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SPR design using feedback

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ABSTRACT

In this paper we derive necessary and sufficient conditions for a square transfer matrix to be rendered Strictly-Positive-Real (SPR) using output feedback. These conditions are then related to the existence of stable inverse systems.

I. INTRODUCTION

The concepts of Positive-Real (PR) and Strictly-Positive-Real (SPR) functions and matrices have been very useful in network theory [1], adaptive and robust control [2]. When dealing with uncertain systems, a nominal SPR transfer function allows for large passive uncertainties without the loss of stability [2,4,5]. It is then important to be able to test a given transfer matrix for its positive-realness and, more importantly, to make a given transfer matrix positive-real. The standard definition of SPR matrices [4], termed strong SPR, is usually difficult to apply. Moreover, it was recently shown [5] that the strong SPR definition is overly restrictive for control theory applications. In this paper we will use the term SPR to denote weak SPR matrices as defined in [5,6] and reviewed in the next section.

On the other hand, if a given transfer matrix is not SPR, the question of whether a feedback controller might make the closed-loop system SPR is of considerable interest. What has been lacking, however, is a set of conditions that will answer the existence question: Given a transfer matrix $P(s)$, does a controller that will make it SPR exist? The conditions should be necessary and sufficient and, moreover, a construction of the controller (when it exists) is desirable. A partial answer to the existence and construction questions was given in [7]. In [9], Sufficient existence conditions were found for the single-input-single-output case. In the present paper, we provide a simple proof of the results in [7] and extend these results to more general cases.

This paper is organized as follows: In section II, we review the available SPR definitions for transfer matrices. In section III, we define the problem and present our results on designing controllers to make a closed-loop system SPR. An example is given in section IV and our conclusions are presented in section V.

II. WHICH SPR?

Consider is a multi-input-multi-output linear time-invariant system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \tag{2.1}$$

where x is an n vector; u is an m vector, y is a p vector, A,B,C, and D are of the appropriate dimensions. The corresponding transfer function matrix is

$$P(s) = C(sI-A)^{-1}B + D \tag{2.2}$$

We will first assume that the system has an equal number of inputs and outputs, i.e. $p = m$ Then define the relative degree n^* as follows: $n^* = 0$ if det $(D) \neq 0$, and $n^* = m$ if det $(D) = 0$ but det $(CB) \neq 0$. A formalism for the poles and zeros of multivariable systems is given in [8] and may be used to justify the definition of n . To simplify our notation we will denote the Hermitian part of a real, rational transfer matrix $T(s)$ by $He[T(s)] = \frac{1}{2}[T(s) + T^T(s^*)]$ where s^* is the complex conjugate of s. A number of definitions have been given for SPR functions and matrices [4,6]. It appears that the most useful definition for control applications is the following [5]

Definition 2.1: An $m x m$ matrix $T(s)$ of proper real rational functions which is not identically zero is (weak) SPR if
1) All elements of $T(s)$ are analytic in the right half plane Re$(s) \geq 0$, and
2) The matrix $He[T(s)]$ is positive definite for Re$(s) > 0$.

□

The more standard definition of SPR matrices advocated in [4] is more restrictive than definition 2.1. In fact, a long held view was that strong SPR was needed to prove the Meyer-Kalman-Yakubovitch (MKY) lemma, which is after all, the major application of SPR concepts in control systems. As shown in [5] however, the weak SPR definition is just as useful in this regard and will therefore be adopted in this paper. Note that, from minimum real-part arguments given in [1], condition 2) of definition 2.1 is equivalent to $He[T(jw)] > 0$ for all w.

III. SPR USING FEEDBACK

The question addressed in this paper is to find conditions on (2.1) so that a feedback controller will render the closed-loop system SPR. The result of theorem 3.1 first appeared in [7] for the case of static output feedback, i.e. $u = -\gamma Ky + r$. The corresponding closed-loop system is then given by

$$\dot{x} = (A - \gamma BKC)x + Br$$
$$y = Cx \tag{3.1}$$

or in the frequency-domain

$$Y(s) = [I + \gamma KP(s)]^{-1}KP(s)R(s) \tag{3.2}$$

We present a simple frequency domain proof, to show the existence of K and γ that will render the closed-loop system SPR.

Theorem 3.1: Let system (2.1) be stabilizable and detectable and let its relative degree be $n^* = m$. Then there exists a nonsingular K and a positive scalar γ such that the closed-loop system (3.2) is SPR, if and only if $P(s)$ is minimum phase.
Proof: *Sufficiency:* Consider the closed-loop transfer function

$$T(s) = [I + \gamma KP(s)]^{-1}KP(s) \tag{1}$$

or

$$T(s) = [P^{-1}(s)K^{-1} + \gamma I]^{-1} \tag{2}$$

Since $P(s)$ is minimum phase with a relative degree $n^* = m$, its inverse $P^{-1}(s)$ will be given by

$$P^{-1}(s) = sL + P_1(s) \tag{3}$$

where $P_1(s)$ is proper and stable, and det $(L) \neq 0$. In fact, det$(CB) \neq 0$ and $L = (CB)^{-1}$. On the other hand, since $P(s)$ is minimum phase, $P_1(s)$ can not have any poles in Re $(s) \geq 0$. It is now obvious that $T(s)$ will be stable if and only if $W(s) = [P^{-1}(s)K^{-1} + \gamma I]$ has no zeros in Re$(s) \geq 0$. Let K be given by

$$K = (CB)^{-1} \tag{4}$$

then

$$W(s) = sI + P_1(s)CB + \gamma I \tag{5}$$

and

$$He[W(j\omega)] = He[P_1(j\omega)CB] + \gamma I \tag{6}$$

Since $P_1(j\omega)$ has no poles on the $j\omega-$ axis, $He[W(j\omega)]$ may be made positive-definite by a large enough positive scalar γ. This then implies that $W(s)$ is weakly SPR. Since $T(s)$ is the inverse of $W(s)$, it is also weakly SPR [4].

Necessity: Suppose now that a nonsingular K and a γ were found to make the closed-loop system $T(s)$ SPR and that $D = 0$. Then

$$W(s) = [P^{-1}(s)K^{-1} + \gamma I] \tag{7}$$

is also SPR. Writing $P^{-1}(s)$ as in (3), with $L = (CB)^{-1}$ we get

$$W(s) = (CB)^{-1}K^{-1} + P_1(s)K^{-1} + \gamma I \tag{8}$$

Since $W(s)$ is SPR, $P_1(s)$ must be stable, hence $P(s)$ must be minimum-phase.

□

The choice of $K = (CB)^{-1}$ in the proof of the theorem is by no means unique. In fact, it is sufficient to choose $K = P(CB)^{-1}$ where P is any symmetric positive-definite matrix. Next, note that the condition det $(CB)\neq 0$ (or that $P(s)$ has a relative degree $n^* = m$), also reveals that the system (2.1) has an inverse obtained by cascading one differentiator and a dynamical system [3]. Note that the inverse system is given by (3) in theorem 3.1 or in state-space by

$$\begin{aligned} \dot{x} &= [A - B(CB)^{-1}CA]x + B(CB)^{-1}\dot{y} \\ u &= -(CB)^{-1}CAx + (CB)^{-1}\dot{y} \end{aligned} \tag{3.3}$$

Now recall that the invertibility of the system (2.1) may still be inferred even though det (CB)=0. In fact, a sufficient condition for the inverse to exist is that the first nonzero matrix in the sequence, $D, CB, CAB, CA^2B, ..., CA^{n-1}B$, be nonsingular [3]. It is then obvious that for a nonzero D matrix, the condition for $T(s)$ to be SPR is that D be invertible and $P(s)$ be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a static output feedback if its high frequency gain is nonsingular. On the other hand, the following general result may be established.

Theorem 3.2: Suppose that (2.1) is both stabilizable and detectable, and det $(CA^iB)\neq 0$ where CA^iB is the first nonzero matrix in the sequence $D, CB, CAB, CA^2B, ..., CA^{n-1}B$. Then the closed-loop system from r to $\frac{d^i y}{dt^i}$ given by

$$T_i(s) = KCA^i(sI - A + \gamma BKCA^i)^{-1}B \tag{3.4}$$

is SPR if and only if $P(s)$ is minimum phase.
Proof: Given system (2.1), let us define an output z_i by

$$z_i = \frac{d^i y}{dt^i} = y^{(i)} \tag{1}$$

then

$$z_{i+1} = CA^{i+1}x + CA^iBu \tag{2}$$

If det $(CA^iB) \neq 0$, the inverse system of $P(s)$ is given by

$$P^{-1}(s) = (CA^iB)^{-1}s^{i+1} + P_2(s) \tag{3}$$

where $P_2(s)$ is stable. Repeating the arguments of theorem 3.1 and using the controller

$$u = -\gamma K y^{(i)} + r \tag{4}$$

we obtain the desired result.

□

Next suppose that one has access to the full state vector, under what conditions can a feedback controller produce an SPR transfer function between the input and an appropriate output? It is simple to see that we can choose C such that det $(CB)\neq 0$ and $C(sI-A)^{-1}B$ minimum-phase, then use output feedback from $y = Cx$ to make the closed-loop system SPR. This is equivalent to the MKY Lemma [2] where $A - BK$ is stable and $C = B^TP$, P being the solution of a Lyapunov equation.

IV. EXAMPLE

Consider the following open-loop transfer matrix

$$P(s) = \frac{s+1}{s^4+2s^3+2s^2-s-1}\begin{bmatrix} s^2+s+1 & -s(s+1) \\ -s(s+1) & 2s^2+2s-1 \end{bmatrix}$$

The inverse of the above transfer matrix is given by

$$P^{-1}(s) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} s + \frac{1}{s+1}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The closed-loop transfer function $T(s)$ given in (3.2) is made SPR by the following choices of γ and K

$$\gamma > 1 \; ; \; K = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

V. CONCLUSIONS

In this paper we found necessary and sufficient conditions for a MIMO transfer function to be rendered SPR using output feedback. These results generalize a previously published result and establish a connection with the invertibility problem. The design is useful when a passive uncertainty enters the system such as in the Lure's problem [4].

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