On Refined Neutrosophic Algebraic Structures

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Abstract.The objective of this paper is to develop refined neutrosophic algebraic structures. In particular, we study refined neutrosophic group and we present some of its elementary properties.

Keywords:neutrosophic logic, neutrosophic set,refined neutrosophic algebraic structures, refined neutrosophic group,refined neutrosophic numbers.

1 Introduction

In neutrosophic logic, each proposition is approximated to have the percentage of truth in a subset (T), the percentage of indeterminacy in a subset (I), and the percentage of falsity in a subset (F), where T, I, F are standard or non-standard subsets of the non-standard unit interval $]^-0, 1^+[$.

The concept of neutrosophic numbers of the form $a + bI$, where I is the indeterminacy with $I^n = I$, and, a and b are real or complex numbers, was introduced by Kandasamy and Smarandache in 2003. In the same year, Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures by combining the indeterminate element I with the elements of a given algebraicstructure $(X, *)$ to form a new algebraic structure $(X(I), *) =<X, I>$ generated by X and I and they called it a neutrosophic algebraic structure. Some of the neutrosophic algebraic structures developed and studied by Kandasamy and Smarandache include neutrosophic groupoids, neutrosophic semigroups, neutrosophic groups, neutrosophic loops, neutrosophic rings, neutrosophic fields, neutrosophic vector spaces, neutrosophic modules, neutrosophic bigroupoids, neutrosophic bisemigroups, neutrosophic bigroups, neutrosophic biloops, neutrosophic N-groups, neutrosophic N-semigroups, neutrosophic N-loops, and so on.

In [5], Smarandache introduced the refined neutrosophic logic and neutrosophic set where it was shown that it is possible to split the components $<T, I, F>$ into the form $<T_1, T_2, \dots, T_p; I_1, I_2, \dots, F_r; F_1, F_2, \dots, F_s>$. Also in [6], Smarandache extended the neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $a + b_1I_1 + b_2I_2 + \dots + b_nI_n$, where $a, b_1, b_2, \dots, b_n$ are real or complex numbers and considered the refined neutrosophic set based on these refined neutrosophic numbers.

2 Refined Neutrosophic Algebraic Structures

Consider the split of the indeterminacy I into two indeterminacies I_1 and I_2 defined as follows:

I_1= contradiction (true (T) and false (F)), (1)

I_2 = ignorance (true (T) or false) (F). (2)

It can be shown from (1) and (2) that:

$I_1^2 = I_1$; (3)

$I_2^2 = I_2$; (4)

$I_1I_2 = I_2I_1 = I_1$. (5)

Now, let X be a nonempty set and let I_1 and I_2 be two indeterminacies. Then the set

$X(I_1, I_2) =< X, I_1,I_2> = \{(x, yI_1, zI_2):x, y, z \in X\}$(6)

is called a refined neutrosophic set generated by X, I_1 and I_2, and (x, yI_1, zI_2) is called a refined neutrosophic element of $X(I_1, I_2)$. If + and . are ordinary addition and multiplication, I_k with k = 1, 2 have the following properties:

(1) $I_k+I_k+\cdots+I_k=nI_k$.

(2) $I_k+(-I_k)=0$.

(3) $I_k.I_k.\cdots.I_k= I_k^n=I_k$ for all positive integer $n>1$.

(4) $0.I_k=0$.

(5) I_k^{-1} is undefined and therefore does not exist.

If $* : X(I_1, I_2) \times X(I_1, I_2) \rightarrow X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2),*)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$. If $(X(I_1, I_2),*)$ and $(Y(I_1, I_2),*')$ are two refined neutrosophic algebraic structures, the mapping $\varphi: (X(I_1, I_2),*) \rightarrow (Y(I_1, I_2),*')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \varphi((a, bI_1, cI_2)) *' \varphi((d, eI_1, fI_2))$.

(2) $\varphi(I_k) = I_k$ $\forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Definition 2.1.

Let $(X(I_1, I_2), +, .)$ be any refined neutrosophic algebraic structure where + and . are ordinary addition and multiplication respectively. For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$(a, bI_1, cI_2) + (d, eI_1, fI_2)=(a + d, (b + e)I_1, (c + f)I_2)$,(7)

$(a, bI_1, cI_2).(d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1,$
$(af + cd + cf)I_2).$ (8)

Definition 2.2.

Let $(G, *)$ be any group. We call the couple $(G(I_1, I_2), *)$ a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non-commutative refined neutrosophic group.

$(G(I_1, I_2), *)$ is called a finite refined neutrosophic group if the elements in $G(I_1, I_2)$ are countable. Otherwise, $G(I_1, I_2)$ is called an infinite refined neutrosophic group. If the number of elements in $G(I_1, I_2)$ is n, we call n the order of $G(I_1, I_2)$ and we write $o(G(I_1, I_2)) = n$. For an infinite refined neutrosophic group $G(I_1, I_2)$, we write $o(G(I_1, I_2)) = \infty$.

Example 1.

$(\mathbb{Z}(I_1, I_2), +)$, $(\mathbb{R}(I_1, I_2), +)$, $(\mathbb{C}(I_1, I_2), +)$, $(\mathbb{R}(I_1, I_2), .)$ and $(\mathbb{C}(I_1, I_2), +.)$ are commutative refined neutrosophic groups.

Example 2.

$\left(M^{\mathbb{R}}_{2\times 2}(I_1, I_2, .)\right)$, where
$$M^{\mathbb{R}}_{2\times 2}(I_1, I_2, .) = \left\{\begin{bmatrix} w & x \\ y & z \end{bmatrix} : w, x, y, z \in \mathbb{R}(I_1, I_2)\right\}$$
is a non-commutative refined neutrosophic group.

Example 3.

Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$.

Then $(\mathbb{Z}2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Theorem 2.3.

(1) Every refined neutrosophic group is a semi group but not agroup.
(2) Every refined neutrosophic group contains a group.

Corollary 2.4.

Every refined neutrosophic group $(G(I_1, I_2), +)$ is a group.

Theorem 2.5.

Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *)$ be two refined neutrosophic groups. Then $G(I_1, I_2) \times H(I_1, I_2) = \{(x, y) : x \in G(I_1, I_2), y \in H(I_1, I_2)\}$ is a refined neutrosophic group.

Definition 2.6.

Let $(G(I_1, I_2), *)$ be a refined neutrosophic group and let $A(I_1, I_2)$ be a nonempty subset of $G(I_1, I_2)$. $A(I_1, I_2)$ is called a refined neutrosophic sub-group of $G(I_1, I_2)$ if $(A(I_1, I_2), *)$ is a refined neutrosophic group. It is essential that $A(I_1, I_2)$ contains a proper subset which is a group.

Otherwise, $A(I_1, I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1, I_2)$.

Example 4.

Let $G(I_1, I_2) = (\mathbb{Z}(I_1, I_2), +)$ and let $A(I_1, I_2) = (3\,\mathbb{Z}(I_1, I_2), +)$. Then $A(I_1, I_2)$ is a refined neutrosophic subgroup of $G(I_1, I_2)$.

Example 5.

Let $G(I_1, I_2) = (\mathbb{Z}_6(I_1, I_2), +)$ and let
$A(I_1, I_2) = \{(0, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2),$
$(0, 2I_1, 0), (0, 0, 2I_2), (0, 2I_1, 2I_2),$
$(0, 3I_1, 0), (0, 0, 3I_2), (0, 3I_1, 3I_2),$
$(0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 4I_2),$
$(0, 5I_1, 0), (0, 0, 5I_2), (0, 5I_1, 5I_2)\}$.

Then $A(I_1, I_2)$ is a pseudo refined neutrosophic subgroup of $G(I_1, I_2)$.

Theorem 2.7.

Let $\{A_k(I_1, I_2)\}_1^n$ be a family of refined neutrosophic subgroups(pseudo refined neutrosophic subgroups) of a refined neutrosophic group $G(I_1, I_2)$. Then $\bigcap_1^n A_k(I_1, I_2)\}$ is a refined neutrosophic subgroup (pseudo refined neutrosophic subgroup) of $G(I_1, I_2)$.

Definition 2.9.

Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be any two refined neutrosophic sub-groups (pseudo refined neutrosophic subgroups) of a refined neutrosophic group $(G(I_1, I_2), +)$. We define the sum $A(I_1, I_2) + B(I_1, I_2)$ by the set

$A(I_1, I_2) + B(I_1, I_2) = \{a + b : a \in A(I_1, I_2),$
$b \in B(I_1, I_2)\}$ (9)

which is a refined neutrosophic subgroup (pseudo refined neutrosophic subgroup) of $G(I_1, I_2)$.

Theorem 2.9.

Let $A(I_1, I_2)$ be any refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), +)$ and let $B(I_1, I_2)$ be any pseudo refined neutrosophic subgroup $(G(I_1, I_2), +)$. Then:

(1) $A(I_1, I_2) + A(I_1, I_2) = A(I_1, I_2)$.
(2) $B(I_1, I_2) + B(I_1, I_2) = B(I_1, I_2)$.
(3) $A(I_1, I_2) + B(I_1, I_2)$ is a refined neutrosophic subgroup of $G(I_1, I_2)$.

Definition 2.10.

Let $(G(I_1, I_2), *)$ and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. The mapping $\varphi : (G(I_1, I_2), *) \rightarrow (H(I_1, I_2), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi(x * y) = \varphi(x) *' \varphi(y)$.

(2) $\varphi(I_k) = I_k \ \forall x, y \in G(I_1, I_2)$ and $k = 1, 2$.

The image of φ is defined by the set

$Im\varphi = \{y \in H(I_1, I_2) : y = \varphi(x),$ (10)

for some $x \in G(I_1, I_2)\}$.

If $G(I_1, I_2)$ and $H(I_1, I_2)$ are additive refined neutrosophic groups, then the kernel of the neutrosophic homomorphism $\varphi : (G(I_1, I_2), +) \rightarrow (H(I_1, I_2), +)$ is defined by the set

$Ker\varphi = \{x \in G(I_1, I_2) : \varphi(x) = (0, 0, 0)\}$.

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of φ have the same definitions as those of the classical cases.

Example 6.

Let $(G(I_1, I_2), *)$ and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\varphi : G(I_1, I_2) \times H(I_1, I_2) \rightarrow G(I_1, I_2)$ be a mapping defined by $\varphi(x, y) = x$ and let $\psi: G(I_1, I_2) \times H(I_1, I_2) \rightarrow H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then φ and ψ are refined neutrosophic group homomorphisms.

Theorem 2.11.

Let $\varphi : (G(I_1, I_2), *) \rightarrow (H(I_1, I_2), *')$ be a refined neutro-sophic group homomorphism. Then $Im\varphi$ is a neutrosophic subgroup of $H(I_1, I_2)$.

Theorem 2.12.

Let $\varphi : (G(I_1, I_2), +) \rightarrow (H(I_1, I_2), +)$ be a refined neutro-sophic group homomorphism. Then $Ker\varphi$ is a subgroup of G and not a neutrosophic subgroup of $G(I_1, I_2)$.

Example 7.

Let $\varphi : \mathbb{Z}_2(I_1, I_2) \times \mathbb{Z}_2(I_1, I_2) \rightarrow \mathbb{Z}_2(I_1, I_2)$ be a neutrosophic group homomorphism defined by $\varphi(x, y) = x$ for all $x, y \in \mathbb{Z}_2(I_1, I_2)$. Then

$Im\varphi = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2),$
$(0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$.

$Ker\varphi = \{((0, 0, 0), (0, 0, 0)), ((0, 0, 0), (1, 0, 0)), ((0, 0, 0), (0, I_1, 0)),((0, 0, 0), (0, I_1, I_2)), ((0, 0, 0), (0, 0, I_2)), ((0, 0, 0), (1, I_1, 0)),((0, 0, 0), (1, 0, I_2)), ((0, 0, 0), (1, I_1, I_2))\}$.

Conclusion

By splitting the usual indeterminacy I into two indeterminacies I_1 and I_2, we have developed a new neutrosophic set $X(I_1, I_2)$ called a refined neutrosophic set and we have generated a new neutrosophic algebraic structure $(X(I_1, I_2), *)$ from X, I_1 and I_2 which we called a refined neutrosophic algebraic structure. In particular, we have studied refined neutrosophic group and we have presented some of its elementary properties.

Using the same approach as in this paper, other refined neutrosophic algebraic structures involving rings, fields, vector spaces, modules, group rings, loops, hypergroups, hyperrings, algebras, and so on could be developed. We hope to look into these in our future papers.

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References

[1] A.A.A. Agboola, A.D. Akinola, and O.Y. Oyebola, Neutrosophic Rings I, Int. J. of Math. Comb. 4 (2011), 1-14.
[2] A.A.A. Agboola, E.O. Adeleke, and S.A. Akinleye, Neutrosophic Rings II, Int. J. of Math. Comb. 2 (2012), 1-8.
[3] A.A.A. Agboola, Akwu A.O., and Y.T. Oyebo, Neutrosophic Groups and Neutrosopic Subgroups, Int. J. of Math. Comb. 3 (2012), 1-9.
[4] F. Smarandache (2003), A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (3rd edition), American Research Press, Rehoboth, http://fs.gallup.unm.edu/eBook-Neutrosophic4.pdf.
[5] F. Smarandache, n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, USA, 4 (2013), 143-146.
[6] F. Smarandache, (T,I,F)-Neutrosophic Structures, Neutrosophic Sets and Systems, 8 (2015), 3-10.
[7] W.B. Vasantha Kandasamy and F. Smarandache (2006), Neutrosophic Rings, Hexis, Phoenix, Arizona, USA, http://fs.gallup.unm.edu/NeutrosophicRings.pdf

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