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Weighted Inequalities for Dyadic Operators Over Spaces of Homogeneous Type

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DISSERTATION

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Dedication

To all of the incredible mathematics teachers that have touched my life —

and all other teachers of the world.

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Abstract

A so-called *space of homogeneous type* is a set equipped with a quasi-metric and a doubling measure. We give a survey of results spanning the last few decades concerning the geometric properties of such spaces, culminating in the description of a system of dyadic cubes in this setting whose properties mirror the more familiar dyadic lattices in $\mathbb{R}^n$. We then use these cubes to prove a result pertaining to weighted inequality theory over such spaces. We develop a general method for extending Bellman function type arguments from the real line to spaces of homogeneous type. Finally, we uses this method to extend some recent results in the theory of weighted dyadic operators.

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Glossary

(X, ρ, μ) or X	A space of homogeneous type.
ρ	A quasi-metric.
κ_0	The quasi-triangle constant.
μ	A measure, usually a doubling measure.
κ_1	The doubling measure constant.
$\mathcal{A}$	The collection of all atoms of (X, μ).
$\mathscr{D}^{\mathbb{R}}$	The standard dyadic lattice for the real numbers
I_ℓ, I_r	The left and right children of a dyadic intervarl $I \subseteq \mathbb{R}$.
$\mathscr{D}^X$	A dyadic lattice over a space of homogeneous type.
$\tilde{\mathscr{D}}$	An honest dyadic system.
Q	A dyadic cube.
$Q_\pm$	The two children of an honest dyadic cube Q.
$\widehat{Q}$	The parent cube for Q.
$\mathrm{Quad}_{\mathscr{D}}(x)$	The dyadic quadrant of $\mathscr{D}$ which contains a point x.

$\mathrm{Quad}_{\mathscr{D}}(X)$	Collection of all quadrants of $\mathscr{D}$ ofer X.
A_p	Collection of weights which satisfy the Mukenhoupt A_p inequality.
RH_q	Collection of weights which satisfy the Reverse Hölder q inequality.
C_t	Collection of weights which satisfy the C_t inequality.
$Db(\mathscr{D})$	Collection of weights which satisfy a dyadic doubling inequality.
$\mathbb{1}_S$	Characteristic or indicator function for a set S.
h_Q^i	The i^{th} Haar function associated to a dyadic cube Q.
$\langle f \rangle_S$	The mean of a function f over a set S.
$\mathfrak{B}$	A Bellman function
$\mathbf{x}, (u, v)$	A point $\mathbf{x} \in \mathbb{R}^{d_1+d_2}$ equal to (u, v) with $u \in \mathbb{R}^{d_1}$ and $v \in \mathbb{R}^{d_2}$.

Chapter 1

Introduction

The tendency to try to extend knowledge from familiar settings to more abstract and exotic ones is ubiquitous in mathematics. Mathematicians have since time immemorial been concerned with generalizing favorite concepts and ideas beyond their original scope. Natural numbers generalize to integers, classical algebra generalizes to abstract algebra, and the geometry of antiquity generalizes to modern topological spaces and beyond. In this way, provided the generalizations are done appropriately, we can gain insight into the goings-on of more abstract settings without necessarily reinventing the wheel.

This dissertation is concerned with continuing in that tradition. Our aim here is to take a close look at recent developments in the field of abstract harmonic analysis, specifically of dyadic weighted inequalities. We will look at how over the last two decades many results have been generalized beyond the comfortable world of $\mathbb{R}^n$ and similarly well structured vector spaces. We then build on these results.

1.1 Spaces of Homogeneous Type

For the purposes of analysis, two concepts are necessary for any generalization of Euclidean space: *distance* and *volume*. Without a working version of each of these concepts, not much meaningful analysis is possible. One would be hard pressed to define a theory of integration without first defining measure, for instance. While normal $\mathbb{R}^n$ has extra structure besides – most notably, that of a vector space – these structures are not necessary for definitions to remain meaningful. In this way we keep what is necessary and dispose of the fluff.

A space of homogeneous type is one such flavor of generalization. They were first introduced in [13], by Coifman and Weiss. Spaces of homogeneous type, or SHTs as we will often call them, are characterized by quasi-metrics and doubling measures.

A quasi-metric is a generalization of a metric. Quasi-metrics satisfy all the axioms of a metric excluding the triangle inequality. Instead, they satisfy a weaker inequality which includes a multiplicative constant, $\kappa_0 \geq 1$:

$$\rho(x, z) \leq \kappa_0(\rho(x, y) + \rho(y, z)). \tag{1.1}$$

A measure μ is said to be *doubling* when it satisfies that for some constant $\kappa_1 \geq 1$,

$$\mu(B(x, 2r)) \leq \kappa_1 \cdot \mu(B(x, r)), \tag{1.2}$$

where by $B(x, r)$ we mean the open ball with respect to the quasi-metric ρ centred at $x \in X$ with radius $r > 0$

Spaces of Homogeneous type appear to be very good generalizations of Euclidean space for the purposes of extending many results of analysis, in particular the theory of Calderón-Zygmund singular operators.

We briefly note that spaces of homogeneous type as we have defined them are not "maximally general" with respect to the the distance/volume paradigm. For example, in [1] the authors consider quasi-metrics to lack not only the triangle inequality axiom, but the axiom of reflexivity as well. Also, the assumption of doubling on the measure seems to be very specialized. Further generalizations are natural to consider and there is indeed some interest in investigating them, however this is beyond the scope of this dissertation. We point the interested reader to the paper by Martell et. al. [35] or the new paper by Volberg and Zorin-Kranich [49] for weighted inequality theory results concerning *non-doubling* measures. Also, the paper [41] by Nazarov, Treil, and Volberg.

1.1.1 Dyadic Systems and Haar Bases

The use of dyadic collections has proven to be fruitful for harmonic analysis techniques. At its most fundamental, a dyadic system is a highly organized hierarchy of partitions of Euclidean space which satisfies certain properties. The collection of dyadic intervals in $\mathbb{R}$ serves as the most basic example and the example from which dyadic systems derive their name.

Intimately tied to the notion of dyadic systems is the Haar basis of functions. There is a natural extension of the basics of Dyadic theory, and thus the Haar functions, to spaces of homogeneous type. As with the traditional settings, these constructs can be very helpful when seeking proofs of results of both the dyadic and continuous variety.

1.1.2 Weights

A weight is a locally integrable function which is positively-valued almost everywhere. Weight theory has a rich history indeed. We will, over the course of this document, give the necessary ideas and point the reader to the excellent textbooks [17] and [18] by Grafakos, or Duoandikoetxea's textbook [14] for a more serious overview.

1.2 Chapter Summaries

Here we give a brief summary of each of the proceeding chapters.

In Chapter 2 we will look at the basic properties of spaces of homogeneous types. Much of the content of this chapter could be considered the low hanging fruit of the theory and has been covered elsewhere. Nevertheless, we attempt to provide a broader, more extensive look than what has been written by others. We also include several examples of SHTs, as well as some things which one might expect to be SHTs but are not (failures). This chapter will serve as necessary background information.

In Chapter 3 we give a generalization of dyadic systems from $\mathbb{R}$ to SHTs. We start with an overview of the main result of [25] which guarantees the existence of a dyadic system. In this chapter we will introduce the concept of a dyadic quadrant. We close the chapter with a utility definition, that of "honest" dyadic systems, and give a constructive proof for their existence.

In Chapter 4, we collect some basic results about weight theory which are used in several other chapters. This chapter is very short, and is intended to be more of a quick reference.

In Chapter 5 we prove our first weighted dyadic inequality, Gehring's inequality. This is an example of a "open" property or "self-improvement" property. In [5] it was shown that this inequality *cannot* be extended to SHTs in general, however in this chapter we show that, surprisingly, a dyadic version *does* hold.

In Chapter 6 we work with the Haar bases for SHTs. We first give a good idea of what a proper "Haar-like" basis would look like, and then show that we can in fact construct one. The ending of the chapter deals with proving that our proposed Haar basis satisfies the requirements we would have for it.

In Chapter 7 we give a method for extending Bellman function style proofs from $\mathbb{R}$ to SHTs. This is a particular proof technique which has become popular with some mathematicians. There has been some interest in extending the Bellman technique to SHTs, but there are some subtle difficulties in doing so. Here we address these difficulties and show how the technique can be extended. The so-called "Good Bellman Lemma" is the main result of the chapter and can be used to accomplish this in general.

In Chapter 8 we give some applications of the previous chapters. Our aim here is to extend previous $\mathbb{R}$-centric results to the setting of SHTs. We prove bounds for the paraproduct and t-Haar multipliers in spaces of homogeneous type. This serves as a useful template for future applications of the work in this dissertation.

In the appendix we will stash away some of the interesting proofs that the above chapters rely on, but which would have interrupted the flow of the text if included within them.

Chapter 2

Introduction to Spaces of Homogeneous Type

In this chapter we will introduce the idea of spaces of homogeneous type. We begin with the required definitions of a quasi-metric and a doubling measure. We will then move on to build up a foundation of theorems which will come in handy in the later chapters.

2.1 Basics and Definitions

This section introduces the main definitions needed for any discussion pertaining to spaces of homogeneous type; that of quasi-metrics and doubling measures.

2.1.1 Quasi-metric Spaces

For any set X, we can consider a metric on X as an abstraction of the narXiv:1606.03461 otion of distances between the elements of X. Metrics satisfy three axioms: positive-definite, reflexive, and the triangle inequality. Generalizations of metrics can be considered where one of these axioms is relaxed. For our purposes, we consider instead *quasi-metrics*: functions which satisfy all axioms of a metric except that the triangle inequality is weaker.

Definition 2.1.1 (Quasi-metric)**.** Let X be any set. A *quasi-metric on* X is a function $\rho : X \times X \to \mathbb{R}$ with the following properties:

- (*Positive-Definite*) For every $x, y \in X$, $\rho(x, y) \geq 0$. Furthermore, $\rho(x, y) = 0$ if and only if $x = y$.
- (*Reflexive*) For every $x, y \in X$, $\rho(x, y) = \rho(y, x)$.
- (*Quasi-Triangle Inequality*) There exists a constant $\kappa_0 \geq 1$ so that for every $x, y, z \in X$,

$$\rho(x, y) \leq \kappa_0 \cdot (\rho(x, z) + \rho(z, y)). \tag{2.1}$$

We call the constant κ_0 the *quasi-triangle constant.* Notice that if $\kappa_0 = 1$ then ρ is actually a metric.[1]

If X is a set and ρ is a quasi-metric on X, then we call the pair (X, ρ) a *quasi-metric space.*

Remark 2.1.2. Another well studied generalization of a metric is the *pseudo-metric*, where the axiom of positive-definiteness has been relaxed. For a pseudo-metric p, $p(x, y) = 0$ does not necessarily imply that $x = y$.

Example 2.1.3 (A Finite Quasi-Metric Space)**.** Let $X := \{\heartsuit, \clubsuit, \spadesuit\}$. Define the function $d : X \times X \to \mathbb{R}$ as

$$d(\heartsuit, \heartsuit) = d(\clubsuit, \clubsuit) = d(\spadesuit, \spadesuit) := 0 \tag{2.2}$$

$$d(\heartsuit, \clubsuit) = d(\clubsuit, \heartsuit) := 5 \tag{2.3}$$

$$d(\clubsuit, \spadesuit) = d(\spadesuit, \clubsuit) := 3 \tag{2.4}$$

$$d(\heartsuit, \spadesuit) = d(\spadesuit, \heartsuit) := 4. \tag{2.5}$$

[1]Some authors insist that $\kappa_0 > 1$ be strict, but we will not.

It is not difficult to verify that d defined in this way is a metric on X, making (X, d) a metric space. However, consider the similar function $\rho : X \times X \to \mathbb{R}$ defined as

$$\rho(\heartsuit,\heartsuit) = \rho(\clubsuit,\clubsuit) = \rho(\spadesuit,\spadesuit) := 0 \tag{2.6}$$

$$\rho(\heartsuit,\clubsuit) = \rho(\clubsuit,\heartsuit) := 10 \tag{2.7}$$

$$\rho(\clubsuit,\spadesuit) = \rho(\spadesuit,\clubsuit) := 3 \tag{2.8}$$

$$\rho(\heartsuit,\spadesuit) = \rho(\spadesuit,\heartsuit) := 4. \tag{2.9}$$

Clearly, ρ fails to be a metric since

$$\rho(\heartsuit,\clubsuit) = 10 > 7 = \rho(\heartsuit,\spadesuit) + \rho(\spadesuit,\clubsuit) \tag{2.10}$$

which violates the triangle inequality. However, ρ is a quasi-metric on X, with $\kappa_0 \geq 10/7$.

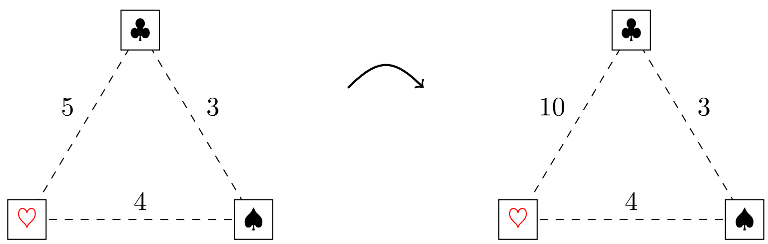


Figure 2.1: Here we see the space from Example 2.1.3 with distances represented as the sides of a triangle. On the left we have the $\heartsuit\spadesuit\clubsuit$ metric space, and on the right is the $\heartsuit\spadesuit\clubsuit$ quasi-metric space.

We define open balls in the obvious way.

Definition 2.1.4 (Open Balls)**.** Let (X, ρ) be a quasi-metric space. For all $x \in X$ and $r \in \mathbb{R}$ with $r > 0$ define the set $B_\rho(x, r) = \{y \in X \mid \rho(x, y) < r\}$ the *open ρ-ball* centred at x with radius r.

We will usually write B instead of B_ρ when the quasi-metric under consideration is well understood.

Naïvely, this definition seems entirely benign. In the next section, we will look at some of the issues that this definition introduces.

2.1.2 Topological Concerns

Recall that given a metric space (X, d), d induces a topology τ_d on X.

Definition 2.1.5 (Topology Induced by a Metric, v1)**.** Let (X, d) be a metric space. Then the *topology* τ_d *induced by* d is defined as the topology such that all open sets in τ_d can be realized as arbitrary unions of open d-balls.

Definition 2.1.6 (Topology Induced by a Metric, v2)**.** Let (X, d) be a metric space. Then the *topology* τ_d *induced by* d is defined as the topology such that all open sets in τ_d have the *open property.* That is, $O \in \tau_d$ if and only if for all $x \in O$, there exists $r_x > 0$ such that $B(x, r_x) \subseteq O$.

This induced topology is $T2$, or Hausdorff, in the sence of topological classification (see the book [15] for an introduction to this topic). In the case of standard metrics these two definitions are equivalent.

When generalizing to quasi-metric spaces, a complication arises. Consider this example, presented in [25, Section 2.1]:

Example 2.1.7. Let $X = \{-1\} \cup [0, \infty)$ and ρ be defined as

$$\rho(x, y) := \begin{cases} 1/2 & \text{if} \quad (x, y) = (0, -1) \\ 1/2 & \text{if} \quad (x, y) = (-1, 0) \\ |x - y| & \qquad \text{otherwise} \end{cases} \tag{2.11}$$

Then (X, ρ) is a quasi-metric space, with quasi-triangle constant $\kappa = 2$.

Indeed, looking at the open ball $B(-1, 3/4)$ we see that this ball contains only two points, 0 and -1. However, there is no $r > 0$ so that $B(0, r)$ is in the ball. In other words, *the open ball fails to have the open property!*

Care needs to be taken when saying things like "topology induced by a quasi-metric" since the two definitions are *not* equivalent when replacing d with ρ. If we generalize using

Definition 2.1.5, then open balls are of course open sets. If we instead generalize using Definition 2.1.6, then they may not be. In this document, we will enforce that the topology of open sets induced by ρ will have the open property:

Definition 2.1.8 (Topology Induced by a Quasi-Metric)**.** Let (X, ρ) be a quasi-metric space. Then the *topology* τ_ρ *induced by* ρ is defined as the topology such that all nonempty, open sets in τ_ρ have the *open property.* That is, for $O \in \tau_\rho$ if and only if for for all $x \in O$, there exists $r_x > 0$ such that $B_\rho(x, r_x) \subseteq O$.

It is clear that this collection is indeed a topology, i.e., $\emptyset, X \in \tau_\rho$ and τ_ρ is closed under arbitrary unions and finite intersections.

2.1.3 Doubling Measures

We now discuss doubling measures and spaces of homogeneous type.

Definition 2.1.9 (Doubling Measure)**.** Let (X, ρ) be a quasi-metric space and let μ be a Borel measure defined on X such that the collection of all open balls with respect to ρ are μ-measurable sets. Then μ over X is a *doubling measure* if there exists a constant $\kappa_1 \geq 1$ so that for every $x \in X$ and every $r > 0$

$$\mu(B(x, 2r)) \leq \kappa_1 \cdot \mu(B(x, r)). \tag{2.12}$$

We will refer to κ_1 as the doubling measure constant for μ.

Recall that a measure is a Borel measure if the σ-algebra of measurable sets has the topology of open sets as a basis. The σ-algebra for our doubling measures must therefore be at least as large as the Borel σ-algebra for the topology induced by the quasi-metric ρ. This issue is discussed in detail in [24], an addendum to [23]. There the authors note that the meaning of "Borel measure" can be ambigious, given the open ball problem. For our purposes, it is enough to stipulate that both open balls, as well as open sets (in the sense of Definition 2.1.8) be μ-measurable.

We can now give the major definition of this document:

Definition 2.1.10 (Space of Homogeneous Type)**.** A *Space of Homogeneous Type* (abbr. "SHT") is a tuple (X, ρ, μ) where X is a set, ρ is a quasi-metric on X, and μ is a measure such that

- the σ-algebra of μ-measurable sets is the smallest one that contains both the Borel measurable sets and all open ρ-balls,
- μ is doubling with respect to the ρ-balls.

We will often use the letter X to mean both the set of elements, as well as the tuple (X, ρ, μ), depending on the context. In this way, we can write something such as "let X be a SHT" without needing to refer to the measure and quasi-metric directly.

We'll close with a lemma.

Lemma 2.1.11 (Generalized Doubling)**.** *Let $x \in X$ and let $R > r > 0$. Then there exists a constant $C \geq 1$ which depends only on the quasi-triangle constant for ρ and the ratio R/r such that*

$$\mu(B(x, R)) \leq C \cdot \mu(B(x, r)). \tag{2.13}$$

Moreover, $C = \kappa_1^{\log_2(\lceil R/r \rceil)}$ is sufficient.

Proof. Follows from an iterated application of inequality 2.12. More details can be found in Appendix 3. $\square$

Remark 2.1.12. Some authors use the constant $\kappa_1^{\log_2(1+R/r)}$.

2.1.4 More Definitions

We will now build upon the definitions in the previous section.

Definition 2.1.13. Let X be a set, and ρ, ρ' be two quasi-metrics on X. We say that ρ and ρ' are *equivalent as quasi-metrics*, or just *equivalent*, if there exists constants $0 < a < A < \infty$ such that

$$a \cdot \rho(x, y) \leq \rho'(x, y) \leq A \cdot \rho(x, y) \tag{2.14}$$

for all $x, y \in X$.

Finally, throughout this document we will often use the following convenience definition:

Definition 2.1.14 (Geometric Constant)**.** Let (X, ρ, μ) be a space of homogeneous type with quasi-triangle constant κ_0 and doubling measure constant κ_1. Any constant which depends solely on κ_0 and κ_1 is referred to as a *geometric constant*.

Example 2.1.15 (Geometric Constants)**.** Let X be an SHT. The following are all geometric constants:

- $3 \cdot \kappa_0$
- κ_1/κ_0
- κ_1^α for fixed α

2.1.5 A Discussion of Alternative Definitions

We note here a definition that is usually important when discussing SHTs:

In the new book by Alvarado and Mitrea [1] the authors give a slight variation on the definition of a space of homogeneous type. Unlike here, they consider quasi-metrics to be *equivalence classes* where the equivalence relation is as in Definition 2.1.13. In their formulation, rather than representing a single quasi-metric, ρ instead represents an *equivalence class* of quasi-metrics under the equivalence relation 2.1.13. In their framework, the quasi-triangle inequality is

$$\rho(x, y) \leq \kappa_0 \cdot \max\{\rho(x, z), \rho(z, y)\},$$

i.e. the "quasi ultra-metric" condition. Moreover, they also weaken the requirement of reflexivity on quasi-metrics, i.e., they have that there exists a constant $C \geq 1$ such that for all $x, y \in X$,

$$\rho(x, y) \leq C \cdot \rho(y, x).$$

For these authors, this decision has the primary advantage of eliminating some of the messy details which need to be considered when working in SHTs. In particular, it allows them to easily circumvent the "open ball problem" which we mentioned previously. However, the price paid is one of a lack of arbitrariness. In this document, we try to as much as possible avoid putting restrictions on geometric properties, following T. Hytönen's school. We do this to allow for applications in which the SHT is provided ahead of time.

We will address this more later on when we want to use a theorem from [1] in our context.

2.2 The Quasi-Metric Zoo

In this section we give several examples of quasi-metric spaces, spaces of homogeneous type, as well as examples of things which fail to be either. We will use these examples as insight, which we will then formulate into lemmas in the next section. For more examples, see [13] and Section 1.3 of [1].

2.2.1 SHTs and Quasi-Metric Spaces

First, some examples of SHTs and quasi-metric spaces.

Example 2.2.1. The set $\mathbb{R}^n$ with the usual metric and measure is an SHT.

Definition 2.2.2 (Trivial Measure)**.** If (X, μ) is a measure space such that $\mu(X) \equiv 0$ or $\mu(X) \equiv \infty$ we say that μ is a *trivial measure*.

Example 2.2.3. Let (X, ρ) be any quasi-metric space. Define the measures μ and μ' as

$$\mu(S) := 0, \quad \mu'(S) := \infty \quad \forall S \text{ measurable.} \tag{2.15}$$

Then (X, ρ, μ) and (X, ρ, μ') are both SHTs.

Trivial measures are inherently uninteresting, but we point this detail out because they provide a counterexample to many of the results we present pertaining to properties of SHTs.[2] We will need to be careful to refer to measures as non-trivial for theorems later on.

Example 2.2.4. One interesting example in [1] is the Cantor carpet in $\mathbb{R}^d$ equipped with the usual $\mathbb{R}^d$ distance and d-dimensional Hausdorff measure.

2.2.2 Examples of Failures

Now let us consider a few spaces which are *not* SHTs, in order to get a sense of what types of situations do not arise.

Example 2.2.5. If X is an infinite set, ρ is the discrete metric, and μ is a non-trivial measure, then (X, ρ, μ) is not an SHT. To see this, observe that for any point $x \in X$

$$B(x, 3/4) = \{x\}$$
$$B(x, 3/2) = X.$$

[2] In fact, some authors will stipulate that the measure be nontrivial in the definition of SHT.

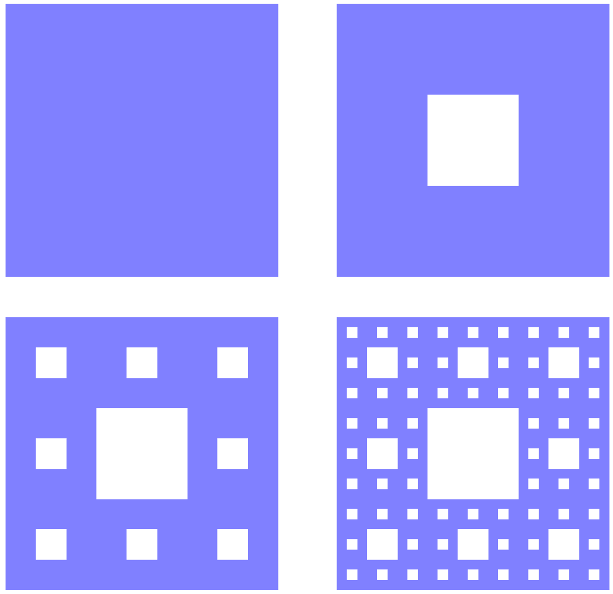


Figure 2.2: The first four iterations of the sets which converge to the $\mathbb{R}^2$ Cantor carpet.

This and the doubling condition together imply that $\mu(X) \leq \kappa_1 \cdot \mu(\{x\})$ for all $x \in X$. Since μ is nontrivial, $\mu(X) > 0$ so

$$\mu(\{x\}) > \frac{\mu(X)}{\kappa_1} > 0$$

for every point in X. But X was infinite, so actually $\mu(X) = \infty$. The doubling inequality thus forces that $\mu(\{x\}) = \infty$ for all x which means μ was trivial anyway.

Example 2.2.6. Let (X, ρ, μ) be defined as $X := \mathbb{Z}$, $\rho(x, y) := |x-y|$, and $\mu(\{x_1, ..., x_n\}) := \sum_{i=1}^n e^{|x_i|}$. Then (X, ρ) is a quasi-metric space (actually a metric space) but μ fails to be a doubling measure. Let us prove this. Let $x, r \in \mathbb{Z}$ with $2x > r > 0$. Then

$$\mu(B(x, r)) = \sum_{i=x-r+1}^{x+r-1} e^i < (2r-1)e^{x+r-1} < 2re^{x+r} \tag{2.16}$$

and

$$\mu(B(x, 2r)) = \sum_{i=x-2r+1}^{x+2r-1} e^i > e^{x+2r-1} > e^{x+2r}. \tag{2.17}$$

However,

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} > \frac{e^r}{2r} \tag{2.18}$$

which is unbounded in r.

Example 2.2.7. Let X be the real numbers with the usual metric, and the Gaussian measure:

$$\mu(S) := \frac{1}{\sqrt{2\pi}} \int_S e^{-x^2}\, dx. \tag{2.19}$$

This measure also fails to be doubling. While it is not difficult to prove this directly, with a little more theory we can get this result quite easily without needing to do any calculations.

The previous examples illustrate something important. Generally exponential behavior destroys any hope of doubling.

Example 2.2.8 (The Bullseye Space)**.** Let $x \in \mathbb{R}^2$ and for $R \geq 2$ define the open bullseye centred at x of radius $R \geq 2$ as

$$Y(x,R) := \{y \in \mathbb{R}^2 \,:\, ||y-x|| < 1 \text{ or } R-1 < ||y-x|| < R\}. \tag{2.20}$$

where $||\cdot||$ is the Euclidean norm.

Observe that the area of any bullseye is equal to $\pi R^2 - \pi(R-1)^2 + \pi = 2\pi R$

Now, define X to be a set made up of bullseyes:

$$X := \bigcup_{n=1}^{\infty} Y(x_n, n+1) \tag{2.21}$$

where the points x_n are chosen so that no two bullseyes overlap:

$$Y(x_n, n+1) \cup Y(x_m, m+1) = \emptyset \quad \forall m \neq n. \tag{2.22}$$

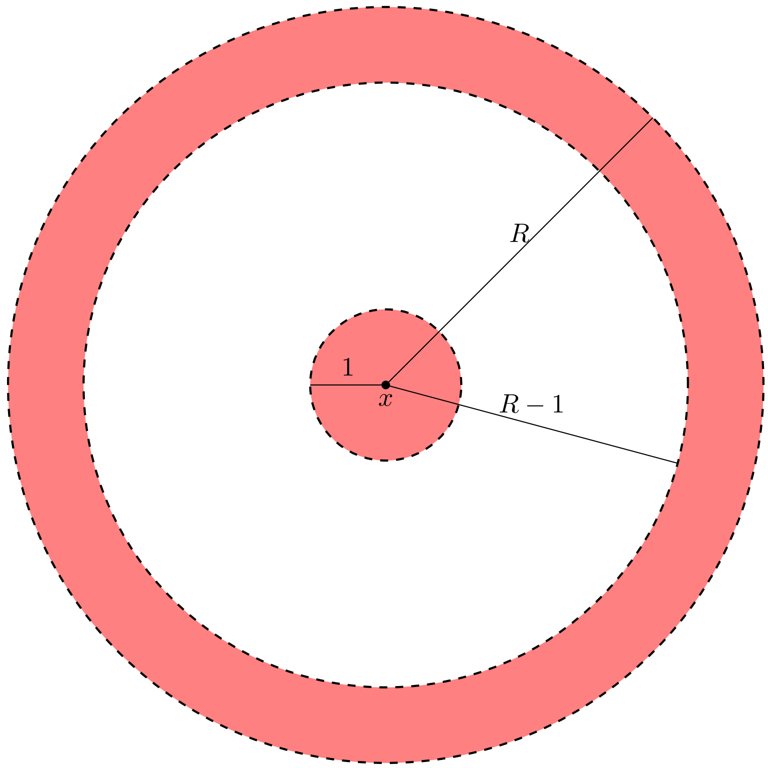


Figure 2.3: A Bullseye

If we consider the metric-measure space $(X, ||\cdot||, m)$ where m is the usual Lebesgue measure on $\mathbb{R}^2$ restricted to measurable subsets of X, then this space is *not* an SHT. This is because for any bullseye centered at x_n,

$$
\begin{aligned}
m(B(x_n,(n+1)/2)) &= \pi \\
m(B(x_n,n+1)) &= 2\pi(n+1)
\end{aligned}
$$

and the ratio of these measures grows to infinity as $n \to \infty$, prohibiting the existence of a doubling measure constant. This is in spite of the fact that $X \subset \mathbb{R}^2$ and $(\mathbb{R}^2, ||\cdot||, m)$ is not only an SHT, but a metric space with doubling measure.

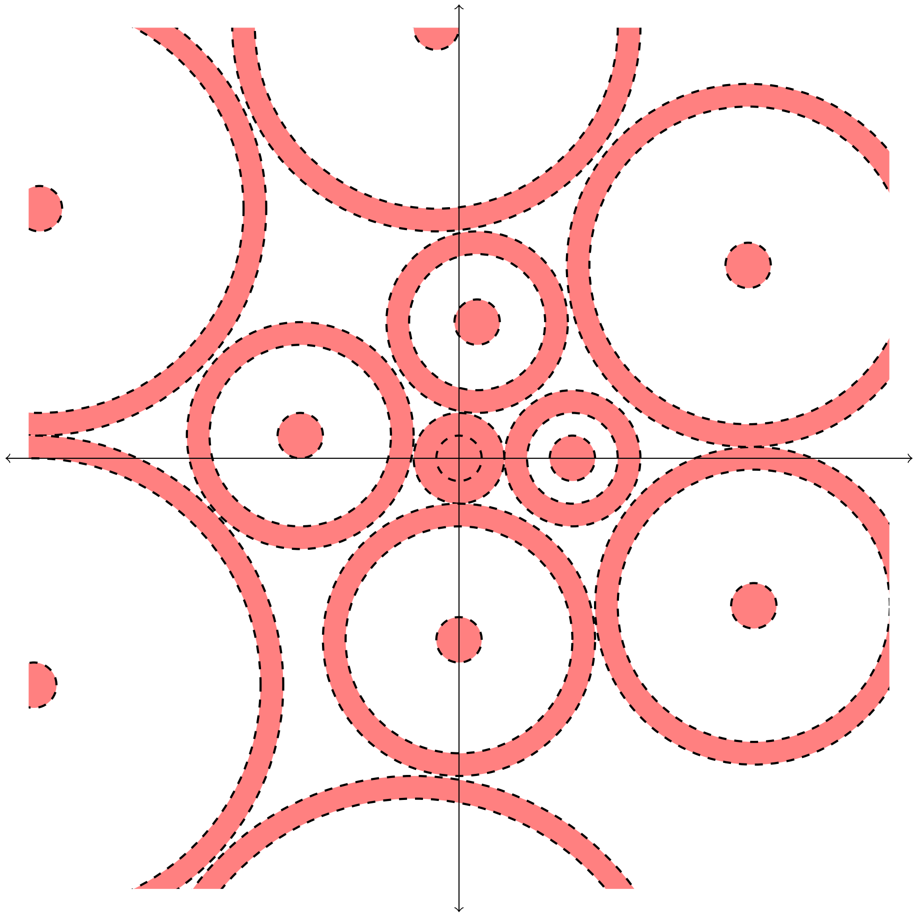


Figure 2.4: The set X, a set made up of tightly packed bullseyes of ever growing radius.

2.3 Geometrically Doubling Quasi-Metric Spaces

In this section we will introduce the concept of geometrically doubling quasi-metric spaces. This definition will be used later on in Chapter 3.

Definition 2.3.1 (Geometrically Doubling Quasi-Metric Spaces)**.** Let (X, ρ) be a quasi-metric space. If there exists a constant $\gamma_0 \in \mathbb{N}$ such that for all $x \in X$ and all $r > 0$, the ball $B(x, r)$ can be covered by at most γ_0 many balls of radius $r/2$, then we say the space X is *geometrically doubling*. In other words, there exists a finite set $\{x_1, x_2, ...x_{\gamma_0}\} \subseteq B(x, r)$

such that

$$B(x,r) \subseteq \bigcup_{n=1}^{\gamma_0} B(x_n, r/2) \quad \forall x, \in X, \, \forall r > 0 \tag{2.23}$$

In [20] (pg. 20), this following is stated:

Theorem 2.3.2 (Geometric Doubling passes to subsets)**.** *Let (X,ρ) be a geometrically doubling quasi-metric space with geometric doubling constant γ_0. Then for any $E \subseteq X$, $(E, \rho|_E)$ is also a geometrically doubling quasi-metric space with geometric doubling constant $\gamma_0^{\log_2 \kappa_0} \cdot \gamma_0$.*

There is no formal proof in [20].

2.3.1 Doubling Measure vs. Geometrically Doubling

It is important to observe the subtle difference between this definition and the idea of a doubling measure, defined in Section 2.1. "Doubling" here is a property of a measure and thus requires a measure space equipped with a quasi-metric. In contrast, "geometric doubling" is a property of a quasi-metric and does not require a measure to be meaningful. In spite of this (and perhaps confounding the confusion), we have the following useful theorem:

Theorem 2.3.3 (Doubling Measure Implies Geometric Doubling)**.** *Let $X = (X, \rho, \mu)$ be an SHT with non-trivial measure. Then ρ is geometrically doubling. Moreover, the geometric doubling bound γ_0 is a geometric constant in the sense of Definition 2.1.14.*

We will withhold the proof of this until the end of the chapter when we have more tools at our disposal. However, referring back to Examples 2.2.6 and 2.2.7, we can see that the converse of Theorem 2.3.3 is *false*. That is, a space with non-doubling measure can still be geometrically doubling.

Remark 2.3.4. The constant γ_0 depends on both κ_0 and κ_1. This means that if we are only considering a quasi-metric space, and not thinking about a measure, γ_0 is not *strictly* a geometric constant. In fact, it is easy to find examples of metrics which have the same quasi-triangle constant but whatever geometric doubling constant you like, for example, $\mathbb{R}^n$.

2.3.2 Disperse Points

We close the section with one more definition which we will need later on.

Definition 2.3.5 (R-disperse points)**.** Let $R > 0$. A set of points $\{x_a\}_{a \in A} \subseteq X$, where A is an index set, is R-*disperse* if $\inf\limits_{a,b \in A; a \neq b} \rho(x_a, x_b) > R$.

It is not impossible for A to be uncountable when we only consider only a quasi-metric space. For example, when $X = \mathbb{R}$ with the discrete metric then all of X is a 2-disperse set in X. However, A must be countable once when X is an SHT. This is a consequence of a theorem in the next section.

2.4 Theorems, Lemmas, and Properties

In this section we will list some useful lemmas pertaining to spaces of homogeneous type.

In the book [1] by R. Alvarado and M. Mitrea, the authors dedicate their first chapter to the geometric properties of spaces of homogeneous type. Many of the results in this section are also found in this book, however the proofs are different and exploit the use of these equivalence classes.

2.4.1 Trivial Measure Lemma

Consider $\mathbb{R}$ with the usual metric and the measure

$$\mu(S) := \int_S \frac{dx}{|x|}. \tag{2.24}$$

where S is any Lebesgue measurable set. This measure has one pretty ugly feature: a bad singularity at zero. In general, we might be worried about measures such as this when thinking about edge cases. However, the next lemma rescues us from this problem.

Lemma 2.4.1 (Trivial Measure Lemma)**.** *Suppose that (X, ρ, μ) is a SHT, then the following are true:*

(a) If there exists a ball $B_0 \subseteq X$ such that $\mu(B_0) = 0$ then $\mu \equiv 0$.

(b) If there exists a ball $B_\infty \subseteq X$ such that $\mu(B_\infty) = \infty$ then $\mu(B) = \infty$ for every ball B.

Proof. For (a), let $B(x,r)$ be a ball with $\mu(B(x,r)) = 0$, and let $U \subseteq X$ be a bounded measurable set. Then there must exists $R \geq r$ such that $[B(x,r) \cup U] \subseteq B(x,R)$. By Lemma 2.1.11,

$$\mu(B(x,R)) \leq C \cdot \mu(B(x,r)) = 0 \tag{2.25}$$

where C is a geometric constant. This implies that $\mu(U) = 0$, showing that all bonded, measurable sets have a μ measure of zero. However X can be covered by such sets (e.g. $X = \bigcup_{n=1}^{\infty} B(x,n)$ for any $x \in X$). Thus, $\mu \equiv 0$.

For (b), let $B(x,r)$ be a ball with $\mu(B(x,r)) = \infty$ and let $B(y,r')$ be another ball. We can choose $R \geq r'$ such that $B(x,r) \subseteq B(y,R)$. Then by basic properties of measure, $\mu(B(y,R)) = \infty$. However, by Lemma 2.1.11, we have that

$$\infty = \mu(B(y,R)) \leq C \cdot \mu(B(y,r')) \tag{2.26}$$

for some constant C that depends on κ_0, κ_1, r, and R. This means that $\mu(B(y,r')) = \infty$. $\square$

An immediate consequence of this is that balls always have positive, finite measure. Measures such as the example above cannot be doubling. In fact, returning to the measure defined in (2.24), we notice that

$$\mu(B(3/2,1)) = \mu((1/2,5/2)) = \ln 5$$

whereas

$$\mu(B(3/2,2)) = \mu((-1/2,7/2)) = \infty$$

since $0 \in (-1/2, 7/2)$.

2.4.2 Finite Measure Lemma

Here we give a proof of a somewhat major result.

Lemma 2.4.2 (Finite Measure Lemma)**.** *Suppose that (X,ρ,μ) is a SHT with non-trivial measure , i.e. $\mu \not\equiv 0,\infty$. Then the following are equivalent:*

- $\mu(X) < \infty$
- *$X \subseteq B(x,r)$ for some $x \in X$ and $r > 0$.*

Using this lemma, it is obvious that the Gaussian measure from Example 2.2.7 cannot be doubling. We will defer the proof of Lemma 2.4.2 for a moment. In order to prove it, we first require two auxiliary lemmas.

Lemma 2.1.11 is well known and can also be found in [25] for example.

Corollary 2.4.3 (Spaces With Equivalent Metrics)**.** *Let (X,ρ,μ) be a SHT, and let ρ be equivalent to another quasi-metric ρ', i.e. that for every $x,y \in X$,*

$$\frac{1}{C}\cdot\rho'(x,y) \le \rho(x,y) \le C\cdot\rho'(x,y) \tag{2.27}$$

for some constant $C \geq 1$. Then (X, ρ', μ) is also an SHT with doubling measure constant $\kappa_1' \leq \kappa_1^{1+\log_2(\lceil 2C^2 \rceil)}$. Moreover, there is a constant $A \geq 1$ such that

$$\frac{1}{A} \cdot \mu(B'(x,r)) \leq \mu(B(x,r)) \leq A \cdot \mu(B'(x,r)) \tag{2.28}$$

where B denotes open balls with respect to ρ and B' denotes the open balls with respect to ρ'.

Proof. First observe that for any $x, y \in X$ and $r > 0$, we have that if $\rho(x,y) < r$ then $\rho'(x,y) < Cr$. Thus, $B(x,r) \subset B'(x,Cr)$. Similarly, $B'(x,r) \subseteq B(x,Cr)$. In particular, the topologies induced by ρ and ρ' are the same.

We now fix $x \in X$ and $r > 0$. Then by Lemma 2.1.11,

$$\mu(B'(x,2r)) \leq \mu(B(x,2Cr)) \leq \kappa_1^{1+\log_2(\lceil 2C^2 \rceil)} \cdot \mu(B(x,r/C)) \leq \kappa_1^{1+\log_2(\lceil 2C^2 \rceil)} \cdot \mu(B'(x,r)).$$

This shows that the proposed κ_1' is sufficient as a doubling measure constant for (X, ρ', μ).

Next, we calculate that

$$\mu(B'(x,r)) \leq \mu(B(x,Cr)) \leq \kappa_1^{\log_2(\lceil C \rceil)} \cdot \mu(B(x,r))$$

and

$$\mu(B(x,r)) \leq \mu(B'(x,Cr)) \leq (\kappa_1')^{\log_2(\lceil C \rceil)} \cdot \mu(B'(x,r))$$

Since $\kappa_1' \geq \kappa_1$, setting $A := (\kappa_1')^{\log_2(\lceil C \rceil)}$ is sufficient to satisfy both inequalities in (2.28). $\square$

Lemma 2.4.4 (Distant Balls Lemma)**.** *Let $x, y \in X$, let $r > 0$ and set $R := \rho(x,y)$. There exists $C \geq 1$ depending only on geometric constants and on the ratio R/r such that*

$$\mu(B(y,r)) \leq C \cdot \mu(B(x,r)). \tag{2.29}$$

Moreover, $C := \kappa_1^{\log_2 \lceil \kappa_0 \frac{R}{r} + \kappa_0 \rceil}$ is sufficient.

Proof. Let $x, y \in X$ and $r > 0$. We wish to cover the ball $B(y, r)$ with a ball centred at x. It will suffice to consider a ball of radius $\kappa_0(R + r)$ (see Figure 2.4.2). Indeed, suppose that $z \in B(y, r)$, then

$$\rho(x, z) \leq \kappa_0(\rho(x, y) + \rho(y, z)) \tag{2.30}$$
$$\leq \kappa_0(R + r) \tag{2.31}$$

implying that $z \in B(x, \kappa_0(R + r))$. Thus, $B(y, r) \subseteq B(x, \kappa_0(R + r))$ meaning that

$$\mu(B(y, r)) \leq \mu(B(x, \kappa_0(R + r)) \tag{2.32}$$
$$\leq \kappa_1^{\log_2 \left\lceil \frac{\kappa_0(R+r)}{r} \right\rceil} \mu(B(x, r)) \tag{2.33}$$

where the last line follows from generalized doubling. $\square$

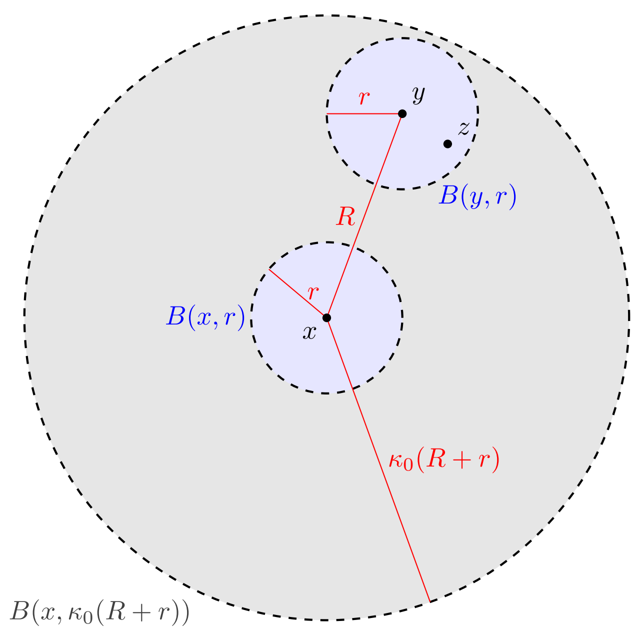


Figure 2.5: Distant balls

We are now ready to do the big proof:

Proof of Finite Measure Lemma. As always, start with the easy direction:

(*Bounded* $\Rightarrow$ *Finite Measure*) This is an immediate consequence of Lemma 2.4.1, that balls always have finite measure.

(*Finite Measure* $\Rightarrow$ *Bounded*) Let $\mu(X) < \infty$ and suppose X is not bounded. That is, suppose that for all $x \in X$ and all $r > 0$ there exists $y \in X$ such that $\rho(x, y) > r$. In other words, X cannot be contained completely inside a ball.

Fix some point $x \in X$ and consider the collection of measurable, pairwise disjoint sets (henceforth called "coronas") C_n defined by

$$C_0 := B(x, \kappa_0) \tag{2.34}$$

$$C_n := B(x, \kappa_0 2^n) \setminus B(x, \kappa_0 2^{n-1}), \quad n \geq 1. \tag{2.35}$$

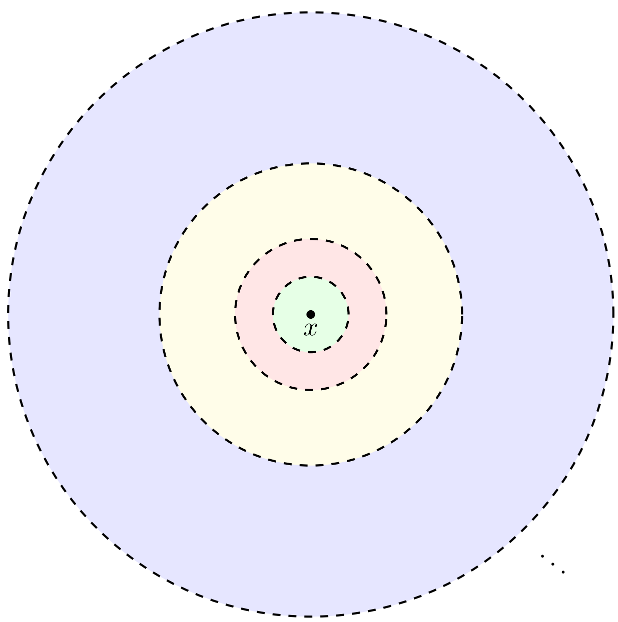


Figure 2.6: The first four coronas

Since X is equal to the disjoint union of all the coronas, and μ is a measure,

$$\mu(X) = \mu\left(\bigcup_{n=0}^{\infty} C_n\right) = \sum_{n=0}^{\infty} \mu\left(C_n\right). \tag{2.36}$$

By assumption this sum is finite so the terms must go to zero as $n \to \infty$, namely, that $\lim_{n\to\infty} \mu(C_n) = 0$.

Next we define a sequence $\{y_n\}_{n=0}^{\infty} \subseteq X$ by

$$y_n = \begin{cases} y \in C_{n+2} & \text{if} \quad C_{n+2} \neq \emptyset \\ x & \text{otherwise} \end{cases} \tag{2.37}$$

Then look at the subsequence $\{y_{n_j}\}_{j=0}^{\infty}$ where $y_{n_j} \neq x$ for all $j > 0$. This subsequence *must* exists and be infinite by our supposition that X is unbounded. Since each y_{n_j} belongs to a corona, we are guaranteed that

$$\kappa_0 2^{n_j+1} \leq \rho(x, y_{n_j}) < \kappa_0 2^{n_j+2} \quad \forall j \in \mathbb{N}. \tag{2.38}$$

Without loss of generality, we suppose that $n_j = j$, that is C_n is non-empty for all n. Had this not been the case, we could proceed by the same argument, but writing n_j instead of n (this is purely for ease of reading).

Now look at the sequence of balls $\{B(y_n, 2^n)\}_{n=1}^{\infty}$. We claim that

$$B(y_n, 2^n) \subseteq \bigcup_{k=-\min\{K_0,n\}}^{K_1} C_{n+k}, \tag{2.39}$$

a finite union of no more than $K_0 + K_1 + 1$ coronas, the number of which depends only on the quasi-triangle constant κ_0 (the exact nature of these constants to be determined later).

Fix n and let $z \in B(y_n, 2^n)$. Then

$$\rho(x,z) \leq \kappa_0(\rho(x,y_n) + \rho(y_n,z)) \leq \kappa_0(\kappa_0 2^{n+2} + 2^n) \leq \kappa_0^2 \cdot 2^{n+3} = \kappa_0 \cdot 2^{n+3+\log_2(\kappa_0)}.$$

This means that z cannot belong to a corona numbering higher than $n+3+\log_2(\kappa_0)$. Setting $K_1 := 3 + \lceil \log_2(\kappa_0) \rceil$ is sufficient. We now break into cases, according to whether or not

$n > K_0$ where $K_0 := \lceil \log_2(\kappa_0) \rceil$. The reason for this choice will become clear when analyzing the second case.

Case 1. ($n \leq K_0$) In this case, the union on line (2.39) is actually equal to $B(x, \kappa_0 2^{n+K_1})$. So z belongs to the union by the calculation above.

Case 2. ($n > K_0$) We need to show that $\rho(x,z) \geq \kappa_0 2^{n-K_0-1}$. Suppose that $\rho(x,z) < \kappa_0 2^{n-K_0-1}$. Then

$$\begin{aligned}\rho(x, y_n) &\leq \kappa_0(\rho(x,z) + \rho(z, y_n) < \kappa_0(\kappa_0 2^{n-K_0-1} + 2^n) \\ &= \kappa_0(2^{n+\log_2(\kappa_0)-K_0-1} + 2^n) < \kappa_0 2^{n+1},\end{aligned}$$

since $K_0 > \log_2(\kappa_0)$. Therefore we have reached a contradiction with the fact that $y \in C_{n+2}$.

Now, since $\lim\limits_{n\to\infty} \mu(C_n) = 0$, the limit of the union (2.39) also goes to zero. This implies by the work above that $\lim_{n\to\infty} \mu(B(y_n, 2^n)) = 0$. However, by the distant balls lemma (2.4.4) we have that

$$\begin{aligned}\mu(B(x, 2^n)) &\leq \kappa_1^{\log_2\left(\kappa_0 \cdot \frac{\rho(x,y_n)}{2^n} + \kappa_0\right)} \cdot \mu(B(y_n, 2^n)) \\ &\leq \kappa_1^{\log_2\left(\kappa_0 \cdot \frac{2^{n+2}}{2^n} + \kappa_0\right)} \cdot \mu(B(y_n, 2^n)) \\ &\leq \kappa_1^{\log_2(5\kappa_0)} \cdot \mu(B(y_n, 2^n)).\end{aligned}$$

This goes to zero as $n \to \infty$ which contradicts our original supposition of X having non-trivial measure.

$\square$

The Finite Measure Lemma is a very useful characterization of what kinds of measures are permitted when we wish to have doubling. With it, we can exclude spaces such as Example 2.2.7 without resorting to explicit calculation. In Chapter 3, we will expand on this lemma.

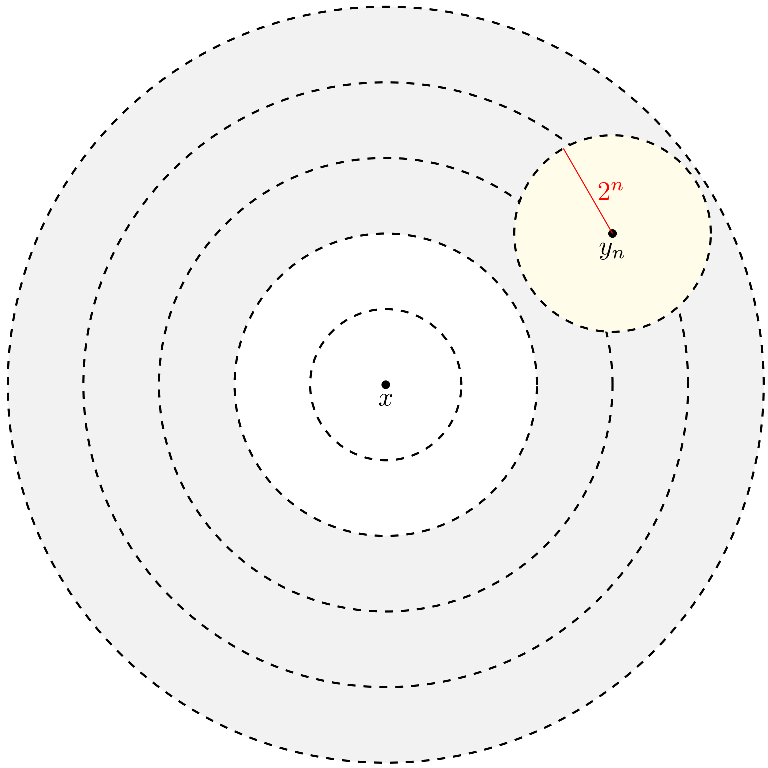


Figure 2.7: The ball $B(y_n, 2^n)$ is contained in $K = K_0 + K_1 + 1$ coronas. We do not draw the circles "to scale" in this figure because they double in radius as they grow and would grow to large to represent in the figure.

2.4.3 The Geometric Doubling Theorem

In this section we prove Theorem 2.3.3. This theorem and proof are due to Coifman and Weiss, found in [13] (pg. 69), but was also proved in [48]. Here we have updated the terminology to be in line with ours, but the logic remains the same.

We use this helper theorem.

Theorem 2.4.5 (Geometric Doubling is Equivalent to Global Maximum on Disperse Points)**.** *Suppose (X, ρ) is a quasi-metric space with quasi-triangle constant κ_0. Then the following are equivalent:*

(a) ρ is geometrically doubling with constant γ_0.

(b) There exists a constant $\gamma_1 \in \mathbb{N}$ such that any ball $B(x, r)$ can contain at most γ_1 many $\frac{r}{2}$-disperse points.

We prefer to prove the easy direction first.

Proof that (b) $\Rightarrow$ (a). Let $B := B(x, r)$ be a ball and let $\{x_1, ..., x_N\} \subseteq B$ be a collection of $\frac{r}{2}$-disperse points which does not permit any more points. That is, for all $y \in B(x, r)$, $\rho(y, x_i) < r/2$ for at least one x_i. We note that $N \leq \gamma_1$. Then the collection $\{B(x_i, r/2)\}_{i=1}^{N}$ is a (perhaps redundant) cover for $B(x, r)$ with no more than $\gamma_0 := \gamma_1$ elements in the collection. □

Proof that (a) $\Rightarrow$ (b). Suppose the desired result is false. That is, we suppose that ρ has the geometric doubling property, but not a bound on disperse points. Then for any $C \in \mathbb{N}$ we can find a ball $B(x, r)$ containing at least C many $r/2$-disperse points.

Begin by fixing $C > \gamma_0^a$ where $a \geq 2$ is an integer exponent to be determined. By supposition, find a ball $B_1 := B(x, r)$ which satisfies that there are C many $r/2$ disperse points contained in B_1 and label these points $\{x_i\}_{i=1}^{C}$. Since ρ is geometrically doubling, we can find balls of radius $r/2$ and centers $\{y_{n,1}\}_{n=1}^{\gamma_0}$ such that

$$B_1 \subseteq \bigcup_{n=1}^{\gamma_0} B(y_{n,1}, r/2),$$

that is that these balls cover B_1. By the pigeon-hole principle, one of these balls must contain more than γ_0^{a-1} of the points x_i. We label this ball B_2, and continue in the fashion finding centers $\{y_{n,2}\}_{n=1}^{\gamma_0}$ so that

$$B_2 \subseteq \bigcup_{n=1}^{\gamma_0} B(y_{n,2}, r/4),$$

and so on, stopping after a steps when we find a ball B_a with radius $r \cdot 2^{-a}$ which contains exactly two of the x_i. Refer to these two points as x_1 and x_2, and the center of B_a as y. Then,

$$\rho(x_1, y) < r \cdot 2^{-a} \quad \text{and} \quad \rho(x_2, y) < r \cdot 2^{-a}.$$

However,

$$\frac{r}{2} < \rho(x_1, x_2) \leq \kappa_0(\rho(x_1, y) + \rho(x_2, y)) < \kappa_0 r \cdot 2^{-a+1}$$

which implies that

$$2^{a-2} < \kappa_0.$$

We can thus choose a large enough to force a contradiction. This means that $C \leq \gamma_1 := \gamma_0^a$. □

Iterating, we will get the following result.

Corollary 2.4.6. *Let (X, ρ) be a quasi-metric space with quasi-triangle constant κ_0. Suppose also that X is geometrically doubling with doubling constant γ_0. Then for any pair of radii, r_1 and r_2, there exists a natural number N depending only on γ_0 and the ratio r_1/r_2 such that any ball $B(x, r_1)$ contains at most N many r_2-disperse points.*

Proof. If $r_2 \geq r_1$ set $N := 1$. Otherwise set $N := \gamma_0 \cdot \lceil \log_2(r_1/r_2) \rceil$ and apply the theorem above repeatedly. Recall that γ_0 depends only on κ_0. □

We are now ready to prove the earlier deferred theorem.

Proof of Theorem 2.3.3. We proved above that geometric doubling is equivalent to bounded dispersiveness, therefore it is enough to show that (X, ρ) has bounded dispersiveness.

Let $B := B(x, r)$ be an arbitrary ball in X. Set $R := r(\kappa_0 + 1/4)$. Suppose $x_1, x_2, ..., x_N$ are some points in B which satisfy that $\rho(x_i, x_j) > r/2$ for $i \neq j$. We will prove that N can be in fact bounded uniformly by γ_0 with respect to x and r.

We first claim that the balls $B(x_i, \frac{r}{4\kappa_0})$ are all disjoint and contained in the larger ball $B(x, R)$, for $R := r\left(\kappa_0 + \frac{1}{4}\right)$ To see this, first observe that if there existed $y \in B(x_i, \frac{r}{4\kappa_0}) \cap B(x_j, \frac{r}{4\kappa_0})$ then

$$\rho(x_i, x_j) \leq \kappa_0(\rho(x_i, y) + \rho(x_j, y)) < \kappa_0\left(\frac{r}{4\kappa_0} + \frac{r}{4\kappa_0}\right) = \frac{r}{2}$$

implying that $x_i = x_j$ since these points are $\frac{r}{2}$-disperse. On the other hand, if $y \in B(x_i, \frac{r}{4\kappa_0})$, then

$$\rho(x, y) \leq \kappa_0(\rho(x, x_i) + \rho(x_i, y)) < \kappa_0\left(r + \frac{r}{4\kappa_0}\right) = r\left(\kappa_0 + \frac{1}{4}\right) = R.$$

Thus, since the balls are disjoint,

$$\bigcup_{i=1}^{N} B\left(x_i, \frac{r}{4\kappa_0}\right) \subseteq B(x, R)$$

implying that

$$\sum_{i=1}^{N} \mu\left(B\left(x_i, \frac{r}{4\kappa_0}\right)\right) \leq \mu(B(x, R))$$

by the basic properties of measure.

Next we claim that the ball $B(x, R)$ is contained in a collection of larger balls $B(x_i, R')$ where $R' := \kappa_0(\kappa_0 + \frac{5}{4})r$. To prove this claim, see that if $y \in B(x, R)$ then for all $i = 1, ..., N$

$$\begin{aligned}\rho(x_i, y) \leq\ & \kappa_0(\rho(x_i, x) + \rho(x, y)) \ < \kappa_0(r + R) \\ &= \kappa_0\left(r + r\left(\kappa_0 + \frac{1}{4}\right)\right) = \kappa_0 r\left(\kappa_0 + \frac{5}{4}\right) =: R'.\end{aligned}$$

Select k so that

$$\mu\left(B\left(x_k, \frac{r}{4\kappa_0}\right)\right) = \min_{j=1,...,N} \mu\left(B\left(x_j, \frac{r}{4\kappa_0}\right)\right),$$

that is, k is the index of the center whose ball has the least measure. Then we have that

$$\begin{aligned}
N \cdot \mu\left(B\left(x_k, \frac{r}{4\kappa_0}\right)\right) &\leq \sum_{i=1}^{N} \mu\left(B\left(x_i, \frac{r}{4\kappa_0}\right)\right) \\
&= \mu\left(\bigcup_{i=1}^{N} B\left(x_i, \frac{r}{4\kappa_0}\right)\right) \\
&\leq \mu(B(x, R)) \\
&\leq \mu(B(x_k, R')) \\
&\leq \gamma_0 \cdot \mu\left(B\left(x_k, \frac{r}{4\kappa_0}\right)\right)
\end{aligned}$$

where the constant γ_0 is given by Lemma 2.1.11. Explicitly,

$$\gamma_0 = \kappa_1^{\log_2(\lceil 4\kappa_0 R'/r \rceil)} = \kappa_1^{\log_2\left(\left\lceil 4\kappa_0^2\left(\kappa_0 + \frac{5}{4}\right)\right\rceil\right)}.$$

Thus γ_0 depends only on κ_0 and κ_1 and is independent of r (and x for that matter). The above calculation therefore gives that

$$N \cdot \mu\left(B\left(x_k, \frac{r}{4\kappa_0}\right)\right) \leq \gamma_0 \cdot \mu\left(B\left(x_k, \frac{r}{4\kappa_0}\right)\right)$$

implying $N \leq \gamma_0$ since μ is assumed to be non-zero for all balls. This completes the proof. $\square$

2.5 Concerning Atoms

Next we consider properties of point masses or atoms. Atoms turn out to be somewhat tricky in the theory of SHTs as they often pose annoying problems. We will be able to better define the troubles with atoms in later chapters. For now we will give their definition and some simple results.

Definition 2.5.1 (Atom)**.** For (X, μ) a measure space, let $x \in X$ be a point with non-zero measure, i.e., $\mu(\{x\}) > 0$. Such a point is called an *atom*. Some authors use the name *point mass*, but we will refrain from doing so here.

We will denote by $\mathcal{A}$ the set of all atoms of X.

The following theorem is due to Macías and Segovia, found in [34].

Theorem 2.5.2 (Atoms are Isolated). *Let (X, ρ, μ) be an SHT whose measure is non-trivial. Then the following are true:*

(a) For all $x \in \mathcal{A}$, x is an isolated point, i.e., there exists a radius $r_x > 0$ such that $B(x, r_x) = \{x\}$.

(b) The set $\mathcal{A}$ is at most countable.

The proof is a simple proof by contradiction. We give the details in Appendix 1.

Corollary 2.5.3. *Let X be an SHT with nontrivial measure. Then $x \in X$ is an atom if and only if x is an isolated point.*

Proof. Follows from Theorem 2.5.2 and the converse of part (1) of Lemma 2.4.1. $\square$

Corollary 2.5.3 will be important in Chapters 3 and 5 when we develop the dyadic theory and Haar theory respectively.

2.6 Subspaces of Homogeneous Type

We will close out the chapter with a short section describing subspaces.

2.6.1 Definition and Simple Examples

We will define subspaces in the logical way

Definition 2.6.1. Let (X, ρ, μ) be a space of homogeneous type. Let Σ be the σ-algebra of μ-measurable sets. Suppose that $Y \subseteq X$ is a measurable subset, and define $\rho_Y := \rho|_{Y \times Y}$ and $\mu_Y := \mu|_{\Sigma_Y}$ where $\Sigma_Y := \{S \in \Sigma \mid S \subseteq Y\}$. If (Y, ρ_Y, μ_Y) is in its own right a space of homogeneous type, then Y is a *subspace of homogeneous type of* X (abbr. subSHT).

We *do not* require that Y has the same geometric constants as X. Since any subset of a quasi-metric space is also a quasi-metric space, the same quasi-triangle constant can be used for subSHTs, however it may be possible to find a smaller one. With regards to the doubling measure constant, it could be either larger or smaller, depending on the subspace.

Example 2.6.2 (Trivial Atomic Singleton)**.** Let X be an SHT with at least one atom $a \in X$. The singleton set $\{a\}$ is a subSHT of X with both geometric constants equal to one, no matter what they were for X.

Example 2.6.3 (Area Below a Parabola)**.** Consider the SHT $\mathbb{R}^2$ equipped with the 1-norm (i.e. taxicab distance) and the usual Lebesgue measure. This space has doubling measure constant $\kappa_1 = 4$. The area beneath the curve $y = x^2$ is a subSHT (see Figure 2.8). However, it requires a larger doubling constant. To see this, observe that for balls centred at zero with radius less than one,

$$\mu(B(0, 2r)) = \int_{-2r}^{2r} x^2\, dx = \frac{16r^3}{3} = 8 \cdot \frac{2r^3}{3} = 8 \cdot \int_{-r}^{r} x^2\, dx = 8 \cdot \mu(B(0, r)). \tag{2.40}$$

This shows that a doubling constant of at least 8 is required.

As we develop more technology in the proceeding chapters, we will further revisit the idea of subspaces of homogeneous type. For now, we should think back to the examples of the bullseye space from earlier in this chapter, which clearly shows that *not every subset is a subspace of homogeneous type.* Full classification of exactly what can and can't be a subSHT is beyond the scope of this document, however we will partially tackle this question in Chapter 3 with the first and second dyadic subspace theorems.

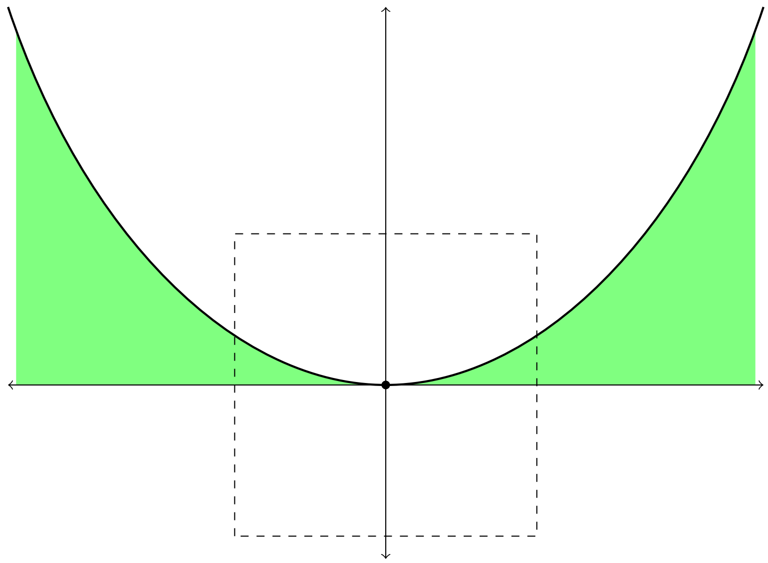


Figure 2.8: The subSHT example of the area beneath a parabola, with a “ball” around the origin.

The Bullseye space example illustrates that not every sub*set* of an SHT is a sub*space*. A natural question would be, “what kinds of subsets are subspaces?” For example, one might be lead to ask about the open balls. In [1], it is shown that this will be true given an additional assumption on the measure. At this time, the author does not know if this is true in general.

In the next chapter, we will be interested in the question of whether or not dyadic cubes are SHTs.

Chapter 3

A Tour of Dyadic Theory in Spaces of Homogeneous Type

In this chapter we will begin the process of developing a dyadic theory in spaces of homogeneous type. Most of this chapter will deal with quasi-metric spaces with the geometric doubling property, and will not require a measure. However, because of Theorem 2.3.3, everything will be applicable to SHTs.

We first review the basics of classical dyadic theory in order to give ourselves a starting ground before moving to the generalized SHT dyadic theory. We then move on to discussing some of the properties of so called "quadrants," which are essentially generalizations of the four familiar quadrants of the plane of $\mathbb{R}^2$. We close the chapter with a construction of a "honest" dyadic system- so called because each cube has exactly two children. This construction will prove to be quite useful in later chapters.

3.1 A Review of Dyadic Theory in $\mathbb{R}^n$

We will open this chapter with an overview of dyadic systems in $\mathbb{R}^n$. This material is all very standard and should be review for most readers. See [43] or [22], for a more extensive introduction to dyadic harmonic analysis.

3.1.1 Dyadic Intervals and Cubes

Definition 3.1.1 (Dyadic Interval)**.** An interval $I \in \mathbb{R}$ is a *dyadic interval* if

$$I = [j \cdot 2^{-k}, (j+1) \cdot 2^{-k}) \tag{3.1}$$

for integers j and k.

Dyadic intervals are the building blocks which our dyadic cubes will be made up of.

Definition 3.1.2 (Dyadic Cube)**.** A set $Q \subset \mathbb{R}^n$ is a *dyadic cube* if Q is the Cartesian product of dyadic intervals of the same length. We call this length Q's *side length*, and denote it by $\ell(Q)$.

We will denote the collection[1] of all dyadic cubes[2] as $\mathscr{D}(\mathbb{R}^n)$ or simply $\mathscr{D}$ if the underlying space is understood.

3.1.2 Basic Observations About Dyadic Systems

Let us now make some observations about dyadic intervals/squares/cubes:

[1]We will try to stick with the convention of referring to sets of sets, such as the dyadic intervals, as *collections*.

[2]In the special case of $\mathbb{R}^2$ it might make more sense to refer dyadic *squares* instead of cubes.

- **Cubes are organized into generations.** For any integer k, the collection $\mathscr{D}_k := \{Q \in \mathscr{D} \mid \ell(Q) = 2^{-k}\}$ forms a partition of $\mathbb{R}^n$.

- **Cubes are mutually nested.** For any two cubes Q and Q', exactly one of the following is true: $Q \subseteq Q'$, $Q' \subsetneq Q$, or $Q \cap Q' = \emptyset$.

- **Cubes have unique parents.** For every cube $Q \in \mathscr{D}_k$, there exists a unique cube $\widehat{Q} \in \mathscr{D}_{k-1}$ such that $Q \subseteq Q'$. We refer to $\widehat{Q}$ as Q's *parent*.

- **Cubes have a set number of children.** For any cube $Q \in \mathscr{D}_k$ there are exactly 2^n cubes belonging to $\mathscr{D}_{k+1}$ which are subsets of Q. We refer to these cubes as Q's *children*, and denote by $\mathrm{ch}(Q)$ the set of all Q's children. We may sometimes refer to two cubes which have the same parent as *siblings*. In the particular case where $n = 1$ and cubes are intervals, we call the children of I the left and right children and denote them as I_ℓ and I_r respectively.

  Also note that for any cube Q, $\ell(Q) = \ell(\widehat{Q})/2$.

Remark 3.1.3. As a matter of convention, we include a minus sign in the definition of the collection of the generations $\mathscr{D}_k$. This ensures that cubes shrink as the generation increases, which more closely aligns with our intuition about the words “child” and “parent.”

The usefulness of dyadic cubes in harmonic analysis cannot be understated. These cubes have been instrumental in the proving of many important theorems such as the Calderón-Zygmund Decomposition, itself central to the study of singular integral operators (see [17], [18], and [14]). It is for this reason that developing a dyadic theory for SHTs was desired.

We will introduce a little more notation here. For $Q \in \mathscr{D}$, by $\mathscr{D}(Q)$ we mean the set of all cubes which are subsets of Q, i.e. Q's descendants. Furthermore, for $j \in \mathbb{Z}$, by $\mathscr{D}_j(Q)$ we mean the cubes in $\mathscr{D}(Q)$ which are exactly j generations below Q. For example, if $Q \in \mathscr{D}_{10}$ then $\mathscr{D}_6(Q)$ would be equal to $\{Q' \in \mathscr{D}_{16} \mid Q' \subseteq Q\}$.

3.2 Dyadic Theory in Quasi-metric Spaces

Because of the usefulness of dyadic cubes to the field of harmonic analysis over $\mathbb{R}$ and $\mathbb{R}^n$, it was natural to ask if a similar type of structure could exist within the realm of SHTs. In this section we will be citing the relevant theorems which allow us to make use of dyadic grids.

3.2.1 Historical Perspective

M. Christ is widely credited as being the first to fully formalize a construction of a dyadic lattice on spaces of homogeneous type in the 1990 paper [10]. While earlier partial constructions exist, such as in [46] by Sawyer and Wheeden, it was the so called "Christ cubes" which first had the nice properties outlined in Section 3.1.2. However, Christ's construction had a few drawbacks. First, Christ's construction was not over the entirety of X and omitted infinitely many points, although the set of all points omitted had μ-measure zero. Secondly, the proof relied on the use of the axiom of choice to locate the dyadic centers. Lastly, the theorem requires a space of homogeneous type, and not simply a quasi-metric space. Nevertheless, Christ's theorem is still widely cited, and is for example the basis for the dyadic theory presented in [1] and [20]

We will be basing our theory on the cubes of T. Hytönen and A. Kairema, which is the main result of [25] (see also [29]). This construction has the benefit that every point of X belongs to exactly one cube in each generation, an advantage over the Christ cubes. Furthermore, this theorem is totally independent of measure. All that is required to construct the dyadic lattice is that (X, ρ) be a quasi-metric space which is geometrically doubling.[3]

[3]Recall the result of Chapter 2 that any SHT is also a geometrically doubling measure space.

3.2.2 Hytönen-Kairema Cubes

We now present the modern theorem for dyadic cubes over SHTs. As stated above, this construction has several advantages over the original Christ cubes which we will soon see.

The construction follows by first proving the existence of a set of points called *dyadic centers*, which forms the skeleton of the system of dyadic cubes.

Proposition 3.2.1 (Dyadic Centres Exist [25]). *Let (X, ρ) be a quasi-metric space which is geometrically doubling. Let $0 < c_0 < C_0$ and $\delta \in (0, 1)$ be constants. For every $k \in \mathbb{Z}$ there exists a set of points $\{z_\alpha^k\}_{\alpha \in I_k}$, I_k being an index set at most countable, which satisfies that*

$$\rho(z_\alpha^k, z_\beta^k) \geq c_0 \delta^k; \;\; \forall \alpha \neq \beta \in I_k \tag{3.2}$$

$$\inf_{\alpha \in I_k} \rho(x, z_\alpha^k) < C_0 \delta^k \;\; \forall x \in X \tag{3.3}$$

i.e., that every $x \in X$ is no further than $C_0\delta^k$ from some z_α^k, but the set is $c_0\delta^k$-disperse. We will call the collection of all such sets dyadic points *or* dyadic centers.

That the dyadic centers exist is consequence of maximality. For details, see [25, Subsection 2.21]. We give Proposition 3.2.1 because the next theorem depends on it. This is the major result which tells us that there exists at least one dyadic lattice. The proof can be found in [25, Section 2, pg. 4-9].

Theorem 3.2.2 (Dyadic Cube Existence). *Suppose that constants $0 < c_0 < C_0$ and $\delta \in (0, 1)$ satisfy that*

$$12\kappa_0^3 C_0 \delta \leq c_0. \tag{3.4}$$

Given a collection of dyadic points, we can construct families of sets $\{\tilde{Q}_\alpha^k\}_{\alpha \in I_k}$, $\{Q_\alpha^k\}_{\alpha \in I_k}$, and $\{\overline{Q}_\alpha^k\}_{\alpha \in I_k}$ – called open, half-open, and closed dyadic cubes *respectively – such that:*

(a) $\tilde{Q}_\alpha^k$ and $\overline{Q}_\alpha^k$ are respectively the interior and closure of Q_α^k;

(b) If $\ell \geq k$ then either $Q^k_\alpha \subseteq Q^\ell_\beta$ or $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$;

(c) $X = \bigsqcup_\alpha Q^k_\alpha$ (the disjoint union) for all $k \in \mathbb{Z}$;

(d) $B(z^k_\alpha, c_1\delta^k) \subseteq Q^k_\alpha \subseteq B(z^k_\alpha, C_1\delta^k) =: B(Q^k_\alpha)$ where $c_1 := (3\kappa_0^2)^{-1}c_0$ and $C_1 := 2\kappa_0 C_0$;

(e) If $k \leq \ell$ and $Q^\ell_\beta \subseteq Q^k_\alpha$ then $B(Q^\ell_\beta) \subseteq B(Q^k_\alpha)$.

Let us dissect this theorem a little. Property (a) gives us that the open, closed, and half-open cubes are named appropriately. Property (b) is the familiar pairwise nestedness. Property (c) tells us that for every k the collection $\{Q^k_\alpha\}_{\alpha \in I_k}$ forms a true partition of X. So far, these are the same types of properties that we would come to expect, given our understanding of dyadic systems in $\mathbb{R}^n$. Properties (d) and (e) are particular to SHTs. These properties define an interior and exterior ball associated to each cube, and give that exterior balls have similar nesting to cubes.

The point $z^k_\alpha \in Q^k_\alpha$ is called the center of Q^k_α. Likewise ,we can generalize the notion of side length by declaring that $\ell(Q^k_\alpha) = C_1\delta^k$, i.e. the radius of Q^k_α's exterior ball.

From the above properties, we can deduce this familiar result:

Proposition 3.2.3. *Let $k \in \mathbb{Z}$ and $\alpha \in I_k$. Then there exists a unique $\beta \in I_{k-1}$ such that $Q^k_\alpha \subseteq Q^{k-1}_\beta$.*

Definition 3.2.4 (Dyadic Systems)**.** The collection of dyadic cubes from Theorem 3.2.2 is called a *dyadic system* (also *lattice, structure, grid*) on X. We denote by $\mathscr{D}(X)$ (or just $\mathscr{D}$) the set of all such cubes. For each $k \in \mathbb{Z}$ the family $\mathscr{D}_k := \{Q^k_\alpha\}_{\alpha \in I_k}$ is called the k^{th} generation. For any cube Q belonging to generation $\mathscr{D}_k$ we call the unique cube belonging to generation $\mathscr{D}_{k-1}$ given by Proposition 3.2.3 Q's *parent*, and denote it by $\widehat{Q}$. If two cubes have the same parent, we refer to them as *siblings*.

We will reuse the notation of $\mathscr{D}(Q)$ and $\mathscr{D}_j(Q)$ to mean the same here as they did in $\mathbb{R}^n$.

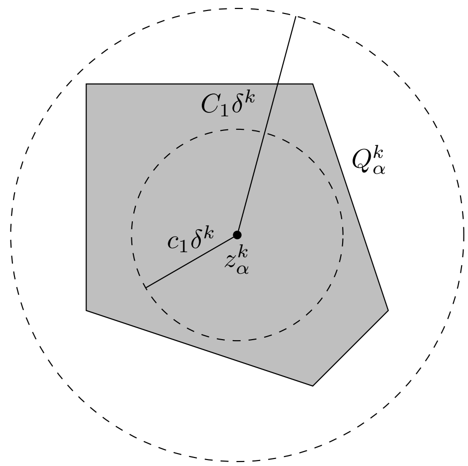


Figure 3.1: A typical cube $Q_\alpha^k \in \mathscr{D}$, with its inner and outer balls $B(z_\alpha^k, c_1\delta^k)$ and $B(z_\alpha^k, C_1\delta^k)$.

Remark 3.2.5. The constants c_0, C_0, and δ are parameters which are restrained by κ_0 via inequality (3.4). For this reason, we consider them geometric constants. Likewise, c_1 and C_1 are also geometric constants since they are defined in part (d) of Theorem 3.2.2.

Remark 3.2.6 (Cube Equivalence)**.** We consider cubes to be more than just subsets of X. They also carry with them their generation. With this in mind, it is totally possible for two cubes to be equal *as sets* but not *as cubes*. For example, if Q is a cube with no siblings, then $\widehat{Q} \subseteq Q$ and $Q \subseteq \widehat{Q}$ but $Q \neq \widehat{Q}$ because they belong to different generations. This distinction will matter more later on and we will be careful to say precisely what we mean when talking about cube equivalence.

As remarked above, dyadic structures are defined entirely with respect to a quasi-metric space. There is no dependence on the measure, or even a requirement that one be defined. However, we do have this final result to close out this section:

Corollary 3.2.7 (Dyadic Cubes in SHTs)**.** *Let (X, ρ, μ) be an SHT with non-trivial measure, and let constants $0 < c_0 < C_0$ and $\delta \in (0,1)$ satisfy (3.4). Then a system of dyadic cubes such as in Theorem 3.2.2 can be constructed for X.*

Proof. By Theorem 2.3.3, (X, ρ) is a quasi-metric space with the geometric doubling property. Therefore, the dyadic centers exist by Proposition 3.2.1, which fulfills the hypothesis for Theorem 3.2.2. □

Corollary 3.2.8 (Parent Cubes)**.** *Let X be an SHT with at least two points. Let $\mathscr{D}$ be a dyadic lattice for X an SHT. There exists a constant $Dbl(\mathscr{D}) \geq 2$ so that*

$$\mu(\widehat{Q}) \leq Dbl(\mathscr{D}) \cdot \mu(Q). \tag{3.5}$$

for all $Q \in \mathscr{D}$.

This result is a corollary of Lemma 2.1.11. See Appendix 3 for the details of the proof. We call the constant from (3.5) the *dyadic doubling constant for $\mathscr{D}$.* The reason $Dbl(\mathscr{D}) \geq 2$ is clear enough: There must be a cube Q with more than one child cube. If Q_+ and Q_- are two of Q's children, then $\mu(Q_+) + \mu(Q_-) \geq \mu(Q)$ implies either $\mu(Q)/\mu(Q_+) \geq 2$ or $\mu(Q)/\mu(Q_-) \geq 2$.

3.2.3 Boundedness on the Number of Children

For each $Q \in \mathscr{D}$, denote by $\mathrm{ch}(Q)$ the set of all children of Q and denote by $N(Q) := \#\,\mathrm{ch}(Q)$ the number of children of Q.

Theorem 3.2.9. *There exists $N_{ch} \in \mathbb{N}$ such that $N(Q) \leq N_{ch}$ for all $Q \in \mathscr{D}$.*

This theorem can be found in several places, including [1], [20], and [25]. We will give a proof here as well.

Proof. It is sufficient to show that $N(Q^k_\alpha)$ can be uniformly bounded by some constant not depending on α or k. Let $Q^k_\alpha \in \mathscr{D}_k$ be a cube and let $B(z^k_\alpha, C_1\delta^k)$ be its outer ball. For each child $Q^{k+1}_\beta \in \mathrm{ch}(Q^k_\alpha)$ there exists an inner ball $B(z^{k+1}_\beta, c_1\delta^{k+1})$. Since each child is disjoint and the inner balls are subsets of the children, the set of points z^{k+1}_β are $c_1\delta^{k+1}$-disperse. By

Corollary 2.4.6, the number of such points is bounded by γ_1, a constant depending only on the quasi-triangle constant and the ratio of the radii. This ratio is equal to $c_1\delta^{-1}$, which does not depend on k (or α for that matter) at all. Thus γ_1 is uniform across all generations. Since the number of *centers* is equal to the number of children, we are done. $\square$

3.2.4 An Example

Example 3.2.10. Consider the set $X \subset \mathbb{R}^2$ defined by

$$X = \bigcup_{n=0}^{\infty} \left([n, n+2^{-n}) \times [0, 2^{-n})\right). \tag{3.6}$$

If we look at X with the usual measure and metric on $\mathbb{R}^2$, then the usual dyadic structure on

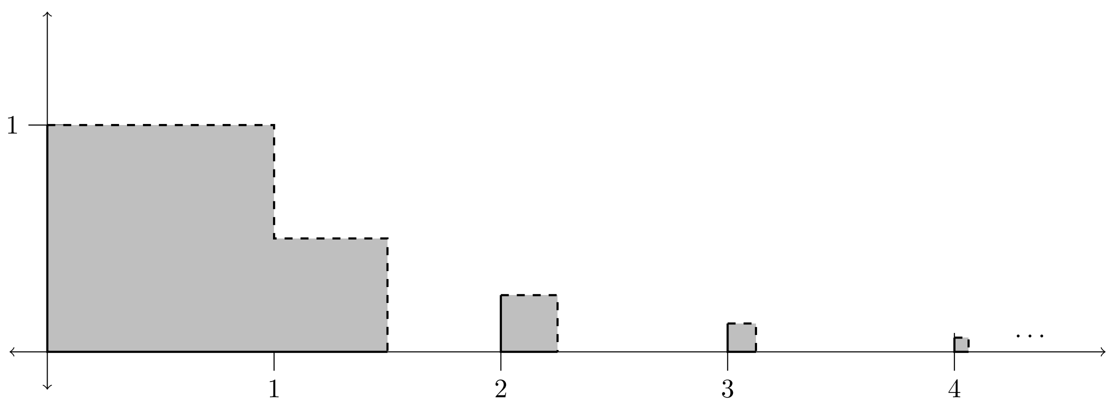


Figure 3.2: The set X in Example 3.2.10.

$\mathbb{R}^2$ can function as a dyadic structure on X as well. That the cubes exist is obvious, however there is a detail to consider with regards to the inner and outer balls. For any isolated square, say $[m, m+2^{-m}) \times [0, 2^{-m})$, with $m > 0$ fixed, the entire square will be a cube for every generation $0 \leq k < m$. Thus, we can use as a center the point $z := (m+2^{-m-1}, m+2^{-m-1})$. It appears that we have a problem with our inner ball $B(z, 2^{-k-1})$, since it looks like it lies *outside* the cube. However, the ball is actually just the entire contents of the cube, since *nothing* lies outside it as far as X is concerned. With this in mind, we can continue to use the same center point until the inner ball actually lies inside the square in generation $k = m$, at which point the normal subcubes and centers kick in for all subsequent generations.

An interesting property of this example is that given any generation $\mathscr{D}_k$, there are infinitely many cubes which have only one child. Moreover, for any natural number m, we can find a cube belonging to generation zero with the property that this cube will have only one descendant for m number of generations.

This shows us that there need not be a bound on the number of generations for a cube to finally split.

3.3 Dyadic Cubes in SHTs

One of the advantages of the dyadic cubes of Hytönen and Kairema is that the construction lives entirely within the realm of quasi-metric spaces. This allows us to have a dyadic lattice on *any* quasi-metric measure space without issue. Of course, in this document we are primarily interested in spaces of homogeneous type, although we will briefly look at a non-doubling result of sorts in the Chapter 5.

We will now look at a few additional properties of dyadic cubes when there is an underlying doubling measure.

3.3.1 Thin Boundaries

Christ's construction of dyadic cubes, having a dependence on the measure, have an extra useful property:

Property 3.3.1 (Christ Cubes Have Thin Boundaries)**.** Let $\mathscr{D}$ be a system of dyadic cubes constructed via the method found in [10]. Then there exist constants $C, \eta > 0$ such that for all cubes Q in generation $\mathscr{D}^k$,

$$\mu(\{x \in Q \mid \rho(x, X \setminus Q) < t\delta^k\}) < Ct^\eta \mu(Q) \tag{3.7}$$

for every $t > 0$.

Christ cubes are automatically also Hytönen-Kairema cubes. Yet, a priori there is no obvious reason that the newer Hytönen-Kairema cubes would have the same property. In [28], Hytönen and Kairema get around this by introducing the notion of *adjacent systems* of cubes. The details of this are not important to this document. We will instead, when necessary, include the assumption that the cubes have thin boundaries.

3.3.2 The First Dyadic Subspace Theorem

We know that in general subsets of SHTs are not themselves subSHTs. However, we know that dyadic cubes are a special type of subset.

Theorem 3.3.2 (First Dyadic Subspace Theorem)**.** *Let X be an SHT and let $\mathscr{D}$ be a dyadic lattice on X such that cubes have thin boundaries as in 3.3.1. Then for every cube $Q \in \mathscr{D}$, Q is a subSHT of X. Moreover, the geometric constants are independent of choice of cube.*

This theorem can be found in [1] (**Proposition 3.24**) and [20] as part of the list of properties of dyadic systems. At the time of this writing, it is not clear if the property of thin-boundaries will automatically exist for any collection of Hytönen-Kairema cubes, or if it needs to be assumed when needed. It would be interesting to try to prove that the assumption is redundant, or to find a counter-example.

3.4 Concerning Quadrants

In the usual dyadic intervals on $\mathbb{R}$, the number zero uniquely has the property that it is not contained in the interior of any dyadic intervals. This effectively splits the real numbers into two "non-interacting" halves: the positive and negative numbers. This generalizes in the expected way where in $\mathbb{R}^2$ every point on the x and y axes has this property, and so on in higher dimensions.

The moral of the situation is that, from the perspective of the dyadic systems, *these non-interacting subsets might as well be totally separate spaces on their own.* In this section we will look at how this idea generalizes further to SHTs.

3.4.1 A Motivating Conundrum

Let us take a look at an example of a measure which might make us feel worried.

Example 3.4.1 (A Half-Bad Measure Space)**.** Let $X = \mathbb{R}$ with the usual metric. For the measure μ, we will use a *half-bad* measure

$$\mu(S) := \int_S m(x), dx\,; \qquad m(x) := \begin{cases} 1 & \text{if} \quad x \geq 0 \\ e^x & \text{if} \quad x < 0 \end{cases}$$

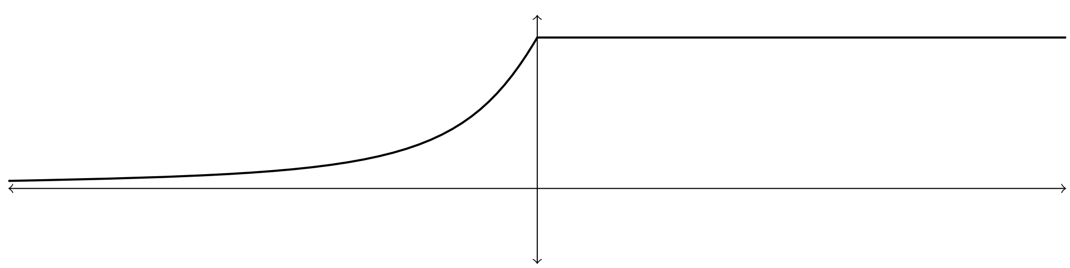


Figure 3.3: Graph of $y = m(x)$.

This measure space is *not* doubling, but at this point we need to show this by direct calculation. It is clear that $\mu(X) = \infty$, so the Finite Measure Lemma does not apply.

However, this space is half-bad, in sense that it has a bad subspace, namely the negative real numbers. Isolated on their own, the negative reals with μ restricted to them is not an SHT.

The idea here is that if a subset of a measure space is bad in some way, then it potentially takes the whole space down with it. However, we know from the Bullseye space example (Example 2.2.8), that it isn't enough to just have a subset which fails to be a subSHT.

We would like to say that things like the half-bad measure space can't be spaces of homogeneous type. We will try now to zoom in on what exactly we mean by this.

3.4.2 Quadrant Definition

We will get the the meat of this section now.

Definition 3.4.2 (Quadrants)**.** Let X be an SHT and let $\mathscr{D}$ be a dyadic lattice on X. For every $x \in X$, define the set

$$\mathrm{Quad}_{\mathscr{D}}(x) := \bigcup_{Q \in \mathscr{D} \mid Q \ni x} Q$$

as *the quadrant in $\mathscr{D}$ which contains x*. Furthermore, define *the collection of quadrants*

$$\mathrm{Quad}_{\mathscr{D}}(X) := \{\mathrm{Quad}(x) \mid x \in X\} / \sim$$

with $\sim$ an equivalence relation defined on quadrants where $\mathrm{Quad}_{\mathscr{D}}(x) \sim \mathrm{Quad}_{\mathscr{D}}(y)$ if and only if $\mathrm{Quad}_{\mathscr{D}}(x) = \mathrm{Quad}_{\mathscr{D}}(y)$, where the equality holds as sets. Equivalently, $\mathrm{Quad}_{\mathscr{D}}(x) \sim \mathrm{Quad}_{\mathscr{D}}(y)$ if and only if there exists a cube $Q \in \mathscr{D}$ which contains both x and y.

When the dyadic lattice is clear (i.e. there is only one in consideration) then we will omit the subscript and just write $\mathrm{Quad}(x)$ and $\mathrm{Quad}(X)$.

Remark 3.4.3. We will use the script letter $\mathfrak{Q}$ to refer to quadrants, to differentiate from the regular Q typically used for cubes.

3.4.3 Examples of Quadrants

At first, it is tempting to tie the number of quadrants to some notion of "dimension" of a space. After all, in $\mathbb{R}^n$ there are the usual 2^n quadrants which we are very familiar with, so we may decide to define $\dim(X) := \log_2(N_q)$ where N_q is the number of quadrants. The following examples show why this is not such a great plan.

Example 3.4.4 (Finitely Many Quadrants)**.** In Figure 3.4, we see a few different dyadic lattices on $\mathbb{R}^2$. We have divided $\mathbb{R}^2$ into N_q "slices" where each slice is the set $\{(r, \theta) \mid r > 0,\ 2\pi j/N_q \leq \theta < 2\pi(j+1)/N_q\}$, $0 \leq j < N_q$, with (r, θ) the polar form. We can then partition the slices into a collection of congruent isosceles triangles. At every generation, we can always divide each triangle into four similar triangles with sides half as long. These triangles are of course our cubes and the slices are the quadrants. Dyadic centers are the midpoints along the line of reflection symmetry. For inner and outer balls $\delta = 1/2$ and the values of c_1 and C_1 depend on the angle.

In this manner, we could easily impose as many or as few quadrants as we like: simply by shrinking the angle and squeezing in more slices. All this, in spite of the fact that $\mathbb{R}^2$ has dimension 2. However, we can not use this particular style to generate countably many quadrants. For that, something with slightly more fineness is needed.

Example 3.4.5 (A Space With Infinitely Quadrants)**.** Let $X \subset \mathbb{R}^3$ be the set $\{(x, y, z) \in \mathbb{R}^3 \mid x = t \cdot \cos(2\pi z), y = t \cdot \sin(2\pi z), t \geq 1\}$, an infinite spiral. Set $\rho(\mathbf{x}, \mathbf{y})$ to be length of the shortest path within X from $\mathbf{x}$ to $\mathbf{y}$ and set μ to be the surface area measure. Then (X, ρ, μ) forms a space of homogeneous type.

For each integer n, the section of X where $n \leq z < n+1$ is in bijection with $\mathbb{R}^2 \setminus B(0,1)$. Thus for each of these slices, the usual dyadic lattice for $\mathbb{R}^2$ excluding the unit disk forms a dyadic lattice for the slice. (In this example we exclude the unit disk because it causes too much warping near the z-axis and destroys cube uniformity for this lattice). As each

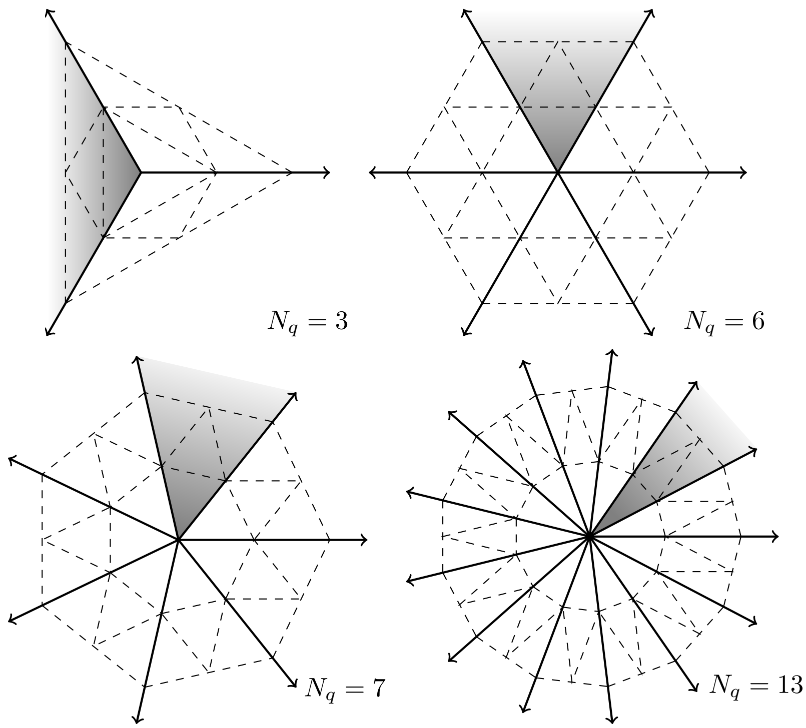


Figure 3.4: Example showing different dyadic structures imposed on $\mathbb{R}^2$, each with a different number of quadrants, (here called N_q). One quadrant is highlighted in each example, for emphasis.

individual slice has four quadrants, and there are infinitely many slices, X with this dyadic lattice has infinitely many quadrants.

3.4.4 A Bounded Subset Theorem

The following theorem is highly intuitive.

Theorem 3.4.6. *Let $\mathfrak{Q}$ be a quadrant in a dyadic system $\mathscr{D}$ over an SHT X. For every bounded subset $S \subseteq \mathfrak{Q}$, there is a cube $Q \in \mathscr{D}$ such that $S \subseteq Q$.*

Proof. We observe that $\text{Quad}(x) = \text{Quad}(y)$ if and only if there exists some dyadic cube Q which contains both x and y. Moreover, if there is a cube Q that contains two points x and

y, then every predecessor cube of Q (cubes belonging to previous generations) also contains x and y.

Let $S \subseteq \mathfrak{Q}$ be bounded. For any $k \in \mathbb{Z}$, we can cover S by finitely many cubes belonging to generation $\mathscr{D}_k$. Fix k arbitrarily and let $\{Q_n\}_{n=1}^{N} \subsetneq \mathscr{D}_k$ be the finite number of cubes which cover S. If $N = 1$ we are done. Otherwise, if $N > 1$, then choose two cubes Q_1 and Q_2 and let $x \in Q_1$ and $y \in Q_2$ with $x, y \in S$. Since the two cubes are disjoint, $x \neq y$. However, $\mathrm{Quad}(x) = \mathrm{Quad}(y) = \mathfrak{Q}$, since $S \subseteq \mathfrak{Q}$, so there must be a cube Q' which is a predecessor to both Q_1 and Q_2 which contains both x and y. If $Q' \in \mathscr{D}_j$ then $j < k$. Moreover, it must take fewer than N unique cubes from $\mathscr{D}_j$ to cover S since x and y now belong to a single cube. We can repeat this process until the number of cubes to cover S equals one. $\square$

3.4.5 Properties of Quadrants

Let us run down some basic properties of quadrants:

Theorem 3.4.7 (Dyadic Quadrant Properties)**.** *Let X be an SHT and let $\mathscr{D}$ be a dyadic structure on X. Let $\mathfrak{Q}$ be a quadrant of $\mathscr{D}$. We have that*

(a) all quadrants are pairwise disjoint.

(b) if X is a bounded space, then X has only one quadrant.

(c) if X is unbounded, then every quadrant is unbounded.

Proof of (a). The proof is a trivial application of the definition of quadrant. $\square$

Proof of (b). Let X be a bounded SHT. Since X is bounded, there must be $r > 0$ so that $B(x, r) = X$ for all $x \in X$ (choose, for example r larger than the diameter of X). Let $k \in \mathbb{Z}$ be small enough so that $c_1\delta^k > r$, where here c_1 and δ are as in Theorem 3.2.2. Then for all

$\alpha \in I_k$, the ball $B(z_\alpha^k, c_1\delta^k)$, which is the inner ball corresponding to the cube Q_α^k, is a super set of X. Fix α and let $\mathfrak{Q}$ be the quadrant that Q_α^k is a member of. Then

$$\mathfrak{Q} \supseteq Q_\alpha^k \supseteq B(z_\alpha^k, c_1\delta^k) \supseteq X.$$

But $\mathfrak{Q} \subseteq X$, so they are equal. Any other quadrant must be a subset of $X \setminus \mathfrak{Q} = \emptyset$, so there aren't any. □

Proof of (c). Let X be an unbounded SHT and suppose for the sake of a contradiction that $\mathfrak{Q}$ is a bounded quadrant of X. Fix $r > 0$ larger than the diameter of $\mathfrak{Q}$. Find k such that $c_1\delta^k > r$, and let Q_α^k be a cube in generation $\mathscr{D}_k$ which is also in $\mathfrak{Q}$. Then for any $x \in \mathfrak{Q}$, $\rho(x, z_\alpha^k) < r < c_1\delta^k$, meaning that x belongs to the inner ball of Q_α^k and thus $x \in Q_\alpha^k$. This actually implies that at the k^{th} generation, and for that matter any previous generation, $\mathfrak{Q}$ contains only one cube.

Now, since X is unbounded, we can find $y \in X$ such that $\rho(y, z_\alpha^k) > C_1\delta^k$. This means that $y \notin Q_\alpha^k$ because it lies outside of Q_α^k's outer ball. Moreover, since $Q_\alpha^k = \mathfrak{Q}$ as sets, $y \notin \mathfrak{Q}$ as well. Find $\ell < k$ small enough so that $\rho(y, z_\beta^\ell) < c_1\delta^\ell$ where z_β^ℓ is the center of the single cube in generation ℓ belonging to $\mathfrak{Q}$. Then $y \in Q_\beta^\ell$ since it is an element of the inner ball of Q_β^ℓ. However, this implies that $\operatorname{Quad}(y) = \mathfrak{Q}$ by definition of quadrant, which is a contradiction. □

3.4.6 Second Dyadic Subspace Theorem

As we saw in the bullseye space example (Example 2.2.8), just because a space is a subset of an SHT does not mean that it is itself an SHT. This applies to quadrants as well, at least a priori. If we want quadrants to be SHTs, we need to show that they cannot become sufficiently sparse in the way that the bullseye space can.

Theorem 3.4.8 (Second Dyadic Subspace Theorem)**.** *Let X be an SHT and $\mathscr{D}$ a dyadic* *lattice on X such that every cube has a thin boundary. Then for every $\mathfrak{Q} \in \mathrm{Quad}(X)$, $\mathfrak{Q}$ is a subSHT of X.*

Proof. For $x \in \mathfrak{Q}$ and $r > 0$ we will denote by $\tilde{B}(x,r) := B(x,r) \cap \mathfrak{Q}$ the open balls in $\mathfrak{Q}$. Once again, the fact that $\rho|_{\mathfrak{Q}\times\mathfrak{Q}}$ is a quasi-metric is trivial, so we only need to verify that μ restricted to the subsets of $\mathfrak{Q}$ has the doubling property.

Let $x \in \mathfrak{Q}$ and $r > 0$. The set $\tilde{B}(x,2r)$ is therefore a bounded set in $\mathfrak{Q}$. By Theorem 3.4.6, there is a cube $Q \subseteq \mathfrak{Q}$ such that $\tilde{B}(x,2r) \subseteq Q$. Thus

$$\mu(\tilde{B}(x,2r)) = \mu(B(x,2r) \cap Q) \leq \tilde{\kappa}_1 \cdot \mu(B(x,r) \cap Q) = \tilde{\kappa}_1 \cdot \mu(\tilde{B}(x,r))$$

where $\tilde{\kappa}_1$ is the doubling constant for Q as a subSHT of X given by Theorem 3.3.2. However $\tilde{\kappa}_1$ is independent of Q, so $\mathfrak{Q}$ can inherit the same doubling measure constant. $\square$

The Second Dyadic Subspace Theorem is deceptively powerful. With it now firmly in our grasp we can dispose of the half-bad measure example, and anything else like it. This gives a very nice test for determining if some measure space is an SHT or not.

Corollary 3.4.9 (No Half-Bad Measures)**.** *Let (X,ρ) be a quasi-metric space which is geometrically doubling and unbounded. Let μ be a non-trivial measure on X. If there exists a dyadic lattice $\mathscr{D}$ on X such that for one of the quadrants $\mathfrak{Q}$ of $\mathscr{D}$, $\mu(\mathfrak{Q}) < \infty$, then (X,ρ,μ) is not an SHT.*

Proof. Suppose the corollary is false and (X,ρ,μ) is an SHT. By the Second Dyadic Subspace Theorem, $\mathfrak{Q}$ is a subSHT of X. Moreover, by Theorem 3.4.7 part (c), $\mathfrak{Q}$ is also unbounded. But by the Finite Measure Lemma (Lemma 2.4.2), $\mu(\mathfrak{Q}) = \infty$, which is a contradiction since it has finite measure by hypothesis. $\square$

3.4.7 What is a Quadrant?

A natural question might be that given a subset $S \subseteq X$, can we find a dyadic system $\mathscr{D}$ which has S as one of its quadrants? This question is interesting and, given that we now have Corollary 3.4.9, could be relevant when determining if certain examples with a little misbehavior are SHTs or not. While contrived example such as the half-bad Example 3.4.1 are easy to discredit, more nefarious examples could be difficult.

In her paper *What is a Cube?* ([26]) A. Kairema tackles a similar question for cubes. The main result of the paper is a list of necessary and sufficient conditions for a subset S to be a dyadic cube in some system on X. It is the authors expectation that essentially similar conditions apply for quadrants, and we remark here that it could be interesting to explore this in the future.

3.4.8 A Final Remark on the Usefulness of Quadrants

With all this talk about quadrants, we should note that this theorem is known:

Theorem 3.4.10 (A. Kairema [29]). *It is possible to impose of the construction of $\mathscr{D}$ that there is only one quadrant.*

The existence of this theorem perhaps raises the question, "*what exactly is the use of talking about quadrants?*"

Our aim in this document is to, as much as possible, remain general with respect to dyadic grids and their underlying spaces. While it is true that Theorem 3.2.2 has a constructive proof, the construction given is not necessarily unique. We would like to be able to say as much as we can without a priori special knowledge about the grid we are using. For this reason, we are shying away from the use of results which require us to put extra restrictions on the grid. We do not, therefore, know that there will be only one quadrant.

In Chapter 5 we will begin to look at Haar bases in spaces of homogeneous type and the framework built up here will be very useful in eliminating some otherwise pesky case. For now, we will mention that the potential for some quadrants to be finite in measure while others are infinite in measure would have caused some difficulties in defining the basis. That we have eliminated this possibility is very useful indeed.

3.5 "Honest" Systems of Dyadic Cubes

One of the major hurtles when dealing with dyadic analysis over spaces of homogeneous type is the thorny issue of the varying number of children in each cube. In $\mathbb{R}^n$, we always know that each cube has the same number of children, and furthermore that this number is a power of 2. In SHTs, as we have seen, this nice property falls by the wayside. The practical cost of this is an unfortunate increase in the amount of bookkeeping when dealing with dyadic focused proofs. To help ease the burden, we have developed a work-around of sorts, which we are calling "honest" dyadic systems. The word "honest" here refers to the prefix "dy-" in that the cubes really do have two children[4]. While the honest systems are not strictly necessary to derive any results, they nevertheless allow for a nice simplification and remove the potential for overly cumbersome notation.

Definition 3.5.1 (Overlapping Grids)**.** Let X be an SHT and let $\mathscr{D}$ and $\mathscr{D}'$ be two dyadic grids on X. If for every $Q \in \mathscr{D}$, it is also true that $Q \in \mathscr{D}'$ then we say that *$\mathscr{D}'$ overlaps $\mathscr{D}$.*

The main lemma looks like this:

Lemma 3.5.2 (Honest Dyadic Cube Existence)**.** *Let $X = (X, \rho)$ denote a quasi-metric which is geometrically doubling and let $\mathscr{D}$ be a dyadic grid over X. There exists a dyadic structure $\tilde{\mathscr{D}}$ on X which overlaps $\mathscr{D}$ and is* honest*, that is, that each cube in $\tilde{\mathscr{D}}$ has no more than two children cubes.*

[4]We do not mean to imply that dyadic systems with more than two children per cube are "dishonest," and apologize in advance to any offended lattices.

What makes honest systems nice to work in is that we always know the number of children each cube has. As we will see in Chapters 5-7, this is very helpful when dealing with generalizing some basic concepts from $\mathbb{R}^n$ to X. For example, in $\mathbb{R}$ we have the familiar operator

$$\Delta_I f := \frac{1}{|I|}\left(\int_{I_\ell} f(x)\,dx - \int_{I_r} f(x)\,dx\right) \ ; \quad I \in \mathscr{D}_k$$

Generalizing Δ_I to $\mathbb{R}^n$ is already a pain. The pain is exacerbated even more in the world of SHTs. The existence of honest systems makes things of this nature trivial.

Proof of Lemma 3.5.2. By the definition of dyadic grids, we have that there exists constants $0 < c_1 < C_1$, $\delta \in (0,1)$, and $N_{ch} \in \mathbb{N}$ so that

1. For any cube $Q \in \mathscr{D}_k$ with center z_Q, $B(z_Q, c_1\delta^k) \subseteq Q \subseteq B(z_Q, C_1\delta^k)$,
2. For any cube $Q \in \mathscr{D}$, $1 \leq N(Q) \leq N_{ch}$.

Fix a cube $Q \in \mathscr{D}$, and suppose that $N(Q) \neq 1$, that is, that Q has more than one child. We will consider this case later. Let u_Q be an enumeration of Q's children, that is, $u_Q : \{1, 2, ..., N(Q)\} \to \text{ch}(Q)$ a bijection. As there are $N(Q)$ children, there are exactly $N(Q)!$ such enumerations. For the time being, we choose any of these without caring about which one. Next we fix $p : \mathbb{Z}_{N(Q)-1} \to \mathbb{Z}_{N(Q)-1}$ a permutation (also a bijection). Define the sets $E_Q^{i,0}$, and $E_Q^{i,1}$ for $i = 1, ..., N(Q)-1$ as

$$E_Q^{i,0} := \bigcup_{j=1}^{p(i)} u(j),$$

$$E_Q^{i,1} := \bigcup_{j=p(i)+1}^{N(Q)} u(j) = Q \setminus E_Q^{i,0}.$$

From these sets, we can define set $F_Q^{i,j}$ for $i = 1, ..., N(Q)-1$ and $j = 0, ..., 2^i - 1$ as

$$F_Q^{i,j} = \bigcap_{\ell=1}^{i} E_Q^{\ell, a_\ell(j)}$$

where $a_\ell(j)$ is the ℓ^{th} digit of j in binary. Alternatively, we could have defined $F_Q^{i,j}$ recursively:

$$\begin{aligned} F_Q^{i,j} &= E_Q^{i,j} && \text{for } i = 1;\ j = 0, 1 \\ F_Q^{i+1,j} &= F_Q^{i,j'} \cap E_Q^{i+1,\ell} && \text{for } i > 1;\ j = 0, ..., 2^i - 1 \end{aligned}$$

where

$$(j', \ell) = \begin{cases} (j, 0) & \text{if} \quad j < 2^i - 1 \\ (j - 2^i, 1) & \text{if} \quad 2^i \leq j < 2^{i+1} - 1 \end{cases}$$

Example 3.5.3. If it exists, the subset $F_Q^{5,11} = E_Q^{5,0} \cap E_Q^{4,1} \cap E_Q^{3,0} \cap E_Q^{2,1} \cap E_Q^{1,1} = E_Q^{5,0} \cap F_Q^{4,11}$ because 11 has the binary representation of 01011_2 to five digits.

We now claim that for each $i = 1, ..., N(Q) - 1$,

$$\#\{F_Q^{i,j} \mid j = 0, ..., 2^i - 1 \text{ and } F_Q^{i,j} \neq \emptyset\} = i + 1 \tag{3.8}$$

First, observe that $F_Q^{1,0} = E_Q^{1,0}$ and $F_Q^{1,0} = E_Q^{1,0}$, so (3.8) is true when $i = 1$. Second, if (3.8) holds for a particular i, then it holds for $i+1$ because at every step, for only one j does both $F_Q^{i,j} \cap E_Q^{i+1,0} \neq \emptyset$ and $F_Q^{i,j} \cap E_Q^{i+1,0} \neq \emptyset$. Inducting on i proves the claim.

With equation (3.8) verified, we can now say that there are exactly $N(Q)$ non-empty $F_Q^{i,j}$ sets when $i = N(Q) - 1$. Clearly, these sets are precisely Q's children.

It is possible that, for some cubes $N(Q) < N_{ch}$. In this case, we define the sets $F_Q^{i,j} = F_Q^{i-1,j}$ for $i > N(Q) - 1$.

We now construct the dyadic structure $\tilde{\mathscr{D}}$. First, set generation $\tilde{\mathscr{D}}_{(N_{ch}-1)k} := \mathscr{D}_k$. We next construct the intermediate generations. For every $i = 1, ..., N_{ch} - 2$, set

$$\tilde{\mathscr{D}}_{(N_{ch}-1)k+i} := \left\{F_Q^{i,j} \mid Q \in D_k\}; j = 0, ..., 2^i - 1; F_Q^{i,j} \neq \emptyset\right\}.$$

This gives us the sets which make up the generations of $\tilde{\mathscr{D}}$. See figure 3.6 for an example of this process.

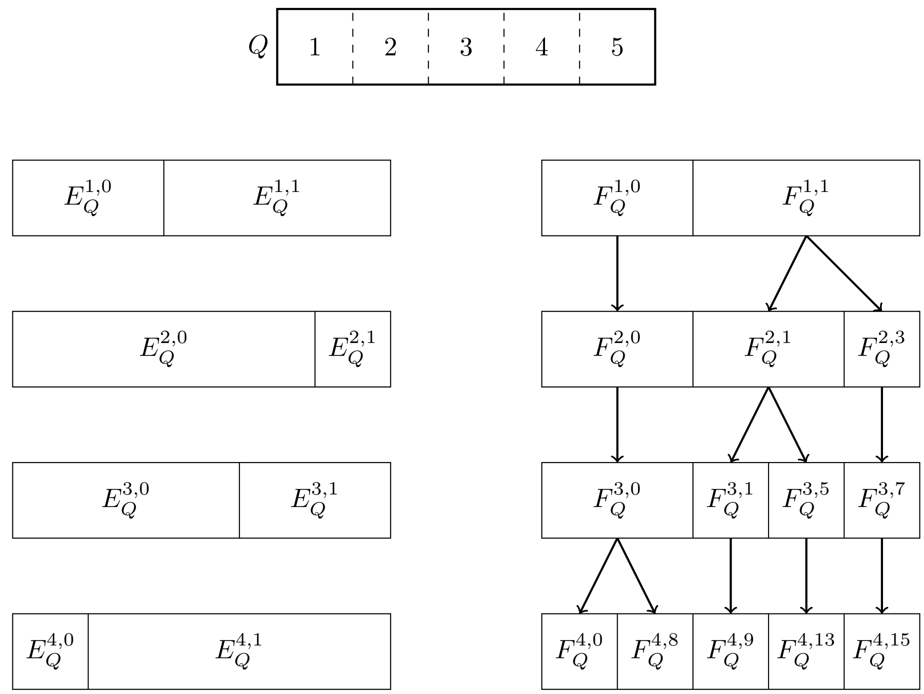


Figure 3.5: In this figure we consider a cube with five children. At the top we have the cube Q with its children labeled by the enumeration u_Q. We then generate the $E_Q^{i,0/1}$ sets via the permutation $p = (2, 4, 3, 1)$ which is shown on the left. On the right, we show how these give rise to the $F_Q^{i,j}$ sets. Notice how at each level we only introduce one more subset by only splitting one previous set. Also note that $F_Q^{i,j} = \emptyset$ for any $F_Q^{i,j}$ not pictured.

We must now find parameters $\tilde{\delta} \in (0,1)$, $0 < \tilde{c}_1 < \tilde{C}_1$, and the centers $\tilde{z}_Q$ so that

$$B(\tilde{z}_Q, \tilde{c}_1 \tilde{\delta}^k) \subseteq Q \subseteq B(\tilde{z}_Q, \tilde{C}_1 \tilde{\delta}^k). \tag{3.9}$$

For every $Q \in \tilde{\mathscr{D}}_k$, Q is a finite union of cubes $Q' \in \mathscr{D}_j$ for some j. Set $\tilde{z}_Q = z_{Q'}$ any of the such Q'. Set $\tilde{\delta} := \delta^{1/(N_{ch}-1)}$, $\tilde{c}_1 = c_1 \cdot \delta$ and $\tilde{C}_1 := 2\kappa_0 \cdot C_1/\delta$. We will now prove that these choices of centers and constants are sufficient.

Let $Q \in \tilde{\mathscr{D}}_k$ with center $\tilde{z}_Q$. Then Q is equal to the union of cubes belonging to $\mathscr{D}_j$ where $j(N_{ch} - 1) \geq k > (N_{ch} - 1)(j - 1)$. Moreover, these cubes are all siblings, i.e., have the same parent in $\mathscr{D}_{j-1}$.

We need to show that (3.9) is satisfied. Let $Q' \in \mathscr{D}_j$ be the cube such that $\tilde{z}_Q = z_{Q'}$.

Then

$$B(\tilde{z}_Q, \tilde{c}_1\tilde{\delta}^k) = B(z_{Q'}, c_1\delta^{k/(N_{ch}-1)+1}) \subseteq B(z_{Q'}, c_1\delta^j) \subseteq Q' \subseteq Q.$$

Now let $x \in Q$. Then

$$\rho(x, \tilde{z}_Q) \le \kappa_0(\rho(x, z_{\widehat{Q'}}) + \rho(z_{\widehat{Q'}}, \tilde{z}_Q)) < 2\kappa_0 C_1\delta^{j-1} \le 2\kappa_0 C_1\delta^{k/(N_{ch}-1)-1} = \tilde{C}_1\tilde{\delta}^k.$$

This verifies (3.9). □

Remark 3.5.4. The proof above constructs an honest system which has *mostly* null generations. It is not difficult to see that this isn't the only way to do this construction. One could, if so desired, reorganize the way the cubes are split to give fewer intermediate generations. However, unless N_{ch} is a power of 2 and all cubes have N_{ch} children, some null generations must be generated. This isn't really a problem for most practical situations and can be handled quite easily, as we will see in Chapters 6 and 7.

Notation 1. If a dyadic system is honest, we will refer to the two child cubes of Q as Q_+ and Q_-, or together as $Q_\pm$.

3.6 Dyadic Lebesgue Differentiation Theorem

We make a note here about the Lebesgue Differentiation Theorem. For (X, ρ, μ) a space of homogeneous type and $f : X \to \mathbb{R}$, we would like to be able to say that

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) = f(x) \quad a.e.\ x \in X \text{ with respect to } \mu.$$

However, it is known that in general this unfortunately fails without making extra assumptions about the measure μ beyond just doubling. This was pointed out to me in an e-mail from David Cruz-Uribe. Surprisingly then we *do* have a version of the Lebesgue Differentiation Theorem for dyadic cubes!

Theorem 3.6.1 (Lebesgue Differentiation Theorem for Dyadic Cubes)**.** *Let X be an SHT and $\mathscr{D}$ be a dyadic lattice for X. For $x \in X$ denote by $Q_k(x)$ the unique cube in generation $\mathscr{D}_k$ which contains x. Let $f : X \to \mathbb{R}$ be an integrable function. Then for every $x \in X$, except possibly on a set with μ-measure zero, we have that*

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} f(y)\, d\mu(y) = f(x).$$

A proof of this theorem can be found in [1], however it relies on the thin boundary property. Another proof can be found in Hytönen's lecture notes, *Martengales and Harmonic Analysis*, [21], Corollary 2.6.

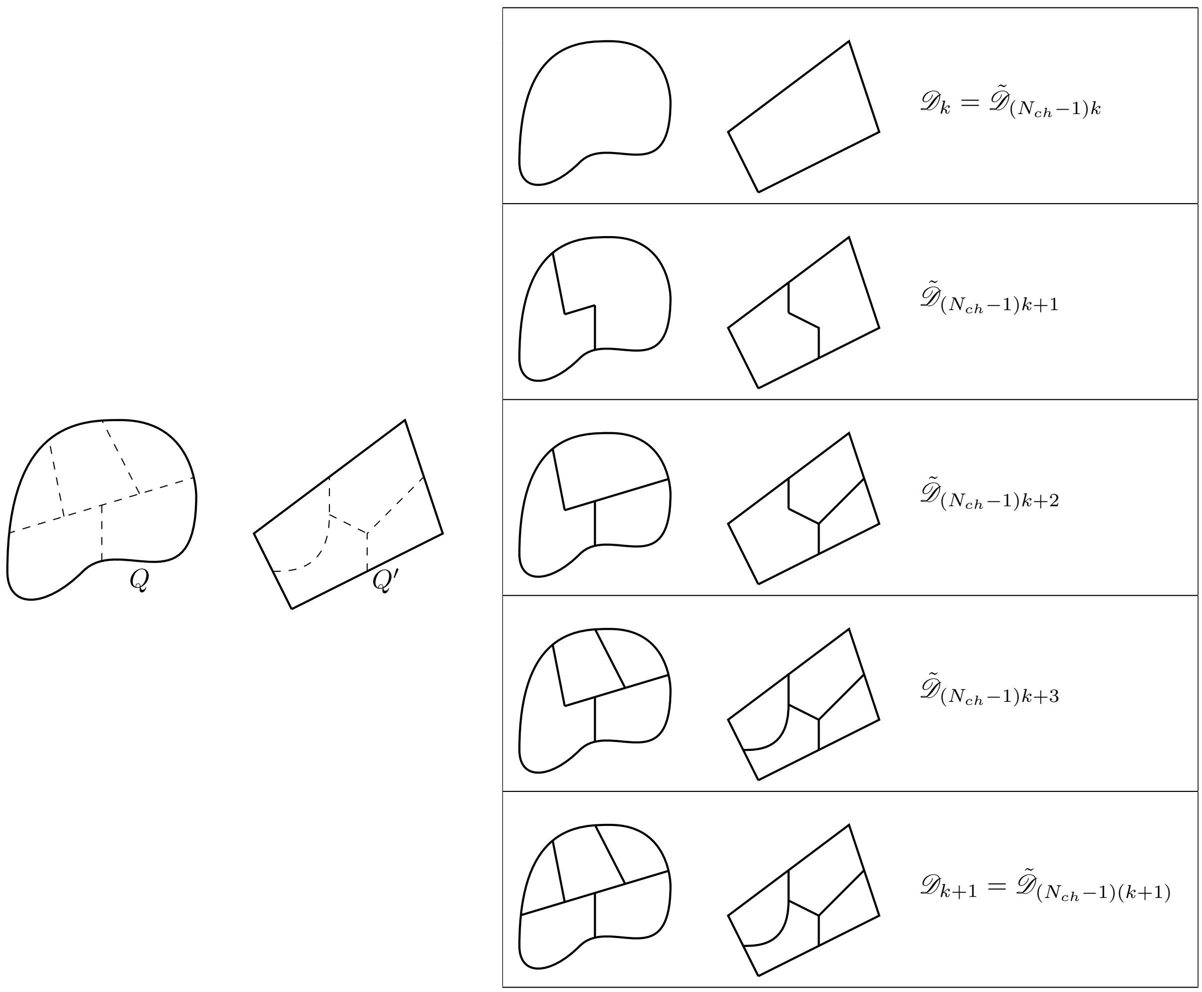


Figure 3.6: On the left are two cubes Q and Q', belonging to the same generation. On the right we see the process by which the honest generations are to be inserted, by splitting exactly one honest cube at a time. Notice that Q' has one null generation where no new cubes are inserted, because it started with four children cubes instead of five.

Chapter 4

The Greats of Weights

Now that we have a solid footing, we can begin to look at weight theory. In this short chapter we will collect the basic definitions and "great" theorems which we will make use of in the later chapters. For a more thorough introduction to weighted theory, see textbooks [17] and [14].

This chapter introduces the first use of the following notation, which will be used extensively through the rest of this document:

Notation 2. For an integrable function f and a μ-measurable set $S \subseteq X$ with $0 < \mu(S) < \infty$, we will use the notation[1] $\langle f \rangle_S$ to denote the μ-average of f in S:

$$\langle f \rangle_S := \frac{1}{\mu(S)} \int_S f(x)\, d\mu(x).$$

It may be more proper to include a "μ" in the notation for the average, but we will omit it to avoid cumbersome subscripts and hope the measure is clear from context.

[1]While we are using the notation of Volberg et. all, other authors, such as Pereyra, prefer "$m_S f$" for the mean.

4.1 Introduction

We will begin with a straightforward definition.

Definition 4.1.1. Let (X, μ) be a measure space. A *weight* is a function $w \in L^1_{loc}(X)$ which is positive almost everywhere with respect to μ.

Notation 3. Let w be a weight defined over a measure space (X, μ). For any measurable set $S \subseteq X$, we mean by $w(S) := \int_S w(x)\, d\mu(x)$.

Obviously, we are interested primarily in weights defined over SHTs. In this section we will introduce the basic definitions and propositions in weight theory.

4.1.1 Weighted L^p

As usual, when (X, μ) is a measure space we can define the Lebesgue L^p-norm, $1 \leq p < \infty$ for real-valued measurable functions as

$$||f||_{L^p} := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.$$

If $||f||_{L^p} < \infty$ then $f \in L^p$.

For w a weight, we can also define weighted L^p-norm:

$$||f||_{L^p(w)} := \left(\int_X |f(x)|^p w(x) \, d\mu(x) \right)^{\frac{1}{p}}.$$

Remark 4.1.2. As in other places, we note that it might be more appropriate to say "$L^p(X, \mu)$" instead of L^p, but we are trying to be terse when a fixed context permits.

Proposition 4.1.3. *Let w be a weight. A function $f \in L^p(w)$ if and only if $fw^{\frac{1}{p}} \in L^p$. Moreover $||f||_{L^p(w)} = ||fw^{\frac{1}{p}}||_{L^p}$.*

We can likewise define a weighted and unweighted inner product.

Definition 4.1.4. Let f and g be real-valued square integrable functions. The L^2 *inner product* of f and g is

$$\langle f, g\rangle := \int_X f(x)g(x)\,d\mu(x).$$

For w a weight, the *weighted inner product* defined for $L^2(w)$ functions is

$$\langle f, g\rangle_w := \int_X f(x)g(x)w(x)\,d\mu(x).$$

4.1.2 The Classes A_p, RH_q, and C_t

We will now introduce three main classes of weights which are of interest to us.

For each of these definitions, we let (X, μ) be a measure space and w be a weight on X. Furthermore, we let $\mathcal{S}$ be a family of μ-measurable subsets of X, each with positive measure.

Definition 4.1.5 (Mukenhoupt Class)**.** Let $1 < p < \infty$. Suppose there exists a constant $C > 0$ such that for all $S \in \mathcal{S}$

$$\langle w\rangle_S \langle w^{\frac{1}{1-p}}\rangle_S^{p-1} \leq C. \tag{4.1}$$

We then say that w belongs to the *Mukenhoupt A_p class*, or simply *A_p class*, with respect to $\mathcal{S}$, written $w \in A_p(\mathcal{S})$ and we denote the smallest such C as $[w]_{A_p(\mathcal{S})}$, called the *A_p characteristic of w.*

Definition 4.1.6 (Reverse Hölder Class)**.** Let $1 < q < \infty$. Suppose there exists a constant $C > 0$ such that for all $S \in \mathcal{S}$

$$\langle w^q\rangle_S^{1/q} \leq C \cdot \langle w\rangle_S. \tag{4.2}$$

We then say that w belongs to the *Reverse Hölder q* class with respect to $\mathcal{S}$, written $w \in RH_q(\mathcal{S})$ and we denote the infimum of all such C as $[w]_{RH_q(\mathcal{S})}$, called the *Reverse Hölder q characteristic of w.*

Definition 4.1.7 (C_t Class)**.** Let $t \in \mathbb{R}$. Suppose there exists a constant $C > 0$ such that for all $S \in \mathcal{S}$

$$\langle w^t \rangle_S \langle w \rangle_S^{-t} \leq C. \tag{4.3}$$

We then say that w belongs to the C_t class with respect to $\mathcal{S}$, written $w \in C_t(\mathcal{S})$ and we denote the infimum of all such C as $[w]_{C_t(\mathcal{S})}$, called the C_t *characteristic* w.

The reverse of inequalities (4.1), (4.2), and (4.3) are always true with $C = 1$ by Hölder's inequality.

Notice that for $p > 1$, $w \in A_p(\mathcal{S})$ if and only if $w \in C_{1/(1-p)}(\mathcal{S})$ with $[w]_{A_p(\mathcal{S})}^{p-1} = [w]_{C_{1/(1-p)}(\mathcal{S})}$. Also for $q > 1$, $w \in RH_q(\mathcal{S})$ if and only if $w \in C_q(\mathcal{S})$ with $[w]_{RH_q(\mathcal{S})}^q = [w]_{C_q(\mathcal{S})}$.

The A_p class will be used in several chapters of this dissertation. In Chapter 5 we will look at an important theorem pertaining to Reverse Hölder weights. In Chapter 7, C_t weights will play an important role for bounding the operators we want to study in that chapter.

4.1.3 Continuous and Dyadic Classes

In particular, if (X, ρ, μ) is a space of homogeneous type and $\mathcal{S}$ is the collection of open ρ-balls, then we say that the classes are *continuous* classes and we simply write A_p, RH_q, and C_t. If $\mathcal{S}$ instead denotes a collection of dyadic cubes $\mathscr{D}$ over X then we say that the classes are *dyadic* classes and write $A_p^{\mathscr{D}}$, $RH_q^{\mathscr{D}}$, and $C_t^{\mathscr{D}}$.

4.1.4 Dyadic Doubling Weights

The last class of weights we define will be dependent on a particular dyadic grid.

Definition 4.1.8. Let X be an SHT, and $\mathscr{D}$ a dyadic lattice over X. Then a weight w is *dyadic doubling* if there exists a constant $C > 0$ such that

$$w(Q) \leq C \cdot w(\widehat{Q})$$

for every $Q \in \mathscr{D}$. Recall that by $\widehat{Q}$ we mean Q's parent.

4.2 Theorems

Here we will list some important facts, propositions, and theorems regarding weights which are interesting and which we will use in several proofs later on. For a concise list of some basic results, see Chapter 1 of D. Panek's Ph.D. dissertation, [42].

4.2.1 Simple Propositions

There are several simple results relating A_p weights and RH_q weights, of which we give a selection here.

Proposition 4.2.1. *For a weight w, $w \in A_p(\mathcal{S})$ if and only if $w^{\frac{1}{1-p}} \in A_{p'}(\mathcal{S})$.*

This proposition is easily verified by manipulating exponents in Definition 4.1.5.

Proposition 4.2.2. *For a weight w, if $w \in A_p(\mathcal{S})$ then $w \in A_{p+\epsilon}$ for all $\epsilon > 0$.*

Proposition 4.2.3. *For a weight w, if $w \in RH_q(\mathcal{S})$ then $w \in RH_{q-\epsilon}$ for all $0 < \epsilon < q - 1$.*

Propositions 4.2.2 and 4.2.3 are simple consequences of Hölder's inequality.[2]

Lastly, we have this basic result:

Proposition 4.2.4. *For a weight w, there exists $p > 1$ such that $w \in A_p$ if and only if there exists $q > 1$ such that $w \in RH_q$. This also holds for $A_p^{\mathscr{D}}$ and $RH_q^{\mathscr{D}}$.*

[2] In the next chapter we will look at a theorem related to Proposition 4.2.3.

4.2.2 A Suite of Maximal Functions

The A_p class was first introduced in [38] by B. Mukenhoupt, as a way to characterize the maximal function.

Definition 4.2.5 (Maximal Functions)**.** Let (X, ρ, μ) an SHT and w a weight over X. For $f \in L^1_{loc}(X)$ define

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y)$$

the *Hardy-Littlewood maximal function*, or *centered maximal function*, and

$$M_w f(x) := \sup_{r>0} \frac{1}{w(B(x,r))} \int_{B(x,r)} |f(y)|\, w(y) d\mu(y)$$

the *weighted, centered maximal function.*

If $\mathscr{D}$ is a dyadic lattice over X, we can define dyadic versions:

$$M^{\mathscr{D}} f(x) := \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y)$$

the *dyadic maximal function*, and

$$M^{\mathscr{D}}_w f(x) := \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)|\, w(y) d\mu(y)$$

the *weighted dyadic maximal function.* Here the supremum is taken over all cubes $Q \in \mathscr{D}$ which contain the point x.

There are other variations of this operator as well, e.g. the uncentered maximal function. In this document, we will focus primarily on the two dyadic versions of the maximal function given above.

In [38], the A_p class was defined for $\mathbb{R}^d$, and shown to be exactly a characterization of the weights w for which the maximal function is bounded in $L^p(w)$:

Theorem 4.2.6 (Mukenhoupt's A_p Weight Characterization ($\mathbb{R}$))**.** *A weight $w \in A_p$, $1 < p < \infty$, if and only if the Hardy-Littlewood maximal function is bounded in $L^p(w)$, i.e., there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{R}} |Mf(x)|^p \, dx\right)^{\frac{1}{p}} < C \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

for all functions f.

We stated here the original $\mathbb{R}$ version of this theorem.

4.2.3 Bound on the Dyadic Maximal Function, SHT Version

We have very nice weighted L^p bounds on the weighted and unweighted dyadic maximal function:

Theorem 4.2.7. *Let X be an SHT, $\mathscr{D}$ a dyadic lattice over X. Let $1 < p < \infty$ and w be a weight defined on X. Then for every $f \in L^p(wd\mu)$,*

$$||M_w^{\mathscr{D}} f||_{L^p(w)} \leq p' ||f||_{L^p(w)}$$

and

$$||M^{\mathscr{D}} f||_{L^p(w)} \leq C[w]_{A_p^{\mathscr{D}}}^{\frac{p'}{p}} ||f||_{L^p(w)} \tag{4.4}$$

where the constant C depends only on p, and geometric constants and p' denotes the Hölder conjugate of p, that is $1/p + 1/p' = 1$.

The $\mathbb{R}$ version of (4.4) is the celebrated Buckley inequality, first demonstrated by S. Buckley in [9]. For proofs of these inequalities in the SHT setting, [25] or [29].

Chapter 5

A First Weighted Inequality: Dyadic Gehring

With all the groundwork laid in the previous three chapters, we are now ready to tackle our first weighted inequality. The content of this chapter can also be found in the paper by the auther and T. Anderson [6].

Gehring's Theorem is an example of a classical so-called "self-improvement" result in the theory of weights. We let w be a Reverse Hölder p weight. It is a trivial consequence of Hölder's inequality that if w satisfies the reverse Hölder p condition, inequality (4.2), for some p, then it likewise satisfies reverse Hölder q for any $1 < q < p$. Surprisingly though, one can show that there exists $\epsilon > 0$ so that w satisfies (4.2) for $p + \epsilon$ as well. This is the well known Gehring Theorem, first proved in the [16], and we say it is a self improvement result because we have slightly improved the exponent.

5.1 Background

We will start by giving a brief rundown of recent results which are related to this chapter's main theorem, before stating the theorem formally.

5.1.1 Notable Related Results

Recent work has gone into proving an analogue to Gehring's Theorem in the more abstract setting of spaces of homogeneous type. In [33], Maasalo showed that the theorem is true in metric spaces with doubling measures provided the measure satisfies a radial decay property. Then in [5], Anderson, Hytönen, and Tapiola showed that the theorem is true for *weak* Reverse Hölder classes in general spaces of homogeneous type. What characterizes these classes as weak is that the domain of integration is enlarged on the right hand side of the inequality. One would hope that the "strong" result would soon follow, however in the same paper the authors constructed an explicit counterexample: a weight over a specific space which satisfies a inequality analogous to (4.2) for $p \leq p_0$ but not for $p > p_0$.

Incidentally, in [27], Pérez, Hytönen , and Rela found *sharp* bounds for this weak RH_q class and used this result to show a bound for the related weak maximal function.

In [31], Katz and Pereyra used a decaying stopping time argument to prove Gehring's Theorem for weights over the real line. In this chapter we adapt this method to show that, in spite of the aforementioned counterexample, a *dyadic* version of the strong Gehring Theorem does indeed hold.

5.1.2 Statement of Gehring's Theorem

The main theorem of this chapter is that Gehring's Theorem holds in the dyadic setting for spaces of homogeneous type.

Theorem 5.1.1 (Main Result)**.** *Let (X, ρ, μ) be a space of homogeneous type with dyadic lattice $\mathscr{D}$. Let $1 < p < \infty$ and let $w \in RH_p^{\mathscr{D}}$. Then there exists ϵ depending only on p, w, and geometric constants such that $w \in RH_{p+\epsilon}^{\mathscr{D}}$.*

5.2 Decaying Stopping Time

The proof of Theorem 5.1.1, which can be found in Section 5.4, relies on a decaying stopping time argument. We introduce the idea here. Throughout this section (X, ρ, μ) is assumed to be a space of homogeneous type, with dyadic structure $\mathscr{D}$.

5.2.1 Dyadic Properties

Let $\mathcal{P}$ denote some property about cubes, i.e., for any given dyadic cube the statement "Q has $\mathcal{P}$" is meaningful. This property may depend on any number of parameters including other cubes.

Primarily, for the purposes of stopping times, we are interested in properties which relate one cube to another. For two dyadic cubes Q and Q' we would say that Q' has $\mathcal{P}$ with respect to Q.

Example 5.2.1. Let X be the real numbers with the usual metric and measure. For two cubes $Q, Q' \in \mathscr{D}$, suppose that Q' has $\mathcal{P}$ with respect to Q if and only if $Q' \subsetneq Q$. Then $[0, 2)$ and $[2, 4)$ have P with respect to $[0, 4)$ and are the two maximal intervals with that property.

Definition 5.2.2 (Admissible Property)**.** Suppose that $\mathcal{P}$ is a property about cubes with respect to another cube. Then we say $\mathcal{P}$ is *admissible* if for all $Q \in \mathscr{D}$, Q *does not* have $\mathcal{P}$ with respect to itself.

Remark 5.2.3. The property from Example 5.2.1 is admissible.

5.2.2 Stopping Time Definition

For a fixed cube $Q \in \mathscr{D}$, we denote by $\mathcal{J}(Q) \subsetneq \mathscr{D}(Q)$ a sub-collection of cubes which are maximal with respect to $\mathcal{P}$. By maximality, we mean that if $Q' \subseteq Q$ has $\mathcal{P}$, then no descendant of Q' will be included in $\mathcal{J}(Q)$, regardless of whether it has $\mathcal{P}$ or not. Formally,

$$\mathcal{J}(Q) := \{Q' \in \mathscr{D}(Q) \mid Q' \text{ has } \mathcal{P} \text{ but } Q'' \text{ does not have } \mathcal{P} \forall Q'' \in \mathscr{D} \text{ with } Q'' \supsetneq Q'\}$$

For an admissible property set $\mathcal{J}_0(Q) := \{Q\}$. We now define the collections $\mathcal{J}_n(Q)$ inductively. Let $n > 0$. Define

$$\mathcal{J}_n(Q) := \bigcup_{Q' \in \mathcal{J}_{n-1}(Q)} \mathcal{J}(Q').$$

Note that $\mathcal{J}_1(Q) = \mathcal{J}(Q)$. The family of collections $\{\mathcal{J}_n(Q)\}_{n \geq 0}$ is called the *stopping time* $\mathcal{J}$ *for* Q.

Definition 5.2.4 (Decaying Stopping Time)**.** Let (X, ρ, μ) be a quasi-metric space equipped with a measure which has dyadic structure $\mathscr{D}$ and let $\mathcal{J}$ be a stopping time. We say that $\mathcal{J}$ is a *decaying stopping time* if and only if there exists $0 < c < 1$ such that for every $Q \in \mathscr{D}$,

$$\sum_{Q' \in \mathcal{J}_1(Q)} \mu(Q') \leq c\mu(Q). \tag{5.1}$$

Remark 5.2.5. Iterating 5.1 gives that

$$\sum_{Q' \in \mathcal{J}_n(Q)} \mu(Q') \leq c^n \mu(Q)$$

provided $\mathcal{J}$ is decaying. Furthermore,

$$\sum_{Q' \in \cup_{n>0} \mathcal{J}_n(Q)} \mu(Q') \leq \frac{\mu(Q)}{1-c}$$

Remark 5.2.6. For the time described in Example 5.2.1, $\mathcal{J}([0,4))=\{[0,2),[2,4)\}$. This, however, is *not* decaying, since $c = 1$.

Remark 5.2.7. What we call in this dissertation *stopping times* is equivalent to the idea of *sparse families* of cubes. In the short book [32] we are given these definitions:

A family of cubes $\mathcal{S} \subseteq \mathscr{D}$ that has the property that for all $Q \in \mathscr{D}$

$$\sum_{Q' \in \mathcal{S}, Q' \subset Q} \mu(Q') \leq \Lambda \cdot \mu(Q) \tag{5.2}$$

is called Λ-Carleson.

For $0 < \eta < 1$, a family of cubes $\mathcal{S} \subseteq \mathscr{D}$ is η-sparse if one can choose pariwise disjoint measurable sets $E_Q \subset Q$ with $\mu(Q) \leq \eta \cdot \mu(E_Q)$, for all $Q \in \mathscr{D}$.

According to LEmma 6.3 in [32], a family is Λ-Carleson if and only if it is Λ^{-1}-sparse.

5.2.3 The Stopping time $\mathcal{J}^w$

Let us now describe a particular stopping time. Suppose that $w \in RH_p^{\mathscr{D}}$ for some $1 < p < \infty$. If Q is a cube, we say that another cube $Q' \subset \mathscr{D}(Q)$ has property $\mathcal{P}^w$ with respect to Q if either $\langle w \rangle_{Q'} \geq \lambda \langle w \rangle_Q$ or $\langle w \rangle_{Q'} \leq \lambda^{-1} \langle w \rangle_Q$ where $\lambda > 1$ is a fixed parameter. While this property depends on a weight w, a parameter λ and a cube Q, we only write $\mathcal{P}^w$ (as opposed to, say, $\mathcal{P}_Q^{w,\lambda}$, in order to avoid over-cluttered notation.

Clearly the following lemma is true.

Lemma 5.2.8. *Property $\mathcal{P}^w$ is admissible.*

Proof. For any cube Q, since $\lambda > 1$, $\langle w \rangle_Q < \lambda \langle w \rangle_Q$ and $\langle w \rangle_Q > \lambda^{-1} \langle w \rangle_Q$. Thus no cube will ever have property $\mathcal{P}^w$ with respect to itself, which implies admissibility. $\square$

We define the stopping time $\mathcal{J}^w$ for Q as the stopping time generated by $\mathcal{P}^w$ with respect to Q. That is,

$$\mathcal{J}_Q^w = \bigsqcup_{n \geq 0} \mathcal{J}_n^w(Q) = \left\{ \mathcal{J}_Q^w \mid n \geq 0 \right\}$$

5.3 Auxilary Lemmas

To prove Theorem 5.1.1 we show the following two lemmas:

Lemma 5.3.1. *If the stopping time $\mathcal{J}^w$ described above is decaying then Theorem 5.1.1 holds.*

Lemma 5.3.2. *The stopping time $\mathcal{J}^w$ is decaying provided the parameter λ is chosen large enough.*

It is thus sufficient to prove Lemmas 5.3.2 and 5.3.1.

5.3.1 Some Useful Facts

The following fact will be useful for both proofs.

Lemma 5.3.3. *Let $Q' \in \mathcal{J}^w(Q)$. Then $\langle w\rangle_{Q'} \leq D\lambda\langle w\rangle_Q$ where $D = Dbl(\mathscr{D})$ is the dyadic doubling constant from Corollary 3.2.8.*

Proof. By the maximality condition for stopping times, since $Q' \in \mathcal{J}^w(Q)$, its parent $\widehat{Q'} \notin \mathcal{J}^w(Q)$. This means that $\lambda^{-1}\langle w\rangle_Q < \langle w\rangle_{\widehat{Q'}} < \lambda\langle w\rangle_Q$. Thus,

$$\langle w\rangle_{Q'} = \frac{1}{\mu(Q')}\int_{Q'} w\,d\mu \leq \frac{1}{\mu(Q')}\int_{\widehat{Q'}} w\,d\mu \leq \frac{D}{\mu(\widehat{Q'})}\int_{\widehat{Q'}} w\,d\mu = D\langle w\rangle_{\widehat{Q'}} < D\lambda\langle w\rangle_Q.$$

□

Corollary 5.3.4. *Suppose $Q' \in \mathcal{J}^w_n(Q)$. Then $\langle w\rangle_{Q'} \leq (D\lambda)^n\langle w\rangle_Q$.*

Proof. Let $Q^0 := Q' \in \mathcal{J}^w_n(Q)$. By definition, there exists $Q^1 \in \mathcal{J}^w_{n-1}$ so that $Q^0 \in \mathcal{J}^w(Q^1)$. Continuing on in this fashion, for all $1 \leq i \leq n$ there exists $Q^i \in \mathcal{J}^w_{n-i}$ so that $Q^{i-1} \in \mathcal{J}^w(Q^i)$. With this notation, $Q^n = Q$. Iterating n times the result of Lemma 5.3.3 gives that

$$\langle w\rangle_{Q'} = \langle w\rangle_{Q^0} \leq D\lambda\langle w\rangle_{Q^1} \leq (D\lambda)^2\langle w\rangle_{Q^2} \leq \cdots \leq (D\lambda)^n\langle w\rangle_{Q^n} = (D\lambda)^n\langle w\rangle_Q.$$

□

The following will also be useful.

Lemma 5.3.5. *For almost every $x \in X$ (with respect to the measure μ), $\lambda^{-1}\langle w\rangle_Q \leq w(x) \leq \lambda\langle w\rangle_Q$ for $x \notin \cup_{Q'\in\mathcal{J}^w(Q)} Q'$.*

Proof. Let $x \in Q$ such that $x \notin Q'$ for all $Q' \in \mathcal{J}^w(Q)$. Let k_0 be Q's generation, i.e. $Q \in \mathscr{D}^{k_0}$ and define Q_x^k as the cube belonging to generation $\mathscr{D}^k$ with $x \in Q_x^k$ for $k \geq k_0$. So $Q_x^k \notin \mathcal{J}^w(Q)$ for all $k \geq k_0$, thus by definition of property $\mathcal{P}^w$,

$$\lambda^{-1}\langle w\rangle_Q \leq \langle w\rangle_{Q_x^k} \leq \lambda\langle w\rangle_Q.$$

By the Lebesgue Differentiation Theorem for dyadic cubes, the limit as $k \to \infty$ of the center expression goes to $w(x)$ a.e. with respect to the measure μ. □

5.4 Proofs

In this section we present the proofs of Lemmas 5.3.2 and 5.3.1, thus establishing Theorem 5.1.1. This proof is in spirit the same as the one first shown in [31], but with care given to the peculiarities of the SHT setting.

Proof of Lemma 5.3.1. Let $Q \in \mathscr{D}$ be any cube. We define the n^{th} "good" and "bad" sets as

$$B_n(Q) := \bigsqcup_{Q'\in\mathcal{J}_n^w(Q)} Q' \ ; \quad n \geq 0,$$

$$G_n(Q) := B_{n-1}(Q) \setminus B_n(Q) \ ; \quad n > 0.$$

Notice that $B_0(Q) = Q = \sqcup_n G_{n>0}(Q)$, up to a set of measure zero. By the Lemma 5.3.2, we can choose $\lambda > 1$ sufficiently large to ensure that $\mathcal{J}^w$ is decaying. So there exists $0 < c < 1$

so that

$$\mu(B_n(Q)) \leq c^n \mu(Q) \quad ; \quad \forall Q \in \mathscr{D}.$$

Our first goal will be to establish that

$$\int_{G_n(Q)} w^p \, d\mu \leq a^{n-1} \int_Q w^p \, d\mu \tag{5.3}$$

for a constant $0 < a < 1$ depending only on p, c, $[w]_{RH_p^{\mathscr{D}}}$, and geometric constants. First, we consider some properties of $G_1(Q)$. We know by Lemma 5.3.5 that

$$\lambda^{-1}\langle w\rangle_Q \leq w(x) \quad \text{a.e. } x \in G_1(Q),$$

and that since $B_1(Q) \sqcup G_1(Q) = B_0(Q) = Q$ and $B_1(Q) \cap G_1(Q) = \emptyset$,

$$\mu(G_1(Q)) \geq (1-c)\mu(Q).$$

Using these two facts, we conclude that

$$\begin{aligned} \int_{G_1(Q)} w^p \, d\mu &\geq \int_{G_1(Q)} \frac{1}{\lambda^p}\langle w\rangle_Q^p \, d\mu = \frac{\mu(G_1(Q))}{\lambda^p}\langle w\rangle_Q^p \geq \frac{(1-c)\mu(Q)}{\lambda^p}\langle w\rangle_Q^p \\ &\geq \frac{(1-c)\mu(Q)}{\lambda^p [w]_{RH_p^d}^p}\langle w^p\rangle_Q = \frac{(1-c)}{\lambda^p [w]_{RH_p^d}^p} \int_Q w^p \, d\mu \end{aligned} \tag{5.4}$$

Notice that the domain of integration for the far left hand side of inequality (5.4) is a subset of the domain of integration of the far right hand side. In fact, $\mu(G_1(Q)) < \mu(Q)$. Set

$$(1-a) := \frac{(1-c)}{\lambda^p [w]_{RH_p^{\mathscr{D}}}^p} \in (0,1).$$

We observe that this constant a depends only on p, c, $[w]_{RH_p^{\mathscr{D}}}$, and geometric constants. In particular, we observe that a is independent of Q. We now extrapolate this result. We observe (in order to abuse) that

$$G_n(Q) = \bigsqcup_{Q' \in \mathcal{J}_{n-1}^w(Q)} G_1(Q').$$

This allows us to easily see that

$$\begin{aligned}\int_{G_n(Q)} w^p\,d\mu &= \sum_{Q'\in\mathcal{J}^w_{n-1}(Q)} \int_{G_1(Q')} w^p\,d\mu \geq \sum_{Q'\in\mathcal{J}^w_{n-1}(Q)} (1-a)\int_{Q'} w^p\,d\mu \\ &= (1-a)\int_{B_{n-1}(Q)} w^p\,d\mu.\end{aligned}$$

With this, we now have that

$$\begin{aligned}\int_{B_n(Q)} w^p\,d\mu &= \int_{B_{n-1}(Q)} w^p\,d\mu - \int_{G_n(Q)} w^p\,d\mu \\ &\leq \int_{B_{n-1}(Q)} w^p\,d\mu - (1-a)\int_{B_{n-1}(Q)} w^p\,d\mu \\ &= a\int_{B_{n-1}(Q)} w^p\,d\mu. \qquad (5.5)\end{aligned}$$

Since $G_n(Q) \subseteq B_{n-1}(Q)$, iterating (5.5) $n-1$ times gives (5.3).

Fix $\epsilon > 0$ (determined later). Using that for almost every $x \in G_n(Q)$, $w(x)^\epsilon \leq \lambda^\epsilon \langle w\rangle_Q^\epsilon$, we can apply Lemma 5.3.3 and Corollary 5.3.4 to get that

$$\begin{aligned}w(x)^\epsilon &\leq \lambda^\epsilon\left[(D\lambda)^{n-1}\langle w\rangle_Q\right]^\epsilon \\ &\leq (D\lambda)^{n\epsilon}\langle w\rangle_Q^\epsilon.\end{aligned}$$

From this, we can then show that

$$\int_Q w^{p+\epsilon}\,d\mu = \sum_{n=1}^\infty \int_{G_n(Q)} w^{p+\epsilon}\,d\mu \leq \langle w\rangle_Q^\epsilon \sum_{n=1}^\infty (D\lambda)^{n\epsilon}\int_{G_n(Q)} w^p\,d\mu \qquad (5.6)$$

$$\leq \langle w\rangle_Q^\epsilon \sum_{n=1}^\infty (D\lambda)^{n\epsilon} a^{n-1}\int_Q w^p\,d\mu. \qquad (5.7)$$

From here, we choose ϵ small enough so that $(D\lambda)^\epsilon < a^{-1}$, which is possible since $0 < a < 1$. Then the sum

$$\sum_{n=1}^\infty (D\lambda)^{n\epsilon} a^{n-1} =: A < \infty.$$

Therefore, dividing both sides by $\mu(Q)$ gives that

$$\langle w^{p+\epsilon}\rangle_Q \leq A\langle w\rangle_Q^\epsilon \langle w^p\rangle_Q \leq A[w]^p_{RH_p^{\mathscr{D}}}\langle w\rangle_Q^{p+\epsilon}.$$

Since the constant A depended only on p, w, and geometric constants we can conclude that $w \in RH^{\mathscr{D}}_{p+\epsilon}$. Moreover,

$$[w]^{p+\epsilon}_{RH^{\mathscr{D}}_{p+\epsilon}} \leq A[w]^{p}_{RH^{\mathscr{D}}_{p}}.$$

$\square$

Proof of Lemma 5.3.2. Fix λ large, precisely how large to be determined later. For now it suffices to enforce that $\lambda > 3$. For a cube $Q \in \mathscr{D}$ let $\mathcal{J}^w$ be the stopping time for Q. Since the property $\mathcal{P}^w$ with respect to Q has two mutually exclusive stopping conditions, we can split $\mathcal{J}^w(Q)$ into two disjoint parts:

$$\mathcal{J}^w(Q) = \{Q' \in \mathscr{D}(Q) : \langle w \rangle_{Q'} \geq \lambda \langle w \rangle_Q\} \sqcup \{Q' \in \mathscr{D}(Q) : \langle w \rangle_{Q'} \leq \lambda^{-1} \langle w \rangle_Q\}$$

where by $\sqcup$ we mean the disjoint union, i.e., the union of two disjoint sets. We let $\{Q_i^\lambda\}_i$ be an enumeration of the subcubes in the first part and $\{Q_i^{1/\lambda}\}_i$ be an enumeration of the subcubes in the second part. We then write Q as the disjoint union of the three subsets

$$Q = B^\lambda \sqcup B^{1/\lambda} \sqcup G \tag{5.8}$$

with "bad parts" $B^\lambda := \sqcup_i Q_i^\lambda$ and $B^{1/\lambda} := \sqcup_i Q_i^{1/\lambda}$ (so called since the mean is either too large or too small on these parts) and "good part" $G := Q \setminus (B^\lambda \cup B^{1/\lambda})$. It follows from Lemma 5.3.5 that

$$\lambda^{-1}\langle w \rangle_Q \leq w(x) \leq \lambda \langle w \rangle_Q \quad \text{a.e. } x \in G \text{ with respect to } \mu.$$

Suppose that the desired lemma is false, that is, suppose that $\mathcal{J}^w$ is *not* decaying. This would imply that for each $0 < c < 1$ we can find a cube $Q \in \mathscr{D}$ such that

$$\sum_{Q' \in \mathcal{J}^w(Q)} \mu(Q') = \mu(Q \setminus G) > c \cdot \mu(Q)$$

implying that

$$(1 - c) > \frac{\mu(G)}{\mu(Q)}.$$

In other words, the ratio of the measure of the good part to the measure of the whole cube can be made arbitrarily small by selecting the appropriate offending cube.

Choose $Q \in \mathscr{D}$ such that $\mu(G) \leq \mu(Q)/(3\lambda)$. Then

$$\begin{aligned}\int_G w\,d\mu &\leq \int_G \lambda\langle w\rangle_Q\,d\mu = \mu(G)\cdot\lambda\langle w\rangle_Q \\ &= \mu(G)\cdot\frac{\lambda}{\mu(Q)}\int_Q w\,d\mu \leq \frac{1}{3}\int_Q w\,d\mu \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\int_{B^{1/\lambda}} w\,d\mu &\leq \mu(B^{1/\lambda})\cdot\lambda^{-1}\langle w\rangle_Q \leq \lambda^{-1}\frac{\mu(B^{1/\lambda})}{\mu(Q)}\int_Q w\,d\mu \\ &\leq \lambda^{-1}\int_Q w\,d\mu < \frac{1}{3}\int_Q w\,d\mu. \end{aligned} \tag{5.10}$$

Inequalities (5.9) and (5.10) together imply that

$$\begin{aligned}\int_{B^\lambda} w\,d\mu &= \int_{Q\setminus(G\cup B^{1/\lambda})} w\,d\mu = \int_Q w\,d\mu - \int_G w\,d\mu - \int_{B^{1/\lambda}} w\,d\mu \\ &> \int_Q w\,d\mu - \frac{1}{3}\int_Q w\,d\mu - \frac{1}{3}\int_Q w\,d\mu = \frac{1}{3}\int_Q w\,d\mu. \end{aligned} \tag{5.11}$$

We can also see that

$$\begin{aligned}\langle w\rangle_{B^\lambda} &= \frac{1}{\mu(B^\lambda)}\sum_i \int_{Q_i^\lambda} w\,d\mu = \frac{1}{\mu(B^\lambda)}\sum_i \mu(Q_i^\lambda)\langle w\rangle_{Q_i^\lambda} \\ &\leq \frac{1}{\mu(B^\lambda)}\sum_i \mu(Q_i^\lambda) D\lambda\langle w\rangle_Q = D\lambda\langle w\rangle_Q \end{aligned} \tag{5.12}$$

where in (5.12) we used Lemma 5.3.3. We use (5.12) and (5.11) to get a lower bound on the measure of B^λ:

$$\mu(B^\lambda) = \frac{1}{\langle w\rangle_{B^\lambda}}\int_{B^\lambda} w\,d\mu \geq \frac{1}{3\langle w\rangle_{B^\lambda}}\int_Q w\,d\mu \geq \frac{1}{3D\lambda\langle w\rangle_Q}\int_Q w\,d\mu = \frac{1}{3D\lambda}\mu(Q) \tag{5.13}$$

We will now use this lower bound to establish a contradiction. Observe that

$$
\begin{aligned}
\int_Q w^p\,d\mu &\geq \int_{B^\lambda} w^p\,d\mu = \sum_i \int_{Q_i^\lambda} w^p\,d\mu \\
&\geq \sum_i \frac{1}{\mu(Q_i^\lambda)^{p-1}} \left(\int_{Q_i^\lambda} w\,d\mu\right)^p && (5.14)\\
&= \sum_i \mu(Q_i^\lambda)\langle w\rangle_{Q_i^\lambda}^p \geq \lambda^p \sum_i \mu(Q_i^\lambda)\langle w\rangle_Q^p && (5.15)\\
&= \lambda^p \mu(B^\lambda)\langle w\rangle_Q^p \geq \frac{1}{3D}\lambda^{p-1}\mu(Q)\langle w\rangle_Q^p && (5.16)
\end{aligned}
$$

where in (5.14) follows from the Hölder inequality, (5.15) by the definition of B^λ, and (5.16) from (5.13). Dividing both sides by $\mu(Q)$ and taking the $1/p$ power gives that

$$
\langle w^p\rangle_Q^{1/p} \geq \left(\frac{1}{3D}\lambda^{p-1}\right)^{1/p} \langle w\rangle_Q. \tag{5.17}
$$

We thus contradict that $w \in RH_p^{\mathscr{D}}$, provided that λ is chosen large enough so that $\lambda > (3D[w]_{RH_p^{\mathscr{D}}}^p)^{1/(p-1)}$. $\square$

Remark 5.4.1. The preceding proof was a proof by contradiction. While we demonstrated that the decaying constant c does exists, we have no guarantee on the size of this constant.

5.5 Corollaries

It is worth noting that the only time the doubling condition on the weight μ was used was in Lemma 5.3.3. With this in mind we can state the following corollary, replacing the doubling condition on the weight μ with a doubling condition on the weight w.

Corollary 5.5.1 (Gehring for Doubling Weights)**.** *Let (X, ρ, μ) be a quasi-metric measure space with μ a measure which may or may not be doubling and some dyadic structure $\mathscr{D}$. Let $1 < p < \infty$ and let $w \in RH_p^{\mathscr{D}}$ be a weight which is dyadic doubling. Then there exists $\epsilon > 0$ such that $w \in RH_{p+\epsilon}^{\mathscr{D}}$.*

We do need to be careful here. In Corollary 5.5.1, we supposed that we have a dyadic lattice available. Remember that ρ being geometrically doubling was a sufficient condition for the existence of a dyadic grid. However, it has not been shown to be a necessary condition. For this reason, we did *not* add the assumption that ρ be geometrically doubling and instead supposed the existence of $\mathscr{D}$ directly.

We are motivated to give the following definition.

Definition 5.5.2 (Continuous Doubling weights)**.** We say that the set Db of *continuous doubling weights* is the set of all weights such that there exists a constant C so that

$$w(B) \leq C \cdot w(2B) \tag{5.18}$$

where $B = B(x, r)$ and $2B = B(x, 2r)$ for $x \in X$ and $r > 0$.

In Chapter 7, we will revisit the idea of a doubling weight.

Remark 5.5.3. It is easy to confuse a doubling weight with a doubling measure. However, these are not the same thing. In fact, as the corollary below implies, there is no causal relationship between these two, i.e., there exist weights which are not doubling over measures which are, and non-doubling measures can support doubling weights. In light of this, it is important to take care when using this terminology.

We will briefly describe another useful corollary. In [30], the authors describe a finite collection of *adjacent dyadic systems* for X, $\{\mathscr{D}^{(j)}\}_{j=1}^{J_0}$ which satisfy certain properties. In particular, they have that

$$Db \cap RH_p = \bigcap_{j=1}^{J_0} \left(RH_p(\mathscr{D}^{(j)}) \cap Db(\mathscr{D}^{(j)}) \right). \tag{5.19}$$

In other words, the continuous reverse Hölder class is equal to the intersection of finitely many dyadic reverse Hölder classes, at least for doubling weights. We can use this result to give the following nice sufficient condition for continuous Gehring:

Corollary 5.5.4 ("Almost Strong" Continuous Gehring for Spaces of Homogeneous Type)**.** *Let w be a continuous doubling weight so that $w \in RH_p$. There exists $\epsilon > 0$ such that $w \in RH_{p+\epsilon}$.*

Proof. Assume that w is continuous doubling and in RH_p. Then by (5.19), $w \in RH_p(\mathscr{D}^j) \cap Db(\mathscr{D}^j)$ for each $j = 1, ..., J_0$. By Corollary 5.5.1, for each j there exists $\epsilon_j > 0$ such that $w \in RH_{p+\epsilon_j}(\mathscr{D}^{(j)})$. Set $\epsilon := \min\{e_j : 1 \leq j \leq J_0\} > 0$. Then $w \in RH_{p+\epsilon}(\mathscr{D}^j) \cap Db(\mathscr{D}^j)$ for all $j = 1, ..., J_0$, that is w belongs to their intersection. Again by (5.19) we conclude that $w \in RH_{p+\epsilon} \cap Db$. In particular w is in $RH_{p+\epsilon}$. $\square$

Remark 5.5.5. In light of Corollary 5.5.4, we can conclude that the counterexample to strong Gehring provided in [5] must fail to be a doubling weight.

Chapter 6

A Haar Basis of Functions in Spaces of Homogeneous Type

The notion of a dyadic structure on any measure space will be intimately tied to the idea of building a basis of functions utilizing the nice properties of dyadic cubes. In this chapter we present one generalization of the notion of a Haar basis of functions to the SHT setting. Dyadic structures are central to the idea of a Haar basis. In order to properly motivate this generalization, we will first briefly recall the definition of a Haar basis in $\mathbb{R}$. We then look at ways of generalizing this to $\mathbb{R}^n$ in order to gain some insight into how we might go about generalizing to SHTs.

This chapter makes heavy use of the *characteristic* or *indicator function* of a set:

Definition 6.0.1 (Characteristic Function)**.** For X a set and $S \subseteq X$, define the function $\mathbb{1}_S : X \to \mathbb{R}$ as

$$\mathbb{1}_S(x) := \begin{cases} 1 & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S \end{cases} \tag{6.1}$$

called the *characteristic function of S*.

Remark 6.0.2. Some authors (in fact, most likely a majority) use the Greek letter χ for the characteristic function. We will prefer the use of $\mathbb{1}$, however, since χ renders a little below the baseline in the typeface used and this tends to clash somewhat with more complicated subscripts. Compare "$\chi_{Q^k_\alpha}$" with "$\mathbb{1}_{Q^k_\alpha}$," for example.

We let $\mathscr{D}$ be the standard set of dyadic intervals on $\mathbb{R}$. Since we will be momentarily dealing in the specific case of $\mathbb{R}$ we will call dyadic sets intervals instead of cubes. For any interval I, we will denote by $|I|$ the length of I.

Definition 6.0.3 (Haar Function Associated to an Interval)**.** Let $I \in D$ and define the *Haar function associated to* I as

$$h_I(x) := \frac{1}{\sqrt{|I|}} \left(\mathbb{1}_{I_r}(x) - \mathbb{1}_{I_\ell}(x)\right). \tag{6.2}$$

Remark 6.0.4. Notice that the function h_I is always positive valued on the right child of I and negative valued on the left child. This was a purely arbitrary choice and the opposite choice could have been made without effecting our analysis.

These functions (which are also called "*Haar Wavelets*") were first defined by A. Haar in 1910 in the paper [19]. In the same paper, he showed that these functions form a basis for square integrable functions.

Theorem 6.0.5 (Haar functions form a basis)**.** *The collection* $\{h_I\}_{I\in\mathscr{D}}$ *forms an orthornormal basis for* $L^2(\mathbb{R})$.

The aim of this Chapter is thus to generalize this result to the setting of spaces of homogeneous type. We will first discuss the difficulties of generalizing by considering a Haar basis in $\mathbb{R}^n$ in order to gain some insight. We then move on to SHTs and give a proof of an analog to Theorem 6.0.5.

The theorems and proofs in this section are heavily based on the book [43] and the paper [30].

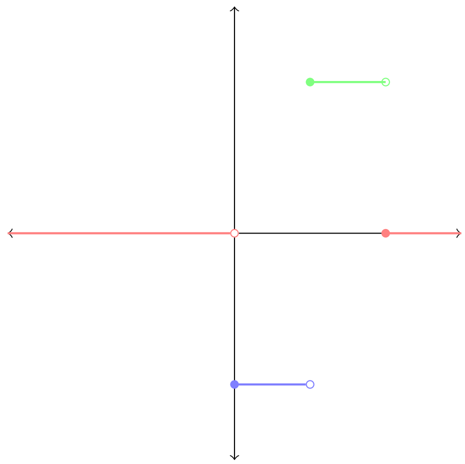


Figure 6.1: A graph of the function $h_{[0,1)}$. Here we use green to denote the positive part, blue to denote the negative part, and red to denote the zero part. In other figures we will continue to use this color scheme for higher dimensional domains.

6.1 A First Look at Generalizing: Haar in $\mathbb{R}^n$

In generalizing the Haar Basis to $\mathbb{R}^n$, we would like to have some basic properties that a "Haar-Like" basis ought to have.

Definition 6.1.1 ("Haar-Like" Basis)**.** Let $\{\varphi_\alpha\}_{\alpha \in A}$ be a complete, orthonormal basis for $L^2(\mathbb{R}^n)$. We will call $\{\varphi_\alpha\}_{\alpha \in A}$ "Haar-Like" if each of the following conditions hold:

(a) $\int_{\mathbb{R}^n} \varphi_\alpha = 0$ for all $\alpha \in A$.

(b) For each $\alpha \in A$, Support$(\varphi_\alpha) = Q$ where Q is some dyadic cube on $\mathbb{R}^n$.

(c) If φ_α is supported on Q, then φ_α is constant on each of Q's children.

(d) If φ_α is supported on Q, then φ_α is positive valued on exactly half of Q's children, and negatively valued on the other half.

A "Haar-like" basis will have the nice properties of single variable Haar functions. Let

Q be a dyadic cube in $\mathbb{R}^n$. We can write

$$Q = \prod_{j=1}^{n} I_j \tag{6.3}$$

where each I_j is a dyadic interval in $\mathbb{R}$. Define

$$h_Q^i(x) := \prod_{j=1}^{n} \left(\sqrt{|I|}\right)^{a_j^{(i)}} h_{I_j}(x)^{a_j^{(i)}}; \quad i = 1, ..., 2^n - 1 \tag{6.4}$$

where $\{a_j^{(i)}\}_{j=1}^n$ is the unique sequence consisting of 1s and 0s so that

$$i = \sum_{j=1}^{n} a_j^{(i)} \cdot 2^{j-1} \tag{6.5}$$

and h_{I_j} is the Haar function on the interval I_j defined above. (I.e. $a_j^{(i)}$ is the j^{th} digit in the binary representation of i.) Here the factor $\left(\sqrt{|I|}\right)^{a_j^{(i)}}$ is to ensure correct normalization of h_Q^i.

It is not difficult to prove that the collection $\{h_Q^i\}_{Q\in\mathscr{D}}^{1\leq i\leq 2^n-1}$ is an orthonormal basis for $\mathbb{R}^n$, given that $\{h_I\}_{\mathscr{D}(\mathbb{R})}$ is. The insight that we gleam from this generalization is that we require more than one Haar function per cube when we are in a higher dimension (see Figures 6.1 and 6.1). In fact, we require $N(Q)-1$ Haar functions (where $N(Q)$ is the number of children that Q has.

6.2 Constructing Haar Functions in Spaces of Homogeneous Type

We now turn our attention to the construction of Haar functions in SHTs. In this section we will make heavy use of the honest cubes described in Section 3.5.

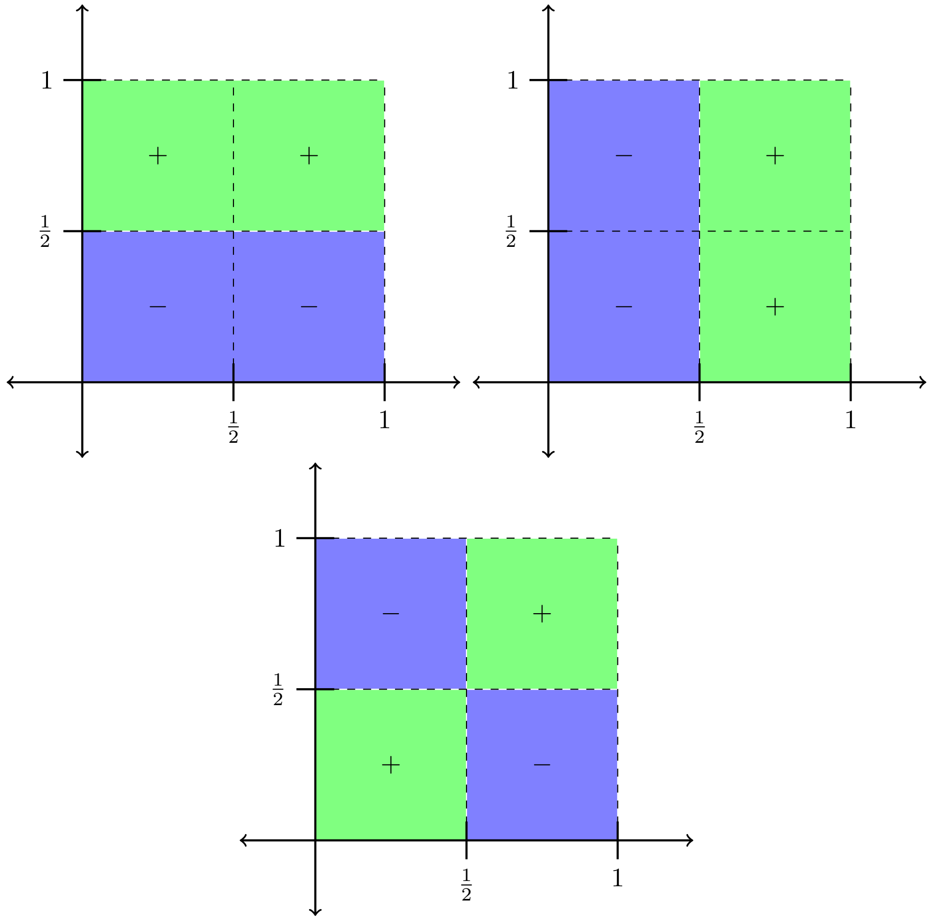


Figure 6.2: The three Haar function associated to a square in $\mathbb{R}^2$.

6.2.1 Defining the Haar Function for an Honest Cube

Let (X, ρ, μ) be an SHT, with dyadic structure $\mathscr{D}$. For the time being we will assume that

(a) X has no atoms,

(b) $\mu(X) = \infty$.

(c) The dyadic structure $\mathscr{D}$ has only one quadrant.

We will deal with erasing these assumptions at the conclusion of the chapter.

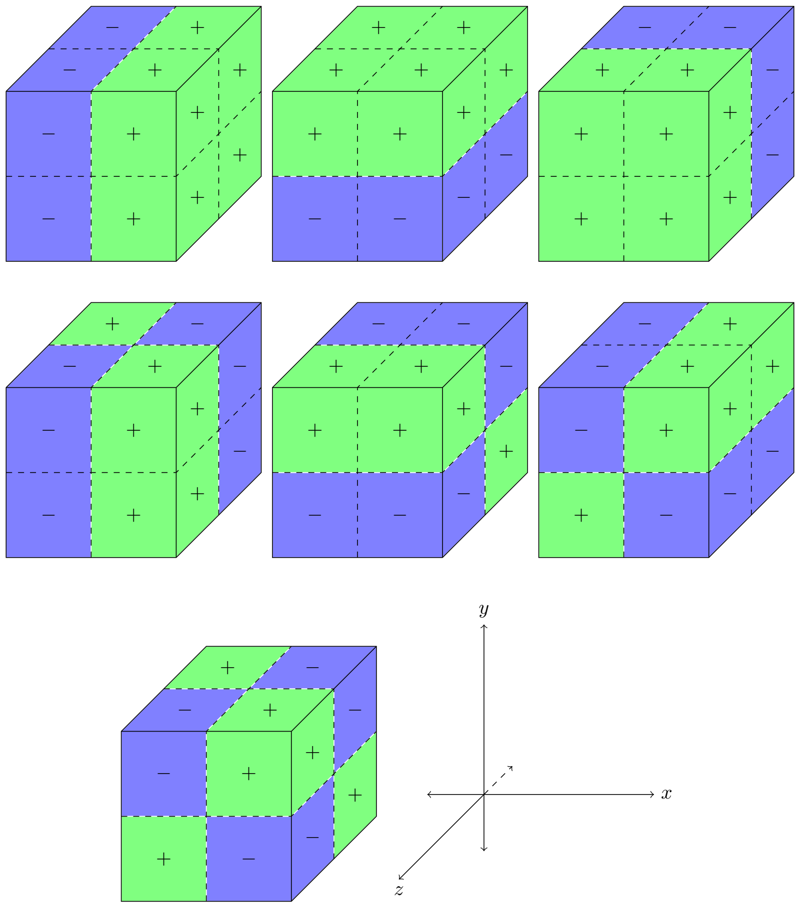


Figure 6.3: The seven Haar functions for a cube in $\mathbb{R}^3$, with the axes for reference.

Let $\tilde{\mathscr{D}}$ be the honest dyadic structure which overlaps with $\mathscr{D}$.

For any honest cube $Q \in \tilde{\mathscr{D}}$ we define the function

$$h_Q(x) := \begin{cases} \lambda_Q^+ \cdot \mathbb{1}_{Q_+}(x) - \lambda_Q^- \cdot \mathbb{1}_{Q_-}(x) & \text{if} \quad N(Q) = 2 \\ 0 & \text{if} \quad N(Q) = 1 \end{cases}$$

where the $\lambda_Q^\pm$ are normalization constants defined by the following relationships:

$$\int_X h_Q = 0 \quad \text{and} \quad \int_X |h_Q|^2 = 1. \tag{6.6}$$

6.2.2 Calculating the λ Coefficients

The relationships in (6.6) imply that

$$\begin{aligned} \lambda_Q^+ \cdot \mu(Q_+) &= \lambda_Q^- \cdot \mu(Q_-), \\ (\lambda_Q^+)^2 \cdot \mu(Q_-) + (\lambda_Q^-)^2 \cdot \mu(Q_+) &= 1. \end{aligned}$$

We solve this system of equations to give expressions for the λs:

$$\lambda_Q^+ = \left(\frac{\mu(Q_-)}{\mu(Q_+)\cdot\mu(Q)}\right)^{1/2} \qquad \lambda_Q^- = \left(\frac{\mu(Q_+)}{\mu(Q_-)\cdot\mu(Q)}\right)^{1/2}. \tag{6.7}$$

Remark 6.2.1. Recall that a "cube" is more than just a set – it also belongs to a generation. In the definition for h_Q we say that $h_Q \equiv 0$ if Q has only one child. It is entirely possible that a cube has only one child for many generations before finally splitting. In this situation, there would be many Haar functions associated to the cube *as a set*. However, only the last one (before it splits) would be non-zero.

6.2.3 The Haar Basis

We can now define the set of Haar functions:

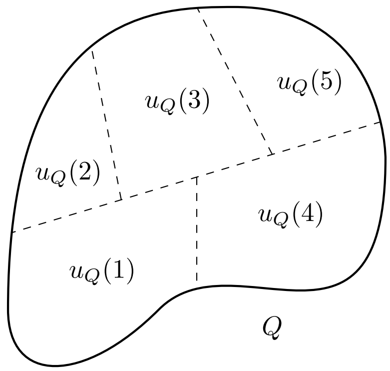
Definition 6.2.2 (Haar basis)**.** Let X be an SHT with any dyadic structure $\mathscr{D}$ and let $\tilde{\mathscr{D}}$ be an honest dyadic structure which overlaps $\mathscr{D}$. For any cube $Q \in \mathscr{D}$ we can define the set

$$\{h_Q^i \mid 1 \le i \le N(Q) - 1\} \tag{6.8}$$

as the collection of non-zero, honest Haar functions supported on more than one of Q's honest descendants. Define the set

$$\{h_Q^i \mid Q \in \mathscr{D}, 1 \le i \le N(Q) - 1\} \tag{6.9}$$

as the *Haar basis* associated to Q with respect to $\mathscr{D}$.



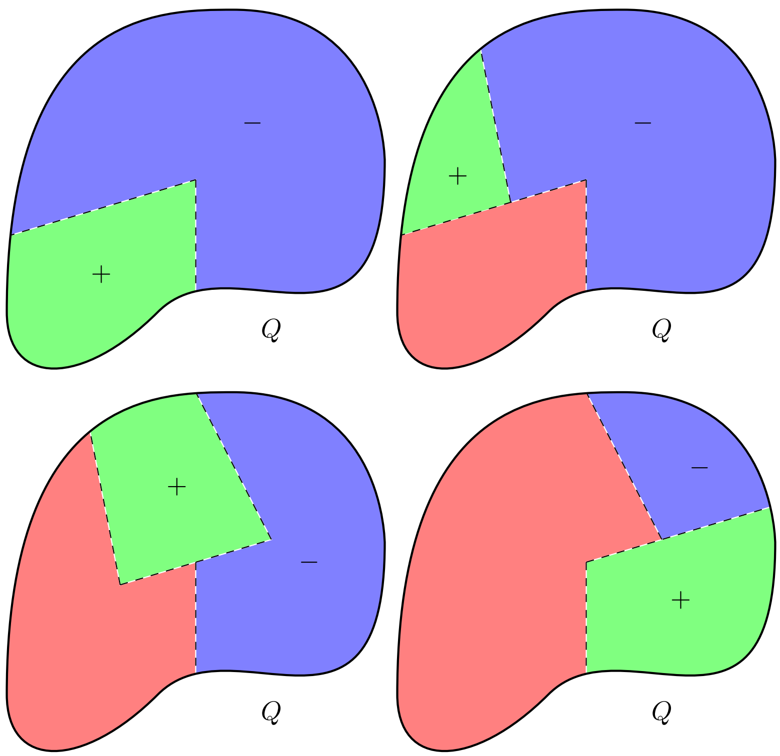


Figure 6.4: *Above*: A non-honest cube Q with its children and a denumeration u_Q which can generate an honest structure. *Below*: The four Haar functions associated to the cube Q with respect to the honest system. Recall that Haar functions take on λ^+ in the green region, λ^- in the blue region, and zero in the red region.

6.3 Different Honest Structures

The definition of a Haar basis given in the previous section has some obvious concerns. In this section we will address the fact that the choice of Haar basis is dependent on the choice of honest dyadic structure.

For $Q \in \mathscr{D}$ denote by S_Q^0 the space of all functions $f : Q \to \mathbb{R}$ constant on the children of Q with mean zero.

Lemma 6.3.1. *Let $Q \in \mathscr{D}$ and let h_Q^i, $1 \leq i \leq N(Q) - 1$. Then* $\operatorname{span}(\{h_Q^i\}) = S_Q^0$.

Proof. Consider the larger space of functions $S_Q \supsetneq S_Q^0$ which are the functions constant on the children of Q. There exists a canonical vector space homomorphism, ψ, between the set S_Q and $\mathbb{R}^{N(Q)}$. We can send $g \in S_Q$ to $\vec{v} \in \mathbb{R}^{N(Q)}$ by way of ψ:

$$g = \sum_{i=1}^{N(Q)} \frac{b_i \mathbb{1}_{u_Q(i)}}{\mu(u_Q(i))} \qquad \overset{\psi}{\mapsto} \qquad \vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N(Q)} \end{bmatrix}$$

Recall that u_Q was defined in Section 3.5 as an enumeration of Q's children. This shows that S_Q has the structure of a $N(Q)$-dimensional vector space.

Now suppose furthermore that $g \in S_Q^0$. Then

$$0 = \int_X g(y)\, d\mu(y) = \int_X \sum_{i=1}^{N(Q)} \frac{b_i \mathbb{1}_{u_Q(i)}(y)}{\mu(u_Q(i))}\, d\mu(y) = \sum_{i=1}^{N(Q)} \frac{b_i}{\mu(u_Q(i))} \int_{u_Q(i)} d\mu(y) = \sum_{i=1}^{N(Q)} b_i.$$

This restricts the element $\vec{v} = \psi(g)$ to a $N(Q) - 1$ dimensional subspace of $\mathbb{R}^{N(Q)}$. Thus S_Q^0, by way of $\psi|_{S_Q^0}$, must also have the structure of a $N(Q) - 1$ dimensional vector space. As in any vector space, a set of $N(Q) - 1$ orthonormal elements constitute a basis. It has already been shown that the Haar functions h_Q^i are such a set. This completes the proof. $\square$

Corollary 6.3.2 (Haar Basis is independent of honest structure)**.** *Let X be an SHT with dyadic structure $\mathscr{D}$. Suppose that $\tilde{\mathscr{D}}_0$ and $\tilde{\mathscr{D}}_1$ are two (perhaps different) honest dyadic structures which overlap $\mathscr{D}$. Set H and G to the Haar basis with respect to $\tilde{\mathscr{D}}_0$ and $\tilde{\mathscr{D}}_1$, respectively. Then* $\operatorname{span}(H) = \operatorname{span}(G)$.

By $\operatorname{span}(H)$, we mean the closure in $L^2(X)$ of H. However, in the proof, we will only need to consider finite linear combinations of functions.

Proof. Set $H = \{h_Q^i\}_{Q\in\mathscr{D}}^{1\leq i\leq N(Q)-1}$ and $G = \{g_Q^i\}_{Q\in\mathscr{D}}^{1\leq i\leq N(Q)-1}$. Without loss of generality, it is enough to show that $\operatorname{span}(H) \subseteq \operatorname{span}(G)$. Let $f \in \operatorname{span}(H)$. This means that there exists constants $\{a_Q^i\}$, $1 \leq i \leq N(Q)-1$ so that

$$f = \sum_{Q\in\mathscr{D}} \left(\sum_{i=1}^{N(Q)-1} a_Q^i \cdot h_Q^i \right)$$

where quality here is in L^2-sense. For each cube Q define $f_Q := \sum_{i=1}^{N(Q)-1} a_Q^i \cdot h_Q^i$. Then $f_Q \in S_Q^0$ for all $Q \in \mathscr{D}$ since it is a linear combination of functions in S_Q^0. Moreover, by the previous lemma there exist constants $\{b_Q^i\}$, $1 \leq i \leq N(Q)-1$, so that

$$f_Q = \sum_{i=1}^{N(Q)-1} b_Q^i \cdot g_Q^i \tag{6.10}$$

since each side of the equation is an element of S_Q^0. Thus,

$$f = \sum_{Q\in\mathscr{D}} f_Q = \sum_{Q\in\mathscr{D}} \left(\sum_{i=1}^{N(Q)-1} b_Q^i \cdot g_Q^i \right) \tag{6.11}$$

which is a function in the span of G. □

The previous corollary establishes that, in some sense, all Haar bases are equivalent. From this point on we will drop the use of "with respect to $\tilde{\mathscr{D}}$" when referring to the Haar basis.

6.4 Main Result

Our goal is to verify that this is a Haar-like basis, i.e. to prove the following theorem:

Theorem 6.4.1 (Haar Like For Honest Cubes)**.** *Let X be an SHT with honest dyadic structure $\tilde{\mathscr{D}}$. The set $\{h_Q\}_{Q\in\tilde{\mathscr{D}}}$*

(a) forms a complete orthonormal basis for $L^2(X)$

(b) is Haar-like.

Once we prove Theorem 6.4.1 we will get this corollary for free:

Corollary 6.4.2. *Let X be an SHT with dyadic structure $\mathscr{D}$. The set $\{h_Q^i\}_{Q\in\mathscr{D}}^{1\le i\le N(Q)-1}$ is a Haar-like basis.*

6.4.1 Proof of Orthonormality and Haar-like

As always, we start with the easy part.

Proof of Orthonormality. Let $Q, Q' \in \mathscr{D}$ be two cubes. By the properties of the cubes, we are in one of three cases:

(Case 1: $Q \cap Q' = \emptyset$) In this case h_Q and $h_{Q'}$ have disjoint support, thus

$$\langle h_Q, h_{Q'}\rangle = 0.$$

(Case 2: $Q' \subsetneq Q$) In this case, Q' is one of Q's descendants and is thus supported entirely on one of Q's . Since h_Q is constant on the children, $\langle h_Q, h_{Q'}\rangle$ is equal to some multiple of the average of $h_{Q'}$. But each of these averages was cooked to be zero, so

$$\langle h_Q, h_{Q'}\rangle = 0.$$

(Case 3: $Q' = Q$) In this case, we are just taking the inner product of h_Q with itself, which was cooked to be equal to one. $\square$

Proof of Haar-like. This follows immediately from how the $\lambda_Q^{\pm}$s were defined. $\square$

That takes care of the easy part!

6.4.2 Proof of Completeness

The proof of completeness is the difficult part.

Notation 4. For $k \in \mathbb{Z}$ and $x \in Q$, let Q_x^k be the unique cube in $\mathscr{D}^k$ which contains x.

Lemma 6.4.3. *Let $f \in L^2(X)$ and $Q \in \tilde{\mathscr{D}}_k$. Then,*

$$\mathrm{Proj}_{S_Q^0}(f)(x) = \frac{1}{\mu\left(Q_x^{k+1}\right)} \int_{Q_x^{k+1}} f \, d\mu - \frac{1}{\mu(Q)} \int_Q f \, d\mu \quad \forall x \in Q \tag{6.12}$$

Proof. Let $f \in L^2(X)$. Since we are dealing with honest cubes, we can break into two cases:

(Case 1: $N(Q) = 1$) Then $Q_x^{k+1} = Q$ so equation (6.12) becomes

$$\frac{1}{\mu\left(Q\right)} \int_Q f \, d\mu - \frac{1}{\mu(Q)} \int_Q f \, d\mu = 0.$$

But S_Q^0 is a zero dimensional vector space since the only function it contains is the zero function. Therefore $\mathrm{Proj}_{S_Q^0}(f) = 0$ as well, verifying this case.

(Case 2: $N(Q) = 2$)

Let $x \in Q$ and let $f \in L^2(X)$. By definition,

$$\mathrm{Proj}_{S_Q^0}(f)(x) = \sum_{i=1}^{N(Q)-1} \langle f, \varphi_i \rangle \varphi_i(x)$$

where $\{\varphi_i\}_{i=1}^{N(Q)-1}$ is an orthonormal basis for S_Q^0. In particular, we could choose the non-trivial, honest Haar functions which satisfy that $x \in Q_+$, i.e. that

$$h_Q(x) = \lambda_Q^+.$$

This gives that

$$\begin{aligned}
\mathrm{Proj}_{S_Q^0}(f)(x) &= \langle f, h_Q\rangle \cdot \lambda_Q^+ = \lambda_Q^+ \int_X f(y)\overline{h_Q(y)}\, d\mu(y) \\
&= \lambda_Q^+ \int_X f(y) \left(\lambda_Q^+ \mathbb{1}_{Q_+}(y) - \lambda_Q^- \mathbb{1}_{Q_-}(y)\right) d\mu(y) \\
&= (\lambda_Q^+)^2 \left(\int_{Q_+} f\, d\mu\right) \; - \; \lambda_Q^+ \cdot \lambda_Q^- \left(\int_{Q_-} f\, d\mu\right).
\end{aligned}$$

Now, looking at the coefficients for these integrals we see that

$$(\lambda^+)^2 = \frac{\mu(Q_-)}{\mu(Q_+)\cdot \mu(Q)}$$

and

$$\lambda_Q^+ \cdot \lambda_Q^- = \left(\frac{\mu(Q_-)}{\mu(Q_+)\cdot\mu(Q)}\right)^{1/2} \left(\frac{\mu(Q_+)}{\mu(Q_-)\cdot\mu(Q)}\right)^{1/2} = \frac{1}{\mu(Q)}.$$

Thus,

$$\begin{aligned}
\mathrm{Proj}_{S_Q^0}(f)(x) &= \frac{\mu(Q_-)}{\mu(Q_+)\cdot\mu(Q)} \int_{Q_+} f\, d\mu \; - \; \frac{1}{\mu(Q)} \int_{Q_-} f\, d\mu \\
&= \frac{\mu(Q_-)}{\mu(Q_+)\cdot\mu(Q)} \int_{Q_+} f\, d\mu \; - \; \frac{1}{\mu(Q)} \int_{Q_-} f\, d\mu \; + \; \frac{1}{\mu(Q)} \int_{Q_+} f\, d\mu \\
&\qquad\qquad - \frac{1}{\mu(Q)} \int_{Q_+} f\, d\mu \\
&= \frac{1}{\mu(Q_+)} \int_{Q_+} f\, d\mu \; - \; \frac{1}{\mu(Q)} \int_Q f\, d\mu. \qquad (6.13)
\end{aligned}$$

□

It will be convenient for us to note that the equation in line (6.13) could be rewritten as

$$\mathrm{Proj}_{S_Q^0}(f)(x) = \frac{1}{\mu(Q)} \int_Q f\, d\mu \; - \; \frac{1}{\mu(\widehat{Q})} \int_{\widehat{Q}} f\, d\mu \qquad \text{for } x \in Q \qquad (6.14)$$

when projecting onto the space $S^0_{\widehat{Q}}$. Here $\widehat{Q}$ denotes a cube's parent.

Now we turn our attention to the following sum:

$$\sum_{Q\in\mathscr{D}} \langle f, h_Q\rangle h_Q(x) \tag{6.15}$$

We would like to say this is equal to f in $L^2(\mu)$ sense.

Theorem 6.4.4. *The sum in equation (6.15) converges to f point-wise almost everywhere.*

Proof. Fix $x \in X$. By Lemma 6.4.3,

$$\begin{aligned}
\sum_{Q\in\mathscr{D}} \langle f, h_Q\rangle h_Q(x) &= \sum_{Q\in\mathscr{D}} \mathrm{Proj}_{S^0_Q}(f)(x)\\
&= \sum_{Q\in\mathscr{D}(x)} \left[\frac{1}{\mu(Q)}\int_Q f\,d\mu \;-\; \frac{1}{\mu(\widehat{Q})}\int_{\widehat{Q}} f\,d\mu\right]\\
&= \lim_{N,M\to\infty} \sum_{k=-M}^{N} \left[\frac{1}{\mu(Q_x^{k+1})}\int_{Q_x^{k+1}} f\,d\mu \;-\; \frac{1}{\mu(Q_x^{k})}\int_{Q_x^{k}} f\,d\mu\right]\\
&= \lim_{N,M\to\infty} \left[\frac{1}{\mu(Q_x^{N+1})}\int_{Q_x^{N+1}} f\,d\mu \;-\; \frac{1}{\mu(Q_x^{-M})}\int_{Q_x^{-M}} f\,d\mu\right]
\end{aligned}$$

where the final line follows from the fact that this is a telescoping sum. It is enough to show that

$$\lim_{N\to\infty} \frac{1}{\mu(Q_x^{N+1})}\int_{Q_x^{N+1}} f\,d\mu = f(x) \tag{6.16}$$

and

$$\lim_{M\to\infty} \frac{1}{\mu(Q_x^{-M})}\int_{Q_x^{-M}} f\,d\mu = 0. \tag{6.17}$$

Line (6.16) follows from the Lebesgue Differentiation Theorem on dyadic cubes, where the limits are in the L^2-sense. For line (6.17), we see that by Cauchy-Schwarz,

$$\left|\int_{Q_x^{-M}} \frac{f}{\mu(Q_x^{-M})}\,d\mu\right| \le \left(\int_{Q_x^{-M}} |f|^2\,d\mu\right)^{\frac{1}{2}} \left(\int_{Q_x^{-M}} \frac{1}{\mu(Q_x^{-M})^2}\,d\mu\right)^{\frac{1}{2}} = \frac{||f||_{L^2(Q_x)}}{\mu(Q)^{\frac{1}{2}}}.$$

Recall that we are under the assumption that there is only a single quadrant in $\mathscr{D}$, and that $\mu(X) = \infty$. Since $f \in L^2(X)$ and $\lim_{k\to-\infty} Q_x^k = \mathrm{Quad}(x) = X$, this goes to zero in the limit. This completes the proof. $\square$

6.5 Removing the Simplifications

Earlier in this chapter we made some simplifications. We will now briefly address each of these and show how they either do not complicate our situation, or can be dealt with via a simple addendum.

6.5.1 Finite Spaces

What happens when $\mu(X) < \infty$. From the finite measure lemma of Chapter 2, we know this imples that X is bounded, i.e. that there exists $R > 0$ and $x_0 \in X$ such that $X \subseteq B(x_0, R)$. Moreover, we know that whatever our dyadic grid, there is a generation $\mathscr{D}_k$ for which there is only one large cube which entirely contains X and all higher generations are similarly just this one cube. This complicates the proof of Theorem 6.4.4 in that equation (6.17) no longer holds. Instead we have that

$$\lim_{M\to\infty} \frac{1}{\mu(Q_x^{-M})} \int_{Q_x^{-M}} f \, d\mu = \langle f \rangle_X.$$

The fix for this is slightly ad hoc, but it does the trick. We simply add into our Haar basis the function $\mathbb{1}_X/\sqrt{\mu(X)}$. Then for any $L^2(X)$ function f, we can write

$$f = f_0 + \frac{\langle f \rangle_X \mathbb{1}_X}{\sqrt{\mu(Q)}}$$

where $\langle f_0 \rangle_X = 0$. We then do the proof of Theorem 6.4.4 for f_0, and line (6.17) still holds.

6.5.2 Multiple Quadrants

If our dyadic lattice has more than one quadrant, then we have a few things to consider. First, it goes worth mentioning again that multiple quadrants can only exists when our space is unbounded, so we are definitely not also in the situation of the previous section. Recall from our discussion in Chapter 2 that quadrants are themselves subSHTs of X when the dyadic lattice has the thin boundary property. In this case we can just run the argument on each quadrant, of which there are at most countably many.

In the case that $\mathscr{D}$ lacks the thin boundaries property, it may be that some quadrants fail to be subSHTs. In particular, it may be that some quadrants while being unbounded are nevertheless finite in measure. This situation forces that we must also include in our basis the characteristic functions for each such quadrant. We can then proceed similarly as in the previous section.

6.5.3 Atoms

Our last consideration is the case when the space X has atoms. It might first seem as though atoms are in some sense the inverse problem of the previous two sections; they introduce an issue for tiny cubes while above we had a problem with large cubes. We may be tempted to segregate atoms into a subset $\mathcal{A}$ and do our construction on $X \setminus \mathcal{A}$, then throw the characteristic function of each atom into our Haar basis. Surprisingly however, this is not actually required and our proof above is still valid – even in a space with atoms.

Recall that atoms are necessarily isolated points in SHTs. This forces that for each atom $a \in \mathcal{A}$, there is a generation k such that the cube $Q_a^k = \{a\}$ and that this is also true for all subsequent generations. As we note in the proof, $\mathrm{Proj}_{S_Q^0}(f)(x) = 0$ when Q has only one child. This has the effect of the causing the telescoping sum to stop telescoping for small cubes when x happens to be an atom. Thus, line (6.16) simply becomes

$$\lim_{N\to\infty} \frac{1}{\mu(Q_x^{N+1})} \int_{Q_x^{N+1}} f \, d\mu = \frac{1}{\mu(Q_a^k)} \int_{Q_a^k} f \, d\mu = f(a).$$

We do not need to invoke the Lebesgue differentiation theorem here. The rest of the proof has no consideration for atoms, so nothing needs to be done.

Chapter 7

Bellman Functions for Spaces of Homogeneous Type

In this chapter, we will begin to explore the Bellman function proof technique. We will not be providing any new Bellman function arguments exactly. Rather our purpose is to attempt to build machinery which will allow us to easily extend Bellman-type results on functions defined over $\mathbb{R}$ to functions defined over spaces of homogeneous type.

We will begin by proving a very useful convexity lemma. We then turn our attention to the main result of the chapter, the so-called Good Bellman Function lemma. This lemma is an adaptation of a Lemma of D. Chung, in which he was able to generalize Bellman arguments from $\mathbb{R}$ to $\mathbb{R}^n$ without difficulty. There are two main complication in extending this to spaces of homogeneous type. First of all, there is the issue that cubes in SHTs do not generally have 2 children. This however can be dealt with by passing first to honest cubes. The second, and more interesting, complication is the issue with convexity. Bellman type arguments rely at their center on proving a "main inequality" which is usually a convexity condition. However, since children intervals in $\mathbb{R}$ always have half the length of their parent, it is enough to consider only midpoint-convexity. For SHTs, even when dealing with honest cubes, this luxury is no longer present. We must therefore first develop the tools to overcome this complication.

Remark 7.0.1. In this chapter we will make use of the following notation convention without repeating its explanation. For a point $\mathbf{x} \in \mathbb{R}^{d_1+d_2}$ we will write $\mathbf{x} = (u, v)$ where $u \in \mathbb{R}^{d_1}$ and $v \in \mathbb{R}^{d_2}$. Similarly if the point has a special subscript or superscript symbol, e.g. $\mathbf{x}^\circ = (u^\circ, v^\circ)$.

7.1 A Generalized Convexity Lemma

In this section we will state and prove a very useful lemma related to function convexity. It is a well known result that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *midpoint-convex* if and only if it is *convex.*

Theorem 7.1.1 (Midpoint Convexity Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.*

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2} \tag{7.1}$$

for all $x, y \in \mathbb{R}$ if and only if

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) \tag{7.2}$$

for every $t \in [0, 1]$.

The proof of this theorem is not very hard and is sometimes given as an exercise in introduction to analysis courses. (See for example [45] Ch. 4 ex. 24, pg. 101).

Our goal in this section is to generalize this result to higher dimensions. In the process, we will be forced to deal with the difficulty of non-convex domains. We will begin with some definitions.

Definition 7.1.2 (Prism)**.** Let $P \subseteq \mathbb{R}^{d_1+d_2}$ be a Lebesgue measurable set so that

$$P = P_B \times P_R \; ; \;\; P_B \subseteq \mathbb{R}^{d_1}, \;\; P_R \subseteq \mathbb{R}^{d_2}. \tag{7.3}$$

Moreover, P_B is any measurable set, and P_R is a d_2-dimensional rectangle, i.e.

$$P_R = \prod_{i=1}^{d_2} I_i \tag{7.4}$$

with each $I_i \subseteq \mathbb{R}$ an interval. We will refer to such a set as a $(d_1 + d_2)$-dimensional prism with d_1-dimensional base B, or more concisely, a d_1-*prism*.

Example 7.1.3. In the colloquial sense, a prism is a polyhedra with rectangular sides and triangular bases. In our terminology this would be a 3-dimensional 2-prism.

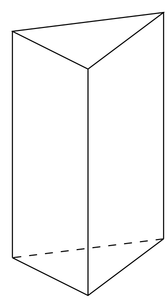


Figure 7.1: A prism

Definition 7.1.4 (Weakly Convex Set)**.** Let $S \subseteq \mathbb{R}^d$ be a (not necessarily convex) set which contains the origin. Suppose there exists a family of matrices $\{A_t\}_{t\in(0,1/2]} \subset \mathbb{R}^{d\times d}$ with the following properties:

- A_t is non-singular for $0 < t < t$.
- A_t is symmetric about the point $t = 1/2$, i.e. $A_t = A_{1-t}$ for all $0 < t < 1$
- The function $a_{\mathbf{x}}(t) = A_t\mathbf{x}$ is continuous for every $\mathbf{x} \in S$

- If u^+, u^-, and u° are three distinct, collinear points contained in S with $u^\circ = tu^+ + (1-t)u^-$ for $t \in (0, 1/2)$, then the line segment $\overline{A_t u^+ A_t u^-}$ is completely contained in S.

We say that the set S is *weakly convex* under the matrices $\{A_t\}$.

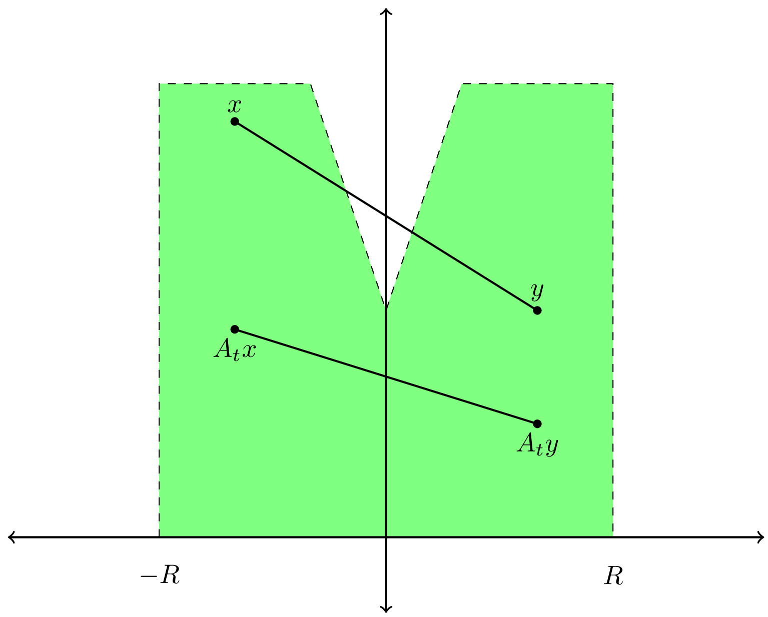


Figure 7.2: This square of side-length $2R$ with a triangular wedge taken out is a weakly convex set.

In Figure 7.2, we have a simple *weakly convex* set which is itself not convex. Here the family of matrices $\{A_t\}$ can be defined as

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{7.5}$$

for every $0 < t < 1$. For more complicated sets, a dependence on t should be expected.

Remark 7.1.5. If S was a convex set, then $A_t = I$ the identity matrix for every t. Thus, convex implies weakly convex.

In the next section we will give some more complicated examples of weakly convex sets. For now, it is enough to note that often the domains of Bellman functions are prisms whose bases are weakly convex sets.

We will use the convention that $\mathbb{R}^+ := \{x \in \mathbb{R} \mid s \geq 0\}$, that is, that $\mathbb{R}^+$ is the non-negative real numbers.

Lemma 7.1.6 (Generalized Midpoint Convexity Lemma). *Let $\Omega \subset \mathbb{R}^{d_1+d_2}$ be a d_1-prism with base Ω_B and rectangle Ω_R. Suppose also that Ω_B is a weakly convex set under the family of matrices $\{A_t\}_{t\in(0,1/2]} \subset \mathbb{R}^{d_1\times d_1}$. For each $t \in (0, 1/2]$, define the block matrices*

$$\tilde{A}_t := \begin{bmatrix} A_t & 0 \\ 0 & I_{d_2} \end{bmatrix} \qquad B_t := \begin{bmatrix} \tilde{A}_t & 0 & 0 \\ 0 & \tilde{A}_t & 0 \\ 0 & 0 & \tilde{A}_t \end{bmatrix}$$

where I_{d_2} is the $d_2 \times d_2$ identity matrix. Let $F : \Omega \to \mathbb{R}^+$ be a continuous function which satisfies a midpoint convexity inequality:

$$F(\mathbf{x}^\circ) \geq \frac{1}{2}\left[F(\mathbf{x}^+) + F(\mathbf{x}^-)\right] + f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-)$$

for all $\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^- \in \Omega$ with $u^\circ = \frac{1}{2}(u^+ + u^-)$ and where $f : \Omega \times \Omega \times \Omega \to \mathbb{R}$ is a positively valued function satisfying

$$f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) = f(\mathbf{x}^\circ, \mathbf{x}^-, \mathbf{x}^+). \tag{7.6}$$

Then for each $t \in (0, 1)$, $F \circ \tilde{A}_t$ satisfies a convexity inequality

$$(F \circ \tilde{A}_{t'})(\mathbf{x}^\circ) \geq t(F \circ \tilde{A}_t)(\mathbf{x}^+) + (1-t)(F \circ \tilde{A}_t)(\mathbf{x}^-) + 2t(f \circ B_t)(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-)$$

where $u^\circ = tu^+ + (1-t)u^-$.

Note that the expression $\tilde{A}_t\mathbf{x}$ is meaningful since

$$\tilde{A}_t\mathbf{x} = \begin{bmatrix} A_t & 0 \\ 0 & I_{d_2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_t u \\ v \end{bmatrix}.$$

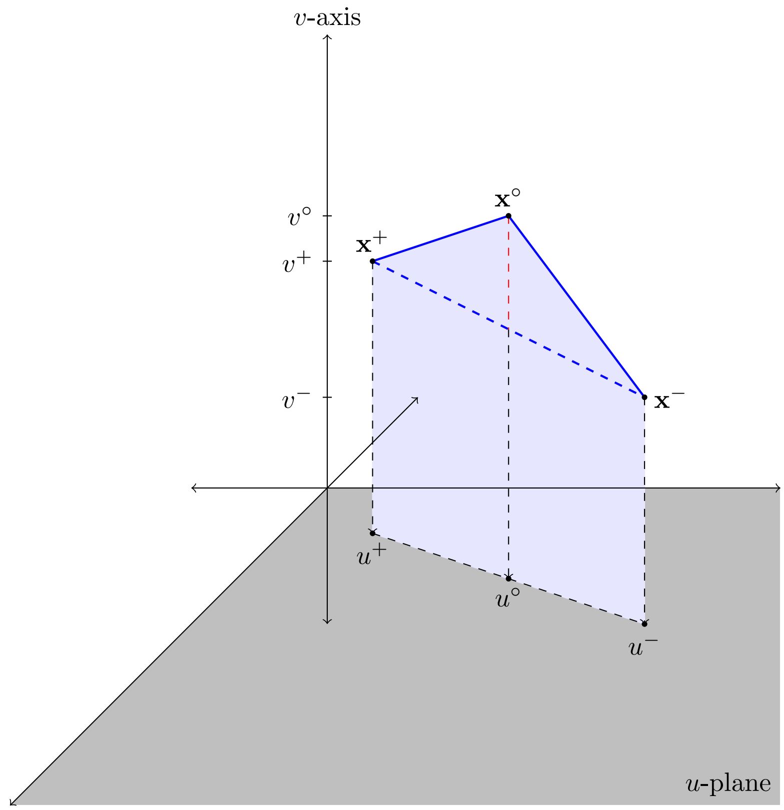


Figure 7.3: A visualization of the points $\mathbf{x}^\circ = (u^\circ, v^\circ), \mathbf{x}^\pm = (u^\pm, v^\pm)$. The points u°, u^+ and u^+ are co-linear and lie in the u-plane. The length of the red dashed segment is equal to the distance between $\mathbf{x}^\circ$ and the true linear interpolation from $\mathbf{x}^+$ and $\mathbf{x}^-$.

Proof. Without loss of generality $t \in (0, 1/2]$. Suppose for the sake of a contradiction that the desired lemma is false. That is, suppose that there exists $t \in (0, 1/2]$ and $\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^- \in \Omega$ so that $u^\circ = tu^+ + (1-t)u^-$ but

$$(F \circ \tilde{A}_t)(\mathbf{x}^\circ) < t(F \circ \tilde{A}_t)(\mathbf{x}^+) + (1-t)(F \circ \tilde{A}_t)(\mathbf{x}^-) + 2t(f \circ B_t)(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-). \tag{7.7}$$

For convenience, set $K := (f \circ B_t)(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-)$.

Let Γ be the union of the two line segments $\overline{\mathbf{x}^+\mathbf{x}^\circ}$ and $\overline{\mathbf{x}^\circ\mathbf{x}^-}$. Note that the projection of Γ onto the subspace $\mathbb{R}^{d_1}$ is actually a line, since u°, u^+, and u^- are collinear. Notice also that $\tilde{A}_t(\Gamma)$ is a line segment completely contained within Ω by the weak convexity of Ω_B.

Parameterize Γ by the function $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \Gamma$ so that

$$\gamma(0) = \mathbf{x}^- \quad \gamma(t) = \mathbf{x}^\circ \quad \gamma(1) = \mathbf{x}^+,$$

γ_1 is linear and γ_2 is piecewise linear. Explicitly,

$$\gamma_1(s) := su^+ + (1-s)u^-$$
$$\gamma_2(s) := \begin{cases} \frac{t-s}{t}v^- + \frac{s}{t}v^\circ & \text{if} \quad 0 \le s \le t \\ \frac{1-s}{1-t}v^\circ + \frac{s-t}{1-t}v^- & \text{if} \quad t \le s \le 1 \end{cases}$$

We next set $L_1 := (F \circ \tilde{A}_t)(\mathbf{x}^+)$, $L_2 := (F \circ \tilde{A}_t)(\mathbf{x}^-)$, and define a function $\varphi_t : [0,1] \to \mathbb{R}$ as

$$\varphi_t(s) := (F \circ \tilde{A}_t \circ \gamma)(s) - \left(sL_1 + (1-s)L_2 - \frac{s(1-s)2K}{1-t} \right).$$

Notice that $\varphi_t(0) = \varphi_t(1) = 0$. Furthermore notice that φ_t is continuous in s, since it is the composition of continuous functions.

We now claim that φ_t satisfies a midpoint convexity inequality (with the same function f as F). Let $0 \le s_1 < s_2 \le 1$ be arbitrary and set $\overline{s} := \frac{1}{2}(s_1 + s_2)$ the midpoint. By the linearity of γ_1 and A_t,

$$A_t\gamma_1(\overline{s}) = \frac{A_t\gamma_1(s_1) + A_t\gamma_1(s_2)}{2}.$$

By the fact that F satisfies a midpoint convexity inequality,

$$\begin{aligned} (F \circ \tilde{A}_t \circ \gamma)(\overline{s}) &= F\left(A_t\gamma_1(\overline{s}), \gamma_2(\overline{s})\right) \\ &= F\left(\frac{A_t\gamma_1(s_1) + A_t\gamma_1(s_2)}{2}, \gamma_2(\overline{s}) \right) \\ &\ge \frac{(F \circ \tilde{A}_t \circ \gamma)(s_1) + (F \circ \tilde{A}_t \circ \gamma)(s_2)}{2} + (f \circ B_t)(\gamma(\overline{s}), \gamma(s_1), \gamma(s_2)). \end{aligned}$$

Moreover, the function

$$p(x) := -\left(sL_1 + (1-s)L_2 + \frac{s(1-s)2K}{1-t} \right)$$

is a downward opening parabola, so it also satisfies the midpoint convexity inequality

$$p(\overline{s}) \geq \frac{p(s_1) + p(s_2)}{2}.$$

Therefore,

$$\begin{aligned}
\varphi_t(\overline{s}) &= (F \circ \tilde{A}_t \circ \gamma)(\overline{s}) + p(\overline{s}) \\
&\geq \frac{(F \circ \tilde{A}_t \circ \gamma)(s_1) + (F \circ \tilde{A}_t \circ \gamma)(s_2)}{2} + \frac{p(s_1) + p(s_2)}{2} \\
&\qquad + (f \circ B_t)(\gamma(\overline{s}), \gamma(s_1), \gamma(s_2)) \\
&= \frac{\varphi_t(s_1) + \varphi_t(s_2)}{2} + (f \circ B_t)(\gamma(\overline{s}), \gamma(s_1), \gamma(s_2)).
\end{aligned}$$

We proceed by plugging t into φ_t and see that

$$\begin{aligned}
\varphi_t(t) &= (F \circ \tilde{A}_t \circ \gamma)(t) - (tL_1 + (1-t)L_2) - 2tK \\
&= (F \circ \tilde{A}_t)(\mathbf{x}^\circ) - t(F \circ \tilde{A}_t)(\mathbf{x}^+) - (1-t)(F \circ \tilde{A}_t)(\mathbf{x}^-) - 2tK \\
&< 2tK - 2tK = 0
\end{aligned}$$

by our original supposition, (7.7). Since φ_t is a continuous function of s defined on a closed interval, it attains a minimum on that interval. Moreover, this minimum must be negative, since $\varphi_t(t) < 0$. Define $c := \inf\{s \in [0,1] \mid \varphi_t(s) = \min\{\varphi_t\}\}$. Since $\varphi_t(0) = \varphi(1) = 0$, $0 < c < 0$. Choose δ small enough so that $(c-\delta, c+\delta) \subset [0,1]$. Then

$$\varphi(c) < \varphi_t(c-\delta) \text{ and } \varphi_t(c) \leq \varphi_t(c+\delta).$$

Then, by the fact that φ_t satisfies a midpoint convexity inequality,

$$\varphi_t(c) < \frac{\varphi_t(c-\delta) + \varphi_t(c+\delta)}{2} \leq \varphi_t(c) - (f \circ B_t)(\gamma(c), \gamma(c-\delta), \gamma(c+\delta)).$$

This is a contradiction, since f is a positively valued function. Thus, our original supposition is false, and the lemma is proved. $\square$

Corollary 7.1.7. *Suppose that F satisfies the midpoint convexity condition from Lemma 7.1.6. Suppose further that there exists three functions $c_1, C_1, c_2 : (0,1) \to (0,\infty)$ such that for all $t \in (0,1)$*

$$c_1(t) = c_1(1-t); \quad C_1(t) = C_1(1-t); \quad c_2(t) = c_2(1-t)$$

and

$$\begin{aligned} c_1(t)F(\mathbf{x}) \le F(A_t u, v) \le C_1(t)F(\mathbf{x}) \\ c_2(t)f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) \le f(A_t u^\circ, v^\circ, A_t u^+, v^+, A_t u^-, v^-) \end{aligned}$$

$\mathbf{x}, \mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^- \in \Omega$, and $u^\circ = tu^+ + (1-t)u^-)$. Then

$$F(\mathbf{x}^\circ) \ge \frac{tc_1(t)}{C_1(t)}F(\mathbf{x}^+) + \frac{(1-t)c_1(t)}{C_1(t)}F(\mathbf{x}^-) + \frac{t'c_2(t)}{C_1(t)}f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-). \tag{7.8}$$

where $t' = \min\{t, 1-t\}$. Moreover, if Ω_B is a convex set, then

$$F(\mathbf{x}^\circ) \ge tF(\mathbf{x}^+) + (1-t)F(\mathbf{x}^-) + t'f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-). \tag{7.9}$$

Proof. Because of symmetry, we can without loss of generality assume that $t \in (0, 1/2]$. Then

$$\begin{aligned} C_1(t)F(\mathbf{x}^\circ) &\ge F(A_t u^\circ, v^\circ) \\ &\ge tF(A_t u^+, v^+) + (1-t)F(A_t u^-, v^-) + 2tf(A_t u^\circ, v^\circ, A_t u^+, v^+, A_t u^-, v^-) \\ &\ge tc_1(t)F(\mathbf{x}^+) + (1-t)c_1(t)F(\mathbf{x}^-) + 2tc_2(t)f(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-). \end{aligned}$$

Divide both sides by $C_1(t)$ completes the proof of (7.8). To prove (7.9), it is enough to observe that for convex Ω_B, the matrix A_t can be chosen to be the identity matrix for all t. This allows use to choose c_1, C_1 and c_2 to be identically equal to 1, which proves the inequality. □

7.2 Bellman Function Primer

We will now devote some time to discussing the Bellman function proof technique for inequalities in harmonic analysis. For a more complete overview, we recommend reading the excellent introduction by Nazarov, Treil, and Volberg, [39].

Since we are describing a proof technique, it would be difficult to give a complete formal definition. Nevertheless, Bellman function style proofs all follow a similar flavor which is easy to spot once you get used to it. To offer a famous quote from Supreme Court Justice Potter Stewart, "I know it when I see it."

In this spirit, we will build up enough of a framework to state the main result of this chapter, Theorem 7.3.1 The Good Bellman function theorem.

7.2.1 Dyadic Inequalities

We now turn our attention to the types of inequalities we will be proving.

Definition 7.2.1 (Dyadic Inequality (Special Case))**.** Let (X, ρ, μ) be a space of homogeneous type with no atoms. Let $\mathscr{D}$ be a dyadic lattice over X. A *special dyadic inequality* is any inequality of the form

$$\sum_{R \in \mathscr{D}(Q)} g(R) \leq \mu(Q) \cdot G(Q) \quad \forall Q \in \mathscr{D} \tag{7.10}$$

where g and G are positively valued functions which take as their inputs dyadic cubes.

We first give the definition without allowing for atoms. This is because atoms present a somewhat tricky problem that, while it won't stop us in our tracks, does require consideration. Consider, for example, a space X with a single atom a. We know from Theorem 2.5.2 that a is an isolated point, which means at some generation the set containing only a must be a cube. Furthermore, the cube $\{a\}$ has only itself as a child, grandchild, great-grandchild, and so on. Thus, the dyadic inequality becomes

$$\mu(\{a\}) \cdot G(\{a\}) \geq \sum_{R \in \mathscr{D}(\{a\})} g(R) = \sum_{i=0}^{\infty} g(\{a\}) = \begin{cases} 0 & \text{if} \quad g(\{a\}) = 0 \\ \infty & \text{if} \quad g(\{a\}) > 0 \end{cases},$$

implying that $g(\{a\}) = 0$ ($G \neq \infty$ for any cube, atom or no). However, for many cases (including the ones we will see later) there is no reason to think that g should be zero for atoms.

There is a second consideration as well, but it is less significant. Remember that in the definition of honest dyadic cubes we said that it is sometimes convenient to consider the empty set as the second child of cubes with only one child. The functions g and G, however, may not have meaningful definitions when taking the empty set as an input.

With these issues in mind, we give a more general definition of a dyadic inequality.

Definition 7.2.2. Dyadic Inequality (General Case) Let (X, ρ, μ) be a space of homogeneous type and let $\mathcal{A}$ be the set of atoms of X. Let $\mathscr{D}$ be a dyadic lattice over $X \setminus \mathcal{A}$. A *general dyadic inequality* is any inequality of the form

$$\sum_{R \in \mathscr{D}(Q)} g(R) + \sum_{a \in Q \cap \mathcal{A}} g(a) \leq \mu(Q) \cdot G(Q) \quad \forall Q \in \mathscr{D} \tag{7.11}$$

where $g, G : (\mathscr{D} \cup \mathcal{A} \cup \{\emptyset\}) \to [0, \infty)$ are positively valued functions, such that $G(\emptyset) = g(\emptyset) = 0$.

Recall that when X has atoms, any dyadic structure over X will eventually have those atoms as cubes. From now on, we will write dyadic inequalities as (7.10), and mean them to be of the form (7.11), that is, cubes which happen to be atoms are only represented once in the sum, even though they are in infinitely many generations.

Remark 7.2.3. Notice that sums such as (7.10) and (7.11) sum over cubes in $\mathscr{D}$. Remember, since $\mathscr{D}$ was defined as the union of collections of generations, we do consider cubes which are *the same set* but belong to *different generations* to be different cubes. For this reason, it is possible that terms are repeated in the sums.

If X is a space and $\mathscr{D}_1$ and $\mathscr{D}_2$ are two dyadic structures over X so that for every cube $Q \in \mathscr{D}_1$, $Q \in \mathscr{D}_2$, then any dyadic inequality which holds for sums over $\mathscr{D}_2$ also holds for sums over $\mathscr{D}_1$. In particular, this means that it is sufficient to prove a dyadic inequality for an honest structure if we desired the inequality for a general structure since every dyadic lattice supports an overlapping honest lattice.

7.2.2 Bellman Functions Are Here!

We now introduce the idea of what a Bellman function actually is. For simplicity, we will stick with Bellman function over $\mathbb{R}$ for the moment and move back into the SHT setting once the ideas are established.

Recall that a *weight* w is a real valued function $X \to \mathbb{R}$ which is positive almost everywhere with respect to μ and locally integrable. We let $\mathscr{W}(\mathbf{x})$ be a collection of weights parameterized by a real vector $\mathbf{x}$. An element of $\mathscr{W}(\mathbf{x})$ could be either a single weight or a tuple of weights, depending on our needs.

Example 7.2.4. The following might be examples of typical weight collections:

- $\mathscr{W}(x) = \{w \text{ a weight over } \mathbb{R} : \langle w \rangle_I \leq x \ \ \forall I = [a,b) \subset \mathbb{R}\}$
- $\mathscr{W}(\mathbf{x}) = \{(w,u) \text{ a pair of weights over } X : \langle w \rangle_I \leq \mu(I)x_1,\ \langle \log u \rangle_I \leq x_2 \ \ \forall I \in \mathscr{D}\}$

In the first example $\mathscr{W}$ is parameterized by a single value, and elements of $\mathscr{W}$ are single weights. In the second example, elements of $\mathscr{W}$ are pairs of weights which are parameterized by the vector $\mathbf{x} \in \mathbb{R}^2$.

A *Bellman function* will be a function satisfying some properties:

- (Domain) $\mathfrak{B}$'s domain is a well defined set Ω. For our purposes, we will also impose that Ω be a prism with a weakly convex base, although this requirement may not be strictly necessary in all cases.

- (Main Inequality) $\mathfrak{B}$ satisfies a midpoint convexity condition such as in Lemma 7.1.6. That is

$$\mathfrak{B}(\mathbf{x}^\circ) \geq \frac{1}{2}\left[\mathfrak{B}(\mathbf{x}^+) + \mathfrak{B}(\mathbf{x}^-)\right] + b(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) \tag{7.12}$$

  for $\mathbf{x}^\circ, \mathbf{x}^\pm \in \Omega$ satisfying a relationship such as in the previous section. Here the function $b : \Omega \times \Omega \times \Omega \to \mathbb{R}^+$ is a positively valued function which satisfies a symmetry condition $b(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) = b(\mathbf{x}^\circ, \mathbf{x}^-, \mathbf{x}^+)$.

- (Range Bound) $0 \leq \mathfrak{B}(\mathbf{x}) \leq B(\mathbf{x})$ for a real valued function B defined over the same domain Ω.

In this setting, the functions $\mathfrak{B}$ and b will be serving the role of F and f from the previous section.

It is worth nothing that many Bellman proofs (see for example [47]) are interested in finding the sharpest possible constant for a particular inequality. In in this situation, we additionally impose that $\mathfrak{B}$ is a function which is defined as a supremum taken over a weight collection $\mathscr{W}$ of the form

$$\mathfrak{B}(\mathbf{x}) = \sup_{W \in \mathscr{W}(\mathbf{x})} \frac{1}{|I|} \sum_{J \in \mathscr{D}(I)} g_W(J). \tag{7.13}$$

Here the weight collection depends on $\mathbf{x}$, W is either a single weight or a tuple of weights and $g_W : \mathscr{D} \to \mathbb{R}^+$ is a non-negative, real valued function that takes as its input a dyadic interval and is parameterized by W.

We will not spend any more time on this flavor of Bellman function proof, and only mention it here in passing.

7.2.3 Providing Inequalities via Reduction on Scales

The final ingredient to the Bellman recipe is some form of mapping of dyadic intervals to points in $\mathfrak{B}$'s domain Ω which we denote by $\mathbf{x}_I$. This mapping could be anything, but in practice it will be required that u_I, u_{I_r} and u_{I_ℓ} be collinear points in Ω_B, Ω's base, and that u_I be the midpoint of the line segment $\overline{u_{I_r}u_{I_\ell}}$ (recall the $\mathbf{x} = (u, v)$ convention). It is not in general necessary that the line segment itself be entirely contained in Ω_B, but it may be needed in some cases.

A Bellman function $\mathfrak{B}$ will provide a dyadic inequality over $\mathbb{R}$ if $B(\mathbf{x}_I) = G(I)$ and $b(\mathbf{x}_I, \mathbf{x}_{I_\ell}, \mathbf{x}_{I_r}) = |I|^{-1} \cdot g(I)$, where G and g satisfy (7.11). The proof generally follows what is known as a "*reduction on scales*" argument. For a particular interval $I \in \mathscr{D}$,

$$\begin{aligned} G(I) = B(\mathbf{x}_I) \geq \mathfrak{B}(\mathbf{x}_I) &\geq \frac{1}{2}\left[\mathfrak{B}(\mathbf{x}_{I_\ell}) + \mathfrak{B}(\mathbf{x}_{I_r})\right] + b(\mathbf{x}_I, \mathbf{x}_{I_\ell}, \mathbf{x}_{I_r}) \\ &\geq \frac{1}{4}\left[\mathfrak{B}(\mathbf{x}_{(I_\ell)_\ell}) + \mathfrak{B}(\mathbf{x}_{(I_\ell)_r}) + \mathfrak{B}(\mathbf{x}_{(I_r)_\ell}) + \mathfrak{B}(\mathbf{x}_{(I_r)_r})\right] + \\ &\qquad \frac{1}{2}\left[b(\mathbf{x}_{I_\ell}, \mathbf{x}_{(I_\ell)_\ell}, \mathbf{x}_{(I_\ell)_r}) + b(\mathbf{x}_{I_r}, \mathbf{x}_{(I_r)_\ell}, \mathbf{x}_{(I_r)_r})\right] + b(\mathbf{x}_I, \mathbf{x}_{I_\ell}, \mathbf{x}_{I_r}) \\ &\geq \cdots \\ &\geq 2^{-N-1} \sum_{R \in \mathscr{D}_N(I)} \mathfrak{B}(\mathbf{x}_R) + \sum_{j=0}^{N} 2^{-j} \sum_{R \in \mathscr{D}_j(I)} b(\mathbf{x}_R, \mathbf{x}_{R_\ell}, \mathbf{x}_{R_r}). \end{aligned}$$

All terms in both the single and the double sums are non-negative, so sending $N \to \infty$ will give that

$$G(I) \geq \sum_{j=0}^{\infty} 2^{-j} \sum_{R \in \mathscr{D}_j(I)} b(\mathbf{x}_R, \mathbf{x}_{R_\ell}, \mathbf{x}_{R_r}) \tag{7.14}$$

since we can just throw out the first sum involving the $\mathfrak{B}$s. We can rewrite (7.14) to get that

$$G(I) \geq \sum_{j=0}^{\infty} 2^{-j} \sum_{R \in \mathscr{D}_j(I)} |R|^{-1} g(R) = |I|^{-1} \sum_{R \in \mathscr{D}(I)} g(R)$$

which proves a dyadic inequality.

7.3 The Good Bellman Function Theorem

In this section we use the previously developed "honest dyadic cubes" to prove a meta-theorem about Bellman function type proofs.

We start with a short warning. In this theorem we will make use of a slight abuse of notation. In the previous section, we used the notation $\mathbf{x}_I$ to mean a point inside a domain Ω which depends somehow on an interval. We will now make the somewhat sneaky substitution x_Q for a SHT dyadic cube instead of I, and assume that this makes sense in the most obvious way. For example, if $\mathbf{x}_I := \langle w \rangle_I$ the mean of a weight $w : \mathbb{R} \to \mathbb{R}$, we intend $\mathbf{x}_Q$ to be equal to $\langle w' \rangle_Q$ the mean of some other weight $w' : X \to \mathbb{R}$ with respect to the measure μ. Since both these quantities make sense in their respective contexts, the notation used is not inappropriate and in the spirit of remaining terse we will use it. Nevertheless, we note that it would probably be more explicit write something such at $\mathbf{x}(Q, \mu)$ or similar.

One final point. In the $\mathbb{R}$ case, the points u_I, u_{I_ℓ}, and u_{I_r} were necessarily collinear with u_I the midpoint. The reason for this *almost always* follows from the fact that the length of any dyadic interval is exactly half that of its parent. It was not strictly necessary that this be the cause of the midpoint relationship, however, just that it be the case. As we extend to the SHT setting, we will be more explicit. It will no longer be the case that u_Q be the midpoint of the line segment $\overline{u_{Q_+} u_{Q_-}}$. Instead, we will have that $u_q = \alpha^+ u_{Q_+} + \alpha^- u_{Q_-}$ where $\alpha^\pm := \mu(Q_\pm)/\mu(Q)$. This enforces that $\alpha^+ + \alpha^- = 1$.

7.3.1 Statement and Proof

We are ready to state the main result of the chapter. The idea here is that given dyadic inequality over $\mathbb{R}$ whose proof relies on a Bellman style argument which satisfies certain conditions, we can extend this result to SHTs without needing to jump through the hoops of reproving the theorerm. More formally:

Theorem 7.3.1 (Good Bellman Function Theorem). *Let $\mathscr{D}^{\mathbb{R}}$ be the standard dyadic lattice on $\mathbb{R}$. Let $\mathfrak{B}$ be a Bellman function defined over a domain Ω which satisfies the main inequality (7.12). Suppose that Ω is a prism whose base is weakly convex under the family of matrices $\{A_t\}_{t\in(0,1/2]}$. Suppose that $\mathfrak{B}$ provides the dyadic inequality over $\mathbb{R}$*

$$\sum_{J\in\mathscr{D}^{\mathbb{R}}(I)} g(J) \leq |I|\cdot G(I)$$

for all $I \in \mathscr{D}^{\mathbb{R}}$. In other words,

- *there is mapping of dyadic intervals to points in Ω which respects the midpoint requirement,*
- *we have that $G(I) = B(\mathbf{x}_I)$, $g(I) = |I|\cdot b(\mathbf{x}_I, \mathbf{x}_{I_\ell}, \mathbf{x}_{I_r})$,*
- *the function b satisfies the symmetry condition $b(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) = b(\mathbf{x}^\circ, \mathbf{x}^-, \mathbf{x}^+)$,*
- *the desired inequality follows from a reduction on scales argument.*

Suppose further that there exists continuous functions $c_1, C_1, c_2 : (0,1) \to (0,\infty)$ such that

$$c_1(t) = c_1(1-t); \quad C_1(t) = C_1(1-t); \quad c_2(t) = c_2(1-t)$$

and

$$c_1(t)\mathfrak{B}(\mathbf{x}) \leq \mathfrak{B}(A_t u, v) \leq C_1(t)\mathfrak{B}(\mathbf{x})$$
$$c_2(t) b(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) \leq b(A_t u^\circ, v^\circ, A_t u^+, v^+, A_t u^-, v^-)$$

for every $\mathbf{x}, \mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^- \in \Omega$, with $u^\circ = tu^+ + (1-t)u^-$).

Let X be a space of homogeneous type with honest dyadic structure $\mathscr{D}^X$. If the points $\mathbf{x}_Q \in \Omega$ for all $Q \in \mathscr{D}^X$ and u_Q, u_{Q_+}, u_{Q_-} are collinear in Ω_B with

$$u_Q = \frac{\mu(Q_+)}{\mu(Q)} u_{Q_+} + \frac{\mu(Q_-)}{\mu(Q)} u_{Q_-}$$

then there exists constants $D \geq 1, 0 < \delta \leq 1$ not depending on Q such that $\mathfrak{B}$ provides an extension to an analogue dyadic inequality

$$\sum_{k=0}^{\infty} \delta^k \sum_{R \in \mathscr{D}_k^X(Q)} \tilde{g}(R) \leq D \cdot \mu(Q) \cdot \tilde{G}(Q)m \tag{7.15}$$

where $\tilde{g} = g$, $\tilde{G} = G$ when $X = \mathbb{R}$.

Recall that G and g were functions which mapped dyadic cubes to positive real numbers. By saying $\tilde{G} = G$ and $\tilde{g} = g$ when $X = \mathbb{R}$, we mean that $\tilde{G}$ and $\tilde{g}$ are extensions of G and g which are *independent of the underlying space* X. For example, if we defined $g(I) = \int_I 2\,dx$, then it would have the extension $\tilde{g}(Q) = \int_Q 2\,d\mu(x)$. Clearly, in the case where $X = \mathbb{R}$, this would indeed give that $\tilde{g} = g$ as functions.

Proof. By assumption, the Bellman function $\mathfrak{B}$ satisfies the hypotheses of Corollary 7.1.7, thus it also satisfies its conclusion. Fix $Q \in \mathscr{D}^X$ and set $\alpha_Q^{\pm} := \mu(Q_{\pm})/\mu(Q)$. Note that $\alpha_Q^+ + \alpha_Q^- = 1$ and that $\epsilon \leq \alpha_Q^{\pm} \leq 1 - \epsilon$ where $\epsilon := \inf_{Q \in \mathscr{D}^X} \mu(Q)/\mu(\widehat{Q}) = Dbl(\mathscr{D})^{-1}$ Set the constants

$$\delta := \inf_{t \in [\epsilon, 1/2]} \frac{c_1(t)}{C_1(t)}, \tag{7.16}$$

$$\beta := \inf_{t \in [\epsilon, 1/2]} \frac{c_2(t)}{C_1(t)} \tag{7.17}$$

where $\epsilon := \inf_{Q \in \mathscr{D}^X} \mu(Q)/\mu(\widehat{Q}) = Dbl(\mathscr{D})^{-1}$. Then $0 < \delta, \beta \leq 1$ necessarily. We have that

$$\begin{aligned} \tilde{G}(Q)\mu(Q) &\geq \mu(Q)\mathfrak{B}(\mathbf{x}_Q) \\ &\geq \frac{c_1(\alpha_Q^+)}{C_1(\alpha_Q^+)}(\alpha_Q^+\mu(Q)\mathfrak{B}(\mathbf{x}_{Q_+}) + \alpha_Q^-\mu(Q)\mathfrak{B}(\mathbf{x}_{Q_-})) \\ &\quad + \frac{c_2(\alpha_Q^+)}{C_1(\alpha_Q^+)}\mu(Q)b(\mathbf{x}_Q, \mathbf{x}_{Q_+}, \mathbf{x}_{Q_-}). \end{aligned}$$

Recall that since c_1, C_1 and c_2 are symmetric about the point $t = 1/2$, all three functions agree at α_Q^+ and α_Q^-. It therefore did not matter which of $\alpha_Q^{\pm}$ we choose to plug into these functions, so we used α_Q^+. Continuing on,

$$\begin{aligned}\tilde{G}(Q)\mu(Q) &\geq \delta(\mu(Q_+)\mathfrak{B}(\mathbf{x}_{Q_+}) + \mu(Q_-)\mathfrak{B}(\mathbf{x}_{Q_-})) + \beta\mu(Q)b(\mathbf{x}_Q, \mathbf{x}_{Q_+}, \mathbf{x}_{Q_-}) \\ &\geq \delta^2(\mu((Q_+)_+)\mathfrak{B}(\mathbf{x}_{(Q_+)_+}) + \\ &\quad + \mu((Q_-)_+)\mathfrak{B}(\mathbf{x}_{(Q_-)_+} \\ &\quad + \beta\delta\mu(Q_+)b(\mathbf{x}_{Q_+}, \mathbf{x}_{(Q} \\ &\quad + \beta\mu(Q)b(\mathbf{x}_Q, \mathbf{x}_{Q_+}, \mathbf{x}_{Q} \\ &= \delta^2 \sum_{R\in\mathscr{D}_2^X(Q)} \mu(R)\mathfrak{B}(\mathbf{x}_R) + \end{aligned}$$

where we used here that $\mathfrak{B}$ satisfies the corollary to the Generalized Convexity Lemma. Iterating this inequality N times gives that

$$\begin{aligned}\tilde{G}(Q)\mu(Q) &\geq \delta^N \sum_{R\in\mathscr{D}_N^X(Q)} \mu(R)\mathfrak{B}(\mathbf{x}_R) + \beta\sum_{k=0}^{N-1}\delta^k \sum_{R\in\mathscr{D}_k^X(Q)} b(\mathbf{x}_R, \mathbf{x}_{R_+}, \mathbf{x}_{R_-})\mu(R) \\ &\geq \beta\sum_{k=0}^{N-1}\delta^k \sum_{R\in\mathscr{D}_k^X(Q)} \tilde{g}(R).\end{aligned}$$

Setting $D := 1/\beta$ and sending $N \to \infty$ gives the desired inequality. $\square$

Corollary 7.3.2. *Theorem 7.3.1 still holds if we remove the requirement of $\mathscr{D}^X$ to be honest, as long as $x_Q \in \Omega$ for every honest cube Q contained in an overlapping honest structure.*

Proof. We let $\tilde{\mathscr{D}}^X$ be an honest dyadic structure for X which overlaps $\mathscr{D}^X$. Then the inequality (7.15) holds for $\tilde{\mathscr{D}}^X$. Since each cube in $\mathscr{D}^X$ is also a cube in $\tilde{\mathscr{D}}^X$ and $\tilde{g}$ is always non-negative, (7.15) also holds for $\mathscr{D}^X$. $\square$

The constants D and δ will not depend on Q, but they *do* depend on the dyadic doubling constant of the honest structure for X and on the particular Bellman function used in the proof. This leaves the door open to potentially finding a better constant, if desired, by choosing a different Bellman function or honest structure.

Finally, we state a nice special case of the Good Bellman Function Lemma, where the domain Ω happens to be convex (and not just weakly convex).

Corollary 7.3.3 (Convex Domain Special Case). *Let $\mathscr{D}^{\mathbb{R}}$ be the standard dyadic lattice on $\mathbb{R}$. Let $\mathfrak{B}$ be a Bellman function defined over a domain Ω which satisfies the main inequality (7.12). Suppose that Ω is a prism with a convex base Ω_B. Suppose that $\mathfrak{B}$ provides the dyadic inequality over $\mathbb{R}$*

$$\sum_{J \in \mathscr{D}^{\mathbb{R}}(I)} g(J) \leq |I| \cdot G(I)$$

for all $I \in \mathscr{D}^{\mathbb{R}}$. (See the statement of Theorem 7.3.1.)

Let X be a space of homogeneous type with dyadic structure $\mathscr{D}^X$. If the points $\mathbf{x}_Q \in \Omega$ for all $Q \in \mathscr{D}^X$ and u_Q, u_{Q_+}, u_{Q_-} are collinear in Ω_B with

$$u_Q = \frac{\mu(Q_+)}{\mu(Q)} u_{Q_+} + \frac{\mu(Q_-)}{\mu(Q)} u_{Q_-} \tag{7.18}$$

then $\mathfrak{B}$ provides an extension to an analogue dyadic inequality

$$\sum_{R \in \mathscr{D}^X(Q)} \tilde{g}(R) \leq \mu(Q) \cdot \tilde{G}(R) \tag{7.19}$$

where $\tilde{g} = g$, $\tilde{G} = G$ when $X = \mathbb{R}$.

Proof. Since the domain Ω has a convex base, it is also weakly convex. Thus, the functions c_1, c_2, and C_1 from the original proof can all be set identically to one. This forces that δ and D also equal one. Plugging into (7.15) yields (7.19). $\square$

7.4 A Two Weight Application

We close the chapter by giving a relatively simple application of the Good Bellman Function Lemma. The Bellman function here is a function of two varaibles. We will stick to the original notation found in [39] and use x and y as our varaiables, in contrast to the more cumbersome notation we needed for arbitrary dimensions. Also in this paper, u and v were weights, so they will play that role here as well.

7.4.1 A result for weights over $\mathbb{R}$.

We recall an example of a two-weight theorem for weights over $\mathbb{R}$:

Theorem 7.4.1. *Let u, v be two positive functions such that for any interval $I \in \mathscr{D}$,*

$$\langle u \rangle_I \cdot \langle v \rangle_I \leq 1 \tag{7.20}$$

Then for any $I \in \mathscr{D}$,

$$\frac{1}{|I|} \sum_{J \in \mathscr{D}(I)} |\langle u \rangle_{J_\ell} - \langle u \rangle_{J_r}| \cdot |\langle v \rangle_{J_\ell} - \langle v \rangle_{J_r}| \cdot |J| \leq 16\sqrt{\langle u \rangle_I \langle v \rangle_I}. \tag{7.21}$$

For a complete proof, see [39]. Here, we will cover details of the proof which are pertinent to this discussion. This dyadic property is provided by the Bellman function

$$\mathfrak{B}(x, y) = 4 \cdot (4\sqrt{xy} - xy)\,. \tag{7.22}$$

over the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } xy < 1\} \tag{7.23}$$

with the range

$$0 \leq \mathfrak{B}(x, y) \leq 16\sqrt{xy} =: B(x, y) \tag{7.24}$$

and main inequality

$$\mathfrak{B}(x, y) \geq \frac{1}{2}\left(\mathfrak{B}(x^+, y^+) + \mathfrak{B}(x^-, y^-)\right) + |x^+ - x^-| \cdot |y^+ - y^-|. \tag{7.25}$$

In this proof, $b(\mathbf{x}^\circ, \mathbf{x}^+, \mathbf{x}^-) := |x^+ - x^-| \cdot |y^+ - y^-|$, a function which is clearly symmetric in the second and third variables (and incidentally does not depend on the first at all). As this is a Bellman function proof, the details follow from a reduction on scales argument.

7.4.2 Extending to SHTs

We will now prove an analogue theorem for weights over spaces of homogeneous type.

Theorem 7.4.2. *Let X be a SHT with honest dyadic structure $\mathscr{D}$. Let $u, v : X \to \mathbb{R}$ be two positive functions such that for any $Q \in \mathscr{D}$,*

$$\langle u \rangle_Q \cdot \langle v \rangle_Q \leq 1 \tag{7.26}$$

Then for any $Q \in \mathscr{D}$,

$$\frac{1}{\mu(Q)} \sum_{k=0}^{\infty} \delta^k \sum_{R \in \mathscr{D}_k(Q)} |\langle u \rangle_{R_+} - \langle u \rangle_{R_-}| \cdot |\langle v \rangle_{R_+} - \langle v \rangle_{R_-}| \cdot \mu(R) \leq 16 \cdot D \sqrt{\langle u \rangle_Q \langle v \rangle_Q}.$$

where $D = (16/3) \cdot Dbl(\mathscr{D})^2$ and $\delta = 45/(64 \cdot Dbl(\mathscr{D}))$.

In order to apply our Good Bellman Function Lemma, we have a few things that we need to demonstrate. We will state the lemmas here, and give proofs in the final section of the chapter.

Lemma 7.4.3. *The domain $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x, 0 < y, xy < 1\}$ is weakly convex under the family of matrices*

$$A_t := \begin{bmatrix} t/4 & 0 \\ 0 & t/4 \end{bmatrix}$$

Note that actually Ω is a 2-dimensional 2-prism, since *any* set in $\mathbb{R}^d$ is a d-dimensional d-prism. Thus, in this case, $\tilde{A}_t = A_t$, so we will drop the tilde.

Lemma 7.4.4. *The function $\mathfrak{B}(x, y) = 4(4\sqrt{xy} - xy)$ satisfies the pair of inequalities*

$$c_1(t)\mathfrak{B}(x, y) \leq (\mathfrak{B} \circ A_t)(x, y) \leq C_1(t)\mathfrak{B}(x, y)$$

for A_t as in Lemma 7.4.3 and

$$c_1(t) = 15t/64$$
$$C_1(t) \equiv 1/3.$$

Normally, we would need a similar inequality for the function b as well, but in this case it will be trivial so we will not state this as a lemma. Indeed, we do not even need an inequality since

$$\begin{aligned} b(A_t\mathbf{x}_1, A_t\mathbf{x}_2) &= b\left(\frac{t}{4}x_1, \frac{t}{4}y_1, \frac{t}{4}x_2, \frac{t}{4}y_2\right) = \left|\frac{t}{4}x_1 - \frac{t}{4}y_1\right| \cdot \left|\frac{t}{4}x_2 - \frac{t}{4}y_2\right| \\ &= \frac{t^2}{16}|x_1 - y_1| \cdot |x_2 - y_2| = \frac{t^2}{16} \cdot b(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

so the function $c_2(t) := t^2/16$ suffices.

Lemma 7.4.5. *The extension of the mapping $\{\mathbf{x}_I\}_{I\in\mathscr{D}^{\mathbb{R}}}$ to $\{\mathbf{x}_Q\}_{Q\in\mathscr{D}^X}$ is completely contained within the domain Ω. Moreover, For an honest cube Q with children Q_+ and Q_-, the points $\mathbf{x}_Q$, $\mathbf{x}_{Q_+}$ and $\mathbf{x}_{Q_-}$ are collinear with*

$$\mathbf{x}_Q = t\mathbf{x}_{Q_+} + (1-t)\mathbf{x}_{Q_-} \tag{7.27}$$

where $t = \mu(Q_+)/\mu(Q)$.

7.4.3 Proof of the Extended Result

We will prove Lemma 7.4.3 in the appendix, and prove Lemmas 7.4.4 and 7.4.5 here.

Proof of Lemma 7.4.4. We let $\mathfrak{B}(x,y) = 4(4\sqrt{xy} - xy)$ implying $(\mathfrak{B}\circ A_t)(x,y) = 4(t\sqrt{xy} - 16^{-1}t^2xy)$. We have by Lemma 7.4.3 that Ω is weakly-convex under the family $\{A_t\}$, which among other things guarantees that $A_t : \Omega \to \Omega$. We can therefore say that Ω's definition forces that $\mathfrak{B}$ and $(\mathfrak{B}\circ A_t)$ are always positive.

The desired result is equivalent to showing that

$$c_1(t) \le \frac{(\mathfrak{B}\circ A_t)(x,y)}{\mathfrak{B}(x,y)} = \frac{t - 16^{-1}t^2\sqrt{xy}}{4-\sqrt{xy}} \le C_1(t).$$

The left-hand inequality follows from

$$\frac{t - 16^{-1}t^2\sqrt{xy}}{4-\sqrt{xy}} \ge t\frac{1 - 16^{-1}\sqrt{xy}}{4-\sqrt{xy}} \ge \frac{15t}{64} =: c_1(t).$$

The right-hand inequality follows from

$$\frac{t - 16^{-1}t^2\sqrt{xy}}{4 - \sqrt{xy}} \leq \frac{1}{3} \equiv: C_1(t).$$

□

Proof of Lemma 7.4.5. In the original proof in [39], $\mathbf{x}_I$ was defined as $(\langle u \rangle_Q, \langle v \rangle_Q)$ for the two weights u and v. Thus, when extending the theorem to SHTs, we define $\mathbf{x}_Q := (\langle u \rangle_Q, \langle v \rangle_Q)$. We need to verify that $\mathbf{x}_Q \in \Omega$ for a $Q \in \mathscr{D}^X$. This clear from the fact that u and v are weights, and that $\langle u \rangle_Q \langle v \rangle_Q < 1$ by assumption.

Next, we need that $\mathbf{x}_Q$ and $\mathbf{x}_{Q_\pm}$ are three collinear points which satisfy (7.27) with $t = \mu(Q_+)/\mu(Q)$. This is clear from m

$$\begin{aligned} t\mathbf{x}_{Q_+} + (1-t)\mathbf{x}_{Q_-} &= \frac{\mu(Q_+)}{\mu(Q)}\mathbf{x}_{Q_+} + \left(1 - \frac{\mu(Q_+)}{\mu(Q)}\right)\mathbf{x}_{Q_-} \\ &= \frac{\mu(Q_+)}{\mu(Q)}\left(\langle u \rangle_{Q_+}, \langle v \rangle_{Q_+}\right) + \frac{\mu(Q_-)}{\mu(Q)}\left(\langle u \rangle_{Q_-}, \langle v \rangle_{Q_-}\right) \\ &= \frac{1}{\mu(Q)}\left(\int_{Q_+} u\, d\mu + \int_{Q_-} u\, d\mu, \int_{Q_+} v\, d\mu + \int_{Q_-} v\, d\mu\right) \\ &= (\langle u \rangle_Q, \langle v \rangle_Q) = \mathbf{x}_Q. \end{aligned}$$

□

With all lemmas established, we conclude the chapter with the proof of Theorem 7.4.2.

Proof of 7.4.2. The original proof over $\mathbb{R}$ follows from a Bellman-function argument, so we need only confirm the additional hypotheses. These were verified in Lemmas 7.4.3, 7.4.4, and 7.4.5. Thus, by the Good Bellman Function Lemma, we have the desired inequality. Moreover,

$$D = \left[\inf_{t \in [Dbl(\mathscr{D})^{-1}, 1/2]} \frac{c_2(t)}{C_1(t)}\right]^{-1} = \left[\inf_{t \in [Dbl(\mathscr{D})^{-1}, 1/2]} \frac{3t^2}{16}\right]^{-1} = \frac{16 \cdot Dbl(\mathscr{D})^2}{3}$$

and

$$\delta = \inf_{t\in[Dbl(\mathscr{D})^{-1},1/2]} \frac{c_1(t)}{C_1(t)} = \inf_{t\in[Dbl(\mathscr{D})^{-1},1/2]} \frac{45}{64t} = \frac{45}{64 \cdot Dbl(\mathscr{D})}$$

as required. □

As we can see from this example, The Good Bellman Function Lemma is indeed very powerful.

Chapter 8

Fun With Paraproducts and t-Haar Multipliers

In this chapter we will apply the results of the preceding chapters. Our main application will be to prove bounds for particular dyadic operators: the t-Haar multiplier and the dyadic paraproduct. These bounds were shown first for $t = -1/2, 1/2, 1$ in [31], then for more gereral t in [8] and [11], respectively.

We will proceed by, via our Good Bellman Function Lemma, extend several Bellman function type arguments originally from O. Beznosova [8] so that we can use their results in our setting of SHTs.

8.1 Preliminaries

Here we give some preliminary tools necessary for this chapter.

8.1.1 The A_2 Theorem

The A_2 Conjecture gives generic bounds for Calderón-Zygmund operators on weighted L^2 and claims that this bound depends linearly on the A_2 characteristic of the weight. This celebrated result is now a Theorem, having been proved in 2012 by T. Hytönen.

Theorem 8.1.1 (A_2 Theorem [21][1]). *Let T be a Calderón-Zygmund Operator for functions on $\mathbb{R}^n$. Then for any $w \in A_2$,*

$$||T||_{L^2(w)} \leq C(n,T)[w]_{A_2}. \tag{8.1}$$

where the constant $C(n,T)$ depends on the dimension and the operator.

The particular bounds demonstrated here for the paraproduct and t-Haar multiplier are examples of this result.

8.1.2 Carleson Sequences in Spaces of Homogeneous Type

In this chapter several lemmas make use of the idea of a Carleson sequence. We give the definition now.

Definition 8.1.2 (v-Carleson Sequence ($\mathbb{R}$)). Let $v : \mathbb{R} \to \mathbb{R}$ be a weight. A sequence of non-negative real numbers $\{\lambda_I\}_{I\in\mathscr{D}}$ is called a v*-Carleson sequence* if and only if there exists a constant B such that for every $I \in \mathscr{D}$,

$$\sum_{J\in\mathscr{D}(J)} \lambda_J \leq B \cdot \int_J v(x)\,dx.$$

Here B is called $\{\lambda_Q\}_{I\in\mathscr{D}}$'s *intensity*. In the case where $v \equiv 1$ we call $\{\lambda_I\}$ a Carleson sequence.

We extend this notion to SHTs in the obvious way:

[1]Extended to SHTs in [7] by T. Anderson and A. Vagharshakyan.

Definition 8.1.3 (v-Carleson Sequence (X)). Let X be a space of homogeneous type with dyadic structure $\mathscr{D}$, and let $v : X \to \mathbb{R}$ be a weight. A sequence of non-negative real numbers $\{\lambda_Q\}_{Q\in\mathscr{D}}$ is called a v-*Carleson sequence* with intensity B if and only if there exists a constant B such that for every $Q \in \mathscr{D}$,

$$\sum_{Q'\in\mathscr{D}(Q)} \lambda_{Q'} \leq B \cdot \int_Q v(x)\, d\mu(x). \tag{8.2}$$

In the case where $v \equiv 1$ we call $\{\lambda_I\}_{I\ in\mathscr{D}}$ a Carleson sequence.

It is worth reminding the reader that this definition is with respect to the measure μ. Since in oiur context the underlying measure μ is always understood to be present, we will keep with the notation and defnintions as they are here.

Theorem 8.1.4. *If $\{\lambda_Q\}_{Q\in\mathscr{D}}$ is a v-Carleson sequence, and X has non-atomic isolated points, then $\lambda_Q = 0$ for any cube Q which is a finite union of such points.*

Proof. Let R be such a cube. Then any descendant of R is also such a cube, and $\mu(R) = 0$. Since $\{\lambda_Q\}$ is v-Carleson, (8.2) holds. But the right hand side of (8.2) equals zero since

$$\int_R v(x)\, d\mu(x) = \sum_{x\in R} v(x)\mu(\{x\})$$

and the right hand side is a finite sum of terms which are all zero. So, each term of the left hand side of (8.2) must be zero, in particular, the term λ_R. □

We note, therefore, that without loss of generality we can ignore non-atomic isolated points.

No such consideration needs to be made for atoms, since their measure is non-zero. For this reason we do *not* at this point need to assume that sums such as (8.2) include terms corresponding to cubes which are atoms only once, as was the case in the definition of a dyadic inequality we developed in Chapter 6. However, we may still wish to do so, if the situation requires it.

Finally, we will state the follwing useful Lemma.

Lemma 8.1.5. *If $\{\alpha_Q\}_{Q\in\mathscr{D}}$ and $\{\beta_Q\}_{Q\in\mathscr{D}}$ are Carelson sequences with respect to μ with intensities A and B respectively, then $\{\sqrt{\alpha_Q\beta_Q}\}_{Q\in\mathscr{D}}$ is a Carleson sequence with respect to μ with intensity $\sqrt{AB}$.*

The proof is a straightforward application of the Cauchy-Schwarz inequality.

8.1.3 Dyadic BMO

We will now give a brief overview of the class BMO^d.

Notation 5. For $f, g \in L^2(X)$, we write $\langle f, g\rangle$ to mean the $L^2(X)$ inner product, i.e,

$$\langle f, g\rangle := \int_X f(x)g(x)\,d\mu(x).$$

For a weight w, we also write $\langle f, g\rangle_w$ to be the $L^2(w)$ inner product:

$$\langle f, g\rangle_w := \int_X f(x)g(x)w(x)\,d\mu(x).$$

Definition 8.1.6 (Dyadic Bounded Mean Oscillation)**.** Let X be a SHT without non-atomic isolated points, and let $\mathscr{D}$ be an honest dyadic system on X. A function $b : X \to \mathbb{R}$ is in the class *dyadic bounded mean oscillation*, written $b \in BMO^{\mathscr{D}}$, if

$$||b||_{BMO^{\mathscr{D}}} := \sup_{Q\in\mathscr{D}} \frac{1}{\mu(Q)} \int_Q |b(x) - \langle b\rangle_Q|\,d\mu(x) < \infty. \tag{8.3}$$

Alternatively we could define

$$||b||_{BMO^{\mathscr{D}}} := \left(\sup_{Q\in\mathscr{D}} \frac{1}{\mu(Q)} \sum_{R\in\tilde{\mathscr{D}}(Q)} |\langle b, h_R\rangle|^2\right)^{\frac{1}{2}} < \infty. \tag{8.4}$$

The alternatives (8.3) and (8.4) are not *equal*, but are *equivalent* in the sense of semi-norms. See D. Chung's PhD Dissertation [11] for why this is so. The proof is for $\mathbb{R}^n$ but translates to the setting of SHTs nicely.

Remark 8.1.7. While the quantity in the previous definition is often called the BMO-norm of b, it is actually not strictly a norm because it equals zero for functions which are constant on the quadrants of X.

Remark 8.1.8. In light of (8.4), we actually know that $\{|\langle b, h_Q\rangle|^2\}_{Q\in\mathscr{D}}$ is a Carleson sequence with intensity $||b||^2_{BMO^d}$.

8.2 Operator Definitions

In this section, we formally define the operators π_b and T^t_w. Later in this chapter, we demonstrate that these operators are bounded on weighted $L^2(w)$ for an A_2 weight w. This bound will be linear $[w]_{A_2}$, exactly as predicted by the A_2 theroem.

8.2.1 Paraproduct

Definition 8.2.1 (Dyadic Paraproduct)**.** Let X be a space of homogeneous type, and let $\mathscr{D}$ be an honest dyadic system of cubes. Let $b \in BMO^{\mathscr{D}}$. Define the operator $\pi_b : L^2(X) \to L^2(X)$ as

$$\pi_b f(x) := \sum_{Q\in\mathscr{D}} \langle b, h_Q\rangle \langle f\rangle_Q h_Q(x)$$

where $\{h_Q\}_{Q\in\mathscr{D}}$ are the Haar functions constructed in Chapter 6.

The other operators we will look at in this chapter are the t-Haar multiplier.

8.2.2 t-Haar Multiplier

Definition 8.2.2 (t-Haar Multiplier)**.** Let X be a space of homogeneous type, and let $\mathscr{D}$ be an honest dyadic system of cubes. Given a function $f \in L^2(X)$ and a weight $w : X \to \mathbb{R}$, formally define the operator T_w^t as

$$T_w^f t(x) := \sum_{Q\in\mathscr{D}} \left(\frac{w(x)}{\langle w\rangle_Q}\right)^t \langle f, h_Q\rangle h_Q(x)$$

In [37], J. Moreas looks at bounds for t-Haar multipliers with complexity, defined over $\mathbb{R}$. These would generalize to SHTs in the following way:

Definition 8.2.3 (t-Haar Multiplier With Complexity (a, b))**.** Let X be an SHT with honest dyadic structure $\mathscr{D}$. Given a weight $w : X \to \mathbb{R}$ and $a, b \in \mathbb{N}$, define the operator $T_w^t : L^p \to L^p$ as

$$T_{t,w}^{m,n} f(x) := \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_n(Q)\\ S\in\mathscr{D}_m(Q)}} c_{R,S}^Q \left(\frac{w(x)}{\langle w\rangle_Q}\right)^t \langle f, h_R\rangle h_S(x)$$

where $|c_{R,S}^Q| \le \sqrt{\mu(R)\mu(S)}/\mu(Q)$.

It is clear that Definition 8.2.2 is a specific case of Definition 8.2.3, when $m = n = 0$ and $c_{R,S}^Q \equiv 1$. In this document, we will not be looking at the complexity case. However, J. Moreas and the author are presently working together on this generalization in a forthcoming paper.

8.3 Lemmas

In this section we will first state the needed lemmas in their original form for $\mathbb{R}$, and then give the extended versions applicable to SHTs.

8.3.1 $\mathbb{R}$ Versions

We now state some lemmas for the setting of $\mathbb{R}$. We will provide restatements for SHTs and proofs in subsequent sections.

Lemma 8.3.1 (Weighted Carleson Lemma [37]). *Let v be a dyadic doubling weight. Then $\{\lambda_I\}_{I\in\mathscr{D}}$ is a v-Carleson sequence with intensity B if and only if for all non-negative v-measurable functions F on the line*

$$\sum_{I\in\mathscr{D}} \lambda_I \inf_{x\in I} F(x) \leq B \int_{\mathbb{R}} F(x)v(x)\,dx.$$

Lemma 8.3.2 (Little Lemma [8]). *Let v be a weight such that v^{-1} is a weight as well. Let $\{\lambda_I\}_{I\in\mathscr{D}}$ be a Carleson sequence with intensity B. Then $\{\lambda_I/\langle v^{-1}\rangle_I\}_{I\in\mathscr{D}}$ is a v-Carleson with intensity $4B$.*

In the next lemma, for a weight w and dyadic interval I, we define $\Delta_I w := |\langle w\rangle_{I_\ell} - \langle w\rangle_{I_r}|$.

Lemma 8.3.3 ($\alpha\beta$-Lemma [36]). *Let u, v be weights then for any $I \in \mathscr{D}$ and any $\alpha, \beta \in (0, \frac{1}{2})$*

$$\frac{1}{|I|}\sum_{J\in\mathscr{D}(I)} \left(\frac{|\Delta_J u|^2}{\langle u\rangle_J^2} + \frac{|\Delta_J v|^2}{\langle v\rangle_J^2}\right) |J|\langle u\rangle_J^{\alpha}\langle v\rangle_J^{\beta} \leq C_{\alpha,\beta}\langle u\rangle_I^{\alpha}\langle v\rangle_I^{\beta}$$

with $C_{\alpha,\beta} = 36\cdot\min\{\alpha - 2\alpha^2, \beta - 2\beta^2\}^{-1}$.

For any weight w over $\mathbb{R}$ we define the weighted Haar function on the dyadic interval I as

$$h_I^w(x) := \frac{1}{w(I)}\left(\sqrt{\frac{w(I_\ell)}{w(I_r)}}\mathbb{1}_{I_r}(x) - \sqrt{\frac{w(I_r)}{w(I_\ell)}}\mathbb{1}_{I_\ell}(x)\right)$$

Finally, we have this result of $\mathbb{R}$, which we will later extend to SHTs.

Proposition 8.3.4 (Weighted/Unweighted Haar Identity). *For any weight w and any dyadic interval I, there exists α_I^w and β_I^w such that*

$$h_I := \alpha_I^w \cdot h_I^w(x) + \frac{\beta_I^w}{\sqrt{|I|}}\cdot \mathbb{1}_I(x)$$

where

$$|\alpha_I^w| \leq \sqrt{\langle w\rangle_I} \quad \text{and} \quad |\beta_I^w| \leq \frac{|\Delta_I w|}{\sqrt{\langle w\rangle_I}}.$$

8.3.2 SHT Versions

We now state and prove SHT versions of the lemmas. In this section, we will continue to ssume that $\mathscr{D}$ is honest.

Recall that a weight w is dyadic doubling if there existsts a constant $D \geq 1$ such that for all cubes Q, $w(\widehat{Q}) \leq Dw(Q)$.

Lemma 8.3.5 (Weighted Carleson Lemma for SHTs)**.** *Let $v : X \to \mathbb{R}^+$ be a dyadic doubling weight with respect to $\mathscr{D}$. Then $\{\lambda_Q\}_{Q\in\mathscr{D}}$ is a v-Carleson sequence with intensity B if and only if for all non-negative σ-measurable functions F*

$$\sum_{Q\in\mathscr{D}} \lambda_Q \inf_{x\in Q} F(x) \leq B \int_X F(x)v(x)\, d\mu(x). \tag{8.5}$$

where the measure σ is defined as

$$\sigma(E) := \int_E v\, d\mu.$$

The logic of the proof is identical to what is found in [37], with minor change accounting for the SHT setting. We reproduce it more or less verbatim here with only a few typographical changes to match our previous notations.

Proof. $(\Rightarrow)$ We assume that $F \in L^1(v)$ otherwise the statement is automatically true. For a cube Q we define the value $\gamma_Q := \inf_{x\in Q} F(x)$ and the function $\chi : \mathscr{D} \times \mathbb{R} \to \{0, 1\}$ as

$$\chi(Q,t) := \begin{cases} 1 & \text{if} \quad t < \gamma_Q \\ 0 & \quad \text{otherwise} \end{cases}$$

Note that $\chi(Q, t) \leq F(x)$ for all $x \in Q$ and

$$\int_0^\infty \chi(Q,t)\, dt = \int_0^{\gamma_Q} 1\, dt = \gamma_Q.$$

We can now write

$$\sum_{Q\in\mathscr{D}} \lambda_Q\gamma_Q = \sum_{Q\in\mathscr{D}} \int_0^\infty \chi(Q,t)\, dt\, \lambda_Q = \int_0^\infty \left(\sum_{Q\in\mathscr{D}} \chi(Q,t)\lambda_Q \right) dt \tag{8.6}$$

where (8.6) follows from the Lebesgue Monotone Convergence Theorem. Define the set $E_t := \{x \in X \mid F(x) > t\}$. Since F is a *sigma*-measurable function, E_t is a σ-measurable set for all t. Moreover, by Chebychev's Inequality, $\sigma(E_t) < \infty$ for all t. If for some cube $\chi(Q,t) = 1$ then $Q \subset E_t$. By properties of dyadic cubes, there exists a maximal collection of disjoint dyadic cubes which are contained in E_t. Call this collection $\mathcal{P}_t$. Then for any t,

$$\sum_{Q\in\mathscr{D}} \chi(Q,t)\lambda_Q \leq \sum_{Q\subset E_t} \lambda_Q = \sum_{Q\in\mathcal{P}_t} \sum_{R\in\mathscr{D}(Q)} \lambda_R \leq B \sum_{Q\in\mathcal{P}_t} \sigma(Q) \leq B\sigma(E_t) \tag{8.7}$$

where we used in (8.7) that $\{\lambda_Q\}$ is a v-Carleson sequence with intensity B. Therefore,

$$\sum_{Q\in\mathscr{D}} \gamma_Q\lambda_Q \leq B\int_0^\infty \sigma(E_t)\,dt = B\int_X F(x)v(x)\,d\mu(x).$$

To see why this is so, recall that for any measurable function f and measure η,

$$f(x) = \int_0^\infty \mathbb{1}_{E_t(f)}\,d\eta(t).$$

Applying this fact to the function F with measure σ gives the desired result.

($\Leftarrow$) Assume that (8.5) is true. In particular then, it holds for $F(x) = \mathbb{1}_Q(x)/\mu(Q)$. Since $\inf_{x\in R} F(x) = 0$ if $R \cap Q = \emptyset$ and $1/\mu(Q)$ otherwise,

$$\frac{1}{\mu(Q)} \sum_{R\in\mathscr{D}(Q)} \lambda_R \leq \sum_{R\in\mathscr{D}(Q)} \lambda_R \inf_{x\in R} F(x) \leq \int_X F(x)v(x)\,d\mu(x) = \langle v\rangle_Q.$$

which implies that

$$\sum_{R\in\mathscr{D}(Q)} \lambda_R \leq \int_Q v\,d\mu = \sigma(Q).$$

□

The little lemma and the $\alpha\beta$-lemma, being proved via the Bellman technique, can be extended to the SHT setting by our Good Bellman Theorem.

Lemma 8.3.6 (Little Lemma SHT). *Let v be a weight such that v^{-1} is a weight as well. Let $\{\lambda_Q\}_{Q\in\mathscr{D}}$ be a Carleson sequence with intensity B. Then $\{\lambda_Q/\langle v^{-1}\rangle_Q\}_{Q\in\mathscr{D}}$ is a v-Carleson with intensity $4B\langle m\rangle_Q v$.*

Proof. In the original proof found in [8], the Bellman function

$$\mathfrak{B}(u,v,l) = u - \frac{1}{v(1+l)}$$

was used to provide the inequality, defined on the domain $\Omega = \{(u,v,l) \in \mathbb{R}^3 \mid uv \geq 1 \text{ and } 0 \leq l \leq 1\}$. This domain is a 3-dimensional 2-prism where $\Omega_B = \{(u,v) \in \mathbb{R}^2 \mid uv \geq 1\}$ and $\Omega_R = [0,1] \subset \mathbb{R}$. Clearly, Ω_B is a convex set.

Since we have the special case of the Good Bellman Function Lemma, Corollary 7.3.3, we need only check the remaining hypotheses to apply it. Namely, we need to show that the points $\mathbf{x}_Q$ lie in the domian Ω and that the points u_Q, u_{Q_+}, and u_{Q_-} are colinear in Ω_B. (Recall $Q_\pm$ are either the two children of Q if $\mathscr{D}$ is honest, or the two children of Q in an overlapping honest structure $\tilde{\mathscr{D}}$ if $\mathscr{D}$ is not honest.) We also much check that the function b in the the main inequality satisfies the required symmerty condition.

We tackle the symmetry condition first. In [8], the main inequality given is that

$$\mathfrak{B}(\mathbf{x}^\circ) \geq \frac{1}{2}\left(\mathfrak{B}(\mathbf{x}^+) + \mathfrak{B}(\mathbf{x}^-)\right) + \frac{1}{4v}\alpha$$

where v is the second component of $\mathbf{x}^\circ$ (i.e. $\mathbf{x}^\circ = (u,v,l)$) and α is such that

$$\mathbf{x}^\circ - \frac{\mathbf{x}^+ + \mathbf{x}^-}{2} = (0,0,\alpha).$$

Thus, the function $b := \alpha/(4v)$ is symmetric in the variables $\mathbf{x}^+$ and $\mathbf{x}^-$.

We see in the proof of Lemma 7.1 in [8] that for a dyadic interval $I \in \mathscr{D}^{\mathbb{R}}$,

$$\mathbf{x}_I := \left(\langle w \rangle_I, \langle w^{-1} \rangle_I, (|I|B)^{-1} \sum_{J \in \mathscr{D}(I)} \lambda_I \right) \in \Omega.$$

(Note that here we have translated the original's notation into that used by this document, but the meaning is the same.) It is clear that extending this definition of $\mathbf{x}_I$ to $\mathbf{x}_Q$ is permittable, and that $\mathbf{x}_Q \in \Omega$. Moreover,

$$\begin{aligned} u_Q &= (\langle w\rangle_Q, \langle w^{-1}\rangle_Q) \\ &= \frac{\mu(Q_+)}{\mu(Q)}(\langle w\rangle_{Q_+}, \langle w^{-1}\rangle_{Q_+}) + \frac{\mu(Q_-)}{\mu(Q)}(\langle w\rangle_{Q_-}, \langle w^{-1}\rangle_{Q_-}) \\ &= \frac{\mu(Q_+)}{\mu(Q)}u_{Q_+} + \frac{\mu(Q_-)}{\mu(Q)}u_{Q_-} \end{aligned}$$

which checks the linearity requirement. By the Convex Domain Special Case of Good Bellman Function Theorem, we have the inequality

$$\sum_{Q\in\mathscr{D}} \frac{\lambda_Q}{\langle v^{-1}\rangle_Q} \leq 4B\langle v\rangle_Q \cdot \left(\frac{1}{2}\sup_{Q\in\mathscr{D}} \mu(\widehat{Q})/\mu(Q)\right) \tag{8.8}$$

which is what we wished to show. □

Lemma 8.3.7 ($\alpha\beta$-Lemma SHT)**.** *Let u, v be weights then for any $Q \in \mathscr{D}$ and any $\alpha, \beta \in (0, \frac{1}{2})$*

$$\frac{1}{\mu(Q)} \sum_{R\in\mathscr{D}(Q)} \left(\frac{|\Delta_R u|^2}{\langle u\rangle_R^2} + \frac{|\Delta_R v|^2}{\langle v\rangle_R^2}\right) \mu(R)\langle u\rangle_R^\alpha \langle v\rangle_R^\beta \leq C_{\alpha,\beta}\langle u\rangle_Q^\alpha \langle v\rangle_Q^\beta \tag{8.9}$$

with $C_{\alpha,\beta} = 36\min\{\alpha - 2\alpha^2, \beta - 2\beta^2\}^{-1}$ where D is the same as the previous lemma.

It is worth pointing out that the operator Δ_Q is taken to mean the obvious thing:

$$\Delta_Q w := \begin{cases} \langle w\rangle_{Q_+} - \langle w\rangle_{Q_-} & \text{if} \quad N(Q) = 2 \\ 0 & \text{if} \quad N(Q) = 1. \end{cases}$$

Recall that we already assumed that $\mathscr{D}$ was an honest structure, so the Δ_Q defined here is not ambiguous.

Proof. In the proof of the $\mathbb{R}$ version of this lemma, found in [37], the Bellman function

$$\mathfrak{B}(x, y) := x^\alpha y^\beta$$

is used over the domain $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ to provide the required inequality. This domain is a convex set, thus it is a 2 dimensional 2 prism with a convex base. We proceed similarly as in the previous proof, needing only to check that the the ponts $\mathbf{x}_Q \in \Omega$ and that the linearity condition is held.

In the proof of Lemma 3.7 in [37] the mapping $\mathbf{x}_I := (\langle u \rangle_I, \langle v \rangle_I)$ is used. It is clear that extending this mapping to $\mathbf{x}_Q$ will be permitted and that $\mathbf{x}_Q \in \Omega$. Moreover,

$$\begin{aligned}
\mathbf{x}_Q &= (\langle u \rangle_Q, \langle v \rangle_Q) \\
&= \frac{\mu(Q_+)}{\mu(Q)}(\langle u \rangle_{Q_+}, \langle v \rangle_{Q_+}) + \frac{\mu(Q_-)}{\mu(Q)}(\langle u \rangle_{Q_-}, \langle v \rangle_{Q_-}) \\
&= \frac{\mu(Q_+)}{\mu(Q)}\mathbf{x}_{Q_+} + \frac{\mu(Q_-)}{\mu(Q)}\mathbf{x}_{Q_-}
\end{aligned}$$

which proves the linearity condition. By the Good Bellman Function Theorem, the $\mathbb{R}$ version of the $\alpha\beta$-Lemma is extendable to SHTs to give the desired result. $\square$

8.3.3 Weighted Haar Functions

Lastly, we define the idea of a weighted Haar function over an SHT in the obvious way:

Definition 8.3.8 (Weighted Haar Function)**.** The *weighted Haar function* h_Q^w with honest dyadic grid $\mathscr{D}$ is defined as

$$h_Q^w := \kappa_Q^+ \mathbb{1}_{Q_+} - \kappa_Q^- \mathbb{1}_{Q_-} \tag{8.10}$$

where

$$\kappa_Q^\pm := \sqrt{\frac{w(Q_\mp)}{w(Q) \cdot w(Q_\pm)}}$$

Notice that when $w \equiv 1$ that $\kappa_Q^\pm = \sqrt{\mu(Q_\mp)/(\mu(Q)\mu(Q_\pm))} = \lambda_Q^\pm$ from (6.7).

Proposition 8.3.9 (Weighted/unweighted Haar Identity SHT)**.** *For any weight w and any honest dyadic cube Q, there exists α_Q^w and β_Q^w such that*

$$h_Q := \alpha_Q^w \cdot h_Q^w(x) + \frac{\beta_Q^w}{\sqrt{\mu(Q)}} \cdot \mathbb{1}_Q(x) \tag{8.11}$$

where

$$|\alpha_Q^w| \leq \sqrt{\langle w\rangle_Q} \quad \textit{and} \quad |\beta_Q^w| \leq \frac{|\Delta_Q w|}{\langle w\rangle_Q}.$$

Proof. Fix Q and w. To simplify notation we will write $\alpha := \alpha_Q^w$, $\beta := \beta_Q^w$, $h := h_Q$ and $h^w := h_Q^w$.

Solve (8.11) for h^w to get that

$$h^w(x) = \frac{h(x)}{\alpha} - \frac{\beta \cdot \mathbb{1}_Q(x)}{\alpha\sqrt{\mu(Q)}}.$$

Using the axioms for Haar functions, we know that

$$\int_X h^w(x)w(x)\,d\mu(x) = 0 \quad \text{and} \quad \langle h^w, h^w\rangle_w = 1 \tag{8.12}$$

Starting with the first part of (8.12), we substitute to get that

$$\begin{aligned} 0 &= \int_X \left[\frac{h(x)}{\alpha} - \frac{\beta \cdot \mathbb{1}_Q(x)}{\alpha\sqrt{\mu(Q)}}\right] w(x)\,d\mu(x) \\ \frac{\beta \cdot w(Q)}{\sqrt{\mu(Q)}} &= \int_X h(x)w(x)\,d\mu(x) = \int_X \left[\lambda^+\mathbb{1}_{Q_+}(x) - \lambda^-\mathbb{1}_{Q_-}(x)\right] w(x)\,d\mu(x) \\ &= \lambda^+ w(Q_+) - \lambda^- w(Q_-) \end{aligned}$$

implying that

$$\beta = \frac{1}{w(Q)}\left[\sqrt{\frac{\mu(Q_-)}{\mu(Q_+)}}w(Q_+) - \sqrt{\frac{\mu(Q_+)}{\mu(Q_-)}}w(Q_-)\right] = \frac{\sqrt{\mu(Q_+)\mu(Q_-)}}{w(Q)}\Delta_Q w \tag{8.13}$$

$$|\beta| \leq \frac{\mu(Q_+)+\mu(Q_-)}{2w(Q)}|\Delta_Q w| = \frac{|\Delta_Q w|}{2\langle w\rangle_Q} \tag{8.14}$$

where in the last line we used that the geometric mean is bounded by the arithmetic mean. Notice also that $|\beta| < 1$.

For the bound on α, we use the seccond part of (8.12) to get that

$$1 = \int_X \left[\frac{h(x)}{\alpha} - \frac{\beta \cdot \mathbb{1}_Q(x)}{\alpha\sqrt{\mu(Q)}}\right]^2 w(x)\,d\mu(x)$$

implying that

$$\begin{aligned}
\alpha^2 &= \int_X \left[h(x) - \frac{\beta \cdot \mathbb{1}_Q(x)}{\sqrt{\mu(Q)}}\right]^2 w(x)\, d\mu(x) \\
&= \int_Q h(x)^2 w(x)\, d\mu(x) - \frac{2\beta}{\sqrt{\mu(Q)}} \int_Q h(x) w(x)\, d\mu(x) + \frac{\beta^2}{\mu(Q)} \int_Q w(x)\, d\mu(x) \\
&= (\lambda^+)^2 w(Q_+) + (\lambda^-)^2 w(Q_-) - \frac{2\beta}{\sqrt{\mu(Q)}} \big(\lambda^+ w(Q_+) - \lambda^- w(Q_-)\big) + \beta^2 \langle w \rangle_Q \\
&= \frac{\mu(Q_-)}{\mu(Q)} \langle w \rangle_{Q_+} + \frac{\mu(Q_+)}{\mu(Q)} \langle w \rangle_{Q_-} - \frac{2\beta\sqrt{\mu(Q_+)\mu(Q_-)}}{\mu(Q)} (\langle w \rangle_{Q_+} - \langle w \rangle_{Q_-}) + \beta^2 \langle w \rangle_Q \\
&= \frac{\mu(Q_-)}{\mu(Q)} \langle w \rangle_{Q_+} + \frac{\mu(Q_+)}{\mu(Q)} \langle w \rangle_{Q_-} - \beta^2 \langle w \rangle_Q \\
&\leq (1 - \beta^2) \langle w \rangle_Q < \langle w \rangle_Q
\end{aligned}$$

where in the last line we used (8.13). □

8.4 Bound on the Paraproduct

We now use the above Lemmas to prove the desired bound for the SHT paraproduct.

Theorem 8.4.1. *Let (X, ρ, μ) be an SHT. There exists a constant $C > 0$ such that for any $b \in BMO^d$, and π_b be the dyadic paraproduct as defined in 8.2.1. Then for $w \in A_2^d$,*

$$||\pi_b f||_{L^2(w)} \leq CD[w]_{A_2^d} ||b||_{BMO^d} ||f||_{L^2(w)}$$

with D, as above, equal to the reciprocal of the dyadic doubling constant. By $||f||_{L^2(w)}$ we mean $\left(\int_X f(x)^2 w(x)\, d\mu(x)\right)^{1/2}$.

To prove this theorem, we will essentially need to check that the proof for the $\mathbb{R}$ version of the bound transfers over to SHTs.

Proof. If $||f||_{L^2(w)} = \infty$ then the inequality is trivially true.

Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$. By duality, it is enough to show that

$$\left|\langle \pi_b(fw), gw^{-1}\rangle\right| \leq C[w]_{A_2^d}||b||_{BMO^d}||f||_{L^2(w)}||g||_{L^2(w^{-1})} \tag{8.15}$$

for some constant C. Expanding the left hand side of (8.15) gives

$$\left|\langle \pi_b(fw), gw^{-1}\rangle\right| = \left|\left\langle \sum_{Q\in\mathscr{D}} b_Q \langle fw\rangle_Q h_Q, gw^{-1}\right\rangle\right|$$

where here we write $b_Q := \langle b, h_Q\rangle$. Using the weighted/unweighted Haar identity, we write $h_Q = \alpha_Q^{w^{-1}} h_Q^{w^{-1}} + \beta_Q^{w^{-1}} \mathbb{1}_Q/\sqrt{\mu(Q)}$. Then

$$\begin{aligned}\left|\langle \pi_b(fw), gw^{-1}\rangle\right| &= \sum_{Q\in\mathscr{D}} b_Q \langle |f|w\rangle_Q \left|\left\langle \alpha_Q^{w^{-1}} h_Q^{w^{-1}} + \beta_Q^{w^{-1}} \mathbb{1}_Q/\sqrt{\mu(Q)}, gw^{-1}\right\rangle\right| \\ &\leq \Sigma_1 + \Sigma_2\end{aligned}$$

where

$$\begin{aligned}\Sigma_1 &:= \sum_{Q\in\mathscr{D}} |b_Q| \langle |f|w\rangle_Q |\langle h_Q^{w^{-1}}, gw^{-1}\rangle| \sqrt{\langle w^{-1}\rangle_Q} \\ \Sigma_2 &:= \sum_{Q\in\mathscr{D}} |b_Q| \langle |f|w\rangle_Q |\langle \mathbb{1}_Q, gw^{-1}\rangle| \frac{|\Delta_Q w^{-1}|}{\langle w^{-1}\rangle_Q \sqrt{\mu(Q)}}\end{aligned}$$

using the bounds for $\alpha_Q^{w^{-1}}$ and $\beta_Q^{w^{-1}}$. We will now estimate Σ_1 and Σ_2 separately.

(*Estimating* Σ_1) We use the fact that $M_w^{\mathscr{D}} f(x) \geq \langle |f|w\rangle_Q / \langle w\rangle_Q$ for every $x \in Q$ and that $\langle h_Q^{w^{-1}}, gw^{-1}\rangle = \langle h_Q^{w^{-1}}, g\rangle_{w^{-1}}$ to write that

$$\begin{aligned}\Sigma_1 &\leq \sum_{Q\in\mathscr{D}} |b_Q| \left(\inf_{x\in Q} (M_w^{\mathscr{D}} f)(x)\right) \left|\langle h_Q^{w^{-1}}, g\rangle_{w^{-1}}\right| \langle w\rangle_Q \sqrt{\langle w^{-1}\rangle_Q} \\ &\leq \sum_{Q\in\mathscr{D}} |b_Q| \left(\inf_{x\in Q} (M_w^{\mathscr{D}} f)(x)\right) \left|\langle h_Q^{w^{-1}}, g\rangle_{w^{-1}}\right| \frac{[w]_{A_2}}{\sqrt{\langle w^{-1}\rangle_Q}}\end{aligned}$$

where in the second line we used that $\langle w\rangle_Q \langle w^{-1}\rangle_Q \leq [w]_{A_2}$. Applying Cauchy-Schwarz gives that

$$\Sigma_1 \leq [w]_{A_2} \left(\sum_{Q\in\mathscr{D}} b_Q^2 \frac{\inf_{x\in Q}(M_w^{\mathscr{D}} f)(x)^2}{\langle w^{-1}\rangle_Q}\right)^{1/2} \left(\sum_{Q\in\mathscr{D}} \langle h_Q^{w^{-1}}, g\rangle_{w^{-1}}^2\right)^{1/2}$$

Since $\{h_Q^{w^{-1}}\}_{Q\in\mathscr{D}}$ forms an orthonormal family for $L^2(w^{-1})$,

$$\left(\sum_{Q\in\mathscr{D}} \langle h_Q^{w^{-1}}, g\rangle_{w^{-1}}^2\right)^{1/2} \leq ||g||_{L^2(w^{-1})}.$$

Now, since $b \in BMO^d$, $\{b_Q^2\}_{Q\in\mathscr{D}}$ is a Carleson sequence with intensity $||b||_{BMO^d}^2$. By the Little Lemma for SHTs, we therefore get that $\{b_Q^2/\langle w^{-1}\rangle_Q\}_{Q\in\mathscr{D}}$ is a w-Carleson sequence with intensity $||b||_{BMO^d}^2$. We apply the weighted Carleson Lemma for SHTs to thus say that

$$\left(\sum_{Q\in\mathscr{D}} b_Q^2 \frac{\inf_{x\in Q}(M_w^{\mathscr{D}} f)(x)^2}{\langle w^{-1}\rangle_Q}\right)^{1/2} \leq 2D||b||_{BMO^d}\left(\int_X (M_w^{\mathscr{D}} f)(x)^2 w(x)\, d\mu(x)\right)^{1/2}$$

Since $M_w^{\mathscr{D}}$ is bounded in $L^2(w)$ with bound not dependent on w,

$$\Sigma_1 \leq 2D\cdot [w]_{A_2}\cdot ||b||_{BMO^d}\cdot ||M_w^{\mathscr{D}}||_{L^2(w)}\cdot ||g||_{L^2(w^{-1})}.$$

(*Estimating* Σ_2) We start with

$$\begin{aligned}
\Sigma_2 &= \sum_{Q\in\mathscr{D}} |b_Q| \langle |f|w\rangle_Q |\langle \mathbb{1}_Q, gw^{-1}\rangle \frac{|\Delta_Q w^{-1}|}{\langle w^{-1}\rangle_Q\sqrt{\mu(Q)}} \\
&= \sum_{Q\in\mathscr{D}} |b_Q| \langle |f|\rangle_Q^w \langle g\rangle_Q^{w^{-1}} \sqrt{\mu(Q)\langle w\rangle_Q^2\left(\Delta_Q w^{-1}\right)^2} \\
&= \sum_{Q\in\mathscr{D}} |b_Q| \sqrt{\mu(Q)\langle w\rangle_Q^2\left(\Delta_Q w^{-1}\right)^2}\cdot \inf_{x\in Q} M_w^{\mathscr{D}} f(x) M_{w^{-1}}^{\mathscr{D}} g(x).
\end{aligned}$$

In the last line we used the fact that for all $x\in Q$, $\langle |f|\rangle_Q^w \leq M_w^{\mathscr{D}} f(w)$. We now claim that $\{\mu(Q)\langle w\rangle_Q^2\left(\Delta_Q w^{-1}\right)^2\}_{Q\in\mathscr{D}}$ is a Carleson sequence with intensity $[w]_{A_2}^2$, with C a positive constant. The proof of this can be found in [37]

By Lemma 8.1.5, $\{|b_Q|^2\}$ is also a Carleson sequence with intensity $2D||b||_{BMO}^2$. Thus, we have that $\left\{|b|\sqrt{\mu(Q)\langle w\rangle_Q^2\left(\Delta_Q w^{-1}\right)^2}\right\}_{Q\in\mathscr{D}}$ is a Carleson sequence wiuth respect to μ with intensity $C||b||_{BMO^d}[w]_{A_2}$. Therefore, by the weighted Carleson Lemma

$$\Sigma_2 \leq ||b||_{BMO^d}[w]_{A_2}\int_X (M_w^{\mathscr{D}} f)(x)(M_{w^{-1}}^{\mathscr{D}} g)(x)\, d\mu(x)$$

implying that

$$\int_X (M_w^{\mathscr{D}} f)(x)(M_{w^{-1}}^{\mathscr{D}} g)(x)\, d\mu(x) = \int_X (M_w^{\mathscr{D}} f)(x)(M_{w^{-1}}^{D} g)(x)\sqrt{\frac{w(x)}{w(x)}}\, d\mu(x)$$
$$\leq ||M_w^{\mathscr{D}} f||_{L^2(w)} ||M_w^{\mathscr{D}} g||_{L^2(w^{-1})}$$

where in the last line we used Cauchy-Schwarz. Therefore,

$$\Sigma_2 \leq [w]_{A_2} ||b||_{BMO^d} ||M_w^{\mathscr{D}}|| \cdot ||M_{w^{-1}}^{\mathscr{D}}|| \cdot ||f||_{L^2(w)} ||g||_{L^2(w^{-1})}$$

The estimates for Σ_1 and Σ_2 together complete the proof. □

8.5 Bound on the t-Haar Multiplier

Here we will prove the L^2 bound for the t-Haar Multiplier without complexity:

Theorem 8.5.1. *Let X be an SHT with honest dyadic lattice $\mathscr{D}$. Let $t \in \mathbb{R}$ and w be a weight over X with $w \in C_{2t}^{\mathscr{D}}$ and $w^{2t} \in A_2^{\mathscr{D}}$. Then the weighted t-Haar multiplier, T_w^t is bounded from $L^2(X) \to L^2(X)$, with*

$$||T_w^t f||_2 \leq C[w]_{C_{2t}^{\mathscr{D}}}^{\frac{1}{2}} [w^{2t}]_{A_2^{\mathscr{D}}}^{\frac{1}{2}} ||f||_2$$

for all $f \in L^2(X)$.

Here the Haar functions used in T_w^t are those associated with $\mathscr{D}$.

Proof. By duality, it is enough to show that

$$|\langle T_w^t f, g\rangle| \leq C[w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} ||f||_2 ||g||_2$$

for $f, g \in L^2(X)$. Bound the inner product on the left-hand side

$$|\langle T_w^t f, g\rangle| \leq \sum_{Q \in \mathscr{D}} \frac{|\langle f, h_Q\rangle||\langle g w^t, h_Q\rangle|}{\langle w\rangle_Q^t}. \tag{8.16}$$

Decompose h_Q as in (8.11) with respect to w^{2t} instead of w, $h_Q = \alpha_Q^{w^{2t}} h_Q^{w^{2t}} + \beta_Q^{w^{2t}} \mathbb{1}_Q / \sqrt{\mu(Q)}$. Here $|\alpha_Q^{w^{2t}}| \leq \sqrt{\langle w^{2t}\rangle_Q}$ and $|\beta_Q^{w^{2t}}| \leq \frac{|\Delta_Q w^{2t}|}{\langle w^{2t}\rangle_Q}$. We now can write (8.16) as two sums: $|\langle T_w^t f, g\rangle| \leq \Sigma_1 + \Sigma_2$ where

$$\Sigma_1 := \sum_{Q\in\mathscr{D}} \frac{\sqrt{\langle w^{2t}\rangle_Q}}{\langle w\rangle_Q^t} |\langle f, h_Q\rangle| \cdot |\langle g w^t, h_Q^{w^{2t}}\rangle| \tag{8.17}$$

$$\Sigma_2 := \sum_{Q\in\mathscr{D}} \frac{\sqrt{\mu(Q)}}{\langle w\rangle_Q^t} \cdot \frac{\Delta_Q(w^{2t})}{\langle w^{2t}\rangle_Q} |\langle f, h_Q\rangle| \cdot \langle |g| w^t\rangle_Q \tag{8.18}$$

(*Bounding* Σ_1) We first observe that

$$\frac{\sqrt{\langle w^{2t}\rangle_Q}}{\langle w\rangle_Q^t} \leq [w]_{C_{2t}}^{\frac{1}{2}}.$$

Seccond, that

$$|\langle g w^t, h_Q^{w^{2t}}\rangle| = |\langle g, w^{-t} h_Q^{w^{2t}}\rangle_{w^{2t}}|.$$

We use these facts and apply Cauchy-Schwarz to (8.17) to get that

$$\begin{aligned}\Sigma_1 &\leq [w]_{C_{2t}}^{\frac{1}{2}} \left(\sum_{Q\in\mathscr{D}} |\langle f, h_Q\rangle|\right)^{\frac{1}{2}} \left(\sum_{Q\in\mathscr{D}} |\langle g w^{-1}, h_Q^{w^{2t}}\rangle_{w^{2t}}|\right)^{\frac{1}{2}} \\ &= [w]_{C_{2t}}^{\frac{1}{2}} ||f||_{L^2} ||g w^{-t}||_{L^2(w^{2t})} = [w]_{C_{2t}}^{\frac{1}{2}} ||f||_{L^2} ||g||_{L^2}.\end{aligned}$$

(*Bounding* Σ_2)

To bound Σ_2 we start by seeing that

$$\begin{aligned}\Sigma_2 &\leq \sum_{Q\in\mathscr{D}} \langle w\rangle_Q^{-t} \left[\mu(Q)\left(\frac{\Delta_Q(w^{2t})^2}{\langle w^{2t}\rangle_Q} + \frac{\Delta_Q(w^{-2t})^2}{\langle w^{-2t}\rangle_Q}\right)\right]^{\frac{1}{2}} \cdot |\langle f, h_Q\rangle| \cdot \frac{\langle |g w^{-t}| w^{2t}\rangle_Q}{\langle w^{2t}\rangle_Q} \\ &= \sum_{Q\in\mathscr{D}} \langle w\rangle^{-t} \left[\mu(Q)\langle w^{2t}\rangle_Q \langle w^{-2t}\rangle_Q \left(\frac{\Delta_Q(w^{2t})^2}{\langle w^{2t}\rangle_Q^2} + \frac{\Delta_Q(w^{-2t})^2}{\langle w^{-2t}\rangle_Q^2}\right)\right]^{\frac{1}{2}} \cdot |\langle f, h_Q\rangle| \\ &\quad \cdot \frac{\langle |g w^{-t}| w^{2t}\rangle_Q}{\langle w^{2t}\rangle_Q} \langle w^{2t}\rangle_Q^{\frac{1}{2}} \langle w^{-2t}\rangle_Q^{-\frac{1}{2}}.\end{aligned}$$

Set $\lambda_Q := \mu(Q)\langle w^{2t}\rangle_Q \langle w^{-2t}\rangle_Q \left(\frac{\Delta_Q(w^{2t})^2}{\langle w^{2t}\rangle_Q^2} + \frac{\Delta_Q(w^{-2t})^2}{\langle w^{-2t}\rangle_Q^2}\right)$ and $F(x) := M^{\mathscr{D}}_{w^{2t}}(|g|w^{-t})(x)$ where $M^d_{w^{2t}}$ denotes the weighted dyadic maximal function. Using the fact that for all $Q \in \mathscr{D}$, $\inf_{x\in Q} F(x) \geq \langle |gw^{-t}|w^{2t}\rangle_Q / \langle w^{2t}\rangle_Q$ gives

$$\Sigma_2 \leq \sum_{Q\in\mathscr{D}} \langle w\rangle_Q^{-t} \lambda_Q^{\frac{1}{2}} \cdot |\langle f, h_Q\rangle| \cdot \inf_{x\in Q} F(x) \cdot \langle w^{2t}\rangle_Q^{\frac{1}{2}} \langle w^{-2t}\rangle_Q^{-\frac{1}{2}}.$$

We now apply Cauchy-Schwarz inequality

$$\Sigma_2 \leq \left(\sum_{Q\in\mathscr{D}} \langle w\rangle_Q^{-2t} \langle w^{2t}\rangle_Q \frac{\lambda_Q}{\langle w^{-2t}\rangle_Q} \inf_{x\in Q} F(x)\right)^{\frac{1}{2}} \left(\sum_{Q\in\mathscr{D}} |\langle f, h_Q\rangle|^2\right)^{\frac{1}{2}}$$

$$\|eq[w]_{C^{\mathscr{D}}_{w^{2t}}} \left(\sum_{Q\in\mathscr{D}} \frac{\lambda_Q}{\langle w^{-2t}\rangle_Q} \inf_{x\in Q} F(x)\right)^{\frac{1}{2}} ||f||_{L^2(d\mu)}.$$

It is known that λ_Q is a Carleson sequence with intensity $C[w^{2t}]_{A_2^{\mathscr{D}}}$. We apply the Little Lemma to get that $\lambda_Q/\langle w^{-2t}\rangle_Q$ is a w^{-2t}-Carleson sequence with intensity $4C[w^{2t}]_{A_2^{\mathscr{D}}}$. By the weighted Carleson Lemma,

$$\begin{aligned}
\Sigma_2 &\leq 2C[w]^{\frac{1}{2}}_{C\mathscr{D}_{w^{2t}}} [w^{2t}]^{\frac{1}{2}}_{A_2^{\mathscr{D}}} \left(\int_X F(x) \cdot w^{2t}(x)\, d\mu(x)\right)^{\frac{1}{2}} \cdot ||f||_{L^2(d\mu)} \\
&\leq 2C[w]^{\frac{1}{2}}_{C^{\mathscr{D}}_{w^{2t}}} [w^{2t}]^{\frac{1}{2}}_{A_2^{\mathscr{D}}} \cdot ||gw^{-t}||_{L^2(w^{2t}d\mu)} \cdot ||f||_{L^2(d\mu)} \\
&=\leq 2C[w]^{\frac{1}{2}}_{C^{\mathscr{D}}_{w^{2t}}} [w^{2t}]^{\frac{1}{2}}_{A_2^{\mathscr{D}}} \cdot ||g||_{L^2(d\mu)} \cdot ||f||_{L^2(d\mu)}.
\end{aligned}$$

Here the constants on each line could be changing, but are independent of f, g, w and t.

The given bounds for Σ_1 and Σ_2 together prove the desired theorem. □

Appendix A

Other Proofs

Here we will give a few extra proofs for things which were claimed in the main text but which did not need to be proved there.

A.1 Proof of Theorem 2.5.2

In this section we give the proof that atoms are isolated points of SHTs and that there can only be countably many atoms. This is a reproduction of the proof given in [33]. See that paper for the details.

Proof that Atoms are Isolated. Let X be an SHT with an atom a, that is, $\mu(\{a\}) > 0$. Suppose for the sake of a contradiction that a is not an isolated point, that is, for every $r > 0$ the ρ-ball $B(a, r)$ contains a point x such that $x \neq a$. It is possible to construct a sequence of points $\{x_n\}$ and associeated radaii $\{r_n\}$ such that

- $B(x_n, r_n) \cap B(x_m, r_m) = \emptyset$ for $n \neq m$
- $a \in B(x_n, Rr_n)$ for all n

where $R > 0$ is a fixed geometric constant. The details of this construction are in [33]. These two facts together give that for some geometric constant C,

$$\mu(B(a, 2\kappa)) \geq \sum_{n=1}^{\infty} \mu(B(x_n, r_n)) \geq C \sum_{n=1}^{\infty} \mu(B(x_n, Rr_n)) \geq C \sum_{n=1}^{\infty} \mu(\{a\}) = \infty.$$

This is a contradiction since no ball can have infinite measure. □

Proof of Countable Many Atoms. Let X be an SHT and denote by $\mathcal{A}$ the set of all atoms. Fix any point $x_0 \in X$. For every natural number n define the the set $\mathcal{A}_n := B(x_0, n) \cap \mathcal{A}$. Clearly $\mu(\mathcal{A}_n) < \mu(B(x_0, n)) \leq \infty$. This implies that for each n, $\mathcal{A}_n$ is no more than countable, since an uncountable collection of atoms would have infinite measure. But $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{A}_n$, so $\mathcal{A}$ is also countable. □

A.2 Alternate Proof for Theorem 3.4.6

Here we present a proof by a different route for The Bounded Subset Theorem from Chapter 3.

Proof. Let $S \subseteq X$ be a bounded set, and fix r_0 greater than the diameter of S[1]. We will prove the contrapositive of what we want to show. Suppose that there is no such cube Q which completely contains S. Then for all generations $\mathscr{D}_k$, we can find $x_k, y_k \in S$ such that $x_k \in Q_\alpha^k$ and $y_k \in Q_\beta^k$ with $\alpha \neq \beta$. We claim that in fact, there is a pair $x, y \in S$ which will satisfy this for all generations. We prove this claim via induction.

First, observe that for a fixed generation $\mathscr{D}_k$, if x and y satisfy that their containing cubes are not equal, then it follows that this must also be so for their respective cubes in the $\mathscr{D}_{k+1}$ generation. This is because the cubes in the $(k+1)^{\text{th}}$ generation are children of the cubes in the k^{th} generation and thus could not possibly be equal to each other.

[1]By *diameter* we mean the standard definition for metric spaces: the supremum of $\rho(x, y)$ for all $x, y \in S$.

Second, for any generation $\mathscr{D}_k$ there are finitely many cubes belonging to $\mathscr{D}_k$ which cover S. This number of required cubes is dependent on κ_0, r_0, k and γ_0 the geometric doubling constant for ρ. Let us say that N_k is the number of cubes from $\mathscr{D}_k$ required to cover S. Then $N_{k-1} \leq N_k$ for all k, since some of the cubes from the k^{th} generation may be siblings. Because N_k must always be positive (in fact we have supposed that $N_k \geq 2$), by the monotone convergence theorem, $N_k \to N$ as $k \to -\infty$. Moreover, $\{N_k\}_{k=-\infty}^{\infty}$ is actually a sequence of integers, implying that eventually it is a constant sequence as $k \to -\infty$. In other words, there is a k_0 such that if $k \leq k_0$ then the same number of cubes from $\mathscr{D}_{k_0}$ and $\mathscr{D}_k$ are required to cover S. Because of this, if $k < k_0$ and Q and Q' are two distinct cubes in the k^{th} generation which are part of the cover of S then they cannot be siblings, for if they were, $N_{k-1} < N$. This implies that $\widehat{Q}$ and $\widehat{Q'}$ must therefore be distinct.

Setting $x := x_{k_0}$ and $y := y_{k_0}$ and inducting in both directions on k proves the claim. With x and y found, we can see that $\mathrm{Quad}(x) \neq \mathrm{Quad}(y)$ since their containing cubes are distinct at all generations. This means that S is not completely contained in a single quadrant, which is what we wished to show. $\square$

A.3 Generalized Doubling Lemmas

For interested readers we give the proof of Corollary 2.1.11. These are not new lemmas, but as far as we know they have not been formally proven until now.

Lemma A.3.1 (Doubling for General Radii). *Let (X, ρ, μ) be a space of homogeneous type. If $x \in X$ and $R > r > 0$ then*

$$\mu(B(x,R)) \leq \kappa_1^{\log_2 \lceil R/r \rceil} \cdot \mu(B(x,r)), \tag{A.1}$$

Proof. By the doubling property,

$$\begin{aligned}
\mu((B(x,R)) &\leq \kappa_1 \cdot \mu(B(x,R/2)) \\
&\leq \kappa_1^2 \cdot \mu(B(x,R/4)) \\
&\leq \cdots \\
&\leq \kappa_1^n \cdot \mu(B(x,R\cdot 2^{-n})) \qquad \text{(A.2)}
\end{aligned}$$

Choose n so that $R \cdot 2^{-n} < r$. □

We next give the proof of the existence of the constant $Dbl(\mathscr{D})$, the dyadic doubling constant.

Proof of Corollary 3.2.8. Let $Q \in \mathscr{D}_k$ be a cube, with parent cube $\widehat{Q} \in \mathscr{D}_{k-1}$ Then there exists balls $B_1 := B(z_1, r_0\delta^k) \subseteq Q$ and $B_2 = B(z_2, R_0\delta^{k-1}) \supseteq \widehat{Q}$. Therefore,

$$\begin{aligned}
\mu(\widehat{Q}) &\leq \mu(B_2) \\
&\leq \kappa_1 \log_2 \left\lceil \frac{\kappa_0(R_0\delta^{k-1} + r_0\delta^k)}{r_0\delta^k} \right\rceil \cdot \mu(B(z_1, R_0\delta^{k-1})) \qquad \text{(A.3)} \\
&\leq \kappa_1 \log_2 \left\lceil \frac{\kappa_0(R_0\delta^{k-1} + r_0\delta^k)}{r_0\delta^k} \right\rceil \cdot \kappa_1^{\log_2(R_0\delta^{k-1}/(r_0\delta^k))} \cdot \mu(B_1) \qquad \text{(A.4)} \\
&\leq \kappa_1 \log_2 \left\lceil \frac{\kappa_0(R_0\delta^{k-1} + r_0\delta^k)}{r_0\delta^k} \right\rceil \cdot \kappa_1^{\log_2(R_0\delta^{k-1}/(r_0\delta^k))} \cdot \mu(Q) \\
&= \kappa_1^{\log_2\lceil R_0/(\delta r_0)\rceil + 1} \cdot \log_2 \left(\frac{\kappa_0(R_0 + r_0\delta)}{r_0\delta} \right\rceil \cdot \mu(Q) \qquad \text{(A.5)}
\end{aligned}$$

where (A.3) follows from the Distant Balls Lemma, and (A.4) follows from doubling for general radii. The constant is a geometric constant since it is dependent entirely on other geometric constants. □

A.4 Proof of Lemma 7.4.3

In the proof of the two weight theorem which closed Chapter 7, we claimed that the Bellman function $\mathfrak{B}$'s domain Ω was weakly convex. Moreover, we claimed that the family of matrices $\{A_t\}_{t\in(0,1/2]}$ was equal to

$$A_t := \begin{bmatrix} t/4 & 0 \\ 0 & t/4 \end{bmatrix}$$

Here, we present the proof of that claim.

Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy < 1\}$. Let $\mathbf{x}, \mathbf{x}^+, \mathbf{x}^- \in \Omega$ such that

$$\mathbf{x} = \alpha_+\mathbf{x}^+ + \alpha_-\mathbf{x}^-$$
$$1 = \alpha_+ + \alpha_-$$
$$\epsilon < \alpha_\pm < 1 - \epsilon$$
$$0 < \epsilon < 1/2$$

so that $\mathbf{x}$, $\mathbf{x}^+$, and $\mathbf{x}^-$ are collinear. Finally, choose $A > 1$ so that the enlarged domain

$$\Omega_A := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy < A\} \tag{A.6}$$

contains the entire line segment $\overline{\mathbf{x}^+\mathbf{x}^-}$. (See Figure).

We observe that the desired result is equivalent to A being no larger than $4/\epsilon$.

We first find an expression for A in terms of $\mathbf{x}$ and $\mathbf{x}^\pm$. Let $\mathbf{x} = (x^\circ, y^\circ)$ and $\mathbf{x}^\pm := (x^\pm, y^\pm)$. The line passing through them has the equation

$$y = y' + m(x - x') \tag{A.7}$$
$$m := \frac{y^+ - y^-}{x^+ - x^-} = \frac{y^\circ - y^-}{x^\circ - x^-} = \frac{y^+ - y^\circ}{x^+ - x^\circ} \tag{A.8}$$

where (x', y') is any point on the line (we already have three such points) and we are free to use whatever expression for m is most convenient for us.

We wish find the smallest A such that $\sup_{(x,y)\in\Omega} xy < A$. Consider the curve

$$y < \frac{A}{x}. \tag{A.9}$$

This curve has a positive second derivative at all points. Any line which passes though the curve twice, necessarily lies above the curve on some interval. In the situation at hand, this means that if there exists $\mathbf{x}, \mathbf{x}^+, \mathbf{x}^- \in \Omega$ so that the line (A.7) passes through the curve (A.9), twice, then the choice of A was too small. Moreover, if the curve *never* touches any such line, then A was too large. We therefore need to find A such that there exist some lines like (A.7) which are tangent to (A.9), but no lines which interesect it twice.

Setting (A.7) equal to (A.9) gives that

$$mx^2 + (y' - mx')x - A = 0 \tag{A.10}$$

We want this quadratic equation to have exactly one solution, so the discriminant must be zero:

$$(y' - mx')^2 + 4mA = 0 \tag{A.11}$$

$$\implies A = \sup_{\mathbf{x},\mathbf{x}^+,\mathbf{x}^-} \frac{-4m}{(y' - mx')^2}. \tag{A.12}$$

At this point we take the time to make an observation. The line segment $\overline{\mathbf{x}^+\mathbf{x}^-}$ can *at most* intersect the curve $xy = 1$ twice. This means that if the line segment exits Ω, then one of the points $\mathbf{x}^\pm$ is separated from $\mathbf{x}$ while the other is not. Without loss of generality we can make some assumptions. First, due to the symmetry of the domain, we can assume that $\mathbf{x}$ lies to the left of the line $x = 1$. Second, we can assume that the point which is separated from $\mathbf{x}$ is also the point which is a further distance away from it. This is not strictly necessary but cases where this is not true will not be able to be maximizers. Third, we will assume that $\alpha_+ > \alpha_-$. With these restrictions in mind, it is clear that m the slope of the line (A.7), must be negative, implying that

$$x^+ < x^\circ < x^- \tag{A.13}$$

$$y^+ > y^\circ > y^-. \tag{A.14}$$

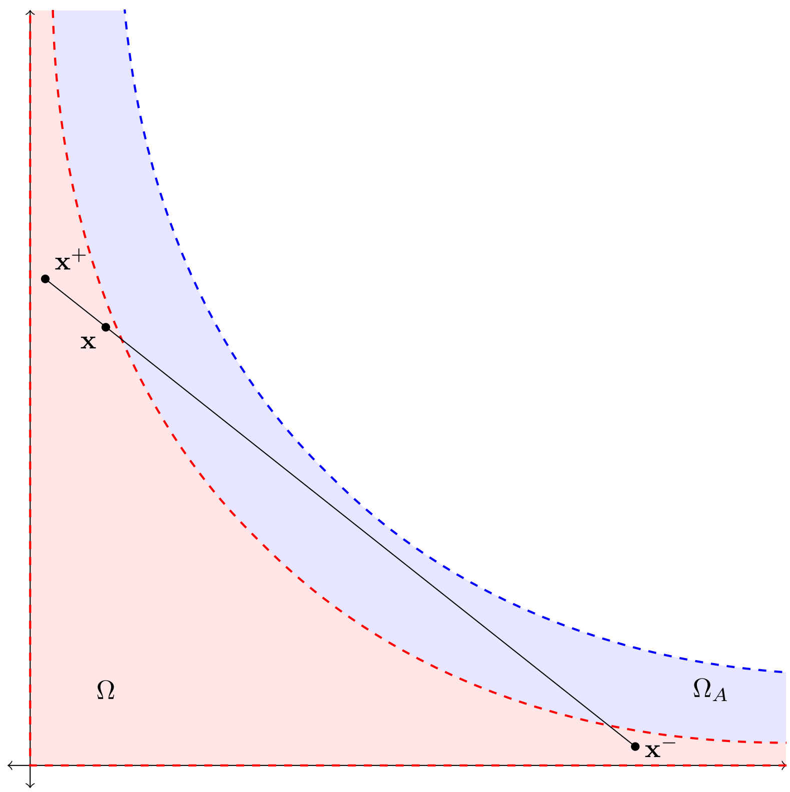


Figure A.1: The domain Ω is shown in red. The three points $\mathbf{x}$, $\mathbf{x}^+$ and $\mathbf{x}^-$ are collinear, but the line segment which connects them goes outside Ω. The enlarged domain Ω_A (the union of the red and blue regions) completely contains the entire line segment.

We now claim the following:

Claim A.4.1. *Fix α_+ and α_-. Suppose that we found a maximizer for A. Then the following must be true:*

- $\mathbf{x}^+$ *lies on the y-axis.*
- $\mathbf{x}$ *and* $\mathbf{x}^-$ *both lie on the curve $xy = 1$*

Claim A.4.2. *A is maximized when $\alpha_+ = 1 - \epsilon$ and $\alpha_- = \epsilon$.*

To see why Claim A.4.1 is true, notice that if we have a supposed maximizer which does not satisfy the claimed constraints then we could *increase* A by slightly perturbing two of the points. The exact way in which this perturbation occurs requires breaking into a large number of cases, but the result is always that we can increase A by moving a point so that it meets the constraint.

The argument for Claim A.4.2 is similar. Notice that we can always increase A by forcing the constants $\alpha^{\pm}$ to be nearer to $1-\epsilon$ and ϵ. This is because it allows us to slide $\mathbf{x}^+$ and $\mathbf{x}$ further away from the x-axis and $\mathbf{x}^-$ further away from the y-axis, which necessarily increases A.

We now find a bound for A in terms of ϵ.

Lemma A.4.3. *The constant $A < 4/\epsilon$.*

Proof. Recall that we have several equivalent expressions for A. We will look at

$$A = \frac{-4m}{(y^\circ - mx^\circ)^2}, \qquad m = \frac{y^\circ - y^-}{x^\circ - x^-} \tag{A.15}$$

By Claims A.4.1 and A.4.2, we can write that

$$\mathbf{x}^+ = (0, y^+), \ \ \mathbf{x} = (x^\circ, 1/x^\circ), \ \text{ and } \ \mathbf{x}^- = (x^-, 1/x^-). \tag{A.16}$$

Furthermore,

$$x^\circ = (1-\epsilon)x^+ + \epsilon x^- = (1-\epsilon)\cdot 0 + \epsilon x^- = \epsilon x^-. \tag{A.17}$$

Thus,

$$m = \frac{\frac{1}{x^\circ} - \frac{1}{x^-}}{x^\circ - x^-} = \frac{-1}{x^\circ x^-}.$$

Plugging into the expression for A, we have that

$$\begin{aligned} A &= \frac{4}{x^\circ x^-(y^\circ + (x^-)^{-1})^2} = \frac{4}{x^\circ x^-}\left(\frac{1}{x^\circ} + \frac{1}{x^-}\right)^{-2} \\ &= \frac{4}{x^\circ x^-}\left(\frac{x^\circ x^-}{x^\circ + x^-}\right)^2 = \frac{4x^\circ x^-}{(x^\circ + x^-)^2}. \end{aligned}$$

Using that $x^\circ = \epsilon x^-$, we now see that

$$A = \frac{4\epsilon(x^-)^2}{((\epsilon+1)x^-)^2} = \frac{4\epsilon}{(\epsilon+1)^2} < \frac{4}{\epsilon}.$$

$\square$

References

[1] R. Alvarado and M. Mitrea, *Hardy spaces of Ahlfors-regular quasi metric spaces: A sharp theory*, Springer, 2015.

[2] H. Amir, *Construction of Haar-type bases on quasi-metric spaces with finite Assouad dimension*, Anal. Acad. Nac. Cs. Ex. F. y Nat. Beunos Aires **54** (2002).

[3] H. Amir, A. Bernadis, and L. Nowak, *Equivalence of Haar bases associated with different dyadic systems*, J. Geom. Anal. **21(2)** (2011), 288–304.

[4] T. Anderson, D. Cruz-Uribe, and K. Moen, *Logarithmic bump conditions for Calderón-Zygmund operators on space of homogeneous type*, Publicacion Mathemàtique **59(1)** (2015), 17–43.

[5] T. Anderson, T. Hytönen, and O. Tapiola, *Weak A_∞ weights and weak reverse Hölder property in space of homogeneous type*, arXiv preprint arXiv:1410.3608 (2014).

[6] T. Anderson and D. Weirich, *A dyadic Gehring inequality in spaces of homogeneous type and applications*, arXiv preprint arXiv:1606.03461 (2016).

[7] T. C. Anderson and A. Vagharshakyan, *A simple proof of the sharp weighted estimate for calderón–zygmund operators on homogeneous spaces*, The Journal of Geometric Analysis **24** (2014), no. 3, 1276–1297.

[8] O. Beznosova, *Bellman functions, paraproducts, Haar multipliers and weighted inequalities*, Ph.D. thesis, University of New Mexico, 2008.

[9] S. Buckley, *Estimates for operator norms on weighted spaces and reverse jensen inequalities.*, Trans. of the Ameri. Math. Soc. **340(1)** (1993), 253–272.

[10] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601–628.

[11] D. Chung, *Commutators and dyadic paraproducts on weighted Lebesgue spaces*, Ph.D. thesis, University of New Mexico, 2010.

[12] ———, *Weighted inequalities for multivaiable dyadic parapriducts*, arXiv preprint arXiv:1001.1461v2 (2010).

[13] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Springer-Verlag Berlin Heidelberg, 1971.

[14] J. Z. Duoandikoetxea, *Fourier analysis*, American Mathematical Soc., 2001.

[15] T.W. Gamelin and R.E. Greene, *Introduction to topology*, Dover books on mathematics, Dover Publications, 1983.

[16] F. W. Gehring, *The L^p integrability of partial derivatives of a quasiconformal mapping*, Acta. Math. **130** (1973), 265–277.

[17] L. Grafakos, *Classical harmonic analysis*, Springer, 2008.

[18] ———, *Modern harmonic analysis*, Springer, 2008.

[19] A. Haar, *Zur theorie der orthogonalen funktionensysteme*, Mathematische Annalen **69** (1910), no. 3, 331–371.

[20] S. Hofmann, D. Mitrea, M. Mitrea, and A. Morris, *L^p-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets*, arXiv preprint arXiv:1301.4943 (2013).

[21] T. Hytönen, *Martingales and harmonic analysis*, Lecture notes available at http://www.ctr.maths.lu.se/media/MATP29/2016vt2016/maha-eng_1.pdf, 2012.

[22] ———, *Dyadic analysis and weights*, Lecture notes of a course at the University of Helsinki wiki.helsinki.fi/download/attachments/130069291/dyadic.pdf, 2014.

[23] T. Hytönen and P. Auscher, *Orthonormal basis of regular wavelets in spaces of homogeneous type*, Appl. Comput. Harmon. Anal. **34(2)** (2012), 266–296.

[24] ———, *Addendum to orthonormal basis of regular wavelets in spaces of homogeneous type*, Appl. Comput. Harmon. Anal. **39(3)** (2015), 568–569.

[25] T. Hytönen and A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. **126** (2012), 1–33.

[26] ———, *What is a cube?*, Annales Academiæ Scientiarum Fennicæ Mathematica **38** (2013), 405–412.

[27] T. Hytönen, C. Pérez, and E. Rela, *Sharp reverse hölder property for a_∞ weights on spaces of homogeneous type*, Journal of Functional Analysis **263** (2012), no. 12, 3883–3899.

[28] T. Hytönen and O. Tapiola, *Almost lipschitz-continuous wavelets in metric spaces via a new randomization of dyadic cubes*, Journal of Approximation Theory **185** (2014), 12–30.

[29] A. Kairema, *Dyadic systems in doubling metric spaces and applicatinos to positive integral operators*, Ph.D. thesis, University of Helsinki, 2012.

[30] A. Kairema, J. Li, M. C. Pereyra, and L. A. Ward, *Haar basis on quasi-metric measure spaces, and dyadic shift structure theorems for function spaces on product spaces of homogeneous type*, preprint 2015.

[31] N. H. Katz and M. C. Pereyra, *Haar multipliers, paraproducts, and weighted inequalities*, Analysis of Divergence (1999), 145–170.

[32] A.K. Lerner and NAzarov F., *Intuitive dyadic calculus: The basics*, arXiv preprint arXiv:1508.05639, to appear in Expositiones Mathematicae, 2015.

[33] O. E. Maasalo, *The Gehring lemma in metric space*, 2006.

[34] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Avd. in Math **33** (1979), 257–270.

[35] J. M. Martell, L. D. López-Sánchez, and J. Parcet, *Dyadic harmonic analysis beyond doubling measures*, Adv. Math **267** (2014), 43–93.

[36] J. C. Moraes, *Weighted estimates for dyadicoperators with complexity*, Ph.D. thesis, University of New Mexico, 2011.

[37] J. C. Moraes and M. C. Pereyra, *Weighted estimates for dyadic paraproducts and t-haar multipliers with complexity* (m, n), Pub. Mat. **57** (2013), 265–294.

[38] B. Mukenhoupt, *Weighted norm inequalities for the hardy maximal function*, Trans. of the Ameri. Math. Soc. **165** (1972), 207–226.

[39] F. Nazarov, S. Treil, and A. Volberg, *Bellman function in stochastic control and harmonic analysis.*

[40] ———, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces*, arXiv preprint math/9711210 (1997).

[41] ———, *The Tb-theorem on non-homogeneous spaces*, Acta Math. **190** (2003), 151–239.

[42] D. Panek, *On sharp extrapolation theorems*, Ph.D. thesis, University of New Mexico, 2008.

[43] C. M. Pereyra and Ward L., *Harmonic analysis: From fourier to wavelet,*, vol. 63, Student Mathematical Library Series, 2012.

[44] C. Pérez and E. Rela, *A new quantitative two weight theorem for the hardy-littlewood maximal operator*, Proceedings of the American Mathematical Society **143** (2015), no. 2, 641–655.

[45] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill Higher Education, 1976.

[46] E. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. of Math. **114** (1992), 813–874.

[47] V. Vasyunin, *The sharp constant in the reverse Hölder inequality for mukenhoupt weights*, St. Petersburg Math. J. **15 no. 1** (2004), 49–79.

[48] A. Volberg and Konyagin S., *On measures with the doubling condition*, Mathematics of the USSR-Izvestiya **30 no. 3** (1988), 629–638.

[49] A. Volberg and P. Zorin-Kranich, *Sparse domination on non-homogeneous spaces with an application to A_p weights*, 2017.