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# Standard Neutrosophic Soft Theory: Some First Results

Bui Cong Cuong<sup>1</sup>, Pham Hong Phong<sup>2</sup>, and Florentin Smarandache<sup>3</sup>

<sup>1</sup> Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam. E-mail: bccuong@gmail.com

<sup>2</sup> Faculty of Information Technology, National University of Civil Engineering, 55 Giai Phong, Hanoi, Vietnam. E-mail: phongph@nuce.edu.vn

<sup>3</sup> Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA. E-mail: fsmarandache@gmail.com

**Abstract.** The traditional soft set is a mapping from a parameter set to family of all crisp subsets of a universe. Molodtsov introduced the soft set as a generalized tool for modelling complex systems involving uncertain or not clearly defined objects. In this paper, the notion of neutrosophic soft set is reanalysed. The novel theory is a combination of neutrosophic set theory and soft set

theory. The complement, “and”, “or”, intersection and union operations are defined on the neutrosophic soft sets. The neutrosophic soft relations accompanied with their compositions are also defined. The basic properties of the neutrosophic soft sets, neutrosophic soft relations and neutrosophic soft compositions are also discussed.

**Keywords:** Soft sets, Fuzzy soft sets, Intuitionistic fuzzy soft sets, Neutrosophic soft sets, Neutrosophic soft relations

## 1 Introduction

Uncertain data modelling is a complex problem appearing in many areas such as economics, engineering, environmental science, sociology and medical science. Some mathematical theories such as probability, fuzzy set [1], [2], intuitionistic fuzzy set [3], [4], rough set [5], [6], and the interval mathematics [7], [8] are useful approaches to describing uncertainty. However each of these theories has its inherent difficulties as mentioned by Molodtsov [9]. Soft set theory developed by Molodtsov [9] has become a new useful approach for handling vagueness and uncertainty.

Later, Maji et al. [10] introduced several basic operations of soft set theory and proved some related propositions on soft set operations. Ali et al. [11] analysed the incorrectness of some theorems in [10]. Then they proposed some new soft set operations and proved that De Morgan’s laws hold with these new definitions. Maji et al. also [12] gave an application of soft set theory in a decision making problem.

Above works are based on classical soft set. However, in practice, the objects may not precisely satisfy the problems’ parameters, thus Maji et al. [13] put forward the concept of fuzzy soft set by combining the fuzzy set and the soft set, then they [14] presented a theoretical approach of the fuzzy soft set in decision making problem. In [15], they considered the concept of intuitionistic fuzzy soft set. By combining the interval-valued fuzzy set and soft set, Yang et al. [16] proposed the interval-valued fuzzy soft set and then analyzed a decision making problem in the interval-valued fuzzy soft set. Yang et al [17] presented the concept of interval-valued intuitionistic fuzzy soft sets which is an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory.

From philosophical point of view, Smarandache’s neutrosophic set [26] generalizes fuzzy set and intuitionistic fuzzy set. However, it is difficult to apply it to the real applications and needs to be specified. Wang et al. [27] proposed interval neutrosophic sets and some operators of them. Wang et al. [28] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [29] defined the concept of simplified neutrosophic sets, which can be described by three real numbers in the real unit interval  $[0,1]$ , and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013 [18], we presented the definition of picture fuzzy sets, which is a generalization of the Zadeh’s fuzzy sets and Atanassov’s intuitionistic fuzzy sets, and some basic operations on picture fuzzy sets. In [18] we also discussed some properties of these operations, then the definition of the Cartesian product of picture fuzzy sets and the definition of picture fuzzy relations were given. Our picture fuzzy set turns out a special case of neutrosophic set. Thus, from now on, we also regard picture fuzzy set as standard neutrosophic set.

The purpose of this paper is to combine the standard neutrosophic sets and soft models, from which we can obtain neutrosophic soft sets. Intuitively, the neutrosophic soft set presented in this paper is an extension of the intuitionistic fuzzy soft sets [13][15].

The rest of this paper is organized as follows. Section 2 briefly reviews some background on soft sets, fuzzy soft sets, intuitionistic soft sets as well as neutrosophic set. In Section 3, we recall the concept of the standard

neutrosophic sets (SNSs) with some operations on SNSs, then we present the concept of neutrosophic soft sets (NSSs) with some operations. Some properties of these operations are discussed in the Sub-section 3.3. Sub-section 3.4 is devoted to the Cartesian product of NSSs. The neutrosophic soft relations are presented in Section 4. Finally, in Section 5, we draw the conclusion and present some topics for future research.

**2 Preliminaries**

In this section, we briefly recall the notions of soft sets, fuzzy soft sets, intuitionistic fuzzy soft sets as well as neutrosophic sets. See especially [9][10][13][15] for further details and background.

**2.1 Soft sets and some extensions**

Molodtsov [8] defined the soft set in the following way. Let  $U$  be an initial universe of objects and  $E$  be the set of related parameters of objects in  $U$ . Parameters are often attributes, characteristics, or properties of objects. Let  $P(U)$  denotes the power set of  $U$  and  $A \subseteq E$ .

**Definition 2.1.** [8] A pair  $(F, A)$  is called a soft set over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow P(U)$ .

In other words, the soft set is not a kind of set, but a parameterized family of subsets of  $U$  [9][10][16]. For any parameter  $e \in E$ ,  $F(e) \subseteq U$  is considered as the set of  $e$ -approximate elements of the soft set  $(F, A)$ .

Maji et al. [13] initiated the study on hybrid structures involving both fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of (crisp) soft set.

**Definition 2.2** [13] Let  $\mathcal{F}(U)$  be the set of all fuzzy subsets of  $U$ ,  $E$  be the set of parameters and  $A \subseteq E$ . A pair  $(F, A)$  is called a fuzzy soft set over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow \mathcal{F}(U)$ .

It is easy to see that every (crisp) soft set can be considered as a fuzzy soft set. Generally speaking, for any parameter  $e \in E$ ,  $F(e)$  is a fuzzy subset of  $U$  and it is called fuzzy value set of parameter  $e$ . If for any parameter  $e \in A$ ,  $F(e)$  is a subset of  $U$ , then  $(F, A)$  is degenerated to the standard soft set. For all  $x \in U$  and  $e \in E$ , let us denote by  $\mu_{F(e)}(x)$  the membership degree that the object  $x$  holds parameter  $e$ . So then  $F(e)$  can be written as

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x) \right\rangle \mid x \in U \right\}.$$

Before introduce the notion of the intuitionistic fuzzy soft set, let us recall the concept of intuitionistic fuzzy set [3], [4].

Let  $X$  be a fixed set. An intuitionistic fuzzy set (IFS) in  $X$  is an object having the form

$$A = \left\{ \left\langle x, \mu_A(x), \nu_A(x) \right\rangle \mid x \in X \right\},$$

where  $\mu_A(x) \in [0, 1]$  and  $\nu_A(x) \in [0, 1]$  respectively define the degree of membership and the degree of non-membership of the element  $x$  to the set  $A$  such that  $\mu_A(x) + \nu_A(x) \leq 1$  for all  $x \in X$ . The set of all IFSs on  $X$  is denoted by  $IFS(X)$ .

In [15] Maji et al. proposed the concept of intuitionistic fuzzy soft set as follows.

**Definition 2.3** [15] Let  $E$  the set of parameters and  $A \subseteq E$ . A pair  $(F, A)$  is called a intuitionistic fuzzy soft set over  $U$ , where  $F$  is a mapping  $F : A \rightarrow IFS(U)$ .

Clearly, for any parameter  $e \in E$ ,  $F(e)$  is an IFS

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x) \right\rangle \mid x \in U \right\},$$

where  $\mu_{F(e)}$  and  $\nu_{F(e)}$  are the membership and non-membership functions, respectively. If for any parameter  $e \in A$ ,  $\nu_{F(e)}(x) = 1 - \mu_{F(e)}(x)$ , then  $F(e)$  is a fuzzy set and  $(F, A)$  is reduced to a fuzzy soft set.

**2.2 Neutrosophic sets**

**Definition 2.4** [26] A neutrosophic set  $A$  in a on a universe  $X$  is characterized by a truth-membership function  $T_A$ , an indeterminacy-membership function  $I_A$  and a falsity-membership function  $F_A$ . For each  $x \in X$ ,  $T_A(x)$ ,  $I_A(x)$  and  $F_A(x)$  are real standard or nonstandard subsets of  $]0^-, 1+[^+$ , that is  $T_A, I_A$  and  $F_A : X \rightarrow ]0^-, 1+[^+$ .

There is no restriction on the sum of  $T_A(x)$ ,  $I_A(x)$  and  $F_A(x)$ , so  $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ , for all  $x \in X$ .

**Definition 2.5** [26] The complement of a neutrosophic set  $A$  is denoted by  $A^c$  and is defined as  $T_{A^c}(x) = \{1^+\} \odot T_A(x)$ ,  $I_{A^c}(x) = \{1^+\} \odot I_A(x)$ , and  $F_{A^c}(x) = \{1^+\} \odot F_A(x)$  for every  $x$  in  $X$ .

**Definition 2.6** [26] A neutrosophic set  $A$  is contained in the other neutrosophic set  $B$ ,  $A \subseteq B$  if and only if

$$\inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x),$$

$$\inf I_A(x) \geq \inf I_B(x), \sup I_A(x) \geq \sup I_B(x),$$

$\inf F_A(x) \geq \inf F_B(x)$ , and  $\sup F_A(x) \geq \sup F_B(x)$  for every  $x$  in  $X$ .

**Definition 2.7** [26] The union of two neutrosophic sets  $A$  and  $B$  is a neutrosophic set  $C$ , written as  $C = A \cup B$ , whose truth-membership, indeterminacy membership and false-membership functions are related to those of  $A$  and  $B$  by

$$T_C(x) = T_A(x) \oplus T_B(x) \ominus T_A(x) \odot T_B(x),$$

$$I_C(x) = I_A(x) \oplus I_B(x) \ominus I_A(x) \odot I_B(x), \text{ and}$$

$F_C(x) = F_A(x) \oplus F_B(x) \ominus F_A(x) \odot F_B(x)$  for any  $x$  in  $X$ .

**Definition 2.8** [1] The intersection of two neutrosophic sets  $A$  and  $B$  is a neutrosophic set  $C$ , written as  $C = A \cap B$ , whose truth-membership, indeterminacy-membership and false-membership functions are related to those of  $A$  and  $B$  by  $T_C(x) = T_A(x) \odot T_B(x)$ ,  $I_C(x) = I_A(x) \odot I_B(x)$ , and  $F_C(x) = F_A(x) \odot F_B(x)$  for any  $x$  in  $X$ .

**Definition 2.9** [29] Consider a neutrosophic set  $A$  in  $X$  characterized by a truth-membership function  $T_A$ , a indeterminacy-membership function  $I_A$  and a falsity-membership function  $F_A$ . If  $T_A(x)$ ,  $I_A(x)$  and  $F_A(x)$  are singleton values in the real standard  $[0,1]$  for every  $x$  in  $X$ , that is  $T_A, I_A$  and  $F_A : X \rightarrow [0,1]$ . Then, a simplification of the neutrosophic set  $A$  is denoted by

$$A = \left\{ \left\langle x, T_A(x), I_A(x), F_A(x) \right\rangle \mid x \in X \right\},$$

which is called a simplified neutrosophic set.

### 3 Neutrosophic soft sets

In this section, first we recall the definition of the standard neutrosophic sets (SNSs), some basic operations with their properties, then we will present the neutrosophic soft set theory which is a combination of neutrosophic set theory and a soft set theory.

#### 3.1 Standard neutrosophic sets

Intuitionistic fuzzy sets introduced by Atanassov in 1983 constitute a generalization of fuzzy sets (FS) [3]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership of an element in a given set.

A generalization of fuzzy sets and intuitionistic fuzzy sets are the following notion of standard neutrosophic set (SNS).

**Definition 3.1** [18] A SNS  $A$  on a universe  $X$  is an object of the form

$$A = \left\{ \left\langle x, \mu_A(x), \eta_A(x), \nu_A(x) \right\rangle \mid x \in X \right\},$$

where  $\mu_A(x) \in [0,1]$  is called the “degree of positive membership of  $x$  in  $A$ ”,  $\eta_A(x) \in [0,1]$  is called the “degree of neutral membership of  $x$  in  $A$ ” and  $\nu_A(x) \in [0,1]$  is called the “degree of negative membership of  $x$  in  $A$ ”, and  $\mu_A, \eta_A$  and  $\nu_A$  satisfy the

following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X.$$

The expression  $(1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)))$  is termed as “degree of refusal membership” of  $x$  in  $A$ .

Basically, SNSs based models may be adequate in situations when we face human opinions involving more answers of type: *yes, abstain, no* and *refusal*. Voting can be a good example of such a situation as the voters are divided into four groups: *vote for, abstain, vote against* and *refusal of the voting*.

Let  $SNS(X)$  denote the set of all the standard neutrosophic set SNSs on a universe  $X$ .

**Definition 3.2** [18] For  $A, B \in SNS(X)$ , the union, intersection and complement are defined as follows:

- $A \subseteq B \Leftrightarrow \begin{cases} \mu_A(x) \leq \mu_B(x) \\ \eta_A(x) \leq \eta_B(x), \forall x \in X; \\ \nu_A(x) \geq \nu_B(x) \end{cases}$

- $A = B \Leftrightarrow \begin{cases} A \subseteq B; \\ B \subseteq A; \end{cases}$

- $A \cup B \in SNS(X)$  with

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)),$$

$$\eta_{A \cup B}(x) = \min(\eta_A(x), \eta_B(x)), \text{ and}$$

$$\nu_{A \cup B}(x) = \min(\nu_A(x), \nu_B(x)), \forall x \in X;$$

- $A \cap B \in SNS(X)$  with

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)),$$

$$\eta_{A \cap B}(x) = \min(\eta_A(x), \eta_B(x)), \text{ and}$$

$$\nu_{A \cap B}(x) = \max(\nu_A(x), \nu_B(x)), \forall x \in X;$$

- $CoA = A^c = \left\{ \left\langle x, \nu_A(x), \eta_A(x), \mu_A(x) \right\rangle \mid x \in X \right\}.$

In this paper, we denote  $a \wedge b = \min(a, b)$  and  $a \vee b = \max(a, b)$ , for every  $a, b \in \mathbb{R}$ .

**Definition 3.3** [18] Let  $X, Y$  be two universes and  $A \in SNS(X), B \in SNS(Y)$ . We define the Cartesian product of these two SNSs by  $A \times B \in SNS(X \times Y)$  such that

$$\mu_{A \times B}(x, y) = \mu_A(x) \wedge \mu_B(y),$$

$$\eta_{A \times B}(x, y) = \eta_A(x) \wedge \eta_B(y), \text{ and}$$

$$v_{A \times B}(x, y) = v_A(x) \vee v_B(y), \forall (x, y) \in X \times Y.$$

The validation of Definition 3.3 was shown in [18]. Now we consider some properties of the defined operations on SNSs.

**Proposition 3.4** [18] For every  $A, B, C \in SNS(X)$  :

- (a) If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ ;
- (b)  $(A^c)^c = A$ ;
- (c) Operations  $\cap$  and  $\cup$  are commutative, associative and distributive;
- (d) Operations  $\cap, Co$  and  $\cup$  satisfy the law of De Morgan.

**Proof.** See [19][20] for detail proof. □

Convex combination is an important operation in mathematics, which is a useful tool on convex analysis, linear spaces and convex optimization. In this sub-section convex combination firstly is defined with some simple propositions.

**Definition 3.5** [18] Let  $A, B \in SNS(X)$ . For each  $\theta \in [0, 1]$ , the convex combination of  $A$  and  $B$  is defined as follows:

$$C_\theta(A, B) = \left\{ (x, \mu_{C_\theta}(x), \eta_{C_\theta}(x), \nu_{C_\theta}(x)) \mid x \in X \right\},$$

where

$$\begin{aligned} \mu_{C_\theta}(x) &= \theta \mu_A(x) + (1 - \theta) \mu_B(x), \\ \eta_{C_\theta}(x) &= \theta \eta_A(x) + (1 - \theta) \eta_B(x), \text{ and} \\ \nu_{C_\theta}(x) &= \theta \nu_A(x) + (1 - \theta) \nu_B(x), \forall x \in X. \end{aligned}$$

**Proposition 3.6** [18] Let  $A, B \in SNS(X)$  and  $\theta, \theta_1, \theta_2 \in [0, 1]$ , then

- If  $\theta = 1$ , then  $C_\theta(A, B) = A$ ; and if  $\theta = 0$ , then  $C_\theta(A, B) = B$ ;
- If  $A \subseteq B$ , then  $A \subseteq C_\theta(A, B) \subseteq B$ ;
- If  $B \subseteq A$  and  $\theta_1 \leq \theta_2$ , then  $C_{\theta_1}(A, B) \subseteq C_{\theta_2}(A, B)$ .

### 3.2 Neutrosophic soft sets

**Definition 3.7** Let  $SNS(U)$  be the set of all standard neutrosophic sets of  $U$ ,  $E$  be the set of parameters and  $A \subseteq E$ . A pair  $(F, A)$  is called a standard neutrosophic

soft set (or neutrosophic soft set for short) over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow SNS(U)$ .

Clearly, for any parameter  $e \in E$ ,  $F(e)$  is a SNS:

$$F(e) = \left\{ (x, \mu_{F(e)}(x), \eta_{F(e)}(x), \nu_{F(e)}(x)) \mid x \in U \right\},$$

where  $\mu_{F(e)}$ ,  $\eta_{F(e)}$  and  $\nu_{F(e)}$  are positive membership, neutral membership and negative membership functions respectively. If for all parameter  $e \in A$  and for all  $x \in U$ ,  $\eta_{F(e)}(x) = 0$ , then  $F(e)$  will degenerated to be an intuitionistic fuzzy set and then  $(F, A)$  is degenerated to an intuitionistic fuzzy soft set.

We denote the set of all standard neutrosophic soft sets over  $U$  by  $SNS(U)$ .

**Example 1.** We consider the situation which involves four economic projects evaluated by a decision committee according to five parameters: *good finance indicator* ( $e_1$ ), *average finance indicator* ( $e_2$ ), *good social contribution* ( $e_3$ ), *average social contribution* ( $e_4$ ) and *good environment indicator* ( $e_5$ ). The set of economic projects and the set of parameters are denoted  $U = \{p_1, p_2, p_3, p_4\}$  and  $A = \{e_1, e_2, e_3, e_4, e_5\}$ , respectively. So, the attractiveness of the projects to the decision committee can be represented by a SNS  $(F, A)$  :

$$\begin{aligned} F(e_1) &= \left\{ (p_1, 0.8, 0.12, 0.05), (p_2, 0.6, 0.18, 0.16), \right. \\ &\quad \left. (p_3, 0.55, 0.20, 0.21), (p_4, 0.50, 0.20, 0.24) \right\}, \\ F(e_2) &= \left\{ (p_1, 0.82, 0.05, 0.10), (p_2, 0.7, 0.12, 0.10), \right. \\ &\quad \left. (p_3, 0.60, 0.14, 0.10), (p_4, 0.51, 0.10, 0.24) \right\}, \\ F(e_3) &= \left\{ (p_1, 0.60, 0.14, 0.16), (p_2, 0.55, 0.20, 0.16), \right. \\ &\quad \left. (p_3, 0.70, 0.15, 0.11), (p_4, 0.63, 0.12, 0.18) \right\}, \\ F(e_4) &= \left\{ (p_1, 0.7, 0.12, 0.07), (p_2, 0.75, 0.05, 0.16), \right. \\ &\quad \left. (p_3, 0.60, 0.17, 0.18), (p_4, 0.55, 0.10, 0.22) \right\}, \\ F(e_5) &= \left\{ (p_1, 0.60, 0.12, 0.07), (p_2, 0.62, 0.14, 0.16), \right. \\ &\quad \left. (p_3, 0.55, 0.10, 0.21), (p_4, 0.70, 0.20, 0.05) \right\}. \end{aligned}$$

The standard neutrosophic soft set  $(F, A)$  is a parameterized family  $\{F(e_i) \mid i = 1, \dots, 5\}$  of standard neutrosophic sets over  $U$ .

**Definition 3.8** 1) For  $(F, A), (G, B) \in SNS(U)$  over a common universe  $U$ , we say that  $(F, A)$  is a subset of  $(G, B)$ ,  $(F, A) \subseteq (G, B)$ , if the following conditions are satisfied:

(a)  $A \subseteq B$ ;

(b) For all  $e \in A$ ,  $F(e)$  and  $G(e)$  are identical approximations.

2)  $(F, A)$  is termed as a superset of  $(G, B)$ ,  $(F, A) \supseteq (G, B)$ , if  $(G, B)$  is a subset of  $(F, A)$ .

3)  $(F, A)$  and  $(G, B)$  are called to be equal,  $(F, A) = (G, B)$ , if  $(F, A) \subseteq (G, B)$  and  $(G, B) \subseteq (F, A)$ .

It is easy to show that  $(F, A) = (G, B)$  iff  $A = B$  and  $F(e) = G(e)$  for all  $e \in A$ .

**3.3 Some operations and properties**

Now we define some operations on standard neutrosophic soft sets and present some properties.

**Definition 3.9** The complement of a NSS  $(F, A)$ ,  $(F, A)^c$ , is defined by  $(F, A)^c = (F^c, A)$ , where  $F^c : A \rightarrow P(U)$  is a mapping given by  $F^c(e) = (F(e))^c$ , for all  $e \in A$ .

**Definition 3.10** If  $(F, A), (G, B) \in NSS(U)$ , then “ $(F, A)$  and  $(G, B)$ ” is a NSS denoted by  $(F, A) \wedge (G, B)$  and defined by  $(F, A) \wedge (G, B) = (H, A \times B)$ , where  $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$  for all  $(\alpha, \beta) \in A \times B$ , that is

$$\begin{aligned} \mu_{H(\alpha, \beta)}(x) &= \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)), \\ \eta_{H(\alpha, \beta)}(x) &= \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x)), \text{ and} \\ \nu_{H(\alpha, \beta)}(x) &= \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)), \forall x \in U. \end{aligned}$$

**Definition 3.11** If  $(F, A), (G, B) \in NSS(U)$ , then “ $(F, A)$  or  $(G, B)$ ” is a NSS denoted by  $(F, A) \vee (G, B)$  and defined by  $(F, A) \vee (G, B) = (H, A \times B)$ , where  $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$  for all  $(\alpha, \beta) \in A \times B$ , that is

$$\begin{aligned} \mu_{H(\alpha, \beta)}(x) &= \max(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)), \\ \eta_{H(\alpha, \beta)}(x) &= \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x)), \text{ and} \end{aligned}$$

$$\nu_{H(\alpha, \beta)}(x) = \min(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)), \forall x \in U.$$

**Theorem 3.1** Let  $(F, A), (G, B) \in NSS(U)$ , then we have the following properties:

- (1)  $((F, A) \wedge (G, B))^c = (F, A)^c \vee (G, B)^c$ ;
- (2)  $((F, A) \vee (G, B))^c = (F, A)^c \wedge (G, B)^c$ .

**Proof.** (1) Assume that  $(F, A) \wedge (G, B) = (H, A \times B)$ . Then

$$((F, A) \wedge (G, B))^c = (H, A \times B)^c = (H^c, A \times B).$$

For any  $(\alpha, \beta) \in A \times B, x \in U$ , we have

$$\begin{aligned} H(\alpha, \beta)(x) &= \left( \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)), \right. \\ &\quad \left. \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x)), \right. \\ &\quad \left. \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)) \right), \end{aligned}$$

which implies

$$\begin{aligned} H^c(\alpha, \beta)(x) &= \left( \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)), \right. \\ &\quad \left. \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x)), \right. \\ &\quad \left. \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)) \right). \end{aligned} \tag{1}$$

On the other hand,

$$(F, A)^c \vee (G, B)^c = (F^c, A) \vee (G^c, B).$$

Let us assume that  $(F^c, A) \vee (G^c, B) = (K, A \times B)$ . We obtain

$$\begin{aligned} K(\alpha, \beta)(x) &= \left( \max(\mu_{F^c(\alpha)}(x), \mu_{G^c(\beta)}(x)), \right. \\ &\quad \left. \min(\eta_{F^c(\alpha)}(x), \eta_{G^c(\beta)}(x)), \right. \\ &\quad \left. \min(\nu_{F^c(\alpha)}(x), \nu_{G^c(\beta)}(x)) \right). \end{aligned}$$

Since  $\mu_{F^c(\alpha)} = \nu_{F(\alpha)}, \eta_{F^c(\alpha)} = \eta_{F(\alpha)}, \nu_{F^c(\alpha)} = \mu_{F(\alpha)},$   
 $\mu_{G^c(\beta)} = \nu_{G(\beta)}, \eta_{G^c(\beta)} = \eta_{G(\beta)}, \nu_{G^c(\beta)} = \mu_{G(\beta)},$

$$\begin{aligned} K(\alpha, \beta)(x) &= \left( \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)), \right. \\ &\quad \left. \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x)), \right. \\ &\quad \left. \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)) \right). \end{aligned} \tag{2}$$

Combining (1) and (2), the proof is completed.

(2) The proof is similar to (1). □

**Theorem 3.2** Let  $(F, A), (G, B), (H, C) \in NSS(U)$ , then we have the following properties:

- a)  $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$  ;
- b)  $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$  .

**Proof.** (1) Assume that

$$(G, B) \wedge (H, C) = (I, B \times C),$$

We have

$$I(\beta, \gamma)(x) = \left( \min(\mu_{G(\beta)}(x), \mu_{H(\gamma)}(x)), \min(\eta_{G(\beta)}(x), \eta_{H(\gamma)}(x)), \max(\nu_{G(\beta)}(x), \nu_{H(\gamma)}(x)) \right),$$

$$\forall (\beta, \gamma) \in B \times C, x \in U.$$

We assume that

$$(F, A) \wedge ((G, B) \wedge (H, C)) = (K, A \times B \times C).$$

In other words,

$$(K, A \times B \times C) = (F, A) \wedge (I, B \times C).$$

By definition of  $\wedge$  operator for two NSSs,

$$K(\alpha, \beta, \gamma)(x) = \left( \min(\mu_{F(\alpha)}(x), \min(\mu_{G(\beta)}(x), \mu_{H(\gamma)}(x))), \min(\eta_{F(\alpha)}(x), \min(\eta_{G(\beta)}(x), \eta_{H(\gamma)}(x))), \max(\nu_{F(\alpha)}(x), \max(\nu_{G(\beta)}(x), \nu_{H(\gamma)}(x))) \right),$$

or

$$K(\alpha, \beta, \gamma)(x) = \left( \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x), \mu_{H(\gamma)}(x)), \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(x), \eta_{H(\gamma)}(x)), \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x), \nu_{H(\gamma)}(x)) \right).$$

By a similar argument, we get

$$((F, A) \wedge (G, B)) \wedge (H, C) = (K, A \times B \times C).$$

This concludes the proof of a).

The proof of b) is analogous. □

**Definition 3.12** The intersection of two NSSs  $(F, A), (G, B) \in NSS(U)$ , denoted by  $(F, A) \cap (G, B)$ , is a NSSs  $(H, C)$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases} \quad (3)$$

**Definition 3.13** The union of two NSSs  $(F, A), (G, B) \in NSS(U)$ , denoted by  $(F, A) \cup (G, B)$ , is a NSSs  $(H, C)$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases} \quad (3)$$

**Theorem 3.3.** Let  $(F, A), (G, B) \in NSS(U)$ , then we have the following properties:

- a)  $((F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c$  ;
- b)  $((F, A) \cup (G, B))^c = (F, A)^c \cap (G, B)^c$  .

**Proof.** a) Assume that  $(F, A) \cap (G, B) = (H, C)$ , with  $C = A \cup B$ , then

$$((F, A) \cap (G, B))^c = (H, C)^c = (H^c, C).$$

By Definition 3.13,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

It implies

$$H^c(e) = \begin{cases} F^c(e) & \text{if } e \in A \setminus B \\ G^c(e) & \text{if } e \in B \setminus A \\ F^c(e) \cup G^c(e) & \text{if } e \in A \cap B \end{cases} \quad (5)$$

Similarly, we denote  $(F, A)^c \cup (G, B)^c = (K, C)$  with  $C = A \cup B$ . Since  $(K, C) = (F^c, A) \cup (G^c, B)$ ,

$$K(e) = \begin{cases} F^c(e) & \text{if } e \in A \setminus B \\ G^c(e) & \text{if } e \in B \setminus A \\ F^c(e) \cup G^c(e) & \text{if } e \in A \cap B \end{cases} \quad (6)$$

From (5) and (6), we get  $H^c = K$ . Hence,

$$((F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c.$$

b) Similarly, we have b). □

### 3.4 Cartesian product of neutrosophic soft sets

**Definition 3.14** Let  $O_1 \in SNS(X_1)$  and  $O_2 \in SNS(X_2)$ . The Cartesian product of these two NSSs is  $O_1 \times O_2 \in SNS(X_1 \times X_2)$  defined as

$$\begin{aligned} \mu_{O_1 \times O_2}(x, y) &= \mu_{O_1}(x) \wedge \mu_{O_2}(y), \\ \eta_{O_1 \times O_2}(x, y) &= \eta_{O_1}(x) \wedge \eta_{O_2}(y), \text{ and} \\ \nu_{O_1 \times O_2}(x, y) &= \nu_{O_1}(x) \vee \nu_{O_2}(y), \forall (x, y) \in X_1 \times X_2. \end{aligned}$$

It is easy to check the validation of Definition 3.15.

**Theorem 3.4** For  $O_1, O_2 \in SNS(X_1), O_3 \in SNS(X_2)$ , and  $O_4 \in SNS(X_3)$ :

- a)  $O_1 \times O_3 = O_3 \times O_1$ ;
- b)  $(O_1 \times O_3) \times O_4 = O_1 \times (O_3 \times O_4)$ ;
- c)  $(O_1 \cup O_2) \times O_3 = (O_1 \times O_3) \cup (O_2 \times O_3)$ ;
- d)  $(O_1 \cap O_2) \times O_3 = (O_1 \times O_3) \cap (O_2 \times O_3)$ .

**Proof.** a) and b) are straightforward. We consider c) and d).

c) We have

$$\begin{aligned} \mu_{O_1 \cup O_2}(x) &= \mu_{O_1}(x) \vee \mu_{O_2}(x), \\ \eta_{O_1 \cup O_2}(x) &= \eta_{O_1}(x) \wedge \eta_{O_2}(x), \text{ and} \\ \nu_{O_1 \cup O_2}(x) &= \nu_{O_1}(x) \wedge \nu_{O_2}(x), \forall x \in X_1. \end{aligned}$$

Thus,

$$\begin{aligned} \mu_{(O_1 \cup O_2) \times O_3}(x, y) &= (\mu_{O_1}(x) \vee \mu_{O_2}(x)) \wedge \mu_{O_3}(y), \\ \eta_{(O_1 \cup O_2) \times O_3}(x, y) &= (\eta_{O_1}(x) \wedge \eta_{O_2}(x)) \wedge \eta_{O_3}(y), \text{ and} \\ \nu_{(O_1 \cup O_2) \times O_3}(x, y) &= (\nu_{O_1}(x) \wedge \nu_{O_2}(x)) \vee \nu_{O_3}(y), \\ &\forall (x, y) \in X_1 \times X_2. \end{aligned}$$

Using the properties of the operations  $\wedge$  and  $\vee$  we obtain

$$\begin{aligned} \mu_{(O_1 \cup O_2) \times O_3}(x, y) &= (\mu_{O_1}(x) \wedge \mu_{O_3}(y)) \vee (\mu_{O_2}(x) \wedge \mu_{O_3}(y)) \\ &= \mu_{O_1 \times O_3}(x, y) \vee \mu_{O_2 \times O_3}(x, y) \\ &= \mu_{(O_1 \times O_3) \cup (O_2 \times O_3)}(x, y), \\ \eta_{(O_1 \cup O_2) \times O_3}(x, y) &= (\eta_{O_1}(x) \wedge \eta_{O_3}(y)) \wedge (\eta_{O_2}(x) \wedge \eta_{O_3}(y)) \\ &= \eta_{O_1 \times O_3}(x, y) \wedge \eta_{O_2 \times O_3}(x, y) \\ &= \eta_{(O_1 \times O_3) \cap (O_2 \times O_3)}(x, y), \\ \nu_{(O_1 \cup O_2) \times O_3}(x, y) &= (\nu_{O_1}(x) \vee \nu_{O_3}(y)) \wedge (\nu_{O_2}(x) \vee \nu_{O_3}(y)) \\ &= \nu_{O_1 \times O_3}(x, y) \wedge \nu_{O_2 \times O_3}(x, y) \\ &= \nu_{(O_1 \times O_3) \cap (O_2 \times O_3)}(x, y), \forall (x, y) \in X_1 \times X_2. \end{aligned}$$

The proof is given.

The proof of d) is analogous. □

Now we give the definition of the Cartesian product of neutrosophic soft sets.

**Definition 3.15** Let  $X_1, X_2$  be two universes,  $E$  be the set of parameters,  $A, B \subseteq E$ . Then the Cartesian product of  $\langle F, A \rangle \in NSS(X_1)$  and  $\langle G, B \rangle \in NSS(X_2)$  is denoted by  $\langle F, A \rangle \times \langle G, B \rangle$  and defined by  $\langle H, A \times B \rangle$ , where

$$\begin{aligned} H(\alpha, \beta)(x, y) &= \left( \min(\mu_{F(\alpha)}(x), \mu_{G(\beta)}(y)), \right. \\ &\quad \min(\eta_{F(\alpha)}(x), \eta_{G(\beta)}(y)), \\ &\quad \left. \max(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(y)) \right), \end{aligned}$$

$$\forall (\alpha, \beta) \in A \times B, \forall (x, y) \in X_1 \times X_2.$$

**Theorem 3.5** Let  $X_1, X_2, X_3$  be three universes,  $E$  be the set of parameters,  $A_1, A_2, B, D \subseteq E$ . For  $\langle F_1, A_1 \rangle, \langle F_2, A_2 \rangle \in NSS(X_1), \langle G, B \rangle \in NSS(X_2)$  and  $\langle H, D \rangle \in NSS(X_3)$ , we have:

- a)  $\langle F_1, A_1 \rangle \times \langle G, B \rangle = \langle G, B \rangle \times \langle F_1, A_1 \rangle$ ;
- b)  $(\langle F_1, A_1 \rangle \times \langle G, B \rangle) \times \langle H, D \rangle = \langle F_1, A_1 \rangle \times (\langle G, B \rangle \times \langle H, D \rangle)$ ;
- c)  $(\langle F_1, A_1 \rangle \cup \langle F_2, A_2 \rangle) \times \langle G, B \rangle = (\langle F_1, A_1 \rangle \times \langle G, B \rangle) \cup (\langle F_2, A_2 \rangle \times \langle G, B \rangle)$ ;
- d)  $(\langle F_1, A_1 \rangle \cap \langle F_2, A_2 \rangle) \times \langle G, B \rangle = (\langle F_1, A_1 \rangle \times \langle G, B \rangle) \cap (\langle F_2, A_2 \rangle \times \langle G, B \rangle)$ ;



$$= (\langle F_1, A_1 \rangle \times \langle G, B \rangle) \cap (\langle F_2, A_2 \rangle \times \langle G, B \rangle).$$

**Proof.** The proof of a) and b) is omitted.

c) Use Definition 3.14, if  $\langle F', A_1 \cup A_2 \rangle = \langle F_1, A_1 \rangle \cup \langle F_2, A_2 \rangle$ , then for all  $\alpha \in A_1 \cup A_2$  :

$$H'(\alpha) = \begin{cases} F_1(\alpha) & \text{if } \alpha \in A_1 \setminus A_2 \\ F_2(\alpha) & \text{if } \alpha \in A_2 \setminus A_1 \\ F_1(\alpha) \cup F_2(\alpha) & \text{if } \alpha \in A_1 \cap A_2 \end{cases}.$$

Let assume that  $\langle K, (A_1 \cup A_2) \times B \rangle = \langle F', A_1 \cup A_2 \rangle \times \langle G, B \rangle$ .

For all  $(x, y) \in X_1 \times X_2$ , there are following three cases :

\* Case 1:  $(\alpha, \beta) \in (A_1 \setminus A_2) \times B$ .

$$\begin{aligned} \mu_{K(\alpha, \beta)}(x, y) &= \min(\mu_{F_1(\alpha)}(x), \mu_{G(\beta)}(y)), \\ \eta_{K(\alpha, \beta)}(x, y) &= \min(\eta_{F_1(\alpha)}(x), \eta_{G(\beta)}(y)), \text{ and} \\ \nu_{K(\alpha, \beta)}(x, y) &= \max(\nu_{F_1(\alpha)}(x), \nu_{G(\beta)}(y)). \end{aligned}$$

\* Case 2:  $(\alpha, \beta) \in (A_2 \setminus A_1) \times B$ .

$$\begin{aligned} \mu_{K(\alpha, \beta)}(x, y) &= \min(\mu_{F_2(\alpha)}(x), \mu_{G(\beta)}(y)), \\ \eta_{K(\alpha, \beta)}(x, y) &= \min(\eta_{F_2(\alpha)}(x), \eta_{G(\beta)}(y)), \text{ and} \\ \nu_{K(\alpha, \beta)}(x, y) &= \max(\nu_{F_2(\alpha)}(x), \nu_{G(\beta)}(y)). \end{aligned}$$

\* Case 3:  $(\alpha, \beta) \in (A_2 \cap A_1) \times B$ .

$$\begin{aligned} \mu_{K(\alpha, \beta)}(x, y) &= \min(\mu_{F_1(\alpha) \cup F_2(\alpha)}(x), \mu_{G(\beta)}(y)) \\ &= \min(\max(\mu_{F_1(\alpha)}(x), \mu_{F_2(\alpha)}(x)), \mu_{G(\beta)}(y)) \\ &= \max(\min(\mu_{F_1(\alpha)}(x), \mu_{G(\beta)}(y)), \min(\mu_{F_2(\alpha)}(x), \mu_{G(\beta)}(y))), \\ \eta_{K(\alpha, \beta)}(x, y) &= \min(\eta_{F_1(\alpha) \cup F_2(\alpha)}(x), \eta_{G(\beta)}(y)) \\ &= \min(\min(\eta_{F_1(\alpha)}(x), \eta_{F_2(\alpha)}(x)), \eta_{G(\beta)}(y)) \\ &= \min(\eta_{F_1(\alpha)}(x), \eta_{F_2(\alpha)}(x), \eta_{G(\beta)}(y)), \text{ and} \\ \nu_{K(\alpha, \beta)}(x, y) &= \max(\nu_{F_1(\alpha) \cup F_2(\alpha)}(x), \nu_{G(\beta)}(y)) \\ &= \max(\min(\nu_{F_1(\alpha)}(x), \nu_{F_2(\alpha)}(x)), \nu_{G(\beta)}(y)) \end{aligned}$$

$$= \min(\max(\nu_{F_1(\alpha)}(x), \nu_{G(\beta)}(y)), \max(\nu_{F_2(\alpha)}(x), \nu_{G(\beta)}(y))).$$

Let us denote  $\langle H_1, A_1 \times B \rangle = \langle F_1, A_1 \rangle \times \langle G, B \rangle$  and  $\langle H_2, A_2 \times B \rangle = \langle F_2, A_2 \rangle \times \langle G, B \rangle$ . We have:

$$\mu_{H_1(\alpha, \beta)}(x, y) = \min(\mu_{F_1(\alpha)}(x), \mu_{G(\beta)}(y)),$$

$$\eta_{H_1(\alpha, \beta)}(x, y) = \min(\eta_{F_1(\alpha)}(x), \eta_{G(\beta)}(y)),$$

$$\nu_{H_1(\alpha, \beta)}(x, y) = \max(\nu_{F_1(\alpha)}(x), \nu_{G(\beta)}(y)),$$

$$\forall (\alpha, \beta) \in A_1 \times B \text{ and } (x, y) \in X_1 \times X_2;$$

$$\mu_{H_2(\alpha, \beta)}(x, y) = \min(\mu_{F_2(\alpha)}(x), \mu_{G(\beta)}(y)),$$

$$\eta_{H_2(\alpha, \beta)}(x, y) = \min(\eta_{F_2(\alpha)}(x), \eta_{G(\beta)}(y)),$$

$$\nu_{H_2(\alpha, \beta)}(x, y) = \max(\nu_{F_2(\alpha)}(x), \nu_{G(\beta)}(y)),$$

$$\forall (\alpha, \beta) \in A_2 \times B \text{ and } (x, y) \in X_1 \times X_2.$$

We consider,

$$\langle K', (A_1 \times B) \cup (A_2 \times B) \rangle = \langle H_1, A_1 \times B \rangle \cup \langle H_2, A_2 \times B \rangle.$$

Again, we have following three cases:

\* Case 1:  $(\alpha, \beta) \in (A_1 \times B) \setminus (A_2 \times B) = (A_1 \setminus A_2) \times B$ . We have:

$$\begin{aligned} \mu_{K'(\alpha, \beta)}(x, y) &= \mu_{H_1(\alpha, \beta)}(x, y) \\ &= \min(\mu_{F_1(\alpha)}(x), \mu_{G(\beta)}(y)); \\ \eta_{K'(\alpha, \beta)}(x, y) &= \eta_{H_1(\alpha, \beta)}(x, y) \\ &= \min(\eta_{F_1(\alpha)}(x), \eta_{G(\beta)}(y)); \\ \nu_{K'(\alpha, \beta)}(x, y) &= \nu_{H_1(\alpha, \beta)}(x, y) \\ &= \max(\nu_{F_1(\alpha)}(x), \nu_{G(\beta)}(y)). \end{aligned}$$

\* Case 2:  $(\alpha, \beta) \in (A_2 \times B) \setminus (A_1 \times B) = (A_2 \setminus A_1) \times B$ .

$$\begin{aligned} \mu_{K'(\alpha, \beta)}(x, y) &= \mu_{H_2(\alpha, \beta)}(x, y) \\ &= \min(\mu_{F_2(\alpha)}(x), \mu_{G(\beta)}(y)); \\ \eta_{K'(\alpha, \beta)}(x, y) &= \eta_{H_2(\alpha, \beta)}(x, y) \end{aligned}$$

$$= \min(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y));$$

$$v_{K'(\alpha, \beta)}(x, y) = v_{H_2(\alpha, \beta)}(x, y)$$

$$= \max(v_{F_2(\alpha)}(x), v_{G(\alpha)}(y)).$$

\* Case 3:  $(\alpha, \beta) \in (A_1 \times B) \cap (A_2 \times B) = (A_2 \cap A_1) \times B$ .

$$\mu_{K'(\alpha, \beta)}(x, y) = \mu_{(H_1(\alpha, \beta)) \cup (H_2(\alpha, \beta))}(x, y)$$

$$= \max(\mu_{H_1(\alpha, \beta)}(x, y), \mu_{H_2(\alpha, \beta)}(x, y))$$

$$= \max(\min(\mu_{F_1(\alpha)}(x), \mu_{G(\alpha)}(y)), \min(\mu_{F_2(\alpha)}(x), \mu_{G(\alpha)}(y)));$$

$$\eta_{K'(\alpha, \beta)}(x, y) = \eta_{(H_1(\alpha, \beta)) \cup (H_2(\alpha, \beta))}(x, y)$$

$$= \min(\eta_{H_1(\alpha, \beta)}(x, y), \eta_{H_2(\alpha, \beta)}(x, y))$$

$$= \min(\min(\eta_{F_1(\alpha)}(x), \eta_{G(\alpha)}(y)), \min(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)))$$

$$= \min(\eta_{F_1(\alpha)}(x), \eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)); \text{ and}$$

$$v_{K'(\alpha, \beta)}(x, y) = v_{(H_1(\alpha, \beta)) \cup (H_2(\alpha, \beta))}(x, y)$$

$$= \min(v_{H_1(\alpha, \beta)}(x, y), v_{H_2(\alpha, \beta)}(x, y))$$

$$= \min(\max(v_{F_1(\alpha)}(x), v_{G(\alpha)}(y)), \max(v_{F_2(\alpha)}(x), v_{G(\alpha)}(y))).$$

We then obtain  $K = K'$  which completes the proof of c). The proof of d) is analogous.  $\square$

## 4 Standard neutrosophic soft relations

### 4.1 Standard neutrosophic relations

Fuzzy relations are one of the most important notions of fuzzy set theory and fuzzy system theory. The Zadeh's composition rule of inference [2] is a well-known method in approximation theory and inference methods in fuzzy control theory. Intuitionistic fuzzy relations were received many results [21][22]. Xu [24] defined some new intuitionistic preference relations, such as the consistent intuitionistic preference relation, incomplete intuitionistic preference relation and studied their properties. Thus, it is necessary to develop new approaches to issues, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation. In this section we shall present some preliminary results on standard neutrosophic relations.

#### 4.1.1 Standard neutrosophic relations

Let  $X, Y$  and  $Z$  be ordinary non-empty sets. A standard neutrosophic relation is defined as follows.

**Definition 4.1** [18] A standard neutrosophic relation (SNR)  $R$  between  $X$  and  $Y$  is a SNS on  $X \times Y$ , i.e.

$$R = \left\{ \left( (x, y), \mu_R(x, y), \eta_R(x, y), v_R(x, y) \right) \mid (x, y) \in X \times Y \right\},$$

where  $\mu_R, \eta_R, v_R : X \times Y \rightarrow [0, 1]$  satisfy the condition

$$\mu_R(x, y) + \eta_R(x, y) + v_R(x, y) \leq 1, (x, y) \in X \times Y.$$

We will denote by  $SNR(X \times Y)$  the set of all SNRs between  $X$  and  $Y$ .

**Definition 4.2** [18] Let  $R \in SNR(X \times Y)$ , the inverse relation  $R^{-1}$  of  $R$  is a SNR between  $Y$  and  $X$  defined as

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y), \eta_{R^{-1}}(y, x) = \eta_R(x, y), \text{ and}$$

$$v_{R^{-1}}(y, x) = v_R(x, y), \forall (y, x) \in Y \times X.$$

Now we will consider some simple properties of SNRs.

**Definition 4.3** [18] Let  $R, P \in SNR(X \times Y)$ , for every, we define:

$$a) \quad R \leq P \Leftrightarrow \begin{cases} \mu_R(x, y) \leq \mu_P(x, y) \\ \eta_R(x, y) \leq \eta_P(x, y); \\ v_R(x, y) \geq v_P(x, y) \end{cases}$$

$$R \vee P = \left\{ \left( (x, y), \mu_R(x, y) \vee \mu_P(x, y), \right. \right.$$

$$b) \quad \left. \eta_R(x, y) \wedge \eta_P(x, y), \right.$$

$$\left. v_R(x, y) \wedge v_P(x, y) \right\} \mid (x, y) \in X \times Y \};$$

$$R \wedge P = \left\{ \left( (x, y), \mu_R(x, y) \wedge \mu_P(x, y), \right. \right.$$

$$c) \quad \left. \eta_R(x, y) \wedge \eta_P(x, y), \right.$$

$$\left. v_R(x, y) \vee v_P(x, y) \right\} \mid (x, y) \in X \times Y \};$$

$$d) \quad R^c = \left\{ \left( (x, y), v_R(x, y), \eta_R(x, y), \mu_R(x, y) \right) \mid \right.$$

$$\left. (x, y) \in X \times Y \right\}$$

**Proposition 4.1** [18] Let  $R, P, Q \in SNS(X \times Y)$ . Then

$$a) \quad (R^{-1})^{-1} = R;$$

$$b) \quad R \leq P \Rightarrow R^{-1} \leq P^{-1};$$

- c1)  $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$ ; c2)  $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$ ;  $\forall (x, z) \in X \times Z$ .
- d1)  $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$ ;
- d2)  $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$ ;
- e)  $R \wedge P \leq R$ ,  $R \wedge P \leq P$ ;
- f1) If  $R \geq P$  and  $R \geq Q$  then  $R \geq P \vee Q$ ;
- f2) If  $R \leq P$  and  $R \leq Q$  then  $R \leq P \wedge Q$ .

**Proof.** For the detail proof of this proposition, see [20].

**4.1.2 Composition of standard neutrosophic relations**

In this sub-section we present some compositions of SNRs.

**Definition 4.4** [20] Let  $R \in SNR(X \times Y)$  and  $P \in SNR(Y \times Z)$ . We will call max - min composed relation  $P \circ_1 R \in SNR(X \times Z)$  to the one defined by

$$\begin{aligned} \mu_{P \circ_1 R}(x, z) &= \bigvee_y \{ \mu_R(x, y) \wedge \mu_P(y, z) \}, \\ \eta_{P \circ_1 R}(x, z) &= \bigwedge_y \{ \eta_R(x, y) \wedge \eta_P(y, z) \}, \text{ and} \\ \nu_{P \circ_1 R}(x, z) &= \bigwedge_y \{ \nu_R(x, y) \vee \nu_P(y, z) \}, \forall (x, z) \in X \times Z. \end{aligned}$$

**Definition 4.5** [20] Let  $R \in SNR(X \times Y)$  and  $P \in SNR(Y \times Z)$ . We will call max - prod composed relation  $P \circ_2 R \in SNR(X \times Z)$  to the one defined by

$$\begin{aligned} \mu_{P \circ_2 R}(x, z) &= \bigvee_y \{ \mu_R(x, y) \cdot \mu_P(y, z) \}, \\ \eta_{P \circ_2 R}(x, z) &= \bigwedge_y [ \eta_R(x, y) \cdot \eta_P(y, z) ], \text{ and} \\ \nu_{P \circ_2 R}(x, z) &= \bigwedge_y \{ \nu_R(x, y) + \nu_P(y, z) - \nu_R(x, y) \cdot \nu_P(y, z) \}, \\ \forall (x, z) &\in X \times Z. \end{aligned}$$

**Definition 4.6** [20] Let  $\beta$  be a  $t$ -norm,  $\rho$  be a  $t$ -conorm,  $R \in SNR(X \times Y)$  and  $P \in SNR(Y \times Z)$ . We will call max -  $t$  composed relation  $R \circ_3 P \in PFR(X \times Z)$  to the one defined by

$$\begin{aligned} \mu_{R \circ_3 P}(x, z) &= \bigvee_y \left( \beta \left( \mu_R(x, y), \mu_P(y, z) \right) \right), \\ \eta_{R \circ_3 P}(x, z) &= \bigwedge_y \left\{ \beta \left( \eta_R(x, y), \eta_P(y, z) \right) \right\}, \text{ and} \\ \nu_{R \circ_3 P}(x, z) &= \bigwedge_y \left\{ \rho \left( \nu_R(x, y), \nu_P(y, z) \right) \right\}, \end{aligned}$$

The validation of Definitions 4.5-4.7 were given in [30].

**4.2 Neutrosophic soft relations**

**4.2.1 Some operations on neutrosophic soft relations**

In this sub-section, we give the definition of standard neutrosophic soft relation (SNSR) as a generalization of fuzzy soft relation and intuitionistic fuzzy soft relation. The novel concept is actually a parameterized family of standard neutrosophic relations (SNRs).

In following definitions,  $X, Y$  are ordinary non-empty sets and  $E$  is a set of parameters.

**Definition 4.7** Let  $A \subseteq E$ . A pair  $(R, A)$  is called a standard neutrosophic soft relation (SNSR) over  $X \times Y$  if  $R$  assigns to each parameter  $e$  in  $E$  a SNR  $R(e)$  in  $SNR(X \times Y)$ , that is

$$R : A \rightarrow SNR(X \times Y).$$

The set of all SNSRs between  $X$  and  $Y$  is denoted by  $SNSR(X \times Y)$ .

**Definition 4.8** Let  $A, B \subseteq E$ . The intersection of two SNSRs  $(R_1, A)$  and  $(R_2, B)$  over  $X \times Y$  is a SNSR  $(R_3, C)$  over  $X \times Y$  such that  $C = A \cap B$  and for all  $e \in C$ ,

$$R_3(e) = \begin{cases} R_1(e) & \text{if } e \in A \setminus B, \\ R_2(e) & \text{if } e \in B \setminus A, \\ R_1(e) \wedge R_2(e) & \text{if } e \in A \cap B. \end{cases}$$

This relation is denoted by  $(R_1, A) \cap (R_2, B)$ .

**Definition 4.9** Let  $A, B \subseteq E$ . The union of two SNSRs  $(R_1, A)$  and  $(R_2, B)$  over  $X \times Y$  is a SNSR  $(R_3, C)$  over  $X \times Y$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$R_3(e) = \begin{cases} R_1(e) & \text{if } e \in A \setminus B, \\ R_2(e) & \text{if } e \in B \setminus A, \\ R_1(e) \vee R_2(e) & \text{if } e \in A \cap B. \end{cases}$$

This relation is denoted by  $(R_1, A) \cup (R_2, B)$ .

**4.2.2 Composition of neutrosophic soft relations**

We denote by  $SNSR_{E_1}(X \times Y)$  the set of all SNSRs on  $X \times Y$  with the corresponding parameter set  $E_1$ . Similarly,  $SNSR_{E_2}(Y \times Z)$  denotes the set of all SNSRs on  $Y \times Z$  with the corresponding parameter set  $E_2$ .

**Definition 4.10** Let  $R \in SNSR_{E_1}(X \times Y)$  and  $P \in SNSR_{E_2}(Y \times Z)$ . We will call max - min composed relation  $P \bullet_1 R \in SNSR_{E_1 \times E_2}(X \times Z)$  to the one defined by

$$P \bullet_1 R(e_1, e_2) = \left\{ (x, z), \mu_{P \bullet_1 R}(x, y)(e_1, e_2), \right. \\ \eta_{P \bullet_1 R}(x, z)(e_1, e_2) \\ \left. \nu_{P \bullet_1 R}(x, z)(e_1, e_2) \mid (x, z) \in X \times Z \right\},$$

$\forall (e_1, e_2) \in A_1 \times A_2$ . Where

$$\mu_{P \bullet_1 R}(x, z)(e_1, e_2) = \bigvee_y \left\{ \mu_{R(e_1)}(x, y) \wedge \mu_{P(e_2)}(y, z) \right\},$$

$$\eta_{P \bullet_1 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \eta_{R(e_1)}(x, y) \wedge \eta_{P(e_2)}(y, z) \right\},$$

$$\nu_{P \bullet_1 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \nu_{R(e_1)}(x, y) \vee \nu_{P(e_2)}(y, z) \right\},$$

for all  $(x, z) \in X \times Z, (e_1, e_2) \in A_1 \times A_2$ .

**Definition 4.11** Let  $R \in SNSR_{E_1}(X \times Y)$  and  $P \in SNSR_{E_2}(Y \times Z)$ . We will call max - prod composed relation  $P \bullet_2 R \in SNSR_{E_1 \times E_2}(X \times Z)$  to the one defined by

$$P \bullet_2 R(e_1, e_2) = \left\{ (x, z), \mu_{P \bullet_2 R}(x, y)(e_1, e_2), \right. \\ \eta_{P \bullet_2 R}(x, z)(e_1, e_2) \\ \left. \nu_{P \bullet_2 R}(x, z)(e_1, e_2) \mid (x, z) \in X \times Z \right\},$$

$\forall (e_1, e_2) \in A_1 \times A_2$ . Where

$$\mu_{P \bullet_2 R}(x, z)(e_1, e_2) = \bigvee_y \left\{ \mu_{R(e_1)}(x, y) \cdot \mu_{P(e_2)}(y, z) \right\},$$

$$\eta_{P \bullet_2 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \eta_{R(e_1)}(x, y) \cdot \eta_{P(e_2)}(y, z) \right\},$$

$$\nu_{P \bullet_2 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \nu_{R(e_1)}(x, y) + \nu_{P(e_2)}(y, z) \right. \\ \left. - \nu_{R(e_1)}(x, y) \cdot \nu_{P(e_2)}(y, z) \right\},$$

for all  $(x, z) \in X \times Z, (e_1, e_2) \in A_1 \times A_2$ .

**Definition 4.12** Let  $R \in SNSR_{E_1}(X \times Y)$ ,  $P \in SNSR_{E_2}(Y \times Z)$ ,  $\beta$  is a  $t$ -norm and  $\rho$  is a  $t$ -conorm. We will call max -  $t$  composed relation  $P \bullet_3 R \in SNSR_{E_1 \times E_2}(X \times Z)$  to the one defined by

$$P \bullet_3 R(e_1, e_2) = \left\{ (x, z), \mu_{P \bullet_3 R}(x, y)(e_1, e_2), \right. \\ \eta_{P \bullet_3 R}(x, z)(e_1, e_2) \\ \left. \nu_{P \bullet_3 R}(x, z)(e_1, e_2) \mid (x, z) \in X \times Z \right\},$$

$\forall (e_1, e_2) \in A_1 \times A_2$ . Where

$$\mu_{P \bullet_3 R}(x, z)(e_1, e_2) = \bigvee_y \left\{ \beta \left( \mu_{R(e_1)}(x, y), \mu_{P(e_2)}(y, z) \right) \right\},$$

$$\eta_{P \bullet_3 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \beta \left( \eta_{R(e_1)}(x, y), \eta_{P(e_2)}(y, z) \right) \right\},$$

$$\nu_{P \bullet_3 R}(x, z)(e_1, e_2) = \bigwedge_y \left\{ \rho \left( \nu_{R(e_1)}(x, y), \nu_{P(e_2)}(y, z) \right) \right\},$$

for all  $(x, z) \in X \times Z, (e_1, e_2) \in A_1 \times A_2$ .

The validation of Definitions 4.11-4.13 is trivial by following arguments. For each pair  $(e_1, e_2) \in A_1 \times A_2$ ,  $P \bullet_1 R(e_1, e_2)$  is max - min composition of two SNSRs  $R(e_1)$  and  $P(e_2)$ , i.e.

$$P \bullet_1 R(e_1, e_2) = P(e_2) \circ_1 R(e_1).$$

By the validation of  $\circ_1$ ,  $P \bullet_1 R(e_1, e_2) \in SNR(X \times Z)$  which yields  $P \bullet_1 R \in SNSR_{E_1 \times E_2}(X \times Z)$ . The validation of  $\bullet_2$  and  $\bullet_3$  are also obtained by analogous calculations.

### Conclusion

In 2013, the new notion of picture fuzzy sets was introduced. The novel concept, which is also termed as standard neutrosophic set (SNS), constitutes an importance case of neutrosophic set. Our neutrosophic soft set (NSS) theory is a combination of the standard neutrosophic theory and the soft set theory. In other words, neutrosophic soft set theory is a neutrosophic extension of the intuitionistic fuzzy soft set theory. The complement, "and", "or", union and intersection operations are defined on the NSSs. The standard neutrosophic soft relations (SNSR) are also considered. The basic properties of the NSSs and the SNSRs are also discussed. Some future work may be concerned interval- valued neutrosophic soft sets and interval- valued neutrosophic relations should be considered.

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