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SPR DESIGN USING FEEDBACK

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I. INTRODUCTION

The concepts of Positive-Real (PR) and Strictly-Positive-Real (SPR) functions and matrices have been very useful in network theory [1], adaptive and robust control [2]. When dealing with uncertain systems, a nominal SPR transfer function allows for large passive uncertainties without the loss of stability [3,4,5]. It is then important to be able to test a given transfer matrix for its positive-realness and, more importantly, to make a given transfer matrix positive-real. The standard definition of SPR matrices [4], termed strong SPR, is usually difficult to apply. Moreover, it was recently shown [5] that the strong SPR definition is overly restrictive for control theory applications. In this paper we will use the term SPR to denote weak SPR matrices as defined in [5,6] and reviewed in the next section.

II. WHICH SPR?

Consider a multi-input-multi-output linear time-invariant system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cz + Du (2.1)
\end{align*}
\]

where \(x\) is an \(n\) vector, \(u\) is an \(m\) vector, \(y\) is a \(p\) vector, \(A,B,C,\) and \(D\) are of the appropriate dimensions. The corresponding transfer function matrix is

\[
P(s) = C(sI - A)^{-1}B + D (2.2)
\]

We will first assume that the system has an equal number of inputs and outputs, i.e. \(p = m\). Then define the relative degree \(n^*\) as follows: \(n^* = 0\) if \(\det(D) \neq 0\), and \(n^* = m\) if \(\det(D) = 0\) but \(\det(CB) \neq 0\). A formalism for the poles and zeros of multivariable systems is given in [6] and may be used to justify the definition of \(n^*\). To simplify our notation we will denote the Hermitian part of a real, rational transfer matrix \(T(s)\) by \(\text{He}[T(s)] = \frac{1}{2}[T(s) + T^*(s)]\) where \(s^*\) is the complex conjugate of \(s\). A number of definitions have been given for SPR functions and matrices [4,6]. It appears that the most useful definition for control applications is the following [5]

**Definition 2.1:** An \(m \times m\) matrix \(T(s)\) of proper real rational functions which is not identically zero is (weak) SPR if

1. All elements of \(T(s)\) are analytic in the right half plane \(\text{Re}(s) \geq 0\), and
2. The matrix \(\text{He}[T(s)]\) is positive definite for \(\text{Re}(s) > 0\).

The more standard definition of SPR matrices advocated in [4] is more restrictive than definition 2.1. In fact, a long held view was that strong SPR was needed to prove the Meyer-Kalman-Yakubovitch (MKY) lemma, which is after all, the major application of SPR concepts in control systems. As shown in [5] however, the weak SPR definition is just as useful in this regard and will therefore be adopted in this paper. Note that, from minimum real-part arguments given in [1], condition 2) of definition 2.1 is equivalent to \(\text{He}[T(jw)] > 0\) for all \(w\).

III. SPR USING FEEDBACK

The question addressed in this paper is to find conditions on \(T(\cdot)\) so that a feedback controller will render the closed-loop system SPR. The result of theorem 3.1 first appeared in [7] for the case of static output feedback, i.e. \(u = -\gamma y + r\). The corresponding closed-loop system is then given by

\[
\begin{align*}
\dot{x} &= (A-\gamma BKC)x + Br \\
y &= Cz
\end{align*}
\]

or in the frequency-domain

\[
Y(s) = [I + \gamma KP(s)]^{-1}KP(s)R(s)
\]

We present a simple frequency domain proof, to show the existence of \(K\) and \(\gamma\) that will render the closed-loop system SPR.

**Theorem 3.1:** Let system \((2.1)\) be stabilizable and detectable and let its relative degree be \(n^* = m\). Then there exists a nonsingular \(K\) and a positive scalar \(\gamma\) such that the closed-loop system \((3.2)\) is SPR, if and only if \(P(\cdot)\) is minimum phase.

**Proof:** Sufficiency: Consider the closed-loop transfer function

\[
T(s) = [I + \gamma KP(s)]^{-1}KP(s)
\]

or

\[
T(s) = [P^{-1}(s)K^{-1} + \gamma I]^{-1}
\]

Since \(P(s)\) is minimum phase with a relative degree \(n^* = m\), its inverse \(P^{-1}(s)\) will be given by

\[
P^{-1}(s) = sL + P_1(s)
\]

where \(P_1(s)\) is proper and stable, and \(\det(L) \neq 0\). In fact, \(\det(CB) \neq 0\) and \(L = (CB)^{-1}\). On the other hand, since \(P(\cdot)\) is minimum phase, \(P_1(s)\) can not have any poles in \(\text{Re}(s) \geq 0\). It is now obvious that \(T(s)\) will be stable if and only if \(W(s) = [P^{-1}(s)K^{-1} + \gamma I]\) has no zeros in \(\text{Re}(s) \geq 0\). Let \(K\) be given by

\[
K = (CB)^{-1}
\]

then

\[
W(s) = sI + P_1(s)CB + \gamma I
\]
Since $P_1(j\omega)$ has no poles on the $j\omega$-axis, $He[W(j\omega)]$ may be made positive-definite by a large enough positive scalar $\gamma$. This then implies that $W(s)$ is weakly SPR. Since $T(s)$ is the inverse of $W(s)$, it is also weakly SPR [4].

**Necessity:** Suppose now that a nonsingular $K$ and a $\gamma$ were found to make the closed-loop system $T(s)$ SPR and that $D = 0$. Then

$$W(s) = [P^{-1}(s)K^{-1} + \gamma I]$$

is also SPR. Writing $P^{-1}(s)$ as in (3), with $L = (CB)^{-1}$ we get

$$W(s) = (CB)^{-1}K^{-1} + P_1(s)K^{-1} + \gamma I$$

Since $W(s)$ is SPR, $P_1(s)$ must be stable, hence $P(s)$ must be minimum-phase.

The choice of $K = (CB)^{-1}$ in the proof of the theorem is by no means unique. In fact, it is sufficient to choose $K = P(CB)^{-1}$ where $P$ is any symmetric positive-definite matrix. Next, note that the condition $det(CB)\neq0$ (or that $P(s)$ has a relative degree $n^* = m$), also reveals that the system (2.1) has an inverse obtained by cascading one differentiator and a dynamical system [3]. Note that the inverse system is given by (3) in theorem 3.1 or in state-space by

$$\dot{x} = [A-B(CB)^{-1}CA]x + B(CB)^{-1}y$$

$$u = -(CB)^{-1}CAx + (CB)^{-1}y$$

Now recall that the invertibility of the system (2.1) may still be inferred even though $det(CB)\neq0$. In fact, a sufficient condition for the inverse to exist is that the first nonzero matrix in the sequence, $D, CB, CAB, CAB^2, ..., CAB^{n-1}B$, be nonsingular [3]. It is then obvious that for a nonsymmetric $D$ matrix, the condition for $T(s)$ to be SPR is that $D$ be invertible and $P(s)$ be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a state output feedback if its high frequency gain is nonsingular. On the other hand, the following general result may be established.

**Theorem 3.2:** Suppose that (2.1) is both stabilizable and detectable, and det $(CA'B)\neq0$ where $CA'$ is the first nonzero matrix in the sequence $D, CB, CAB, CAB^2, ..., CAB^{n-1}B$. Then the closed-loop system from $r$ to $\frac{dy}{dt}$ given by

$$T(s) = KCA'(A-I - \gamma BKCA')^{-1}B$$

is SPR if and only if $P(s)$ is minimum phase.

**Proof:** Given system (2.1), let us define an output $z$ by

$$z_1 = \frac{dy}{dt} = y(t)$$

then

$$z_1 = CAx + CBu$$

If $det(CA'B)\neq0$, the inverse system of $P(s)$ is given by

$$P^{-1}(s) = (CA'B)^{-1}s^{-1} + P_2(s)$$

where $P_2(s)$ is stable. Repeating the arguments of theorem 3.1 and using the controller

$$u = -\gamma Ks + r$$

we obtain the desired result.

Next suppose that one has access to the full state vector, under what conditions can a feedback controller produce an SPR transfer function between the input and an appropriate output? It is simple to see that we can choose $C$ such that $det(CB)\neq0$ and $C(x-A)^{-1}B$ minimum-phase, then use output feedback from $y = Cx$ to make the closed-loop system SPR. This is equivalent to the MKY Lemma [2] where $A-BK$ is stable and $C = B^TP$, $P$ being the solution of a Lyapunov equation.

**IV. EXAMPLE**

Consider the following open-loop transfer matrix

$$P(s) = \frac{s+1}{s^2+s^2+2s^2-1}$$

The inverse of the above transfer matrix is given by

$$P^{-1}(s) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} s + \frac{1}{s+1} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

The closed-loop transfer function $T(s)$ given in (3.2) is made SPR by the following choices of $\gamma$ and $K$

$$\gamma > 0 ; \quad K = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

**V. CONCLUSIONS**

In this paper we found necessary and sufficient conditions for a MIMO transfer function to be rendered SPR using output feedback. These results generalize a previously published result and establish a connection with the invertibility problem. The design is useful when a passive uncertainty enters the system such as in the Lure's problem [4].

**REFERENCES**


