6-4-2003

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Recommended Citation

Abdallah, Chaouki T.; F. Amato; M. Ariola; C. Cosentino; and P. Dorato. "Necessary and sufficient conditions for finite-time stability of linear systems." *Proceedings of the 2003 American Control Conference* (2003): 4452-4456. doi:10.1109/ACC.2003.1240541.

Necessary and Sufficient Conditions for Finite-Time Stability of Linear Systems

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Abstract

In this paper we consider the Finite-Time Stability and Finite-time Boundedness problems for linear systems subject to exogenous disturbances. The main results of the paper are some necessary and sufficient conditions, obtained by means of an approach based on operator theory; such conditions improve some recent results on this topic. An example is provided to illustrate the proposed technique.

1 Introduction

When dealing with the stability of a system, a distinction should be made between classical *Lyapunov Stability* and *Finite-Time Stability* (FTS) (or *short-time stability*). The concept of Lyapunov Asymptotic Stability is largely known to the control community; conversely a system is said to be finite-time stable if, once we fix a time-interval, its state does not exceed some bounds during this time-interval. Often asymptotic stability is enough for practical applications, but there are some cases where *large* values of the state are not acceptable, for instance in the presence of saturations. In these cases, we need to check that these unacceptable values are not attained by the state; for these purposes FTS can be used.

Most of the results in the literature are focused on Lyapunov Stability. Some early results on FTS can be found in [6], [8] and [5]. More recently the concept of FTS has been revisited taking advantage of the Linear Matrix Inequalities (LMIs) theory; this has allowed to find less conservative (but still only sufficient) conditions guaranteeing FTS of linear continuous-time systems (see [2], [1]).

Another concept which is strongly related to that one of FTS is Finite-Time Boundedness (FTB), which takes into account possible norm bounded $\mathcal{L}_2$ disturbances affecting the system. Roughly speaking, a system is said to be FTB if its state does not exceed a prespecified bound for all admissible disturbances.

The main goal of this paper is to provide *necessary and sufficient* conditions for FTS and FTB of linear systems; such conditions improve the recent results provided in [2], where, as said, only sufficient conditions were given. Contrarily to most of the previous literature on the subject, which makes use of approaches based on Lyapunov functions, the methodology proposed in this paper is based on operator theory.

The paper is organized as follows: in Section 2 the definition of FTS and FTB is recalled and some preliminary results are stated; in Section 3 necessary and sufficient conditions for FTS and FTB are given together with an illustrative example; finally some conclusions are drawn in Section 4.

2 Notation, Problem Statement and Preliminary Results

We denote by $\mathcal{PC}^1_\Omega$ the space of the uniformly bounded, piecewise continuously differentiable, real matrix-valued functions defined on $\Omega := [0,T] \subset \mathbb{R}$ and by $\mathcal{L}^2_\Omega$ the space of the real vector-valued functions which are square integrable on Ω.

The Euclidean vector norm and the corresponding induced matrix norm are denoted by $|\cdot|$; $\|\cdot\|$ denotes the usual norm in $\mathcal{L}^2_\Omega$.

Given $S : \Omega \mapsto \mathbb{R}^{n\times n}$, we write $S > 0$ (≥ 0) meaning that S is positive definite (semidefinite), i.e. that there exists $\alpha > 0$ such that for all $v \in \mathbb{R}^n$ and for all $t \in \Omega$

$$v^T S(t) v \geq \alpha |v|^2 \qquad (v^T S(t) v \geq 0)\,.$$

Given two matrix-valued functions of the same dimensions S and Z, the notation $S > Z$ $(S \geq Z)$ means that $S - Z > 0 (\geq 0)$. Finally the symbols $S < (\leq) 0$, and $S < (\leq) Z$ have obvious meaning.

Now consider the linear system

$$\dot{x} = Ax + Gw\,, \quad x(0) = x_0\,, \quad t \in \Omega \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$ and $w \in \mathcal{L}^2_\Omega$. We give the following definitions.

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Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

Definition 1 (Finite-Time Stability) *The linear system (1) with $G = 0$ is said to be Finite Time Stable with respect to (Ω, δ, γ) if*

$$|x_0| \leq \delta \Rightarrow |x(t)| < \gamma \qquad \forall t \in \Omega\,.$$

$\Diamond$

Remark 1 (Finite-Time Stability and Asymptotic Stability) *It is worth noting that Asymptotic Stability and FTS are independent concepts: a system which is FTS may not be asymptotically stable, while a asymptotically stable system may not be FTS.* $\Diamond$

Definition 2 (Finite-Time Boundedness) *The linear system (1) with $x_0 = 0$ is said to be Finite-Time Bounded (FTB) with respect to (Ω, d, γ) if*

$$\|w\| \leq d \Rightarrow |x(t)| < \gamma \qquad \forall t \in \Omega\,.$$

$\Diamond$

Finally we consider the case in which the inital state is non-zero and a $\mathcal{L}_2$ input affects the system.

Definition 3 (Finite-Time Boundedness with Non-Zero Initial State) *The linear system (1) is said to be Finite Time Bounded with non-zero initial state (FTBNZ) with respect to $(\Omega, \delta, d, \gamma)$ if for all x_0 with $|x_0| \leq \delta$ the following holds*

$$\|w\| \leq d \Rightarrow |x(t)| < \gamma \qquad \forall t \in \Omega\,.$$

$\Diamond$

In the sequel we shall state necessary and sufficient conditions for FTS and FTB and sufficient conditions for FTBNZ.

Note that, for a given time instant $t \in]0, T]$, the linear system (1) uniquely defines the linear operator

$$\Gamma_t \,:\, \mathbb{R}^n \oplus \mathcal{L}^2_{[0,t]} \mapsto \mathbb{R}^n \,:\, (x_0, w(\cdot)) \mapsto x(t)\,. \qquad (2)$$

Given $x \in \mathbb{R}^n$ and $w \in \mathcal{L}^2_{[0,t]}$, we equip the space $\mathbb{R}^n \oplus \mathcal{L}^2_{[0,t]}$ with the norm

$$\|(x, w)\| := \sqrt{|x|^2 + \|w\|^2}\,. \qquad (3)$$

We denote by $\|\Gamma_t\|$ the norm of the operator Γ_t induced by (3); it is defined as follows

$$\|\Gamma_t\| := \sup_{(x_0,w)\neq(0,0)} \frac{|x(t)|}{\|(x_0, w)\|}\,, \quad t \in]0, T]\,.$$

In what follows the next lemma will be useful.

Lemma 1 *Let us consider system (1) and a number $\beta > 1$; then the following statements are equivalent:*

i) $\|\Gamma_t\| < \beta$ *for all* $t \in]0, T]$*;*

ii) There exists a symmetric $P \in \mathcal{PC}^1_\Omega$ such that

$$\begin{aligned} \dot{P}(t) + A^T P(t) + P(t)A & \\ +\beta^{-2} P(t) G G^T P(t) &< 0\,, \quad t \in \Omega \\ P(T) &\geq I \\ P(0) &< \beta^2 I \end{aligned}$$

Proof: See the appendix. ■

3 Main Results

3.1 Necessary and Sufficient Conditions for FTS and FTB

Theorem 1 *System (1) (with $G = 0$) is FTS with respect to (Ω, δ, γ) iff there exists a symmetric $P \in \mathcal{PC}^1_\Omega$ such that*

$$\dot{P}(t) + A^T P(t) + P(t) A < 0 \quad t \in \Omega \qquad (4a)$$

$$P(T) \geq I \qquad (4b)$$

$$P(0) < \frac{\gamma^2}{\delta^2} I \qquad (4c)$$

Proof: First of all note that in this case the operator (2) reduces to

$$\Gamma_t \,:\, \mathbb{R}^n \mapsto \mathbb{R}^n\,,\; x_0 \mapsto x(t) \quad t \in]0, T]\,.$$

By virtue of Lemma 1, to prove the statement we have to show that FTS of system (1) is equivalent to the fact that $\|\Gamma_t\| < \frac{\gamma}{\delta}$ for all $t \in]0, T]$.

First we prove the sufficiency. Let $t \in]0, T]$ and assume that $\|\Gamma_t\| < \frac{\gamma}{\delta}$; then we have

$$\|\Gamma_t\| := \sup_{x_0 \in \mathbb{R}^n - \{0\}} \frac{|x(t)|}{|x_0|} < \frac{\gamma}{\delta}$$

which in turn guarantees that for all $x_0 \in \mathbb{R}^n - \{0\}$

$$\frac{|x(t)|}{|x_0|} < \frac{\gamma}{\delta}\,.$$

This last inequality implies that for all x_0 with $|x_0| \leq \delta$, $|x(t)|$ is bounded from above by γ; FTS of system (1) follows from the arbitrariness of t.

Conversely, let us assume that system (1) is FTS. Then, since the operator Γ_t is linear, we have (see [4])

$$\begin{aligned}\|\Gamma_t\| &:= \sup_{x_0\in\mathbb{R}^n-\{0\}} \frac{|x(t)|}{|x_0|}\\ &= \sup_{|x_0|=\delta} \frac{|x(t)|}{|x_0|}\\ &= \frac{1}{\delta}\max_{|x_0|=\delta}|x(t)| < \frac{\gamma}{\delta} \quad \forall t\in]0,T].\end{aligned}$$

■

Example 1 *Let us consider the system*

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} x(t). \qquad (5)$$

For this system we exploit the result stated in Theorem 1 in order to evaluate the maximum attainable norm of the state at the time instant $T = 1$, starting from an initial unitary norm condition ($\|x(0)\| = \delta = 1$). For a given value of γ, in order to find numerically a matrix function $P(\cdot)$ solving (4), we split the interval $[0,1]$ into a number of parts, and approximate the solution by a linear behaviour in each sub-interval (by imposing continuity at the extrema of each sub-interval). If a solution is not found, we refine the splitting of the interval $[0,1]$, until the length of the sub-intervals results to be less than a pre-specified value.

In this way we are able to estimate the lower bound of γ for which system (5) is FTS with respect to $([0,1],1,\gamma)$. This estimate of γ evaluated to $\gamma_{est} = 2.72$.

For this simple example we can evaluate exactly the lower bound of γ by computing explicitly

$$\underline{\gamma} := \sup_{\|x_0\|=1} x(T) = 2.712.$$

Note that γ_{est} and $\underline{\gamma}$ are very close. This is not surprising since the condition stated in Theorem 1 is necessary and sufficient, and so it does not introduce any conservativeness. On the other hand by applying the sufficient condition given in [2] we got the following estimate for γ

$$\gamma_{old} = 5.30.$$

We have reported in Figure 1 the time behaviour of the eigenvalues of the solution $P(t)$ of (4) with $\gamma = 2.72$. ◊

By following the same guidelines of Theorem 1 we can prove the following necessary and sufficient condition for FTB.

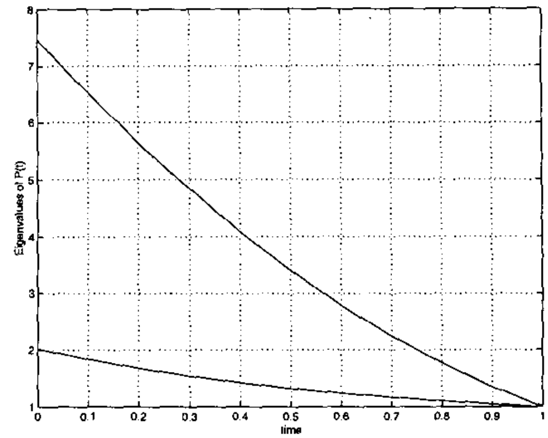

Figure 1: Eigenvalues of the solution $P(t)$

Theorem 2 *System (1) (with $x_0 = 0$) is FTB with respect to (Ω, d, γ) iff there exists a symmetric $P \in \mathcal{PC}^1_\Omega$ such that*

$$\begin{aligned}\dot{P}(t) + A^T P(t) + P(t)A&\\ +(\gamma/d)^{-2}P(t)GG^TP(t) < 0, &\quad t\in\Omega \qquad (6a)\\ P(T) \geq I& \qquad (6b)\\ P(0) < \left(\frac{\gamma}{d}\right)^2 I& \qquad (6c)\end{aligned}$$

3.2 A Sufficient Condition for FTBNZ

The following theorem provides a sufficient condition for FTBNZ.

Theorem 3 *System (1) is FTBNZ with respect to $(\Omega, \delta, d, \gamma)$ if there exists a symmetric $P \in \mathcal{PC}^1_\Omega$ such that*

$$\begin{aligned}\dot{P}(t) + A^T P(t) + P(t)A&\\ +\gamma^{-2}(\delta^2+d^2)P(t)GG^TP(t) < 0 &\quad t\in\Omega \qquad (7a)\\ P(T) \geq I& \qquad (7b)\\ P(0) < \frac{\gamma^2}{\delta^2+d^2} I& \qquad (7c)\end{aligned}$$

Proof: By virtue of Lemma 1, to prove the statement of the theorem we have to show that $\|\Gamma_t\| < \frac{\gamma}{\sqrt{\delta^2+d^2}}$ for all $t \in]0,T]$ implies the FTBNZ of system (1).

Let $t \in]0,T]$; by assumption we have

$$\begin{aligned}\|\Gamma_t\| &:= \sup_{(x_0,w)\neq(0,0)} \frac{|x(t)|}{\|(x_0,w)\|}\\ &< \frac{\gamma}{\sqrt{\delta^2+d^2}}\end{aligned}$$

which guarantees that for all $(x_0, w) \neq (0,0)$

$$\frac{|x(t)|^2}{|x_0|^2+\|w\|^2} < \frac{\gamma^2}{\delta^2+d^2}. \qquad (8)$$

This last inequality implies that for all x_0 with $|x_0| \leq \delta$ and for all w such that $\|w\| \leq d$, $|x(t)|$ is bounded from above by γ. FTBNZ of system (1) follows from the arbitrariness of t. ■

4 Conclusions

In this note necessary and sufficient conditions for Finite Time Stability and Boundedness of linear systems have been provided; such conditions improve the results of [2] and [1]. An illustrative example shows the effectiveness of the proposed methodology.

It is worth noting that, for the sake of presentation simplicity, we have considered time invariant, certain systems, but there is no conceptual difficulty in extending the results contained in this paper to time varying and/or uncertain systems following the guidelines of [2] and [1].

Appendix

In order to prove Lemma 1 we need the following preliminary lemma.

Lemma 2 *Let us consider system* (1) *and a number* $\beta > 1$*; then the following statements are equivalent:*

i) $\|\Gamma_t\| < \beta$ *for all* $t \in]0, T]$;

ii) There exists a symmetric $P \in \mathcal{PC}^1_\Omega$ *and a scalar* $\epsilon > 0$ *such that*

$$\dot{P}(t) + A^T P(t) + P(t)A + \beta^{-2} P(t) G G^T P(t) + \epsilon I = 0\,, \quad t \in \Omega \tag{9a}$$

$$P(T) \geq I \tag{9b}$$

$$P(0) < \beta^2 I \tag{9c}$$

Proof: Let $t \in]0, T]$.

i) $\Rightarrow$ *ii)* Let us augment system (1) with the fictitious output $\tilde{y}$ as follows

$$\dot{x} = Ax + Gw\,, x(0) = x_0 \tag{10a}$$

$$\tilde{y} = \epsilon^{1/2} x\,. \tag{10b}$$

Define the operator

$$\tilde{\Gamma}_t \,:\, \mathbb{R}^n \oplus \mathcal{L}^2_{[0,t]} \mapsto \mathbb{R}^n \,:\, (x_0, w(\cdot)) \mapsto \tilde{y}(t)\,. \tag{11}$$

By using continuity arguments it is clear that Condition i) implies that there exists a sufficiently small ϵ such that

$$\|\tilde{\Gamma}_t\| < \beta\,. \tag{12}$$

Inequality (12) enables us to apply Theorem 1.2 of [7] to the fictitious system (10) which guarantees the existence of a symmetric $P \in \mathcal{PC}^1_{[0,t]}$ such that

$$\dot{P}(\tau) + A^T P(\tau) + P(\tau)A + \beta^{-2} P(\tau) G G^T P(\tau) + \epsilon I = 0\,, \quad \tau \in [0, t] \tag{13a}$$

$$P(t) \geq I \tag{13b}$$

$$P(0) < \beta^2 I \tag{13c}$$

Letting $t = T$ in (13) and $t \to \tau$ the proof follows.

ii) $\Rightarrow$ *i)* The solution $P(\cdot)$ of equation (9a) with terminal condition $P(T) = S \geq I$ can be given the following interpretation. Let us consider the optimal control *maximization* problem (see [3])

$$J(x(t), t) := \max_w \left\{ \int_t^T \Big(\epsilon x^T(\tau) x(\tau) - \beta^2 w^T(\tau) w(\tau) \Big) d\tau + x^T(T) S x(T) \right\}$$
$$\text{s.t.} \quad \dot{x} = Ax + Gw\,. \tag{14}$$

Then the optimal value of the cost index is

$$J(x(t), t) = x^T(t) P(t) x(t)\,; \tag{15}$$

moreover $P(\cdot)$ is non-increasing, in the sense that for $t_2 > t_1$

$$P(t_1) \geq P(t_2)\,. \tag{16}$$

Therefore we have

$$P(t) \geq P(T) \geq I \qquad t \in \Omega\,. \tag{17}$$

By considering the restriction of $P(\cdot)$ to the interval $[0, t]$, $t \in \Omega$, we can conclude that there exists a symmetric $P \in \mathcal{PC}^1_{[0,t]}$ such that (13) holds.

The proof follows by using continuity arguments and applying Theorem 1.2 of [7]. ■

At this point, Lemma 1 follows from Lemma 2 by noticing that $\epsilon > 0$.

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