{3.54}$$

Let us estimate J_1:

$$|J_1|_\infty = |D_i D_k \phi D^{j-1} u_i u_k|_\infty \le |D_i D_k \phi|_\infty |D^{j-1} u_i u_k|_\infty$$

It is not difficult to observe that $|D_i\phi| \leq C\delta^{-1}$ and $|D_iD_k\phi|_\infty \leq C\delta^{-2}$. Also, the use of (3.55) give the estimate for J_1 as

$$|J_1|_\infty \leq C\delta^{-2}(|u|_\infty|\mathcal{D}^{j-1}u|_\infty + t^{-(j-1)/2}|f|_\infty^2).$$

Furthermore, using (3.50) one more time we get

$$|J_1|_\infty \leq C\delta^{-2}t^{-(j-1)/2}|f|_\infty^2.$$

To estimate J_2, we again use (3.55) and proceed as

$$\begin{aligned}|J_2|_\infty &\leq |D_k\phi|_\infty|D^j(u_iu_k)|_\infty \\ &\leq C\delta^{-1}(|u|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2 \\ &\leq C\delta^{-1}(|f|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2)\end{aligned}$$

J_3 has same estimate as J_2. Next, we estimate J_4 with the use of (3.55) one more time to get

$$\begin{aligned}|J_4|_\infty &\leq |\phi|_\infty|D^{j-1}(D_iu_k)(D_ku_i)|_\infty \\ &\leq C(|\mathcal{D}u|_\infty|\mathcal{D}^j u|_\infty + t^{-(j+1)/2}|f|_\infty^2) \\ &\leq C(t^{-1/2}|f|_\infty|\mathcal{D}^j u|_\infty + t^{-(j+1)/2}|f|_\infty^2)\end{aligned}$$

Hence

$$\begin{aligned}\left|\sum_{i,k} D_iD_k(\phi D^{j-1}(u_iu_k))\right|_\infty &\leq C|J_1|_\infty + \cdots |J_4|_\infty \\ &\leq C\delta^{-2}t^{-(j-1)/2}|f|_\infty^2 + C\delta^{-1}(|f|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2) \\ &+ C(t^{-1/2}|f|_\infty|\mathcal{D}^j u|_\infty + t^{-(j+1)/2}|f|_\infty^2)\end{aligned}$$

Note $|D_{l,x}|x-y|^{-1}|_\infty \leq |x-y|^{-2}$ and estimate $|I_1|_\infty$ as

$$\begin{aligned}|I_1|_\infty &\leq C|D_iD_k(\phi D^{j-1}(u_iu_k))|_\infty \int_{B(x,2\delta)} |x-y|^{-2}dy \\ &\leq C\delta\{C\delta^{-2}t^{-(j-1)/2}|f|_\infty^2 + C\delta^{-1}(|f|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2) \\ &+ C(t^{-1/2}|f|_\infty|\mathcal{D}^j u|_\infty + t^{-(j+1)/2}|f|_\infty^2)\}\end{aligned}$$

Choose $\sqrt{\delta} = t$, then

$$|I_1|_\infty \leq C(|f|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty).$$

The estimates on $|I_2|_\infty$ can be obtained in the very similar way as for $|I_1|_\infty$ without integration by parts. Therefore, estimate (3.51) in the theorem 3.18 follows.

To prove the estimate (3.52), we now apply $D_{l,x}$ under the integral sign in (3.54)

$$\begin{aligned} D_{l,x}q_{glb}(x) &= \sum_{i,k} C_0 \int_{|x-y|>\delta} (D_{l,x}D_iD_k(|x-y|^{-1}))(1-\phi)D^{j-1}(u_iu_k)(y)dy \\ &+ \sum_{i,k} C_0 \int_{|x-y|>\delta} (D_iD_k(|x-y|^{-1})(D_{l,x}(1-\phi))D^{j-1}(u_iu_k))))dy \\ &= I_3 + I_4 \end{aligned}$$

We will need the following:

$$\left| \int_{|x-y|>\delta} D_{l,x}D_iD_k(|x-y|^{-1})dy \right| \leq C\delta^{-1}$$

and from (3.55) for $l = (j-1)$ we obtain

$$|D^{j-1}(u_iu_k)|_\infty \leq C|f|_\infty^2 t^{-(j-1)/2}$$

Hence for $\sqrt{\delta} = t$ we have

$$|I_3|_\infty \leq C|f|_\infty^2 t^{-j/2}$$

To estimate I_4, recall that $\phi' \neq 0$ on $[1,2]$, therefore

$$I_4(x) = \sum_{i,k} C_0 \int_{\delta<|x-y|<2\delta} D_iD_k(|x-y|^{-1})(D_{l,x}(1-\phi))(D^{j-1}(u_iu_k)(y)dy$$

Also we will use the fact

$$\left| \int_{\delta<|x-y|<2\delta} D_iD_k(|x-y|^{-1}dy \right| \leq C\ln 2 \leq C,$$

$$|D_{l,x}(1-\phi)|_\infty \leq C\delta^{-1}$$

and same as before

$$|D^{j-1}(u_iu_k)|_\infty \leq C|f|_\infty^2 t^{-(j-1)/2}$$

We now estimate $|I_4|_\infty$:

$$\begin{aligned} |I_4|_\infty &\leq \left| \sum_{i,k} C_0 \int_{\delta<|x-y|<2\delta} D_iD_k(|x-y|^{-1})(D_{l,x}(1-\phi))(D^{j-1}(u_iu_k)(y)dy \right|_\infty \\ &\leq C|D_{l,x}(1-\phi)|_\infty |D^{j-1}(u_iu_k)|_\infty \left| \int_{\delta<|x-y|<2\delta} D_iD_k(|x-y|^{-1}dy \right| \\ &\leq C\delta^{-1}|D^{j-1}(u_iu_k)|_\infty \\ &\leq C\delta^{-1}|f|_\infty^2 t^{-(j-1)/2} \end{aligned}$$

Using $\sqrt{\delta} = t$ we have

$$|I_4|_\infty \le C|f|_\infty^2 t^{-j/2}$$

Hence

$$\begin{aligned} |D_{l,x} q_{glb}| &\le |I_3|_\infty + |I_4|_\infty \\ &\le C t^{-j/2} |f|_\infty^2 \end{aligned}$$

Hence the estimate (3.52) in theorem 3.18 follows. □

3.5 Estimates for the Navier-Stokes Equations

Recall that

$$u_t = \triangle u + Q, \quad Q = -\nabla p - u \cdot \nabla u, \quad u = f \quad \text{at} \quad t = 0.$$

We write $Q = Q_{loc} + Q_{glb}$ with

$$\begin{aligned} Q_{loc} &= -\nabla p_{loc} - \sum_j D_j(u_j u), \\ Q_{glb} &= -\nabla p_{glb}. \end{aligned}$$

Theorem 3.19. (Kriess and Lorenz '001) Consider the Cauchy problem for the Navier-Stokes equations (2.1) and (2.2), where $f \in C_{per}^\infty(\mathbb{R}^3)$ and $\nabla \cdot f = 0$. Then there is a constant $c_0 > 0$, and for every $j = 0, 1, 2 \cdots$ there is a constant K_j such that for an interval

$$0 < t \le \frac{c_0}{|f|_\infty^2}$$

we have

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \le K_j |f|_\infty.$$

The constants c_0 and K_j are independent of t and f.

Using the estimates in lemma 3.17 and the heat equation estimates (3.10), (3.14) and (3.15), we will prove the following:

Lemma 3.20. Set

$$V(t) = |u(t)|_\infty + t^{1/2} |\mathcal{D} u(t)|_\infty, \quad 0 < t < T_f. \tag{3.55}$$

There is a constant $C > 0$, independent of t and f, so that

$$V(t) \le C|f|_\infty + C t^{1/2} \max_{0 \le s \le t} V^2(s), \quad 0 < t < T_f. \tag{3.56}$$

Proof. Using lemma 3.17 with $\delta = t^{1/2}$, we have

$$\begin{aligned} |p_{loc}|_\infty + |u_j u|_\infty &\le C(|u|_\infty^2 + t^{1/2}|u|_\infty |\mathcal{D}u|_\infty, \\ |Q_{loc}| &\le C(t^{-1/2}|u|_\infty^2 + t^{1/2}|\mathcal{D}u|_\infty^2, \\ |Q_{glb}|_\infty &\le Ct^{-1/2}|u|_\infty^2. \end{aligned}$$

Since $u_t = \triangle u + Q_{loc} + Q_{glb}$ and since Q_{loc} is obtained by applying one space derivative to the term p_{loc} and $u_j u$, we obtain from (3.10),(3.14),(3.15) and above estimates

$$\begin{aligned} |u(t)|_\infty &\le |f|_\infty + C \max_{0\le s\le t}(s^{1/2}|u(s)|_\infty^2 + s|u(s)|_\infty|\mathcal{D}u(s)|_\infty) + Ct^{1/2}\max_{0\le s\le t}|u(s)|_\infty^2 \\ &\le |f|_\infty + Ct^{1/2}\max_{0\le s\le t}(|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2) \\ &\le |f|_\infty + Ct^{1/2}\max_{0\le s\le t}V^2(s). \end{aligned}$$

For $v(t) = D_k u(t)$, we have

$$v_t = \triangle v + D_k Q$$

with

$$|Q|_\infty \le C(t^{-1/2}|u|_\infty^2 + t^{1/2}|mcDu|_\infty^2).$$

Therefore, by (3.11) for $j = 1$ and by (3.15),

$$\begin{aligned} |v(t)|_\infty &\le Ct^{-1/2}|f|_\infty + C\max_{0\le s\le t}\{s^{1/2}|Q(s)|_\infty\} \\ &\le Ct^{-1/2}|f|_\infty + C\max_{0\le s\le t}\{s^{1/2}(s^{-1/2}|u(s)|_\infty^2 + s^{1/2}|\mathcal{D}u(s)|_\infty^2)\} \\ &\le Ct^{-1/2}|f|_\infty + C\max_{0\le s\le t}\{|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2\} \end{aligned}$$

which gives

$$t^{1/2}|v(t)|_\infty \le C|f|_\infty + Ct^{1/2}\max_{0\le s\le t}V^2(s).$$

The lemma is proved. $\square$

Lemma 3.20 allows us to estimate $|u(t)|_\infty$ and $|\mathcal{D}u(t)|_\infty$ in terms of $|f|_\infty$ in a small time interval.

Lemma 3.21. Let $C > 0$ denote the constant in (3.57) and set

$$c_0 = \frac{1}{16C^4}$$

Then $T_f > c_0/|f|_\infty^2$ and

$$|u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty < 2C|f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_0}{|f|_\infty^2}. \tag{3.57}$$

Proof. Recall the definition of $V(t)$ from (3.56). If (3.58) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C|f|_\infty$. Using (3.54) we have

$$\begin{aligned} 2C|f|_\infty &= V(t_0) \\ &\leq C|f|_\infty + Ct_0^{1/2} 4C^2 |f|_\infty^2, \end{aligned}$$

thus

$$1 \leq 4C^2 t_0^{1/2} |f|_\infty,$$

thus $t_0 \geq c_0/|f|_\infty^2$. This contradiction proves (3.55) and $T_f > c_0/|f|_\infty^2$. □

PROOF OF THEOREM 3.19

Lemma 3.21 proves theorem 3.19 for $j = 0$ and 1. In this part, we use the induction argument as in the proof of theorem 3.5 to prove theorem 3.19. Let us suppose $j \geq 1$ and assume that for $0 \leq k \leq j-1$

$$t^{k/2} |\mathcal{D}^k u(t)|_\infty \leq K_k |f|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{|f|_\infty^2}. \tag{3.58}$$

Applying D^j to $u_t = \triangle u + Q(s)$, and letting $v := D^j u$, we obtain

$$v_t = \triangle v + D^j Q$$

and the solution

$$v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^j Q(s) ds.$$

We must now estimate

$$|v(t)|_\infty = |\mathcal{D}^j u(t)|_\infty.$$

Note that

$$\begin{aligned} |v(t)|_\infty &= \left| D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^j Q(s) ds \right|_\infty \\ &\leq |D^j e^{\triangle t} f|_\infty + \left| \int_0^t e^{\triangle(t-s)} D^j Q(s) ds \right|_\infty \\ &\leq |T_1|_\infty + |T_2|_\infty. \end{aligned}$$

Use of (3.11) gives us

$$|T_1|_\infty \leq |\mathcal{D}^j e^{\triangle t} f|_\infty \leq C_j t^{-j/2} |f|_\infty$$

where C_j is a constant independent of t and f. Next, we consider T_2:

$$\begin{aligned} T_2 &= \int_0^t e^{\triangle(t-s)} D^j Q(s) ds \\ &= \int_0^{t/2} e^{\triangle(t-s)} D^j Q(s) ds + \int_{t/2}^t e^{\triangle(t-s)} D^j Q(s) ds \\ &= J_1 + J_2. \end{aligned}$$

First, we consider J_1. Applying integration by parts j times, on the interval $[0, t/2]$, the integral exists and we compute

$$\begin{aligned} |J_1|_\infty &= \left| \int_0^{t/2} e^{\triangle(t-s)} D^j Q(s) ds \right|_\infty \\ &= \left| \int_0^{t/2} D^j e^{\triangle(t-s)} Q(s) ds \right|_\infty \\ &\leq \int_0^{t/2} |D^j e^{\triangle(t-s)} Q(s)|_\infty ds \\ &\leq C \int_0^{t/2} (t-s)^{-j/2} |Q(s)|_\infty ds. \end{aligned}$$

Recall from lemma 3.20 and with the induction assumption

$$\begin{aligned} |Q(s)|_\infty &\leq C(s^{-1/2} |u(s)|_\infty^2 + s^{1/2} |\mathcal{D} u(s)|_\infty^2) \\ &\leq C(s^{-1/2} |f|_\infty^2 + s^{1/2} s^{-1} |f|_\infty^2 \\ &\leq C s^{-1/2} |f|_\infty^2. \end{aligned}$$

Then using $\int_0^{t/2} (t-s)^{-j/2} s^{-1/2} ds = C t^{(1-j)/2}$ we obtain

$$|J_1|_\infty \leq C |f|_\infty^2 t^{(1-j)/2}.$$

While estimating J_2, we can only transfer from $D^j Q$ to the heat semigroup one derivative. Moving more derivatives will cause the integral to be non-integrable. Applying integration by parts yields

$$\begin{aligned} J_2 &= \int_{t/2}^t e^{\triangle(t-s)} D^j Q(s) ds \\ &= \int_{t/2}^t D e^{\triangle(t-s)} D^{j-1} Q(s) ds \end{aligned}$$

Note that $|D^{j-1}Q|_\infty \le C(|f|_\infty|\mathcal{D}^j u|_\infty + t^{-j/2}|f|_\infty^2)$ follows from (3.51) and (3.52) . Therefore, J_2 can be estimated as

$$
\begin{aligned}
|J_2|_\infty &= \left| \int_{t/2}^{t} De^{\triangle(t-s)} D^{j-1}Q(s)ds \right|_\infty \\
&\le \int_{t/2}^{t} |De^{\triangle(t-s)} D^{j-1}Q(s)|_\infty ds
\end{aligned}
$$

Using (3.11) for $j = 1$ and using the estimate for $|D^{j-1}Q|_\infty$ obtained above we get

$$
\begin{aligned}
|J_2|_\infty &\le C \int_{t/2}^{t} (t-s)^{-1/2}(|f|_\infty|\mathcal{D}^j u(s)|_\infty + s^{-j/2}|f|_\infty^2)ds \\
&\le C|f|_\infty \int_{t/2}^{t} (t-s)^{-1/2}|\mathcal{D}^j u(s)|_\infty ds + C|f|_\infty^2 \int_{t/2}^{t} (t-s)^{-1/2}s^{-j/2}ds \\
&\le S_1 + S_2.
\end{aligned}
$$

To estimate S_1 and S_2 we need

$$
\int_{t/2}^{t} (t-s)^{-1/2}s^{-j/2}ds = Ct^{(1-j)/2} \quad \text{for some constant} \quad C > 0.
$$

We consider S_1:

$$
\begin{aligned}
S_1 &= C|f|_\infty \int_{t/2}^{t} (t-s)^{-1/2}|\mathcal{D}^j u(s)|_\infty ds \\
&= C|f|_\infty \int_{t/2}^{t} (t-s)^{-1/2}s^{-j/2}s^{j/2}|\mathcal{D}^j u(s)|_\infty ds \\
&\le C|f|_\infty \max_{0\le s\le t} \{s^{j/2}|\mathcal{D}^j u(s)|_\infty\} \int_{t/2}^{t} (t-s)^{-1/2}s^{-j/2}ds \\
&\le C|f|_\infty t^{(1-j)/2} \max_{0\le s\le t} \{s^{j/2}|\mathcal{D}^j u(s)|_\infty\}.
\end{aligned}
$$

Next for S_2:

$$
\begin{aligned}
|S_2|_\infty &\le C|f|_\infty^2 \int_{t/2}^{t} (t-s)^{-1/2}s^{-j/2}ds \\
&\le C|f|_\infty^2 t^{(1-j)/2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|J_2|_\infty &\le S_1 + S_2 \\
&\le C|f|_\infty t^{(1-j)/2} \max_{0\le s\le t} \{s^{j/2}|\mathcal{D}^j u(s)|_\infty\} + C|f|_\infty^2 t^{(1-j)/2}.
\end{aligned}
$$

Thus T_2 has the estimate

$$\begin{aligned}|T_2|_\infty &\le |J_1|_\infty + |J_2|_\infty \\ &\le C|f|_\infty t^{(1-j)/2} \max_{0\le s\le t}\{s^{j/2}|\mathcal{D}^j u(s)|_\infty\} + C|f|_\infty^2 t^{(1-j)/2}.\end{aligned}$$

We finally bound $|v(t)|_\infty$:

$$\begin{aligned}t^{j/2}|v(t)|_\infty &\le t^{j/2} C\left(t^{-j/2}|f|_\infty + |f|_\infty t^{(1-j)/2} \max_{0\le s\le t}\{s^{j/2}|\mathcal{D}^j u(s)|_\infty\} + |f|_\infty^2 t^{(1-j)/2}\right) \\ &\le C|f|_\infty + C|f|_\infty^2 t^{1/2} + C|f|_\infty t^{1/2} \max_{0\le s\le t}\{s^{j/2}|\mathcal{D}^j u(s)|_\infty\}.\end{aligned}$$

Now, as $v = D^j u$, we maximize the resulting estimates $t^{j/2}|D^j u|_\infty$ over all derivatives D^j of order j and derive

$$t^{j/2}|\mathcal{D}^j u|_\infty \le C_j|f|_\infty + C_j|f|_\infty^2 t^{1/2} + C_j|f|_\infty t^{1/2} \max_{0\le s\le t}\{s^{j/2}|\mathcal{D}^j u(s)|_\infty\}.$$

Here, define

$$\psi(t) := t^{j/2}|\mathcal{D}^j u|_\infty$$

We have the estimate

$$\psi(t) \le C_j|f|_\infty + C_j|f|_\infty^2 t^{1/2} + C_j|f|_\infty t^{1/2} \max_{0\le s\le t}\psi(s).$$

Recall the assumption that

$$0 \le t \le \frac{c_0}{|f|_\infty^2}, \qquad c_0 = \frac{1}{16C^4}$$

where C is the constant from lemma 3.20 equation (3.57). Then

$$t^{1/2}|f|_\infty \le \sqrt{c_0}$$

and the term $C|f|_\infty^2$ is bounded by

$$C|f|_\infty^2 t^{1/2} = (C|f|_\infty)(|f|_\infty t^{1/2}) \le C|f|_\infty\sqrt{c_0} = C\sqrt{c_0}|f|_\infty.$$

so that

$$\begin{aligned}\psi(t) &\le C_j|f|_\infty + C\sqrt{c_0}|f|_\infty + C|f|_\infty t^{1/2}\max_{0\le s\le t}\psi(s) \\ &\le (C_j + C\sqrt{c_0})|f|_\infty + C|f|_\infty t^{1/2}\max_{0\le s\le t}\psi(s) \\ &\le C_j|f|_\infty + C_j|f|_\infty t^{1/2}\max_{0\le s\le t}\psi(s).\end{aligned}$$

Thus we have

$$\psi(t) \leq C_j|f|_\infty + C_j|f|_\infty t^{1/2} \max_{0\leq s\leq t} \psi(s) \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{|f|_\infty^2}. \tag{3.59}$$

We note that the constant C_j is a maximum of all constants appearing in the above and is independent of t and f. Fix this constant so that (3.60) holds. Let

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}.$$

We first claim that

$$\psi(t) < 2C_j|f|_\infty \quad \text{for} \quad 0 \leq t < \frac{c_j}{|f|_\infty^2}.$$

Suppose not. Then let $0 < t_0 < c_j/|f|_\infty^2$ denote the smallest time with $\psi(t_0) = 2C_j|f|_\infty$. Then from (3.60)

$$\begin{aligned} 2C_j|f|_\infty = \psi(t_0) &\leq C_j|f|_\infty + C_j|f|_\infty t_0^{1/2} \max_{0\leq s\leq t} \psi(s) \\ &\leq C_j|f|_\infty + C_j|f|_\infty t_0^{1/2} 2C_j|f|_\infty \\ &= C_j|f|_\infty + 2t_0^{1/2} C_j^2 |f|_\infty^2. \end{aligned}$$

Therefore,

$$C_j|f|_\infty \leq t_0^{1/2} 2C_j^2|f|_\infty^2$$

gives

$$1 \leq 2C_j|f|_\infty t_0^{1/2}.$$

This forces

$$t_0 \geq \frac{1}{4|f|_\infty^2 C_j^2} \geq \frac{c_j}{|f|_\infty^2}$$

a contradiction. So we must have

$$t^{j/2}|\mathcal{D}^j u|_\infty \leq 2C_j|f|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_j}{|f|_\infty^2}$$

Then the statement is true for j with $K_j = 2C_j$. Now, suppose that

$$T_j := \frac{c_j}{|f|_\infty^2} < t \leq \frac{c_0}{|f|_\infty^2} =: T_0 \tag{3.60}$$

Then we restart the argument at $t - T_j$. As $T_j < t \leq T_0, 0 < t - T_j \leq T_0 - T_j \leq T_0$.

From lemma 3.21, we have

$$|u(t - T_j)|_\infty \le 2|f|_\infty.$$

and we obtain

$$T_j^{j/2}|\mathcal{D}^j u|_\infty \le 4C_j|f|_\infty \tag{3.61}$$

Finally, for any t with (3.61), we have

$$t^{j/2} \le T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}.$$

Now from (3.62) we have

$$\begin{aligned} t^{j/2}|\mathcal{D}^j u|_\infty &\le T_0^{j/2}|\mathcal{D}^j u|_\infty \\ &\le \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}|\mathcal{D}^j u|_\infty \\ &\le 4C_j\left(\frac{c_0}{c_j}\right)^{j/2} |f|_\infty \\ &= K_j|f|_\infty. \end{aligned}$$

Hence, we have a K_j for which $t^{j/2}|\mathcal{D}^j u|_\infty \le K_j|f|_\infty$, and this completes the proof of the theorem 3.19 which is a goal of this dissertation.

4 The KL Paper in Periodic Case by Eliminating the Pressure

As we discussed in previous sections, the pressure term in the Navier-Stokes equations plays a significant role in obtaining the main estimates in the KL paper. However, in this section we introduce another approach of dealing with the pressure term in the Navier-Stokes equations. Since we are considering the Navier-Stokes equations "in the absence of the boundaries" , i.e on $\mathbb{T}^n$ or on $\mathbb{R}^n$, we will introduce the Helmholtz-Weyl decomposition in the case of torus. The corresponding results for the whole space can be proved similarly. We note here that the KL paper is in $\mathbb{R}^3$, however, we will generalize their work for $n \ge 3$ in this part of our work by the " functional analytic" approach. Next, we are going to introduce the classical theory of the Helmholtz-Weyl decomposition [17] in L^2, for a domain without boundaries to give some insight for readers.

4.1 Helmholtz-Weyl Decomposition

Since solutions of the Navier-Stokes equations are required to be divergence free, it is useful to define and characterize subspaces of the standard spaces that consist of divergence-free functions. We use the following notations in this section:

$$\mathbb{L}^2 := [L^2]^n \quad \text{and} \quad \mathbb{H}^s := [H^s]^n.$$

We will show that any $u \in \mathbb{L}^2$ can be written in a unique way as the sum

$$u = h + \nabla g,$$

where the vector function h is divergence free (in a weak sense) and g belongs to H^1. In other words

$$\mathbb{L}^2 = H \oplus G$$

where H is a space of divergence-free functions (satisfying an appropriate boundary condition) and G is the space of gradient of functions in H^1. For $n = 3$ this theorem is a variant of a well-known result due to Helmholtz (1858) that a C^2 vector field on $\mathbb{R}^3$ that decays sufficiently fast at infinity can be written uniquely as

$$f = \mathrm{curl} h + \nabla g.$$

Since $\nabla \cdot \mathrm{curl} h = 0$ the decomposition of a sufficiently regular function into the curl and the gradient is an example of the decomposition in which we are interested in this section.

For the consideration of the Navier-Stokes equations in the absence of the boundary, i.e on $\mathbb{T}^n$ or $\mathbb{R}^n$, we will prove the validity of the Helmholtz-Weyl decomposition in the case of a torus. The corresponding results for the whole space can be proved similarly. We will also introduce two concepts related to such a decomposition: the Leray projector $\mathbb{P}$ onto divergence-free functions, defined by setting $\mathbb{P}(u) = h$ when $u = h + \nabla g$, and the Stokes operator $A = -\mathbb{P}\triangle$.

4.2 The Helmholtz-Weyl Decomposition on Torus

In the case of torus, we will decompose only the homogeneous space $\dot{\mathbb{L}}^2$ (rather than all of $\mathbb{L}^2$), since we will always include the zero-averaging condition when considering the Navier-Stokes equations in this setting.

In order to prove the decomposition $\dot{\mathbb{L}}^2 = H \oplus G$, we need to define the appropriate spaces H and G in this context. We begin with $H(\mathbb{T}^n)$. If u is given by $u(x) = \widehat{u}_k e^{ik\cdot x}$ then a straightforward computation shows that

$$\mathrm{div} u(x) = i(k \cdot \widehat{u}_k) e^{ik\cdot x}.$$

So this function u is divergence-free if and only if $\widehat{u}_k$ is orthogonal to k. This leads to the following definition.

Definition 4.1. We define the space $H = H(\mathbb{T}^n)$ as

$$\left\{ u \in \dot{\mathbb{L}}^2 : u = \sum_{k\in\dot{\mathbb{Z}}^n} \widehat{u}_k e^{ik\cdot x}, \widehat{u}_k = \overline{\widehat{u}}_{-k} \quad \text{and} \quad k \cdot \widehat{u}_k = 0 \quad \text{for all} \quad k \in \dot{\mathbb{Z}}^3 \right\}$$

and equip H with the $\mathbb{L}^2$-norm.

Since in this definition of the space H, we implicitly consider functions given as the limit of smooth functions, we need to clarify in what sense these limits are "divergence free".

Definition 4.2. Let $\Omega \subseteq \mathbb{R}^n$ a simply connected domain. A function $u \in L^1(\Omega)$ is weakly divergence free if

$$\langle u, \nabla\varphi \rangle = 0 \quad \text{for every} \quad \varphi \in C_c^\infty(\Omega).$$

Lemma 4.3. Each $u \in H(\mathbb{T}^n)$ is weakly divergence free and moreover

$$\langle u, \nabla\varphi \rangle = 0 \quad \text{for all} \quad \varphi \in H^1(\mathbb{T}^n).$$

Proof. If $u \in H$ is given by

$$u(x) = \sum_{l\in\dot{\mathbb{Z}}^n} \widehat{u}_l e^{il\cdot x}$$

and $\varphi_k(x) = e^{-ik\cdot x}$ then

$$\int_{\mathbb{T}^n} u(x) \cdot \nabla\varphi_k(x) dx = \int_{\mathbb{T}^n} \widehat{u}_k e^{ik\cdot x} \cdot \nabla e^{-ik\cdot x} dx = -i \int_{\mathbb{T}^n} \widehat{u}_k \cdot k = 0,$$

since $\widehat{u}_k \cdot k = 0$ for all $k \in \dot{\mathbb{Z}}^n$. The result follows since any $\varphi \in H^1(\mathbb{T}^n)$ can be approximated arbitrarily closely in H^1 by real-valued finite linear combinations of the functions φ_k by the norm in $H^1 = W^{1,2}(\mathbb{T}^n)$. $\square$

Definition 4.4. The space G on the torus $\mathbb{T}^n, n \geq 3$ is defined as

$$G(\mathbb{T}^n) = \{u \in \mathbb{L}^2 : u = \nabla g, \quad \text{where} \quad g \in \dot{H}^1(\mathbb{T}^n)\}.$$

We assume here that g belongs to $\dot{H}^1$ (rather than to H^1) in order to obtain the uniqueness of g in the resulting Helmholtz-Weyl decomposition.

Corollary 4.5. The space H and G are orthogonal in $\mathbb{L}^2(\mathbb{T}^n)$ i.e

$$\langle h, \nabla g \rangle = 0$$

for every $h \in H$ and $\nabla g \in G$.

We can now state the first result of this chapter, which is the existence of the Helmholtz-Weyl decomposition on the torus. In this simple case we are able to prove the existence of such a decomposition with very explicit calculation.

Theorem 4.6. (Helmholtz-Weyl decomposition on $\mathbb{T}^n$) The space $\dot{\mathbb{L}}^2(\mathbb{T}^n)$ can be written as

$$\dot{\mathbb{L}}^2 = H \oplus G,$$

i.e every function $u \in \dot{\mathbb{L}}^2$ can be written in a unique way as

$$u = h + \nabla g \tag{4.1}$$

where the vector-valued function h belongs to H and the scalar function g belongs to $\dot{H}^1$. Moreover, if in addition u belongs to $\dot{\mathbb{H}}^s, s > 0$, then $h \in \dot{\mathbb{H}}^s$ and $g \in \dot{H}^{s+1}$.

Proof. Take $u \in \dot{\mathbb{L}}^2$ and write it in the form

$$u(x) = \sum_{k \in \dot{\mathbb{Z}}^n} \widehat{u}_k e^{ik \cdot x}.$$

Write the vector coefficients $\widehat{u}_k$ as the linear combination of k and a vector w_k perpendicular to k in $\mathbb{C}^n$,

$$\widehat{u}_k = \alpha_k k + w_k$$

Here $\alpha_k = \widehat{u}_k \cdot k/|k|^2 \in \mathbb{C}, w_k \in \mathbb{C}^n$, and $w_k \cdot k = 0$. Notice that

$$|\widehat{u}_k|^2 = |\alpha_k|^2|k|^2 + |w_k|^2. \tag{4.2}$$

We therefore have

$$
\begin{aligned}
u(x) = \sum_{k\in\dot{\mathbb{Z}}^n} \widehat{u}_k e^{ik\cdot x} &= \sum_{k\in\dot{\mathbb{Z}}^n} (\alpha_k k + w_k) e^{ik\cdot x} \\
&= \sum_{k\in\dot{\mathbb{Z}}^n} (-i\alpha_k)\nabla e^{ik\cdot x} + \sum_{k\in\dot{\mathbb{Z}}^n} w_k e^{ik\cdot x} \\
&= \nabla g(x) + h(x)
\end{aligned}
$$

where

$$
g(x) := \sum_{k\in\dot{\mathbb{Z}}^n} (-i\alpha_k) e^{ik\cdot x} \qquad \text{and} \qquad h(x) := \sum_{k\in\dot{\mathbb{Z}}^n} w_k e^{ik\cdot x}.
$$

Since $w_k \cdot k = 0$ for every k, we conclude that h is (weakly) divergence free. Thus to show that we have obtained the required decomposition as in (4.1), we only need to show that $g \in \dot{H}^1$ and $h \in \dot{\mathbb{L}}^2$. To this end, we notice that

$$
||g||^2_{\dot{H}^1} = \sum_{k\in\dot{\mathbb{Z}}^n} |\alpha_k|^2 |k|^2, \quad \text{and} \quad ||h||^2 = \sum_{k\in\dot{\mathbb{Z}}^n} |w_k|^2.
$$

Taking into account (4.2), we get

$$
||h||^2 + ||\nabla g||^2 = ||u||^2.
$$

From this equality (and also directly from Corollary 4.5), it follows that g belongs to $\dot{H}^1$, and h belongs to H. Multiplying (4.2) by $|k|^{2s}$ it follows that if $u \in \dot{\mathbb{H}}^s$ then $g \in \dot{H}^{s+1}$ and $h \in \dot{\mathbb{H}}^s$. Finally, the uniqueness of this representation follows easily, since using Corollary 4.5, we have

$$
\begin{aligned}
(h_1 - h_2) + (\nabla g_1 - \nabla g_2) = 0 &\implies ||h_1 - h_2 + \nabla g_1 - \nabla g_2||^2 = 0 \\
&\implies ||h_1 - h_2||^2 + ||\nabla g_1 - \nabla g_2||^2 = 0,
\end{aligned}
$$

so $h_1 = h_2$ and $\nabla g_1 = \nabla g_2$. Since both g_1 and g_2 have mean zero, it follows that $g_1 = g_2$. We notice that the decomposition of $\mathbb{L}^2(\mathbb{T}^n)$ is now straightforward, since we obviously have

$$
\mathbb{L}^2(\mathbb{T}^n) = \mathbb{R}^n \oplus H(\mathbb{T}^n) \oplus G(\mathbb{T}^n)
$$

$\square$

Theorem 4.7. The space $\mathbb{L}^2(\mathbb{R}^n)$ can be decomposed as

$$
\mathbb{L}^2(\mathbb{R}^n) = H(\mathbb{R}^n) \oplus G(\mathbb{R}^n),
$$

where

$$H(\mathbb{R}^n) := \{u : u \in \mathbb{L}^2(\mathbb{R}^n), \quad \mathrm{div} u = 0\}$$
$$G(\mathbb{R}^n) := \{w \in \mathbb{L}^2(\mathbb{R}^n) : w = \nabla g \quad \text{and} \quad g \in \dot{H}^1(\mathbb{R}^n)\}.$$

Proof. See theorem 2.7 of [17]. □

The Helmholtz-Weyl decomposition allows us to define the Leray projector, the orthogonal projector onto the space of divergence-free functions.

Definition 4.8. On the torus and on the whole space the Leray projector $\mathbb{P}$ is given by

$$\mathbb{P}u = v \iff u = v + \nabla w$$

where $v \in H$ and $\nabla w \in G$. On the whole space we have

$$\mathbb{P} : \mathbb{L}^2(\mathbb{R}^n) \to H(\mathbb{R}^n),$$

while on the torus we prefer to have a restricted domain of $\mathbb{P}$:

$$\mathbb{P} : \dot{\mathbb{L}}^2(\mathbb{T}^n) \to H(\mathbb{T}^n),$$

because on $\mathbb{T}^n$ we will always consider solution of the Navier-Stokes equations with zero mean.

The Leray projector can be computed in a very straightforward way when we consider functions in the absence of boundaries. For example, if u belongs $\dot{\mathbb{L}}^2(\mathbb{T}^n)$ and is given by $u(x) = \sum_{k\in\dot{\mathbb{Z}}^n} \widehat{u}_k e^{ik\cdot x}$ then the Leray projection of u is given by the formula

$$\mathbb{P}u(x) = \sum_{k\in\dot{\mathbb{Z}}^n} \left(\widehat{u}_k - \frac{\widehat{u}_k \cdot k}{|k|^2} k\right) e^{ik\cdot x}. \tag{4.3}$$

In general, the Leray projector is defined by

$$\mathbb{P} = (\mathbb{P}_{ij})_{1\leq i,j\leq n}, \quad \mathbb{P}_{ij} = \delta_{ij} + R_i R_j \tag{4.4}$$

where $R_i = (-\triangle)^{-1/2} D_i$ is the ith Riesz transformation and δ_{ij} is the Kronecer delta function. On the torus (and on the whole space), the Leray projector commutes with any derivatives. This follows from the linearity of $\mathbb{P}$ and the fact that differentiation of a function u given by $u(x) = \widehat{u}_k e^{ik\cdot x}$ reduces to multiplication by a constant.

Lemma 4.9. The Leray projector $\mathbb{P}$ on the torus and on the whole space commutes with any derivative:

$$\mathbb{P}(D_j u) = D_j(\mathbb{P}u), \quad j = 1, 2, \cdots n \quad D_j = \partial/\partial x_j$$

for all $u \in \dot{\mathbb{H}}^1$.

Proof. We consider the case of a torus: Let u be given by $\widehat{u}_k e^{ik\cdot x}$. For any such u we have

$$\mathbb{P}D_j(u) = \mathbb{P}(ik_j u) = ik_j \mathbb{P}u.$$

By the linearity of $\mathbb{P}$ the result holds for any finite combination of such functions. The general case now follows since $\mathbb{P} : \dot{H}^s \to \dot{H}^s, s = 0, 1,$ and $D_j : \dot{\mathbb{H}}^1 \to \dot{\mathbb{L}}^2$ are all continuous. $\square$

4.3 Estimates of the Navier-Stokes Equations

In this section, we will prove the main theorem of the KL paper in periodic case using different approach than the proof provided in previous section for the same theorem. Introduction of the new tool like the use of "the Leray projector" eliminates the role of the pressure in obtaining the required estimates whereas in the KL paper they spend good amount of time in obtaining the estimates on the pressure and its derivatives. Let us recall the Navier-Stokes equations one more time

$$u_t + u \cdot \nabla u + \nabla p = \triangle u, \quad \nabla \cdot u = 0 \tag{4.5}$$

and

$$u(x, 0) = f(x) \tag{4.6}$$

where $f \in C^{\infty}_{per}(\mathbb{R}^n)$. Roughly speaking, we could have relaxed the smoothness on f because the solution of the Navier-Stokes equations immediately becomes smooth when $t > 0$ even for very rough initial data. Therefore it is natural to consider $f \in C^{\infty}_{per}(\mathbb{R}^n)$ in this setting. We also consider $\nabla \cdot f = 0$ for the sake of compatibility.

We transform the Navier-Stokes equations into the abstract differential equation for u

$$u_t = \triangle u - \mathbb{P}(u \cdot \nabla)u \tag{4.7}$$

by eliminating the pressure, where $\mathbb{P}$ is defined by (4.3) is the Leray projector on a torus. Note that (4.7) is obtained by applying the Leray projector with the properties $\mathbb{P}(\nabla p) = 0, \mathbb{P}(\triangle u) = \triangle u$ since $\nabla \cdot u = 0$.

Use the solution operator $e^{\triangle t}$ of the heat equation to transform the abstract differential equation into an integral equation

$$u(t) = e^{\triangle t} f - \int_0^t e^{\triangle(t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds. \tag{4.8}$$

This integral equation has a unique solution which is called a mild solution of the Navier-Stokes equations. In [8], it is proved that such mild solution is indeed a strong (local in time) solution of the Navier-Stokes equations. Next, we introduce an analogous system of the Navier-Stokes equations (4.7) and use it to prove the main theorem of our work.

Since $\mathbb{P}(u \cdot \nabla u) = \sum_j D_j \mathbb{P}(u_j u)$, for $1 \leq i, j \leq n$, it is appropriate to consider the system

$$u_t = \triangle u + D_i \mathbb{P} g(u), \quad t \geq 0 \tag{4.9}$$

with

$$u(x, 0) = f(x) \quad \text{where} \quad f \in C^\infty_{Per}(\mathbb{R}^n) \tag{4.10}$$

and $g : \mathbb{R}^n \to \mathbb{R}^n$ is quadratic in u. It is well-known that the solution of (4.9) and (4.10) is C^∞_{per} in a maximal interval $0 \leq t < T_f$ where $0 < T_f \leq \infty$.

Let us consider u is the solution of the system $u_t = \triangle u + D_i \mathbb{P}(g(u(x,t)))$ and recall g is quadratic in u thus, there is a constant C_g such that we have the following:

$$|g(u)| \leq C_g |u|^2, \quad |g_u(u)| \leq C_g |u|, \quad \text{for all} \quad u \in \mathbb{R}^n. \tag{4.11}$$

In the following lemmas and theorem, we will allow the constant change C, C_j change line to line as per the need for convenience in writing and they are independent of t and the initial function f.

Theorem 4.10. Under the assumptions on f and g mentioned above, the solution of (4.9) and (4.10) satisfies the following:

(a) There is a constant $c_0 > 0$ with

$$T_f > \frac{c_0}{|f|^2_\infty} \tag{4.12}$$

and

$$|u(t)|_\infty \leq 2|f|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{|f|^2_\infty}$$

(b) For every $j = 1, 2 \cdots$, there is a constant $K_j > 0$ with

$$t^{j/2}|\mathcal{D}^j u|_\infty \le K_j |f|_\infty \quad \text{for} \quad 0 < t \le \frac{c_0}{|f|_\infty^2}. \tag{4.13}$$

The constants c_0 and K_j are independent of t and f.

Before proving theorem 4.10, we prove the following auxiliary results. We will also let $F(x,t) = g(u(x,t))$ for the corollary that follows lemma 4.11.

Lemma 4.11. Let for any $F \in C_{per}^\infty(\mathbb{R}^n)$ and for any multiindex α such that $|\alpha| = j$ where $j \ge 1$ then

$$|\mathcal{D}^j e^{\triangle t}\mathbb{P}F|_\infty \le C_j t^{-j/2}|F|_\infty \quad \text{for} \quad t > 0$$

for some constant $C_j > 0$ independent of t and F.

Proof. Let us first denote $e^{\triangle t}F = \theta * F$ where $\theta(x,t)$ is the n dimensional periodic heat kernel given by (3.8). For $\xi \in \mathbb{Z}^n$, recall the Fourier cofficient of $\theta(x,t)$ given by $\mathcal{F}(\theta(x,t))(\xi) = \widehat{\theta(x,t)}(\xi) = e^{-t|\xi|^2}, t > 0$. Now for any $t > 0$, any choice of $k, l \in \{1, 2, \cdots, n\}$ and for any multiindex α such that $|\alpha| = j$, the operator $D^j e^{\triangle t}\mathbb{P}_{kl}$ on the Fourier side is given by

$$\begin{aligned}
\mathcal{F}(D^j e^{\triangle t}\mathbb{P}_{kl}F_l)(\xi) &= (-i\xi)^\alpha \mathcal{F}(e^{\triangle t}\mathbb{P}_{kl}F_l)(\xi) \\
&= C(-i\xi)^\alpha \mathcal{F}(\theta(x,t))(\xi)\mathcal{F}(\mathbb{P}_{kl}F_l)(\xi) \\
&= C(-i\xi)^\alpha e^{-t|\xi|^2}\left(\delta_{kl} - \frac{\xi_k\xi_l}{|\xi|^2}\right)\mathcal{F}(F_l)(\xi) \\
&= C(-i\xi)^\alpha e^{-t|\xi|^2}\delta_{kl}\mathcal{F}(F_l)(\xi) \\
&\quad - C(-i\xi)^\alpha \xi_k\xi_l \mathcal{F}(F_l)(\xi)\int_t^\infty e^{-\tau|\xi|^2}d\tau.
\end{aligned}$$

Using the Fourier expansion, we can write

$$\begin{aligned}
D^j(e^{\triangle t}\mathbb{P}_{kl}F_l)(x) &= C\sum_{\xi\in\mathbb{Z}^n}(-i\xi)^\alpha \delta_{kl}e^{-t|\xi|^2}\mathcal{F}(F_l)(\xi)e^{i\xi\cdot x} \\
&\quad + C\sum_{\xi\in\mathbb{Z}^n}(-i\xi)^\alpha(i\xi_k)(i\xi_l)\mathcal{F}(F_l)(\xi)e^{i\xi\cdot x}\int_t^\infty e^{-\tau|\xi|^2}d\tau \\
&= (-1)^j C\delta_{kl}D^j\sum_{\xi\in\mathbb{Z}^n}e^{-t|\xi|^2}\mathcal{F}(F_l)(\xi)e^{i\xi\cdot x} \\
&\quad + C(-1)^j\int_t^\infty \sum_{\xi\in\mathbb{Z}^n}(i\xi)^\alpha(i\xi_k)(i\xi_l)e^{-\tau|\xi|^2}\mathcal{F}(F_l)(\xi)e^{i\xi\cdot x}d\tau \\
&= (-1)^j C\delta_{kl}D^j e^{\triangle t}F_l + (-1)^j C\int_t^\infty D^j D_k D_l e^{\triangle\tau}F_l d\tau \\
&= I_1 + I_2.
\end{aligned}$$

From (3.11) we have $|I_1|_\infty \le C_j t^{-j/2}|F_l|_\infty$, and

$$\begin{aligned}|I_2|_\infty &\le C_j|F_l|_\infty \int_t^\infty \tau^{-(j+2)/2}d\tau\\ &\le C_j t^{-j/2}|F_l|_\infty.\end{aligned}$$

Therefore

$$\begin{aligned}|D^j e^{\triangle t}\mathbb{P}_{kl}F_l|_\infty &\le |I_1|_\infty + |I_2|_\infty\\ &\le C_j t^{-j/2}|F_l|_\infty.\end{aligned}$$

Hence Lemma 4.11 is proved. $\square$

Corollary 4.12. Let $F \in C^\infty_{per}(\mathbb{R}^n \times [0,T])$ for some $T > 0$ then the solution of

$$u_t = \triangle u + D_i\mathbb{P}F, \quad u = 0 \quad \text{at} \quad t = 0 \tag{4.14}$$

satisfies

$$|u(t)|_\infty \le Ct^{1/2}\max_{0\le s\le t}|F(s)|_\infty. \tag{4.15}$$

Proof. The solution of (4.14) is given by

$$u(t) = \int_0^t e^{\triangle(t-s)}D_i\mathbb{P}F(s)ds$$

and

$$|u(t)|_\infty \le \int_0^t |e^{\triangle(t-s)}D_i\mathbb{P}F(s)|_\infty ds.$$

Using lemma 4.11 for $j = 1$ and with the fact that D_i commutes with the heat semi-group, we obtain

$$|u(t)|_\infty \le \max_{0\le s\le t}|F(s)|_\infty \int_0^t (t-s)^{-1/2}ds.$$

Hence we obtain

$$|u(t)|_\infty \le Ct^{1/2}\max_{0\le s\le t}|F(s)|_\infty.$$

$\square$

Lemma 4.13. Denote u by the solution of (4.9) and (4.10). And let C_g and C denote the constants in (4.11) and (4.15) respectively; set $c_0 = \frac{1}{16C^2C_g^2}$. Then we have $T_f > c_0/|f|^2_\infty$ and

$$|u(t)|_\infty < 2|f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_0}{|f|^2_\infty}. \tag{4.16}$$

Proof. Suppose (4.16) does not hold, then we can find the smallest time t_0 such that $|u(t_0)|_\infty = 2|f|_\infty$. Since t_0 is the smallest time, so we have $t_0 < c_0/|f|^2_\infty$. Now by (3.10) and (4.15), we have

$$\begin{aligned}
2|f|_\infty &= |u(t_0)|_\infty \\
&\le |f|_\infty + Ct_0^{1/2} \max_{0\le s\le t_0} |g(s)|_\infty \\
&\le |f|_\infty + CC_g t_0^{1/2} \max_{0\le s\le t_0} |u(s)|^2_\infty \\
&\le |f|_\infty + CC_g t_0^{1/2} 4|f|^2_\infty.
\end{aligned}$$

This gives

$$1 \le 4CC_g t_0^{1/2}|f|_\infty,$$

therefore $t_0 \ge 1/(16C^2C_g^2|f|^2_\infty)$ which is a contradiction. Therefore (4.16) must hold. The estimate $T_f > c_0/|f|^2_\infty$ is valid since $\limsup_{t\to T_f} |u(t)|_\infty = \infty$ if T_f is finite. $\square$

PROOF OF THEOREM 4.10

Lemma 4.13 provides the proof of part (a) of the theorem 4.10. Next we prove the estimate (4.13) of part (b) of the theorem 4.10 using induction on j. Let $j > 1$ and assume

$$t^{k/2}|\mathcal{D}^k u(t)|_\infty \le K_k|f|_\infty, \quad \text{for} \quad 0 \le t \le \frac{c_0}{|f|^2_\infty} \quad \text{and} \quad 0 \le k \le j-1. \tag{4.17}$$

One more time, let us denote $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ by D^j if $|\alpha| = j$ for any multiindex α. Let us apply D^j to the equation $u_t = \triangle u + D_i\mathbb{P}g(u)$ where g is quadratic in u to obtain

$$\begin{gathered}
v_t = \triangle v + D^{j+1}\mathbb{P}(g(u)), \quad v := D^j u, \\
v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^{j+1}\mathbb{P}(g(u(s)))ds.
\end{gathered}$$

Using (3.11) we get

$$t^{j/2}|v(t)|_\infty \le C|f|_\infty + t^{j/2}\left|\int_0^t e^{\triangle(t-s)} D^{j+1}\mathbb{P}(g(u(s)))ds\right|_\infty. \tag{4.18}$$

We split the integral into

$$\int_0^{t/2} + \int_{t/2}^t =: I_1 + I_2.$$

and obtain

$$
\begin{aligned}
|I_1(t)| &= \left| \int_0^{t/2} D^{j+1} e^{\triangle(t-s)} \mathbb{P}(g(u(s)))ds \right|_\infty \\
&\leq \int_0^{t/2} |D^{j+1} e^{\triangle(t-s)} \mathbb{P}(g(u(s)))ds|_\infty ds.
\end{aligned}
$$

Using the inequality in lemma 4.11 along with the equality (4.11), we get

$$
\begin{aligned}
|I_1(t)|_\infty &\leq C \int_0^{t/2} (t-s)^{-(j+1)/2} |g(u(s))|_\infty ds \\
&\leq C|f|_\infty^2 t^{(1-j)/2}.
\end{aligned}
$$

The integrand in I_2 has singularity at $s = t$. Therefore, we can move only one derivative from $D^{j+1}\mathbb{P}g(u)$ to the heat semigroup.(If we move two or more derivatives then the singularity becomes non-integrable.) Thus we have

$$
|I_2(t)|_\infty = \left| \int_{t/2}^{t} D e^{\triangle(t-s)} D^j \mathbb{P}(g(u(s)))ds \right|_\infty .
$$

Since the Leray projector commutes with any derivatives in a domain without boundary, therefore

$$
|I_2(t)|_\infty = \left| \int_{t/2}^{t} D e^{\triangle(t-s)} \mathbb{P}(D^j g(u(s)))ds \right|_\infty .
$$

If we use lemma 4.11 for $j = 1$, we obtain

$$
|I_2(t)|_\infty \leq \int_{t/2}^{t} (t-s)^{-1/2} |D^j g(u(s))|_\infty ds. \tag{4.19}
$$

Since $g(u)$ is quadratic in u, therefore

$$
|D^j g(u)|_\infty \leq C|u|_\infty |\mathcal{D}^j u|_\infty + \sum_{k=1}^{j-1} |\mathcal{D}^k u|_\infty |\mathcal{D}^{j-k} u|_\infty .
$$

By induction hypothesis (4.17) we have

$$
\sum_{k=1}^{j-1} |\mathcal{D}^k u(s)|_\infty |\mathcal{D}^{j-k} u(s)|_\infty \leq C s^{-j/2} |f|_\infty^2 . \tag{4.20}
$$

Integral (4.19) can be estimated as below:

$$
\begin{aligned}
|I_2(t)|_\infty &\leq C \int_{t/2}^{t} (t-s)^{-1/2} \left(C|u(s)|_\infty |\mathcal{D}^j u(s)|_\infty + \sum_{k=1}^{j-1} |\mathcal{D}^k u(s)|_\infty |\mathcal{D}^{j-k} u(s)|_\infty \right) ds \\
&\leq J_1 + J_2.
\end{aligned}
$$

Using (4.20) and $\int_{t/2}^{t}(t-s)^{-1/2}s^{-j/2} = Ct^{(1-j)/2}$ to obtain the following

$$|J_2(t)|_\infty \le C|f|_\infty^2 t^{(1-j)/2}.$$

and

$$\begin{aligned}
|J_1(t)|_\infty &= C\int_{t/2}^{t}(t-s)^{-1/2}|u(s)|_\infty|\mathcal{D}^j u(s)|_\infty ds \\
&\le C|f|_\infty \int_{t/2}^{t}(t-s)^{-1/2}s^{-j/2}s^{j/2}|\mathcal{D}^j u(s)|_\infty ds \\
&\le C|f|_\infty t^{(1-j)/2} \max_{0\le s\le t}\{s^{j/2}\mathcal{D}^j u(s)|_\infty\}.
\end{aligned}$$

We use these bounds to bound the integral in (4.18). We have $v = D^j u$. Then maximizing the resulting estimate for $t^{j/2}|D^j u(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) := t^{j/2}|\mathcal{D}^j u(t)|_\infty$$

we obtain the following estimate

$$\phi(t) \le C|f|_\infty + Ct^{1/2}|f|_\infty^2 + C|f|_\infty t^{1/2}\max_{0\le s\le t}\phi(s) \quad \text{for} \quad 0 \le t \le \frac{c_0}{|f|_\infty^2}.$$

Since $t^{1/2}|f|_\infty \le \sqrt{c_0}$ then $Ct^{1/2}|f|_\infty^2 \le C\sqrt{c_0}|f|_\infty$. Therefore

$$\phi(t) \le C_j|f|_\infty + C_j|f|_\infty t^{1/2}\max_{0\le s\le t}\phi(s) \quad \text{for} \quad 0 \le t \le c_0/|f|_\infty^2. \tag{4.21}$$

Let us fix C_j so that the above estimate holds and set

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}.$$

First, let us prove the following

$$\phi(t) < 2C_j|f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_j}{|f|_\infty^2}.$$

Suppose there is a smallest time t_0 such that $0 < t_0 < c_j/|f|_\infty^2$ with $\phi(t_0) = 2C_j|f|_\infty$. Then using (4.21) we obtain

$$2C_j|f|_\infty = \phi(t_0) \le C_j|f|_\infty + 2C_j^2|f|_\infty^2 t_0^{1/2},$$

thus

$$1 \le 2C_j|f|_\infty t_0^{1/2} \quad \text{gives} \quad t_0 \ge c_j/|f|_\infty^2.$$

which contradicts the assertion. Therefore, we proved the estimate

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le 2C_j|f|_\infty \quad \text{for} \quad 0 \le t \le c_j/|f|_\infty^2. \tag{4.22}$$

If

$$T_j := \frac{c_j}{|f|_\infty^2} < t \le \frac{c_0}{|f|_\infty^2} =: T_0 \tag{4.23}$$

then we start the corresponding estimate at $t - T_j$. Using lemma (4.13), we have $|u(t - T_j)|_\infty \le 2|f|_\infty$ and obtain

$$T_j^{j/2}|\mathcal{D}^j u(t)|_\infty \le 4C_j|f|_\infty. \tag{4.24}$$

Finally, for any t satisfying (4.23),

$$t^{j/2} \le T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}$$

and (4.24) yield

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le 4C_j\left(\frac{c_0}{c_j}\right)^{j/2}|f|_\infty.$$

This completes the proof of Theorem 4.10.

Next, we prove a few lemmas needed for the proof of Theorem 4.16 below, the analogue of Theorem 4.10 in the Navier-Stokes case.

Lemma 4.14. Let us denote u by the solution of the Navier-Stokes equations (4.5) and (4.6) with $f \in C^\infty_{Per}(\mathbb{R}^n)$ and set

$$V(t) = |u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty, \quad 0 < t < T(f). \tag{4.25}$$

There is a constant $C > 0$ independent of t and f so that

$$V(t) \le C|f|_\infty + Ct^{1/2}\max_{0\le s\le t} V^2(t), \quad 0 < t < T(f). \tag{4.26}$$

Proof. Using (3.10) in the integral equation of the solution of the Navier-Stokes equation (4.6) we obtain

$$|u(t)|_\infty \le |f|_\infty + \left|\int_0^t \nabla \cdot e^{\triangle(t-s)}\mathbb{P}(u \otimes u)ds\right|_\infty.$$

The use of the inequality in lemma 4.11 for $j = 1$ yields

$$\begin{aligned}|u(t)|_\infty &\leq |f|_\infty + C\int_0^t (t-s)^{-1/2}|u(s)|_\infty^2 ds \\ &= |f|_\infty + C\int_0^t (t-s)^{-1/2}s^{-1/2}s^{1/2}|u(s)|_\infty^2 ds \\ &\leq |f|_\infty + C\max_{0\leq s\leq t}\{s^{1/2}|u(s)|_\infty^2\}\int_0^t (t-s)^{-1/2}s^{-1/2}ds.\end{aligned}$$

Since $\int_0^t (t-s)^{-1/2}s^{-1/2}ds = C$ which is independent of t, we have the following estimate

$$\begin{aligned}|u(t)|_\infty &\leq |f|_\infty + C\max_{0\leq s\leq t}\{s^{1/2}|u(s)|_\infty^2\} \\ |u(t)|_\infty &\leq |f|_\infty + Ct^{1/2}\max_{0\leq s\leq t} V^2(s).\end{aligned} \tag{4.27}$$

Let $1 \leq k \leq n$ and for $v(t) = D_k u(t)$, we have

$$v_t = \triangle v - D_k\mathbb{P}((u\cdot\nabla)u), \quad v = D_k f \quad \text{at} \quad t = 0.$$

Then

$$v(t) = D_k e^{\triangle t} f - \int_0^t e^{\triangle(t-s)} D_k \mathbb{P}((u\cdot\nabla)u(s))ds. \tag{4.28}$$

Since D_k commutes with the heat semigroup, therefore we can move D_k to $e^{\triangle t}$ and obtain

$$v(t) = D_k e^{\triangle t} f - \int_0^t D_k e^{\triangle(t-s)} \mathbb{P}((u\cdot\nabla)u(s))ds \tag{4.29}$$

We can estimate the integral in (4.29) in the following way:

$$\begin{aligned}\left|\int_0^t D_k e^{\triangle(t-s)}\mathbb{P}((u\cdot\nabla)u(s))ds\right|_\infty &\leq \int_0^t |D_k e^{\triangle(t-s)}\mathbb{P}((u\cdot\nabla)u(s))|_\infty ds \\ &\leq C\int_0^t (t-s)^{-1/2}|(u\cdot\nabla)u(s)|_\infty ds \\ &\leq C\int_0^t (t-s)^{-1/2}s^{-1/2}s^{1/2}|u(s)|_\infty|\mathcal{D}u(s)|_\infty ds \\ &\leq C\max_{0\leq s\leq t}\{s^{1/2}|u(s)|_\infty|\mathcal{D}u(s)|_\infty\}\int_0^t (t-s)^{-1/2}s^{-1/2}ds \\ &\leq C\max_{0\leq s\leq t}\{|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2\}.\end{aligned}$$

Therefore $v(t)$ in (4.29) can be estimated as

$$\begin{aligned}|v(t)|_\infty &\leq Ct^{-1/2}|f|_\infty + C\max_{0\leq s\leq t}\{|u(s)|_\infty^2 + s|\mathcal{D}u(s)|_\infty^2\} \\ t^{1/2}|\mathcal{D}u(t)|_\infty &\leq C|f|_\infty + Ct^{1/2}\max_{0\leq s\leq t} V^2(t).\end{aligned} \tag{4.30}$$

Using (4.27) and (4.30), we have proved lemma 4.14. □

Lemma 4.15. Let u be the same as in the previous lemma. Also let $C > 0$ denote the constant in estimate (4.30) and set

$$c_0 = \frac{1}{16C^4}.$$

Then $T(f) > c_0/|f|_\infty^2$ and

$$|u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty < 2C|f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_0}{|f|_\infty^2}. \tag{4.31}$$

Proof. We prove this lemma by contradiction after recalling the definition of $V(t)$ in (4.25). Suppose that (4.31) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C|f|_\infty$. Use (4.26) and obtain

$$\begin{aligned} 2C|f|_\infty &= V(t_0) \\ &\le C|f|_\infty + Ct_0^{1/2}4C^2|f|_\infty^2, \end{aligned}$$

thus,

$$1 \le 4C^2 t_0^{1/2}|f|_\infty^2,$$

therefore $t_0 \ge c_0/|f|_\infty^2$. This contradiction proves (4.31) and $T(f) > c_0/|f|_\infty^2$. $\square$

Lemma 4.15 derives the bounds (4.13) of theorem 4.10 for the solution of the Navier-Stokes equations for $j = 0$ and $j = 1$. By an induction argument as in the proof of Theorem 4.10 one obtains the following.

Theorem 4.16. Consider the Cauchy problem for the Navier-Stokes equations (4.5) and (4.6), where $f \in C_{per}^\infty(\mathbb{R}^n)$ and $\nabla \cdot f = 0$. Then there is a constant $c_0 > 0$, and for every $j = 0, 1, 2 \cdots$ there is a constant K_j such that for an interval

$$0 < t \le \frac{c_0}{|f|_\infty^2}$$

we have

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le K_j|f|_\infty.$$

The constants c_0 and K_j are independent of t and f.

5 Future Work

We consider the Navier-Stokes equations in the n-dimensional half space $\mathbb{R}^n_+$, for $n \geq 3$;

$$\left.\begin{aligned} u_t + u \cdot \nabla u + \nabla p = -\triangle u \qquad & \text{for} \quad x \in \mathbb{R}^n_+, t > 0, \\ \nabla \cdot u = 0 \qquad & \text{for} \quad x \in \mathbb{R}^n_+, t > 0, \\ u|_{t=0} = f \qquad & \text{for} \quad x \in \mathbb{R}^n_+, \\ u|_{x_n=0} = 0 \qquad & \text{for} \quad t > 0 \end{aligned}\right\} \tag{5.1}$$

where $f \in L^\infty(\mathbb{R}^n_+)$ and we assume $\nabla \cdot f = 0$ in the sense of distribution.

In [2], it is proved that (5.1) has strong (local in time) solution. Our goal will be to obtain the apriori estimates in terms of the maximum norm for the solution of (5.1). Eventually, we would be interested to prove the main theorem of the KL paper in the half space setting.

If we transform (5.1) into the abstract differential equations

$$u_t + Au = -\mathbb{P}(\nabla \cdot (u \otimes u)) \tag{5.2}$$

where $u \cdot \nabla u = \nabla \cdot (u \otimes u)$ and $A = -\mathbb{P}\triangle$ is the Stokes operator and $\mathbb{P}$ is the Leray projector. The Leray projector in halfspace has explicite form and is given by

$$\mathbb{P}g(x) = g(x) + \nabla_x \int_{\mathbb{R}^n_+} \nabla_y \mathcal{N}(x, y) \cdot g(y) dy,$$

when $g_n|_{x_n} = 0$. This integral needs to be treated carefully as per the the assumptions on g. Here,

$$\mathcal{N}(x, y) = G(x - y) + G(x - y^*),$$

where $y^* = (y_1, \cdots, y_{n-1}, -y_n), G(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$, if $n \geq 3$ and $G(x) = \frac{1}{2\pi} \ln |x|$ if $n = 2$, and ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. Using the solution operator of the Stokes equations in $\mathbb{R}^n_+$, the solution of (5.2) is formally expressed in the integral form

$$u(t) = e^{-At}f - \int_0^t e^{-A(t-s)}\mathbb{P}\nabla \cdot (u \otimes u)(s) ds. \tag{5.3}$$

Solonnikov [20] has expressed the solution operator of the Stokes equations in $\mathbb{R}^n_+$ in the integral form

$$e^{-At}f = \int_{\mathbb{R}^n_+} K(x, y, t) \cdot f(y) dy,$$

where $K = (K_{ij})_{i,j=1,2\cdots n}$ is defined by

$$
\begin{aligned}
K_{ij}(x,y,t) &= \delta_{ij}(\theta(x-y,t) - \theta(x-y^*,t)) \\
&\quad + 4(1-\delta_{jn}\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_i}G(x-z)\theta(z-y^*,t)dz. \qquad (5.4)
\end{aligned}
$$

where $\theta(x,t)$ is the n dimensional heat kernel given in section 3.

Our goal will be to prove the KL main theorem (namely theorem 4.10 on this paper) when the initial data is in $L^\infty(\mathbb{R}^n_+)$. More precisely, we try to obtain the estimate of the derivatives of the velocity in terms of the maximum norm for some maximum interval of time. To do that, it is necessary to have the pointwise estimate of the kernel of the operator $D^j e^{-At}\mathbb{P}\mathrm{div}$. Unfortunately, D^j and $e^{-At}\mathbb{P}$ are not commutative. This is the main difficulty in obtaining the desired result in half space case.

References

[1] H. Amann, On the strong solvability of the Navier-Stokes equations, J. Math. Fluid Mech. 2 (2000) 16-98.

[2] H-O. Bae and B.J. Jin Existence of strong mild solution of the Navier-Stokes equations in the half space with non-decaying initial data, J. Korean Math. Soc. 49 (2012), no. 1, 113-138.

[3] J.R Cannon, G.H Knightly, A note on the Cauchy problem for the Navier-Stokes equation,SIAM J. Appls. Math. 18 (1970) 641-644

[4] G.P. Galdi, P. Maremonti, A uniqueness theorem for viscous fluid motions in exterior domains, Arch. Rational Mech. Anal. 91 (1986) 375-384.

[5] Y.Giga, Weak and strong solutions of the Navier-Stokes Initial Value Problem , RIMS Kyoto University (19) 1983, pp. 887-910

[6] Y. Giga, K. Inui, J. Kato, S. Matsui, Remarks on the uniqueness of bounded solutions of the Navier- Stokes equations Hokkaido University Preprint Series in Mathematics, # 497, October 2000.

[7] Y. Giga, K. Inui, J. Kato and S. Matsui, Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations, Nonlinear Analysis, 47 (2001), 4151-4156.

[8] Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, Quaderni di Matematica 4 (1999), 28-68.

[9] R.Iorio, Jr,, & V. Iorio, (2001). Fourier Analysis and Partial Differential Equations (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511623745

[10] J. Kato, The uniqueness of nondecaying solutions for the Navier-Stokes equations. Arch. Rational Mech. Anal. 169 (2003), pp. 159-175

[11] T. Kato, Strong Lp-solutions of the Navier-Stokes equations in $\mathbb{R}^m$; with applications to weak solutions, Math. Z. 187 (1984) 471-480.

[12] H-O. Kreiss and J.Lorenz, Apriori Estimates in Terms of the Maximum Norm for the Solutions of the Navier-Stokes Equations. J. Diff. Eqn., 2004 pp. 216- 231

[13] H.-O. Kreiss and J. Lorenz, Initial-Boundary Value Problems and the Navier-Stokes Equations (Academic Press, London, 1989).

[14] N. Kim, D. Chae, On the uniqueness of the unbounded classical solutions of the Navier-Stokes and associated equations, J. Math. Anal. Appl. 186 (1994) 91-96.

[15] J. Leray, Essai sur le Mouvement D'un Fluide Visqueux Emplissant L'Espace, Acta Math. 63 (1934), pp. 193-248

[16] P. Michael, "A Study of the Pressure Term in the Navier-Stokes Equations." (2013). https://digitalrepository.unm.edu/ math_etds/39

[17] J.C. Robinson, J.L. Rodirog, W. Sadowski,The Three-Dimensional Navier-Stokes Equations: Classical Theory Volume 157 of Cambridge Studies in Advanced Mathemati ,Cambridge University Press, 2016

[18] E.M Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princton University Press, 1993

[19] M. Wiegner, The Navier-Stokes equations-a neverending challenge? DMV Jahresbericht, 101. Band Heft 1, pp. 1-25.

[20] V. A. Solonnikov, On nonstationary Stokes problem and Navier-Stokes problem in a half- space with initial data nondecreasing at infinity, Problemy Matematisheskogo Analiza,

[21] A. Uchiyama, A constructive proof of the Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$ Acta Mathematica 148: (1982) pp.215-241