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Model-based Networked Control for Finite-Time Stability of Nonlinear Systems: The Deterministic Case

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Abstract— **In this paper we analyze model-based networked control systems for a discrete-time nonlinear plant model, operating in the presence of deterministic dropout of state observations. The dropout is modeled using communication network dynamics, and sufficient conditions for finite-time stability are provided. In a companion paper we model the dropout as a stochastic sequence.**

I. INTRODUCTION

In several recent works, the problem of networked control systems (NCS) has been posed and partially investigated [2], [3], [4], [7], [8]. This new problem deals with the possibility of controlling a system remotely via a communication network and as such, instantaneous and perfect signals between controller and plant are not achievable.

In [5] a model for the networked control of linear time invariant systems was proposed. The network is modeled as a sampler placed between the plant and sensors on one side, and the controller on the other side of the network. Utilizing an approximate model of the process at the controller's side, the controller can maintain stability while receiving only periodic updates of the actual state of the plant. Whenever a new update is received, the model plant is initialized with the new information. This idea was utilized in [1], where the system evolved in discrete-time, and state updates were either received or dropped at each sample due to the effect of the network. The characterization of such a dropout is achieved through the use of a dropping sequence that takes on values of 0 or 1 depending on whether a sample was lost or received, respectively. Such dropping sequence is modeled using dynamics a network. Recently in [6], the initial model for a continuous-time plant and a network modeled with a fixed rate sampler was extended to bounded yet random variable sample times driven by a Markov chain.

In this paper, we present an extension of the discrete-time systems in [1] into a nonlinear setting, i.e. our plant and the model used for state estimation are both nonlinear. We utilize the same model of packets being dropped according to dropping sequence, and obtain sufficient conditions to guarantee finite-time stability of the closed-loop system.

The paper is organized as follows: In Section II, we reformulate the model-based networked control problem in the nonlinear setting. Section III describe a model for deterministic packet dropout in which the network dynamics are included in the model through the dropping sequence. We move in section IV to describe the concept of finite-time stability and extended finite-time stability. With the described framework in section V we proceed to develop the analysis tool. Finally in Section VI we present some examples to illustrate our results, and our conclusion in section VII.

A companion paper titled "Model-based Networked Control for Finite-Time Stability of Nonlinear Systems: The Stochastic Case" has also been submitted to the conference and deals with the same problem using stochastic models. The first two sections of the current paper and the companion paper are identical in order to make them as self-contained as possible.

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II. PROBLEM FORMULATION

In [1] a discrete-time model-based control with observation dropouts is proposed for linear discrete-time systems. Our objective in this paper is to propose a similar framework in the case of nonlinear systems, and to study the stability of the closed-loop system. As depicted in Figure 1, discrete-time model-based control is comprised of a plant with the network residing between the sensors of the plant and the actuators.

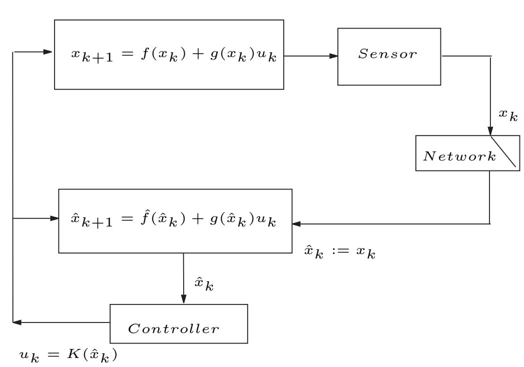

Fig. 1. Model-Based NCS

The network is modeled as a dropping sequence θ_k, where a measurement is dropped if $\theta_k = 0$, and a measurement is received when $\theta_k = 1$. Due to our inability to receive an update of the plant's state at each discrete instant of time, we use an inexact model plant on the controllers side that provides us with the missing measurement. Such a model is given by

$$\hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k)u_k. \tag{1}$$

In order to carry out the analysis, we define the estimation error as $e_k = x_k - \hat{x}_k$, and augment the state vector with e_k so that the closed-loop state vector is given by $z_k = \left(x_k^T;\, e_k^T\right)^T$. The closed-loop system evolves according to

$$\begin{aligned} z_{k+1} &= \begin{pmatrix} f(x_k) \\ (f(x_k) - \hat{f}(x_k)) + .. \\ \quad (1-\theta_k)((\hat{f}(x_k) - \hat{f}(\hat{x}_k))) \end{pmatrix} \\ &+ \begin{pmatrix} g(x_k)K(\hat{x}_k) \\ (g(x_k) - \hat{g}(x_k))K(\hat{x}_k) + .. \\ \quad (1-\theta_k)(\hat{g}(x_k) - \hat{g}(\hat{x}_k))K(\hat{x}_k) \end{pmatrix}. \end{aligned} \tag{2}$$

In the above model $\theta_k \in \{0, 1\}$ is a dropping sequence that indicates the reception ($\theta_k = 1$) or the loss (θ_k=0) of the packet containing the state measurement x_k. If a packet is received, it is used as an initial condition for the next time step in the model, otherwise the previous state of the model is used. We then classify the NCS errors as follows:

(I). Model structure errors

$$\begin{aligned} e_{f1}(x_k) &= f(x_k) - \hat{f}(x_k) && (3) \\ e_{g1}(x_k) &= g(x_k) - \hat{g}(x_k). && (4) \end{aligned}$$

These are the errors between the plant and the model evaluated at the plant's state, and are therefore dependent on the system's structure.

(II). State dependent errors

$$\begin{aligned} e_{f2}(x_k, \hat{x}_k) &= \hat{f}(x_k) - \hat{f}(\hat{x}_k) && (5) \\ e_{g2}(x_k, \hat{x}_k) &= \hat{g}(x_k) - \hat{g}(\hat{x}_k). && (6) \end{aligned}$$

These represent the errors between the model evaluated at the plant's state and at its own state, i.e. the error introduced by the difference in the states.

(III). Structure and state dependent errors

$$\begin{aligned} e_{f3}(x_k, \hat{x}_k) &= f(x_k) - \hat{f}(\hat{x}_k) && (7) \\ e_{g3}(x_k, \hat{x}_k) &= g(x_k) - \hat{g}(\hat{x}_k), && (8) \end{aligned}$$

which include both model structure and state dependent errors.

With the new notation, the system (2) becomes

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f1}(x_k) + e_{g1}(x_k)K(\hat{x}_k) + (1-\theta_k).. \\ \quad (e_{f2}(x_k, \hat{x}_k) + e_{g2}(x_k, \hat{x}_k)K(\hat{x}_k)) \end{pmatrix}$$

Based on the value of θ_k we have two possible situations:

1. for $\theta_k = 1$ the system will be

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f1}(x_k) + e_{g1}(x_k)K(\hat{x}_k) \end{pmatrix} \tag{9}$$

2. for $\theta_k = 0$

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f3}(x_k, \hat{x}_k) + e_{g3}(x_k, \hat{x}_k)K(\hat{x}_k) \end{pmatrix} \tag{10}$$

For the remainder of this paper we use the following compact form to represent the system above, which also highlights the fact that θ_k represents packet dropouts,

$$z_{k+1} = H_1(z_k) + H_2(z_k)(1-\theta_k),\ k \geq 0 \tag{11}$$

with

$$\begin{aligned} H_1(z_k) &= F_1(z_k) + G_1(z_k)K(\hat{x}) && (12) \\ H_2(z_k) &= F_2(z_k) + G_2(z_k)K(\hat{x}) \\ F_1(z_k) &= \begin{pmatrix} f(x_k) \\ e_{f1}(x_k) \end{pmatrix},\ F_2(z_k) = \begin{pmatrix} 0 \\ e_{f2}(x_k, \hat{x}_k) \end{pmatrix} \\ G_1(z_k) &= \begin{pmatrix} g(x_k) \\ e_{g1}(x_k) \end{pmatrix},\ G_2(z_k) = \begin{pmatrix} 0 \\ e_{g2}(x_k, \hat{x}_k) \end{pmatrix} \end{aligned}$$

While the control law has no access to the plant's state, we assume in the analysis of the global system full-state availability (i.e. both x_k and $\hat{x}_k$ available). Moreover, we assume that the control law $u_k = K(\hat{x}_k)$ stabilizes the model plant and in the case of full-state availability, it also stabilizes the plant.

Next we define a particular class of NCS for which we characterize the accuracy of the model in representing the plant's dynamics, and describe how the model discrepancy affects the NCS structure.

Definition 1: A model-based NCS of the form (11), belongs to a class C_{B-NCS} with the bounds $(B_f, B_g, B_{efi}, B_{egi}; B_{h_i})$, $i = 1, 2$ if for all $k \in \mathbb{N}$ and for all $x_k \in S \subset \mathbb{R}^n$, the system structure and error norms are bounded as follows

$$\begin{aligned} &||f(x_k)|| \leq B_f,\ ||g(x_k)u(\hat{x}_k)|| \leq B_g(\hat{x}_k) \qquad (13) \\ &||e_{f1}(x_k)|| \leq B_{ef1},\ ||e_{f2}(x_k, \hat{x}_k)|| \leq B_{ef2}(\hat{x}_k) \\ &||e_{g1}(x_k)u(\hat{x}_k)|| \leq B_{eg1}(\hat{x}_k) \\ &||e_{g2}(x_k, \hat{x}_k)u(\hat{x}_k)|| \leq B_{eg2}(\hat{x}_k) \end{aligned}$$

where B_f, B_{ef1} are constant bounds and $B_g(\hat{x}_k)$, $B_{ef2}(\hat{x}_k)$, $B_{eg1}(\hat{x}_k)$, $B_{eg2}(\hat{x}_k)$ are bounds that depend on the model state. Such NCS are called bounded model-based NCS (B-MB-NCS).

The above definition describes the class of NCS, for which it is possible to define bounds on the plant and the NCS errors, and where such bounds depend only on the model's state.

Next we state a lemma that describes properties of class C_{B-NCS}. In particular the lemma describes how bounds on the norm of the B-MB-NCS errors imply bounds on the norm of the NCS dynamics.

Lemma 1: Consider the NCS (11) and assume the system belongs to class C_{B-NCS}. Then the following bounds hold on the norm of the NCS dynamics for $i, j = \{1, 2\}$, $j \neq i$, $k \in \mathbb{N}$ and for all $x_k \in S \subset \mathbb{R}^n$,

$$H_i^T H_j \leq B_{H_{i,j}}(\hat{x}_k),\ H_i^T H_i \leq B_{H_i}(\hat{x}_k) \tag{14}$$

where the bounds on the vector functions are related to the bounds on the errors as follows:

$$\begin{aligned} B_{H_1}(\hat{x}_k) = \quad & (B_f + B_g(\hat{x}_k)) + (B_{ef1} + B_{eg1}(\hat{x}_k)) \\ & +2(B_f B_g(\hat{x}_k)) + 2(B_{ef1} B_{eg1}(\hat{x}_k)) \\ B_{H_{1,2}}(\hat{x}_k) = \quad & (B_{ef1} B_{ef2}(\hat{x}_k) + B_{eg1}(\hat{x}_k) B_{eg2}^T(\hat{x}_k)) \\ & +B_f B_g(\hat{x}_k) + (B_{ef1} B_{eg2}(\hat{x}_k) \\ & +B_{eg1}(\hat{x}_k) B_{ef2}^T(\hat{x}_k)) \\ B_{H_2}(\hat{x}_k) = \quad & (B_{ef2} + B_{eg2}(\hat{x}_k)) + 2((B_{ef2} B_{eg2}(\hat{x}_k)) \end{aligned}$$

The proof of the above lemma can be found in [13].

Lemma 2: Consider the NCS (11), belonging to class $C_{B-NCS}(B_f, B_g, B_{efi}, B_{egi}; B_{h_i})$, $i = 1, 2$ then for all $x_k \in S \subset \mathbb{R}^n, \forall k \in \mathbb{N}$

$$||x_k|| \leq B_x(\hat{x}) \ \ ||e_k|| \leq B_e(\hat{x}), \ ||z_k|| \leq B_z(\hat{x}) \tag{15}$$

where

$$\begin{aligned} B_x(\hat{x}) &= B_f + B_g(\hat{x}_k) \qquad (16) \\ B_e(\hat{x}) &= B_{ef1} + B_{eg1}(\hat{x}_k) + B_{ef2}(\hat{x}_k) + B_{eg2}(\hat{x}_k) \\ B_z(\hat{x}) &= B_x(\hat{x}) + B_e(\hat{x}) \end{aligned}$$

Proof: The first two inequalities just follow from (2), (13). The second part trivially follows from $||z_k|| = (||x_k|| + ||e_k||) \leq (B_x(\hat{x}) + B_e(\hat{x})) = B_z(\hat{x})$ ■

III. NETWORKED CONTROL SYSTEMS AND NETWORKS

Several studies have been conducted in modeling and controlling Networked-Control Systems (NCS),[5],[1],[2], mostly to study the stability of a system whose control loop has been closed across a network.

The introduction of a network in a control loop brings about problems such as packet drops, delays, and so on. These issues have been analyzed individually although some studies have combined the effects of sampling and delay [5]. However, to the best of our knowledge, the network model itself has not yet been directly incorporated into the NCS model, but only through the effects that arise as a result of the network's conditions.

This is the missing link between Networked-Control System and Network-Control. Models of networks have been developed in Network-Control to study delays and packet drops caused by congestion. Therefore, there is a gap between the network dynamics, covered in Network-Control, and the effects that these dynamics have on a control system, which Networked-Control Systems focuses on. We next provide a complete deterministic model of networked control system including the network dynamics and their effect on the packet dropping.

A. *Deterministic Model for Packet Dropout*

We aim to model the packet dropout by considering the network dynamics. In particular we are interested in the network section that includes the path that a packet is going to follow. This path is composed of a number of n_l links, and with each link is associated an actual traffic, depending on the number and rate of sources that are accessing the path, and on the link physical capacity.

We want to study how the loss of packets affects the stability of the overall system by including the network dynamics in the model. In particular this will allow us to explicitly relate the stability of the system to the capacity of the links involved in the path used by the system, and to the rate of the sources that are accessing such a path. This relation gives us the possibility of eventually designing for the stability of the system by controlling the rate of the sources accessing the path.

Let (L, S) be a network in which each source s_i has an associated rate $r_i(k)$ that is a function of time at which it sends packets trough a set $L_i \subset L$ of links. So through every link l_j a total rate that is the sum of all the rates of n_s sources is given by $R_j(k) = \sum_{i=1}^{n_s} r_i(k)$. Moreover, each link will have a capacity function proportional to the total rate that will indicate the level of occupation of the link $G_j(k) = K_l R_j(k)$, $j = 1, \ldots, n_l$. A link has a limiting capacity beyond which it will drop packets. In particular there is a critical level of leftover capacity $c_i(k)$ above which the link will accommodate packets, and below which it will start dropping them. The packet drop will be modeled by the binary value variable θ_k, as discussed earlier.

We have at every instant of time k

$$\theta_k = \prod_{j=1}^{n_l} \left[\frac{sign(c_i - G_j(k)) + 1}{2} \right] \tag{17}$$

where the function $sign : \mathbb{R} \rightarrow \{-1, 1\}$ is defined as $sign(a) = \begin{cases} 1 & a \geq 0 \\ -1 & a < 0 \end{cases}$.

The complementary variable $\varphi_k = 1 - \theta_k$ can then be obtained as follows

$$\varphi_k = \left[1 - \prod_{j=1}^{n_l} \left[\frac{sign(c_j(k) - G_j(k)) + 1}{2} \right] \right]. \tag{18}$$

With the provided framework we are now able to study the stability of the following dynamical nonlinear time varying system

$$\begin{aligned} z_{k+1} = \ & (F_1(z_k) + G_1(z_k)u_k) + (F_2(z_k) + G_2(z_k)u_k) \\ & \left[1 - \prod_{j=1}^{n_l} \left[\frac{sign(c_i(k) - K \sum_{j=1}^{n_s} r_i(k)) + 1}{2} \right] \right] \end{aligned}$$

where $G_j(k)$ represents the traffic in link j, and $r_i(k)$ are the known sequence of rates for sources accessing the path.

This model of NCS is a discrete-time, time-varying dynamical system that incorporates the system state z_k, and the network dynamics $c_i(k), r_j(k)$. The network is therefore an integral part of the overall system, therefore achieving our goal.

IV. FINITE-TIME STABILITY

We focus on discrete-time dynamical systems described by

$$x_{k+1} = f(x_k), \, x \in \mathbb{R}^n, \, x(0) = x_0 \tag{19}$$

Where x is the system state, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector function. For notational simplicity, we use $x_k = x(k)$. Also from now on we will denote $||.|| \equiv ||.||_2^2$. We are interested in studying the state trajectory of the system in a finite time interval.

Definition 2: [9] The system (19) is finite-time stable (FTS) with respect to the 4-tuple $(\alpha, \beta, N, ||.||)$, $\alpha \leq \beta$ if every trajectory x_k starting in $||x_0|| \leq \alpha$ satisfies the bound $||x_k|| \leq \beta$ for all $k = 1, \ldots, N$.

Next we present a new analysis result for FTS of non-linear discrete-time systems. We consider three classes of systems : a) systems for which the state trajectories always increase in the norm, b) systems for which states always decrease in the norm, and c) systems whose state trajectories behavior's is mixed.

The first step consists of exploring the state trajectories using a discrete version of the continuous-time Bellman-Gronwall inequality [12]. If the state trajectory is always increasing (in the norm) during the time interval of interest, then it is enough to verify that the state at the last time of the interval does not exceed the bound. In the case where the trajectory is always decreasing and it starts inside the bound, the FTS is guaranteed. In the case of a mixed behavior, it is necessary to explore if the trajectory is bounded at each time step. In the next theorem we formulate the conditions for finite-time stability of the system (19).

Theorem 1: The system (19) is finite-time stable with respect to $(\alpha, \beta, N, ||.||)$, $\alpha \leq \beta$, if for a function $V(x_k, k) = V_k \geq 0$ such that $\delta_1 ||x_k|| \leq V_k \leq \delta_2 ||x_k||$, where $\delta_1 > 0$, $\delta_2 > 0$, $\gamma = \delta_1 \beta$, $\gamma_0 = \delta_2 \alpha$, $V_0 \leq \gamma_0$ and $S_\beta = \{x_k : ||x_k|| \leq \beta\}$ we have $\forall k = 0, \ldots, N$, $\forall x_k \in S_\beta$ $\Delta V_k \leq \rho_k V_k$, and one of the following three conditions occur:

- **Case 1:** $(\rho_k \geq 0)$, $\frac{\gamma}{\gamma_0} \geq \prod_{i=0}^{N-1}(1 + \rho_i)$
  The value of $\rho_k \geq 0$ implies that the bounds on the increments of V_k are as a worse case always greater than one, which is the case of monotonically increasing functions.
- **Case 2:** $(0 \geq \rho_k > -1)$ No additional conditions are required.
  The condition $0 \geq \rho_k > -1$ restricts the bounds on the increments of V_k to be always between zero and one, which constrains the function to be monotonically decreasing.
- **Case 3:** $(\rho_k > -1)$, $\frac{\gamma}{\gamma_0} \geq \sup_k \prod_{i=0}^{k-1}(1 + \rho_i)$. The case $\rho_k > -1$ contains the two previous cases, that is the function V_k may be increasing and decreasing.
  Proof: The proof is available in [13] ■

In order to allow more flexibility in our analysis we want to extend the previous definition considering cases in which the state norm may exceed the bound β, but only for a finite number of consecutive steps, after which it needs to contract again below the bound β. The rationale for this is to consider for the deterministic case an equivalent concept to the stochastic one, where the possibility of exceeding the bound for some time is allowed. The proposed extension fits many real situations such as the example of driving a car in a tunnel, where we do not want to hit the tunnel walls, but in case the car is robust enough, we may hit the walls for short periods of time. Another example, may be to consider hot object we need to grab, which even if the temperature is high we can touch it for short time. Therefore we allow a tolerance time within which we can support the object, but after which we need to release it and eventually grab it again. We formalize such a concept with the following definition.

Definition 3: The nonlinear discrete-time system (19) is Extended Finite-Time Stable (EFTS) with respect to $(\alpha, \beta; N, N_o)$, if one of the following holds

(I) for some $k \in [0, N]$ either

$$\{||x_k|| < \beta : k \in [0, N] | \, ||x_0|| \leq \alpha\} \tag{20}$$

or

(II)

$$\{\forall j \in [0, N] : ||x_j|| > \beta, \Rightarrow \min_{j+1 \leq i \leq j+N_o+1} ||x_i|| \leq \beta\}$$

where $N_o < N$ is the number of consecutive steps the system state is allowed to exceed the FT bound.

V. Extended Finite-Time Deterministic Stability Analysis

We consider the deterministic MB-NCS, described in section II

$$z_{k+1} = H_1(z_k) + H_2(z_k)\varphi_k, \; z_k \in \mathbb{R}^{2n}, \; k = 0, 1, \ldots \tag{21}$$

The dropping sequence $\varphi_k = (1 - \theta_k) \in \{0, 1\}$ is defined as in section III.

The NCS described in equation (21) is a deterministic system, and we are interested in investigating its stability over a finite time in the event of packet dropping. In the stochastic case, bounds may be exceeded with low probability. A deterministic definition of EFTS, which also allows bounds to be exceeded, but over limited intervals.

Definition 4: The NCS (21) is EFTDS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$, if the following conditions hold

(I) the system is FTS with respect to (α_x, β_x, N), if no packet dropping occurs

$$\{z_k^T z_k < \beta_z : k \in [0, N] | z_0^T z_0 \leq \alpha_z\} \tag{22}$$

(II) for $\varphi_k = 1$, and some $k \in [0, N]$ either

$$\{z_k^T z_k < \beta_z : k \in [0, N] | z_0^T z_0 \leq \alpha_z\} \tag{23}$$

or

$$\begin{aligned} \{\forall j \in [0, N] \quad &: \quad z_j^T z_j > \beta_z \\ &\Rightarrow \min_{j+1 \leq i \leq j+N_o+1} x_i^T x_i \leq \beta_x\} \end{aligned} \tag{24}$$

where $N_o < N$ is the number of consecutive steps the system state is allowed to exceed the FT bound due to packet dropping.

In particular, FTS for NCS is redefined so that if packet dropping occurs, the system state may exceed the bound β_x for a fixed finite number of consecutive steps N_o. Note that the above definition requires the knowledge of future states to ensure FTS at each step. We will also redefine quadratic FTDS in case it is desired to bound a given quadratic function of the state.

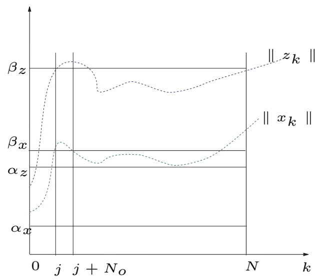

Fig. 2. Extended definition of FTDS for NCS: the global state norm $||z_k||$ may exceed the bound β_z and $||x_k||$ the bound β_x as long as the plant state norm $||x_k||$ contracts in a finite number of steps N_o, back to β_x.

Definition 5: The NCS (21) is quadratically EFTS with respect to $(\gamma_x, \gamma_{x0}; \gamma_z, \gamma_{z0}; N, N_o, M)$, if for the choice of quadratic Lyapunov functions $V_z(z_k, k) = z_k^T M(k) z_k$, $V_x(x_k, k) = x_k^T m_1(k) x_k$ and $V_{\hat{x}}(\hat{x}_k, k) = \hat{x}_k^T m_4(k) \hat{x}_k$, in which $M(k) = M^T(k)$ is a $2n \times 2n$ time-varying matrix, with $m_1(k) > 0,\ m_4(k) > 0$, we have

(I.) for $\varphi_k = 0$

$$\{V_z(z_k, k) < \gamma_z : k \in [0, N] | V_z(z_0, 0) \le \gamma_{z0}\} \quad (25)$$

(II.) for $\varphi_k = 1$ either

$$\{V_z(z_k, k) < \gamma_z : k \in [0, N] | V_z(z_0, 0) \le \gamma_{z0}\} \quad (26)$$

or

$$\begin{aligned} \{\forall j \in [0, N] \quad &: \quad V_z(z_j, j) > \gamma_z \\ &\Rightarrow \min_{j+1 \le i \le j+N_o+1} V_x(x_i, i) \le \gamma_x\} \end{aligned}$$

Theorem 2: Every NCS that is quadratically EFTS with respect to the parameters $(\gamma_x, \gamma_{x0}; \gamma_z, \gamma_{z0}; N, N_o, M)$, is also EFTS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$.

Proof.

The proof easily follows by considering the fact that $\delta_1 ||z_k||^2 \le V_z(z_k, k) \le \delta_2 ||z_k||^2$, $\delta_1(k) = \lambda_{min}\{M(k)\}$, $\delta_2(k) = \lambda_{max}\{M(k)\}$ are the minimum and maximum eigenvalues of $M(k)$, respectively.

In this section, we consider sufficient conditions that will guarantee FTDS for the NCS. In the new setting, if the NCS state exceeds the bound specified at time j, then, in order to predict the future values of the state, it is required to have an estimate of the plant state for the successive $N_o + 1$ steps. This is presented in the following theorem by using the model to predict future states.

the sets of bounded states are denoted by $S_{\gamma_a} = \{a_k : V_a(a_k, k) \le \gamma_a\}$, for $a = z, x, \hat{x}$.

Theorem 3: Consider the class C_{B-NCS} NCS (21), and the state prediction using the model

$$\begin{aligned} \hat{x}_{k+(j+1)} &= \hat{f}(\hat{x}_{k+j}) + \hat{g}(\hat{x}_{k+j}) u_{k+j}, \\ & \qquad k+1 \le j \le k+1+N_o \end{aligned} \quad (27)$$

and assume for all $x_k \in S_{\gamma_x}$ and $k = 1, \dots, N$

$$\begin{aligned} &\Delta V_z \le \Delta V_{B_z} = B_{H_2}(\hat{x}_k) \varphi_k^2 + \\ &2(B_{H_{1,2}}(\hat{x}_k)) \varphi_k + B_{H_1}(\hat{x}_k) - \hat{x}_k^T M(k) \hat{x}_k \\ &\Delta V_x \le \Delta V_{B_x} = B_{H_1}(\hat{x}_k) - \lambda_{min}\{M\} B_z(\hat{x}) \end{aligned} \quad (28)$$

then either

$$\begin{aligned} &[\rho_k V_z(z_k, k) - \Delta V_{B_z}(z_k, k))] \ge 0 \quad (29) \\ &\frac{\gamma_z}{\gamma_{z0}} \ge \sup_{0 \le k \le N} \prod_{j=0}^{k-1} (1 + \rho_j) \end{aligned}$$

or

$$\begin{aligned} &[\rho_k V_z(z_k, k) - \Delta V_{B_z}(z_k, k))] < 0 \\ &\min_{k+1 \le i \le k+N_o+1} [\rho_i^{'} V_x(\hat{x}_i, i) - \Delta V_{B_x}(\hat{x}_i, i)] > 0 \quad (30) \\ &\frac{\beta_{\hat{x}}}{\alpha_{\hat{x}}} \ge \sup_{0 \le k \le N} \prod_{j=0}^{k-1} (1 + \rho_j^{'}) \quad (31) \end{aligned}$$

and finally

$$B_e(\hat{x}_k) + \beta_{\hat{x}} \le \beta_x, \quad \forall \hat{x}_k \in S_{\gamma_{\hat{x}}}, \quad (32)$$

then the NCS (21) is FTDS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$

Proof.

From condition (29) and theorem 1, and considering the the fact that the NCS belongs to class C_{B-NCS}, we can show the FTDS for the NCS. Let us study the case in which $[\rho_k V_z(z_k, k) - \Delta V_{B_z}(z_k, k)] \le 0$, then inequality (30) reduces to

$$\min_{j+1 \le i \le j+N_o+1} [\rho_k V_x(\hat{x}_i, i) - \Delta V_{B_x}(\hat{x}_i, i))] \ge 0,$$

from which it follows that there exists a $j+1 \le i \le N_o+1$ for which $[\rho_i V(\hat{x}_i, i) - \Delta V(\hat{x}_i, i)] \ge 0$, that combined with condition (30) with theorem 1, implies FTS for the model state $\hat{x}$ with respect to $(\alpha_{\hat{x}}, \beta_{\hat{x}}, 1)$. Also since $||x_k|| = ||x_k - \hat{x}_k + \hat{x}_k|| \le ||e_k|| + ||\hat{x}_k|| \le B_e(x_k) + ||x_k||$, then considering the condition (32), and the FTS of $\hat{x}_k$, from which it follows $||x_k|| \le \beta_x$ for at least one $k \in [j+1 \le i \le N_o + 1]$ and moreover FTDS for the NCS.

VI. Examples

Consider the discrete-time Brockett integrator, we investigate in a deterministic setting how packets losses, affect the closed-loop EFTS of the system. Consider again the discrete version of the Brockett integrator [14] and the model

$$\begin{aligned} \hat{x}_1(k+1) &= -23\hat{x}_1(k) - 17u_1(k) \\ \hat{x}_2(k+1) &= -19\hat{x}_2(k) + 3.33u_2(k) \quad (33) \\ \hat{x}_3(k+1) &= -5\hat{x}_3(k) - 8(\hat{x}_1(k)u_2(k) - 7\hat{x}_2(k)u_1(k)) \end{aligned}$$

We study EFTS with respect to $(\alpha_z = 1, \beta_z = 3, \alpha_x = 0.6, \beta_x = 1.5, N = 10, N_o = 2)$. Let us use the controller $u(k) = -\begin{bmatrix} e^{-ak} & 0 \\ 0 & e^{-bk} \end{bmatrix} x(k)$, with parameters $a = 1.3$, $b = 0.7$. Then the conditions of theorem 3 are satisfied if full information is available, i.e. $\varphi_k = 1, \forall k = 0, \dots 10$. In order to simulate the system, we consider the path used to the NCS composed of three links l_1, l_2, l_3, each with limit capacity $c_i(k)$. The links are used by five sources $s_1, \dots, s_5$ as follows $l_1 \rightarrow s_1, s_4$, $l_2 \rightarrow s_1, s_3$, $l_3 \rightarrow s_2, s_3, s_5$ meanwhile the sources send at the rates $r_1(k) = 1(sin(k) + 1)$, $r_2(k) = 3(cos(k) + 1)$, $r_3(k) = 1.7exp^{-k}$, $r_4(k) = 8(cos(k) + 1)$, $r_5(k) = 9exp^{-k}$, from which we can calculate the global rates at each link as $G_1(k) = r_1(k) + r_4(k)$, $G_2(k) = r_1(k) + r_3(k)$, $G_3(k) = r_5(k) + r_2(k) + r_3(k)$.

We study the closed-loop behavior of the NCS as the limit rate of the link, and therefore the amount of packets dropped vary. Starting from initial conditions $x_i(0) = \hat{x}_i(0) = 0.3$, $i = 1, 2, 3$, we first consider a fixe limit capacity $c_i = c = 17$ that will lead to a dropping sequence $\{\varphi_k\}$ of all zeros, that is all the packets are received (and therefore a receiving sequence $\{\theta_k\}$ of all ones). Figure (3) shows the evolution of the system state over time. If we lower the limit

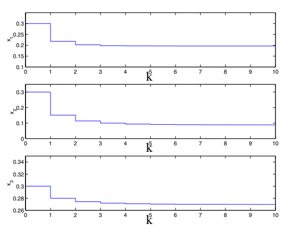

Fig. 3. Brockett integrator controlled through the network with exponential controller with $a_1 = 1.3$, $a_2 = 0.7$, capacity $c = 17$.

capacity to $c = 13$ $packets/second$, the receiving sequence becomes $\theta = [0\,1\,1\,1\,1\,0\,0\,1\,1\,1\,1\,0\,0\,1\,1]$, for which FTDS conditions are still satisfied, as shown in Figure (4). For

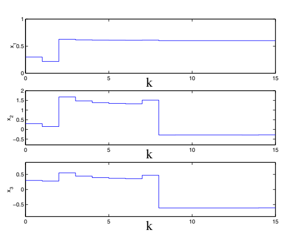

Fig. 4. Brockett integrator controlled through the network with exponential class (a) controller with $a_1 = 1.3$, $a_2 = 0.7$, static capacity $c = 13$

$c = 1$ we obtain a dropping sequence of all ones and the state dynamics are depicted in Figure 5.

VII. Conclusions

We studied $MB - NCS$ with a deterministic model for the packet dropout. This model was realized by including

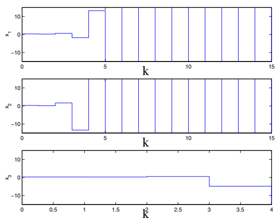

Fig. 5. Brockett integrator controlled through the network with exponential class (a) controller with $a_1 = 1.3$, $a_2 = 0.7$, static capacity $c = 1$

the network in the NCS. This allowed us to obtain a deterministic model for the packet dropping and therefore for the complete NCS. The EFTS for such systems was explored, and in particular redefined for the deterministic NCS. This has allowed the system to possibly exceed the specified bounds for a finite number of steps, which was otherwise unacceptable under the classical FTS definition.

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