10-5-2003

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Recommended Citation
Networked Control Systems: A Sampled-Data Approach

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Abstract—In this paper we present a novel modelling method
for networked control systems, motivated from a sampled-
data approach. We study sufficient conditions that guarantee
exponential stability for the closed-loop system and illustrate our
results via a numerical example.

Index Terms—Networked Control Systems, sampled-data sys-
tems, lifting, exponential stability.

I. INTRODUCTION

Over the past decade, major advancements in the area
of communication and computer networks [9] have made it
possible for control engineers to include them in feedback
systems in order to achieve real-time requirements. This gave
rise to a new paradigm in control systems where instantaneous
flow of the control signals is no longer sufficient, and the
feedback loop is closed through a real-time network. Such
control systems that utilize networks to achieve closed loop
performance are called Networked Control Systems (NCS).
Several examples of NCSs are available in automobile indus-
try, teleoperation of robots, and automated manufacturing
systems. Including the networks into the design of such
systems has made it possible to increase mobility, reduce the
cost of dedicated cabling, and render easier and cheaper
maintenance.

This paper starts by reviewing some basic trends in the
study of stability of networked control systems in Section
II. Then we present our new approach for modelling such
systems in Section III. In Section IV, we address the issue
of stability of such models, using Lyapunov techniques for
discrete-time systems. Finally, we illustrate our results via a
numerical example in Section V.

II. REVIEW OF PREVIOUS WORK

In the past decade, several methods of modelling networked
control systems have been proposed, and the stability of such
models was the main concern of their analysis. In this section
we provide an overview of basic approaches and results.

A. Structural Approach

The authors of [7] present an extended structural analysis
of networked control systems, using an eigenvalue approach.
In their model, the network resides between the sensors
that are attached to the plant, and the actuators. The network is
modelled as a fixed-rate sampling of the continuous plant.
They also present a model plant that provides state estimate,
and the error between the actual plant and the model plant
is used to augment the state-vector. Then, the analysis is
applied to the augmented system in order to obtain necessary
conditions for guaranteeing stability of the closed-loop system.
They analyze the performance of the system when full state
and partial state are available for feedback.

B. Perturbation Approach

In [10], a try-once-discard (TOD) protocol is introduced,
where the next node to transmit data on a multi-node net-
work is decided dynamically based on the highest weighted
error from the last transmission. The goal is to find a
maximum transmission interval that guarantees satisfactory
stability performance. The network resides between the plant
and controller and introduces the error between successive
transmissions. The resulting state-space system is comprised
of the plant state-vector, and the error state-vector. The error
is considered as a perturbation of the original plant, and
methods presented in [5] are utilized to derive conditions for
the stability of the closed-loop system.

C. Delay Approach

Nilsson [8] includes the following cases for modeling the
effects of introducing the network into the control-loop, render-
ing an NCS:
• Constant delay
• Random independent delays
• Random delays governed by an underlying Markov chain
Then for each model he solves an LQG optimal control
problem, to generate a controller that guarantees stability.

D. Hybrid Systems Approach

Zhang et. al [11], [12] utilize results previously derived
for the stability of hybrid systems, to find bounds on the
the network as a constant delay introduced into the full state
feedback as follows:

\[ \dot{x}(t) = A x(t) - BK \tilde{x}(t), \quad t \in [kh + \tau, (k+1)h + \tau) \]

\[ \tilde{x}(t^+) = x(t - \tau), \quad t \in [kh + \tau, k = 0, 1, 2, \ldots] \]  

where \( h \) is the sampling period. Then the trajectory of the
delayed state vector \( x(t - \tau) \) is solved for, in terms of \( x(t) \)

\[ z(t) \]
and \( z(t) \). The bound on the delay \( \tau \) results from imposing Schur stability conditions on the following matrix.

\[
H = \begin{pmatrix}
    e^{Ah} & -E(h)BK \\
    e^{A(h-\tau)} & -e^{-At}(E(h) - E(\tau))BK
\end{pmatrix}
\]

where for a given matrix \( M \),

\[
E(h)M = \int_0^h e^{A(h-n)}Md\eta
\]

An extensive study has recently appeared in [4] where the NCS has limited data rate available in order to maintain stability. The problem is tackled from different perspectives: Variable-rate sampling, various quantization schemes, distributed control, and switching control with sufficient dwell-time. The main objective is to reduce the amount of data to be transmitted via the network.

### III. NEW MODELLING OF NCS

As seen in the previous sections, there are several trends in modeling networked control systems. In this section we are going to introduce yet another modelling method and manipulate it to obtain a generalized LTI sampled-data system. The proposed model allows us to avoid the tedious analysis of the effect of the delay introduced by the network. This is achieved through incorporating the delay into the model of the system, and it is sufficient to study the stability of the overall system, without explicitly addressing the actual value and nature of the delay. Before we introduce the new model, we present few assumptions:

I. The controller and actuators are directly attached to the plant, i.e. no transport delay exists between the controller and plant actuators.

II. The sensors are part of the plant model.

III. The network effect is recognized only between the sensors and controller.

**Proposition 1:** We model the network as a variable-rate ideal sampler \( S_{\tau_k} \), between the plant \( G \) and the controller \( C \), and a corresponding zero-order hold \( H_{\tau_k} \), as shown in Figure 1.

Consider the following plant model,

\[
\begin{align*}
    \dot{z}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\
    z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \\
    y(t) &= C_2z(t)
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input vector, \( w(t) \in \mathbb{R}^l \) is the vector of exogenous inputs, \( z(t) \in \mathbb{R}^p \) is the vector of controlled outputs, and \( y(t) \in \mathbb{R}^q \) is the vector of measurable outputs. Finally,

\[
G = \begin{pmatrix}
    A & B_1 & B_2 \\
    C_1 & D_{11} & D_{12} \\
    C_2 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{pmatrix}
\]

We assume that \( D_{21} = D_{22} = 0 \), i.e. the transfer functions from the control input, \( u(t) \), and from the exogenous input, \( w(t) \), to the measured output, \( y(t) \), are strictly proper. The latter condition provides continuity in the measured output vector [1], i.e. avoiding impulses in the output.

The above framework results in a time-varying system, that has both continuous and discrete signals, hence a hybrid system. The study of such systems is in general complex, and a unified theory for such systems is not yet available [6]. For such reasons, we need to manipulate the model in order to obtain a generalized LTI sampled-data system. In order to do so, we employ the lifting technique [1], [2], and incorporate the ideal sampler and hold devices into the plant model in the following manner:

\[
\begin{align*}
    \tilde{G} &= \left( \begin{array}{c}
    L_{\tau_k} \\
    0
\end{array} \right) \left( \begin{array}{c}
    S_{\tau_k} \\
    0
\end{array} \right) G \left( \begin{array}{c}
    L_{\tau_k}^{-1} \\
    0
\end{array} \right) \left( \begin{array}{c}
    H_{\tau_k} \\
    0
\end{array} \right) \\
    &= \left( \begin{array}{c}
    L_{\tau_k} G_{11} L_{\tau_k}^{-1} \\
    S_{\tau_k} G_{12} L_{\tau_k}^{-1} \\
    S_{\tau_k} G_{21} L_{\tau_k}^{-1} \\
    S_{\tau_k} G_{22} L_{\tau_k}^{-1}
\end{array} \right)
\end{align*}
\]

where \( \tau_k = t_k - t_{k-1} \) is the variable sampling-rate, \( L_{\tau_k} \) and \( L_{\tau_k}^{-1} \) are the lifting and inverse lifting operators, respectively. The transformed system is shown in Figure 2.

Next we present the above transformations mathematically.

i. \( G_{11} \rightarrow \tilde{G}_{11} \)

The transfer function \( G_{11} \) relates \( w(t) \) to \( z(t) \), in continuous time. \( \tilde{G}_{11} \) on the other hand relates \( \tilde{w} \) to \( \tilde{z} \) both being the lifted signals, corresponding to \( w(t) \) and \( z(t) \). Consequently the linear operators of \( \tilde{G}_{11} \) are given as follows:

\[
\begin{align*}
    \tilde{A} &= e^{\tilde{A}_{\tau_k}} \\
    \tilde{B}_1 &= \int_0^{\tau_k} e^{\tilde{A}(\tau_k-n)}B_1w(\eta)d\eta \\
    \tilde{B}_1z(t) &= C_1e^{\tilde{A}t}z(t) \\
    \tilde{B}_1 = \int_0^t e^{\tilde{A}(t-\eta)}B_1w(\eta)d\eta
\end{align*}
\]
In a similar fashion, we transform $BIZ$ and $DIZ$ into $BIZ_1$ and $DIZ_1$, respectively. And $BIZ_1$ relates the discrete input $U_k$ and the lifted output $z_k$.

$BIZ_1 = LT_k e^{A T_k BZ} (7)$

$\gamma(t) = DIZ_0 + CI$ ...

Both transformations follow from (6) and (7).

After applying the above transformations to (4) we obtain an LTI sampled-data system (c), which is shown in Figure 2.

Then we refer back to the usual design (see [2]) to obtain the controller $C$. Assuming that the controller $C$ has been designed, we present stability analysis results of the overall system in the next section.

IV. STABILITY ANALYSIS

In this section we study the stability of the model presented in the previous section. We shall start by deriving the closed-loop system that involves $BIZ_2$ and the controller $C$. Note that we only need to stabilize $BIZ_2$ due to the following theorem.

Theorem 1: [1] The controller $C$ internally stabilizes the hybrid system in figure 2, if and only if it internally stabilizes the discrete-time system $BIZ_2$ in (5).

The plant model of $BIZ_2$ is described as follows,

\[ x_{k+1} = Ax_k + Bz u_k \]
\[ y_k = Cz x_k \]

and the controller $C$ is described by the following state-space realization

\[ v_{k+1} = Ac v_k + Bc y_k \]
\[ u_k = Cc v_k + Dc y_k \]

Combining (8) and (9) we get the following augmented state space realization

\[ s_{k+1} = \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} A & Bz Dc C_2 \\ Bc C_2 & A \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} = H_k s_k \]

Notice that the above system does not take into account the effects of disturbances. Consequently, we shall introduce the effects of disturbances, through $w(t)$ in (3), into (10) as follows

\[ s_{k+1} = H_k s_k + \begin{pmatrix} Bz 1 \bar{w} \\ 0 \end{pmatrix} = H_k s_k + \Gamma_k \]

Before we plunge into the stability analysis, we shall present a general formal definition of exponential stability for discrete-time systems.

Definition 1: The origin of the system $x_{k+1} = A_k x_k$ is exponentially stable if there exists an $\alpha > 0$, and for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$, such that

\[ ||x_k|| \leq \epsilon e^{-\alpha(t_k-t_0)||s_0||} \]

whenever $||s_0|| < \delta(\epsilon)$ and $t_0 > 0$. If $\delta(\epsilon) \to \infty$ then the system is exponentially stable in the large.

The following theorem utilizes results in [3], and specializes them to solve the problem at hand.

Theorem 2: The origin of the closed loop discrete-time system (10) is exponentially stable in the large provided,

i. $\sup_{k \in \mathbb{N}} T_k < \infty$
ii. $||H_k|| < \frac{1}{\sqrt{2}}$, $\forall k \in \mathbb{N}$

Proof. Given $||H_k|| < \alpha < 1, \forall k \in \mathbb{N}$, then there exist a symmetric matrix $P_k > 0$, such that $H_k^T P_k H_k - P_k = -I$. Then $||P_k|| = ||I|| + ||H_k^T P_k H_k|| \leq 1 + \alpha^2 ||P_k|| = 1 \leq \frac{1}{1 - \alpha^2}$, since $0 < \alpha < 1$.

Let $V(s_k) = s(k)^T P_{k-1} s(k)$,

\[ \Delta V = V(s_{k+1}) - V(s_k) \]
\[ = s_{k+1} P_k s_{k+1} - s_k P_{k-1} s_k \]
\[ = s_k^T (H_k^T P_k H_k - P_k) s_k + s_k^T (P_k - P_{k-1}) s_k \]
\[ = -s_k^T I s_k + s_k^T (P_k - P_{k-1}) s_k \]
\[ \leq -||s_k||^2 + \left( \frac{a^2}{1 - a^2} \right) ||s_k||^2 \]
\[ = \left( \frac{2a^2 - 1}{1 - a^2} \right) ||s_k||^2 \]

Since $||P_k - P_{k-1}||_{max} = \frac{1}{1 - a^2} - 1 = \frac{a^2}{1 - a^2}$. For the system to be stable, $\Delta V$ must be less than zero. Therefore, $\left( \frac{2a^2 - 1}{1 - a^2} \right) < 0 \Rightarrow a < \frac{1}{\sqrt{2}}$.

The above result guarantees that the system (10) is stable. Still required to prove that it is exponentially stable. Since $V(s_k) = s(k)^T P_{k-1} s(k)$ then

\[ ||s_k||^2 \leq V(s_k) \leq \frac{1}{1 - a^2} ||s_k||^2 \]

Using (13), $V(s_{k+1}) \leq V(s_k) + \left( \frac{2a^2 - 1}{1 - a^2} \right) ||s_k||^2 \leq (2a^2 - 2) V(s_k)$. But $||s_0||^2 \leq V(s_0) \leq \frac{1}{1 - a^2} ||s_0||^2$ then

\[ V(s_k) \leq (2a^2)^k \left( \frac{1}{1 - a^2} \right) ||s_0||^2 \]

Combining (14) and (15) we get,

\[ ||s_k|| \leq \sqrt{\frac{1}{1 - a^2} (2a^2)^k} ||s_0|| \]

Let $\alpha = \min \{1, -\ln(\sqrt{2}a^2)\}$ and $\epsilon = \sqrt{\frac{1}{1 - a^2}}$, the result follows.

In the above analysis we have ignored the effect of the disturbances on the system. So we are going to extend the result of Theorem 2 to compensate for bounded and vanishing, state-bounded disturbances and in what follows.
Theorem 3: (Bounded Disturbance) Given that the origin of the discrete-time system (10) is exponentially stable, and that \( \|x_0\| \leq \gamma < +\infty \) (bounded-input), then the system (11) has a bounded-state output.

Proof. The proof is simple through analyzing the time progression of the state-vector.

\[
x(k + 1) = \prod_{i=0}^{k} H_i x(0) + \sum_{j=0}^{k} \left( \prod_{i=j+1}^{k} H_i \right) \gamma_j.
\]

Taking the limit of \( k \) on both sides:

\[
\lim_{k \to \infty} x(k + 1) = \lim_{k \to \infty} \left( \prod_{i=0}^{k} H_i \right) x(0) + \lim_{k \to \infty} \sum_{j=0}^{k} \left( \prod_{i=j+1}^{k} H_i \right) \gamma_j
\]

\[
\Rightarrow ||x(\infty)|| \leq ||H|| \cdot \left( \frac{1}{1-a} \right) < \infty.
\]

Since the first limit tends to zero as \( k \to \infty \) and \( ||H|| < a < 1 \Rightarrow we take the maximum of \( H_k = a \) and form a geometric progression whose answer is \( \left( \frac{1}{1-a} \right) \).

Theorem 4: (Vanishing Disturbance) The origin of the closed loop discrete-time system (11) is exponentially stable in the large provided,

i. \( \sup_{k \in \mathbb{N}} H_k < \infty \)

ii. \( ||H_k|| < \frac{1}{2a}, \forall k \in \mathbb{N} \)

iii. \( ||\gamma_k|| < ||s_k||, \forall k \in \mathbb{N} \)

Proof. We follow a similar analysis as in Theorem 2. Let \( ||s_k|| < \gamma ||s_k|| \), where \( \gamma > 0 \).

\[
\Delta V = V(s_{k+1}) - V(s_k)
\]

\[
= (H_k s_k + \Gamma_k)^T P_k (H_k s_k + \Gamma_k) - s_k^T P_{k-1} s_k
\]

\[
= s_k^T (H_k^T P_k H_k - P_k) s_k + s_k^T P_k - P_{k-1} s_k
\]

\[
+ 2 s_k^T H_k^T P_k \Gamma_k + \Gamma_k^T P_k \Gamma_k
\]

\[
\leq \frac{2a^2 - 1}{1 - a^2} ||s_k||^2 + \frac{2a\gamma}{1 - a^2} + \frac{\gamma^2}{1 - a^2} ||s_k||^2
\]

And \( \Delta V \) in (17) is always negative provided that \( \gamma < 1 \). The rest follows as in Theorem 2.

V. Numerical Example

In this section we will consider a numerical example to illustrate the theoretical stability results derived in Section IV, specifically in Theorem 2.

Consider the following scalar continuous-time LTI plant model

\[
\dot{x}(t) = 0.5x(t) + 10u(t)
\]

\[
y(t) = x(t)
\]

whose discrete version is that described in (8). Consider also the following discrete-time LTI controller \( C \)

\[
v_{k+1} = 0.1v_k - 0.5y_k
\]

\[
u_k = -0.5v_k - y_k
\]

Consequently, the closed-loop system matrix \( H_k \) in (10), corresponding to (18) and (19), is given by

\[
H_k = \begin{pmatrix}
0.5r_k - 20(0.5r_k - 1) \\
-10(0.5r_k - 1)
\end{pmatrix}
\]

By Theorem 2, we need to keep the norm of \( H_k \) less than \( \frac{1}{\sqrt{2}} \). Since we fixed the values for the controller parameters, we can vary \( r_k \) to meet the required condition on \( H_k \). The range of \( r_k \) for which the induced Euclidean norm\(^1\) of \( H_k \) is less than \( \frac{1}{\sqrt{2}} \) is shown in Figure 3, where

\[
0.061 < r_k < 0.126.
\]

In order to fully understand the implications of varying the sampling time \( r_k \) on the stability of the system, we will first study the behavior of the closed loop system in (22) given a constant \( r_k \).

The response of the closed-loop system at the boundary of the range given in (21), i.e. \( r_k = 0.126 \), is shown in Figure 4 where the system retains its stability.

We further increase the value to \( r_k \) beyond 0.126 until we hit the first instability point. As seen in Figure 5, the response of the closed loop system diverges for \( r_k = 0.164 \). This conveys the conservativeness of the stability analysis, since the results are sufficient but not necessary.

Finally, we test the system response for a variable sampling time given by

\[
r_k = 0.126 + \epsilon \times U
\]

where \( U \) is a uniformly distributed random number between 0 and 1, and \( \epsilon \in \mathbb{R} \). This representation of \( r_k \) allows us to see how far can we sample randomly beyond the theoretical bound and still maintain stability. As seen in Figure 6, the system diverges for \( \epsilon = 0.076 \).

It is interesting to compare the two results presented in Figures 5 and 6. For the random case, the value of \( r_k \) depends on the outcome of the random number \( U \) given in (22), whose mean is 0.038 for the simulation in Figure 6. Hence, \( \text{average}(r_k) = 0.126 + \text{average}(\epsilon \times U) = 0.164 \) which is the

\[^1\text{The induced Euclidean norm of any matrix } M \text{ is given by } \lambda_{\text{max}}(M^T M)^{1/2}, \text{ where } \lambda_{\text{max}} \text{ denotes the maximum eigenvalue.}\]
same as the fixed $\tau_k$ in Figure 5. Consequently, the random $\tau_k$ behaves like the fixed one on average.

VI. CONCLUSION

In this paper we have presented a new method for modelling Networked Control Systems. The main idea is viewing NCS as a variable-rate, sampled-data system. Then, we utilized some results pertaining to the stability of such sampled-data systems and extended them to the problem at hand. The bounds derived for guaranteeing stability are conservative, and further work should aim at developing new bounds that eliminate that kind of conservativeness.

REFERENCES


