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Abstract: In this paper we study the problem of controlling uncertain nonlinear systems. The uncertainty is considered to be dynamic and additive, and the nominal system is assumed to be feedback-linearizable. It is shown that the state of the uncertain nonlinear system may be stabilized using a controller designed for the nominal system. It is also shown that the conditions for stability depend on the size of the uncertain terms.

I. Introduction

It was shown in [1] that certain nonlinear systems are feedback-equivalent to a controllable linear system. This however, is dependent on the exact description of the nonlinear dynamics. Since such a description is rarely available, it is of interest to try to linearize uncertain nonlinear systems. In [2,3,5,6,] the authors solved this problem by assuming the system to be linear in the uncertain but constant parameters or by assuming two time scales dynamics and adaptively adjusting the linearizing transformations in order to follow a desired linear model.

In this research, we choose instead to study the robustness of the closed-loop system when a linear, time-invariant controller is designed for the linear system and the nonlinear controller found using the usual feedback-linearization approach. It will be shown that the behavior of the closed-loop system is directly related to the size of the uncertainties. In section II the problem is stated and solved. An example is provided in section III, and conclusions are given in section IV.

II. Problem Statement

Given the single-input nonlinear system described by

$$dx/dt = [f(x)+\delta f(x)] + [g(x)+\delta g(x)]u(t) \qquad (2.1)$$

where

$$f(x) + g(x)u(t) \qquad (2.2)$$

is feedback-linearizable [1]. It can be shown that there exists nonlinear transformations

$$y_i = T_i(x), \quad i=1,2,..,n$$

$$v = T_{n+1}(x,u) \qquad (2.3)$$

$$\text{or} \quad y = [y_1 \ \ y_2 \ \dots \ y_n]^T = T(x),$$

such that

$$dy/dt = Ay + bv \qquad (2.4)$$

with (A,b) in the Brunowsky canonical form i.e.

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In addition, u(t) may be obtained from

$$u(t) = \frac{1}{\langle dT_n, g\rangle}(v(t) - \langle dT_n, f\rangle) \qquad (2.5)$$

where $\langle w,v\rangle = \Sigma w_i v_i$ and $dT_n = [\delta T_n/\delta x_1 \ \dots \ \delta T_n/\delta x_n]^T$.

Note: Implicit in equation (2.5) is the condition that $\langle dT_n, g\rangle \neq 0$.

The usual approach to obtain a control u(t) for the nonlinear system (2.2) is to translate the control objectives on x into specifications on y, design v for the controllable linear system (2.4), then apply u in (2.5) to the nonlinear system (2.2). Since our true system is (2.1) which contains some uncertain terms, the following is obtained when u(t) is applied to (2.1)

$$dx/dt = [f(x)+\delta f(x)] + [g(x)+\delta g(x)]\frac{1}{\langle dT_n, g\rangle}(v - \langle dT_n, f\rangle) \qquad (2.6)$$

If one notes that

$$\frac{dy}{dt} = \frac{\delta T}{\delta x}\frac{dx}{dt} = G\frac{dx}{dt}$$

where $G = \delta T/\delta x$ is the Jacobian of the transformation $y=T(x)$, one gets

$$\frac{dy}{dt} = G\left[f + g\frac{1}{\langle dT_n, g\rangle}(v - \langle dT_n, f\rangle)\right] + G\left[\delta f + \delta g\frac{1}{\langle dT_n, g\rangle}(v - \langle dT_n, f\rangle)\right]$$

or

$$\frac{dy}{dt} = Ay + bv + G\left[\delta f + \delta g\frac{1}{\langle dT_n, g\rangle}(v - \langle dT_n, f\rangle)\right] \qquad (2.7)$$

The linear system therefore contains a nonlinear feedback term that is due to the uncertain terms δf and δg. This has been observed in the case of robotic manipulators [9], but in contrast to that work, the nonlinear feedback terms are not necessarily in the range of b. In the following, we give the bound the solution of (2.7) in terms of δf and δg. Let us first assume that for all y such that $\|y\| \le \epsilon$ where $\epsilon > 0$

$$\|G\| \le c_0\|y\| + d_0; \text{ for some } c_0, d_0 \ge 0$$

$$a_0\|y\| \le |\langle dT_n, g\rangle| \le a_1\|y\| + b_1; \text{ for some } a_0, a_1, b_1 \ge 0$$

$$|\langle dT_n, f\rangle| \le b_0\|y\|; \text{ for some } b_0 \ge 0$$

$$\|\delta g\| \le \alpha, \text{ and } \|\delta f\| \le \beta, \text{ for some } \alpha \ge 0, \text{ and } \beta \ge 0. \qquad (2.8)$$

We contend that the above assumptions are reasonable for a sufficiently small ϵ. The following theorem gives bounds on α and β such that the state of system (2.7) is stable under linear control.

Theorem 2.1:

Let v in (2.7) be a state-feedback given by

$$v = Ky = K_1y_1 + K_2y_2 + \dots + K_ny_n \qquad (2.9)$$

such that $A_c = A+bK$ is asymptotically stable. Let $-a$ = maximum eigenvalue of A_c (i.e. the closest to the jw axis),

$$\|K\| \le k$$

$$\|\exp(A_ct)\| \le me^{-at}$$

$$c = [\alpha(b_0+k) + \beta a_0]c_0/a_0$$

$$d = [\alpha(b_0+k) + \beta a_0]d_0/a_0$$

Then, the state y(t) of (2.7) is bounded by

$$\|y(t)\| \le m\|y(0)\|e^{-(a-mc)t} + (md/mc-a)[e^{-(a-mc)t} - 1],$$

if

$$a > mc \text{ i.e. if}$$

$$m[\alpha(b_0+k) + \beta a_0]c_0/a_0 < a. \qquad (2.10)$$

Proof: Using the Total Stability Theorem and detailed in [10]

Note the following

a) The theorem generalizes the study of the stability of a linear system disturbed by a small nonlinearity to that of a nonlinear feedback-linearizable system disturbed by a small nonlinearity.

b) If $\alpha=0$ i.e. $\delta g=0$, condition (2.10) reduces to

$$\beta < (a)/(mc_0) \qquad (2.11)$$

In this case one can place the poles of A_c further to the left (to make "a" large) as to allow a larger β or more uncertainty in δf.

c) If $\beta=0$, i.e. $\delta f=0$, condition (2.10) becomes

$$\alpha < [aa_0]/[(mb_0+k)c_0] \qquad (2.12)$$

In this case one can not increase "a" independently, since "k" is related to the location of eigenvalues of A_c and one has to increase "a/k" which is an interesting problem in its own.

d) In the special case where

$$\delta f\langle dT_n, g\rangle - \delta g\langle dT_n, f\rangle = 0 \qquad (2.13)$$

one has

$$\alpha < (aa_0)/(mc_0k) \qquad (2.14)$$

In order to check condition (2.13), one needs to know δf and δg. In fact, the usefulness of this result is in its reverse interpretation when one knows either δf or δg and the objective is to determine the other uncertainty such that the state y is stable. This approach will be useful in the case where the nonlinear system (2.1) is known but is not feedback-linearizable. One can then choose δf and δg such that the resulting system is feedback-linearizable.

e) If $c_0 = 0$, the theorem is trivially satisfied.

Since one is interested in the behavior of x(t) and not of the fictitious y(t), one needs to study the stability of the nonlinear system (2.1). The following theorem relates the behavior of x to the behavior of y.

Theorem 2.2: [8]

The control law (2.9) which stabilizes the system (2.7) under condition (2.10) results in a stable state trajectory x of the nonlinear system (2.1) where u is given by (2.5) if:

1) Given $\epsilon > 0$, there exists an $\epsilon_1 > 0$ such that $\|x\| \ge \epsilon$ leads to $\|y\| \ge \epsilon_1$, and

2) Given $\epsilon_1>0$, there exists an $\epsilon>0$ such that $\|y\|\geq\epsilon_1$ leads to $\|x\|\geq\epsilon$.

Proof: See [8]

In the general case, where conditions (2.8) are not verified, one can still deduce the boundedness of y(t) and x(t). In order to guarantee the existence and the uniqueness of a solution y(t) to the system (2.7), the nonlinear term is assumed to satisfy a Lipshitz condition [4, p.31] i.e.

$$\left\| \left[G\left(\delta f+\delta g\frac{1}{<dT_n,g>}(v-<dT_n,f>)\right)\right]_1 - \left[G\left(\delta f+\delta g\frac{1}{<dT_n,g>}(v-<dT_n,f>)\right)\right]_2 \right\| \leq \Gamma\|y_1-y_2\| \qquad (2.15)$$

for some Γ, where $F_i = [G(\ldots)]_i$ denotes the nonlinear term evaluated at y_i. In fact, one way to guarantee that the nonlinear vector F(y) satisfies a Lipshitz condition is to verify that all partial derivatives $\delta F_i/\delta y_j$ of every component of F with respect to every component of y exists and is continuous [4, p34]. Then it can be shown that

$$\|\delta F_i/\delta y_j\| \leq \beta \qquad (2.16)$$

and that $\Gamma = n\beta$ is a valid Lipshitz constant. The problem therefore reduces to guaranteeing that all the partial derivatives exist and are continuous. It can be easily shown that this is indeed the case as long as $<dT_n,g> \neq 0$, a condition needed for the feedback-linearization. The stability of y will then depend on the condition that $\Gamma<a$ [7, pp.16-18].

III. EXAMPLE

Given the following nonlinear system

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \left[\begin{bmatrix} 0 \\ g_2(x) \end{bmatrix} + \delta g(x)\right] u; \quad g_2(x) = x_1^2 + x_2^2 \qquad (1)$$

we are concerned with the system in the range

$$U = \{x;\ -5 < x_1, x_2 < 5\} \qquad (2)$$

with

$$T_1 = y_1 = x_1$$

$$T_2 = y_2 = x_2 \qquad (3)$$

and $G = \delta T/\delta x$ is given by

$$G = I_{2x2} \qquad (4)$$

also note

$$<dT_2,g> = g_2(x)$$

$$<dT_2,f> = 0 \qquad (5)$$

using the $\|.\|_2$ for the vectors and the corresponding induced norm for matrices, theorem 2.1 leads to

$$\|y\|_2 = (y_1^2 + y_2^2)^{1/2} \qquad (6)$$

and

$$\|G\|_2 = 1 \qquad (7)$$

Over the range (2)

$$c_0 = 0,\ a_0 = 1,\ b_0 = 0,\ d_0 = 1,\ a_1 = 7.1,\ b_1 = 0, \text{ and } \delta = 0. \qquad (8)$$

According to theorem 2.1, the nonlinear systems will be stabilized with a controller (2.5) with v(t) given by (2.9). Assume v(t) is chosen to place the poles of the linearized system at -2. This leads to the following values

$k = 5.7$, $m = 1.6$, and $a = 2$.

c0 = 0 and (2.10) imply that the linearized system is always stable and that the steady state bound of $\|y(t)\|2$ is given by

$\alpha(km)/(a_0 a) = \alpha 4.6$

IV. CONCLUSIONS

The range of applications of the feedback-linearization has been greatly expanded. In fact, even though the given system is not feedback-linearizable, a perturbed neighbor may be, and the behavior of the original system may be controlled by designing controllers based on the perturbed neighbor. This leads us to believe that feedback-linearizability is a generic property of a large class of nonlinear systems. We have presented bounds on the distance between the given system and its linearizable neighbors which we plan to relax in future research.

REFERENCES

[1] L.R. Hunt, R. Su, and G. Meyer, "Global Transformations of Nonlinear Systems," IEEE Trans. Automat. Control, Vol.AC-28, No. 1, pp. 24-31, 1983.

[2] K. Nam, and A. Arapostathis, "A Model Reference Adaptive Control Scheme for Pure-Feedback Nonlinear Systems," IEEE Trans. Autom. Control, Vol. AC-33, No. 9, pp.803-811, 1988.

[3] S.S. Sastry, and P.V. Kokotovic, " Feedback-Linearization in the Presence of Uncertainties," International Journal of Adaptive Control and Signal Processing, Vol. 2, No. 4, pp. 327-346, Dec. 1988.

[4] S. Lefschetz, Differential Equations: Geometric Approach, Dover Publications Inc., New York, 1977.

[5] D.G. Taylor et. al, "Adaptive Regulation of nonlinear Systems with unmodeled Dynamics," American Control Conf., Atlanta, GA, June 1988.

[6] S. Sastry, and M. Bodson, Adaptive Control: Stability, Convergence, and Robustness, Prentice-Hall Advanced Reference Series, Engineering, Prentice-Hall Inc., Englewood Cliffs, NJ, 1989.

[7] B.D.O. Anderson et. al, Stability of Adaptive Systems: Passivity and Averaging Techniques, The MIT Press, Cambridge, 1986.

[8] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer-Verlag, Berlin, 1959.

[9] M.W. Spong, "Robust Stabilization for a Class of Nonlinear Systems," Theory and Applications of Nonlinear Control Systems, C.I. Byrnes and A. Lindquist, Eds. Elsevier Science Publishers B.V. (North-Holland), pp2.145-165, 1986.

[10] A. Eras, "Robust Control of Unceratin Nonlinear Systems," M.Sc. Thesis, EECE Department, University of New Mexico, Summer 1990.