5-18-1967

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AN INVESTIGATION OF THE KOLMOGOROV-SMIRNOV METHOD OF DATA VALIDATION TO REDUCE BUYER INSPECTION COSTS

By

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A Thesis
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Business Administration

The University of New Mexico

1967

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER
OF
BUSINESS ADMINISTRATION

Brian E. O'Neil
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Date May 18, 1967

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ACKNOWLEDGEMENTS

The subject of this research grew out of many discussions with Mr. M. C. Carter and Mr. D. D. Sheldon, statisticians of Sandia Corporation, Albuquerque, New Mexico. Both of the aforementioned were very enthusiastic about the use of the non-parametric Kolmogorov-Smirnov test as a means of data validation. They had made an extensive search of the available literature and it appeared that this test was both simple to administer and had adequate power of discrimination regardless of the distribution of the data.

Special credit should be given to Dr. Kenneth W. Olm, Professor Ralph L. Edgel and Professor Everett G. Dillman for redirecting my efforts towards obtaining an empirical proof of the validity of the Kolmogorov-Smirnov test by simulating actual conditions through the use of a computer program.

Mr. D. L. Smith, mathematical programmer and Mr. G. J. Lombardi, engineer-statistician, Sandia Corporation, are deserving of special mention. Mr. Smith for putting all of the pieces of the program together so it would run on the CDC 3600 computer, and Mr. Lombardi for his assistance in running the program on the computer.

Credit is also due to A. F. Cone, manager, Quality Control Department, Sandia Corporation for his encouragement and sponsoring of the computer simulation program for the Kolmogorov-Smirnov test.

I should like also to take this opportunity to express appreciation and gratitude to my wife Ina, for her patience, understanding, and encouragement in the development of this study.

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CHAPTER I

INTRODUCTION

The Problem and Purpose of the Study

One of the major functions of any inspection, quality control or quality assurance organization is to provide assurance to management that their products are of adequate quality. Furthermore, management desires this function to be performed at the minimum cost commensurate with its objectives. It has been reported that inspection organizations used sampling inspection prior to 1920 to achieve their objectives but the sampling plans used were arbitrary with little scientific basis. During the 1920's the Western Electric Company made a major breakthrough with the development of the Shewhart control chart and the Dodge-Romig sampling tables. However, it was not until World War II that widespread industrial use of scientific sampling methods and the Shewhart control chart method developed in the United States. These newer and more scientific methods used smaller inspection samples and better control at the machine, thus achieving substantial reductions in cost. The major impetus for the widespread use of these methods was the insistence of the War Department that sampling plans be used on military contracts.[1]

Since World War II many advances and refinements have obviously been made in sampling inspection and methods of controlling quality. Nevertheless, there still appear to be substantial opportunities left for minimizing the cost of obtaining desired quality.

One of the areas of investigation which appears to offer promising opportunities for quality cost reduction is the area of data validation. This concept, promoted by the Department of Defense through DOD Instruction 4155.6, "Department of Defense Quality Assurance Concept and Policy," dated April 14, 1954, established a uniform concept and policy on quality assurance as related to procurement inspection.[2] This policy requires that, after recorded data generated by examination and testing of manufactured product are ascertained to be reliable, optimum use should be made of these data obtained by suppliers in determining the acceptability of supplies.

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[1] P. Peach, Quality Control for Management (Englewood Cliffs: Prentice-Hall, Inc., 1964), p. 31.

[2] U. S. Department of Defense, Quality Control and Reliability Handbook (Interim) H109, Statistical Procedures For Determining Validity of Suppliers Attributes Inspection, Washington, D. C.: Government Printing Office, 1960), p. 1.

Sandia Corporation, a non-profit prime contractor to the U. S. Atomic Energy Commission, has expressed interest in data validation as a possible method for reducing quality verification costs. Data validation as a concept could also have universal application as a means to reduce quality assurance costs in that sector of industry operated for profit.

The purpose of this study was to examine the possibility of using the Kolmogorov-Smirnov test as a method of validating suppliers inspection data instead of the present verification or acceptance inspection practices. These practices require the procurement organization to inspect a sample of the supplier's submitted product and accept or reject on the basis of the sample criteria used. Most companies and government inspection agencies use the sampling plans specified in Mil-Std-105D, Sampling Procedures and Tables For Inspection By Attributes.[3] Duncan points out that these sampling plans are now a common standard in American, British and Canadian industry.[4]

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[3]U. S. Department of Defense, Mil-Std-105D, Sampling Procedures and Tables for Inspection By Attributes, Washington, D. C.: Government Printing Office, 1963), pp. 9-62.

[4]Acheson J. Duncan, Quality Control and Industrial Statistics (Homewood: Richard D. Irvin, Inc., 1965), p. 191.

If suppliers data could be validated by the Kolmogorov-Smirnov method utilizing smaller sized samples than those now commonly used for acceptance inspection, and if it could be shown that data validation would provide equivalent quality protection to the purchaser, then it was believed that worthwhile cost savings could be achieved in the quality control function.

Data validation by the Kolmogorov-Smirnov method would require that suppliers record variables type inspection information. However, in most cases where complex and/or high reliability product is being manufactured this requirement is already being imposed on suppliers.

Hypothesis to be Tested

The hypothesis to be tested (H_I) can be stated as follows: The sample n_1 required for the Kolmogorov-Smirnov test to validate suppliers variables-type inspection and/or test data for acceptance of product with equivalent buyer protection is less than the sample n_2 required by sampling plans in Mil-Std-105D[5] commonly used for most procurement inspection. The alternative hypothesis (H_A) is the

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[5]Mil-Std-105D, *op. cit.*, p. 40.

converse of the above. The two hypotheses are stated in mathematical terms as follows:

$$H_I : n_1 < n_2$$

$$H_A : n_1 \geq n_2$$

If hypothesis H_I could be proved, then logically it follows that the cost of acceptance inspection can be reduced as less inspection will be performed. The actual reduction in inspection costs would not be a strictly linear function as there are some non-variable costs such as set-up time and reporting. However, the actual time to perform inspection is linearly related to units of product.

Method of Solution

At the suggestion of my thesis committee a computer program was developed to provide a Monte Carlo simulation of the problem that would offer empirical proof of the hypothesis.

Monte Carlo method is a name given to that branch of mathematical statistics that is concerned with experiments on random numbers.[6] By utilizing the Monte Carlo approach

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[6]J. M. Hammersley and D. C. Handscomb, Monte Carlo Methods (New York: John Wiley and Sons, Inc., 1964), p. 2.

it was possible to examine more extreme conditions than could be readily found in available data. According to Hammersley and Handscomb, artificial data may be preferable if it is easier to amass or if it permits varying the vital statistics to an extent that nature will not permit.[7]

A wealth of material on the Kolmogorov-Smirnov test was found but very little evidence of the actual power of the test except in the isolated case of a normal distribution with different means but with the same standard deviation. Some apparently unfounded inferences were drawn from this material by different statisticians which is covered in more detail in Chapter II.

Hammersley and Handscomb pointed out that one of the main strengths of theoretical mathematics is its concern with abstraction and generality: one can write symbolic expressions or formal equations which abstract the essence of a problem and reveal its underlying structure. However, this same strength carries with it an inherent weakness: the more general and formal its language, the less is theory ready to provide a numerical solution in

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[7] Ibid.

a particular application.[8]

In seeking to develop a proof for the hypothesis, a comparison was made of (1) two normal distributions with the same means and with different standard deviations, (2) two normal distributions with same standard deviations but with different means, (3) two skewed distributions with same means but with different standard deviations and (4) two skewed distributions with same standard deviations but with different means.

In addition, one comparison was made of two normal distributions with different means and different standard deviations and then two comparisons were made where the shape of the Weibull distribution was varied and comparisons made between two identical shapes except for a shift in the mean.

The skewed distributions were represented by the Weibull distribution with a shape parameter $\beta = 1.5$ except for the two immediate aforementioned non-normal distributions which had shape parameters of $\beta = 1.0$ and $\beta = 2.0$. In Chapter V a more complete explanation is given of the Weibull distribution and its parameters.

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[8] Ibid., p. 3.

The Weibull distribution was selected to represent a skewed distribution as one of the Weibull distributions can approximate to a very close degree any skewed distribution. Sandia Corporation has prepared an unpublished computer program for selecting the parameters of a Weibull distribution to fit any given set of skewed data. Examples of Weibull distributions with a shape parameter $\beta = 1.5$ are shown in Figures 12-17.

Some typical inspection data on previously accepted product were to be matched with a suitable Weibull distribution. However, after reviewing several sets of skewed data, it was found that more information would be gained by selecting a more extreme case represented by a Weibull with a shape parameter of $\beta = 1.5$.

The computer program was set up to randomly select two sets of data using a normal random generator, one set from each of the two different normal distributions, then to rank order the two sets of data, make a cumulative frequency distribution of the rank order data, subtract at each interval one step function from the other, store the absolute value, select the highest absolute value and print out this value which is then compared with k in the Kolmogorov-Smirnov (K-S) statistic k/n. For comparisons of Weibull distributions a different random number generator was used. This generator randomly selects from a rectangular distribution and uses a transform equation to convert the number for the Weibull distribution. Otherwise, the program is the same.

The complete computer program is included in Appendix A.

The initial research procedure called for a test of the hypothesis first with sample sizes of twenty, then if time permitted, sample sizes of fifteen, ten, and perhaps five. As the investigation progressed it became apparent that investigation of smaller sample sizes would not yield any significant information at this time so the decision was made to concentrate on obtaining greater depth of information with sample sizes of twenty.

A total of 15,911 tests were run. As each test involved the generation of 40 samples, 636,440 individual samples were generated.

Operating characteristic curves were plotted for the major comparisons and values were listed in tables for all comparisons to show the relative power of the K-S test. This is covered more fully in Chapters IV and V.

Scope and Limitations

To keep the scope of the investigations within reasonable bounds the following limitations were imposed:

1. Test the hypothesis using data to represent only one quality characteristic, a reasonable limitation because if the test will work for one characteristic it should work for more than one.
2. Compare results of data validation with single sampling plans for normal inspection in Mil-Std-105D,[9] a not unreasonable limitation as most industrial companies and government inspection agencies follow this plan most of the time.
3. Make the assumption that product characteristics will follow either a normal or a skewed distribution, since almost all cases of product characteristics follow either a normal or skewed distribution.
4. Use the Weibull distribution with parameters $\alpha = 1.0$, $\beta = 1.5$ and $\gamma = 0$ to represent a worst case condition of skewed data.
5. Investigate the area where lot sizes would be 151 or greater and arbitrarily use a sample

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[9]Mil-Std-105D, op. cit., pp. 9-12.

size of twenty (20) for making the Kolmogorov-Smirnov test for validation of suppliers data for all lot sizes larger than 151. Single sampling plans for normal inspection in Mil-Std-105D[10] use a simple size of thirty-two (32) for lots of 151 to 280 and progressively increase for lot sizes larger than 280.

Definitions of Terms Used

α Level of significance - the probability of committing a Type I error which is to reject a hypothesis when it is true.

Absolute value - magnitude of a number without regard to its sign always expressed as a positive number.

Acceptable Quality Level (AQL) - the percentage of defective items in a lot that the purchaser or customer is willing to tolerate most of the time.

Attributes inspection - inspection wherein the unit of product is classified as defective or nondefective with respect to a given requirement or set of requirements.

β Level of significance - the probability of committing a Type II error which is to accept a hypothesis

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[10] Mil-Std-105D, op. cit., pp. 9-10.

when it is not true.

Mean - a measure of central location obtained by summing all the X values of the distribution and dividing by the number of readongs, i.e., $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} Xi$. Physically it is the X-coordinate of the center of gravity of the area under the probability density curve for the population.

Normal distribution - sometimes called the Gaussian distribution. It has a probability density function given by $f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma^2}}$ with mean μ and standard deviation σ. It occurs frequently in practical problems and is easy to use because its properties have been thoroughly investigated.

Normal inspection - inspection which is used when there is no statistically significant evidence that the quality of the product being submitted is better or poorer than the specified quality level.

Null hypothesis - the hypothesis of no differences normally designated as H_o. It is usually formulated for the express purpose of being rejected.

Operating characteristic (OC) curve - the curve of a sampling plan which shows the relationship between the quality of submitted inspection lots and the probability of accepting such inspection lots when subjected to that sampling plan.

Parameter - any measurable characteristic of a population. For a given population the characteristic will be a constant.

Population - any set of individuals or objects having some common observable characteristic. Sometimes called a universe.

Probability - if any event may happen in "a" ways and fail to happen in "b" ways, and all these ways are naturally exclusive and equally likely to occur, the probability of the event happening is $a/a + b$, the ratio of the number of ways favorable to the event to the total number of ways.

Probability function - the probability function of a discrete variate is the function $P(X)$ giving the probability of the variate assuming any given value X.

Probability density function - $P(X) = d\,F(X)/dX$ where $F(X)$ is the cumulative distribution function of a continuous variate X or roughly speaking, $P(X)$ means the probability of obtaining a value X or infinitely near X.

Quality characteristic - a physical, chemical visual or any other measurable property of a product or material.

Random sample - when every individual in the population has an equal and independent chance of being chosen.

Sample - one or more units of product selected at random from the material or process represented.

Significant - means that a result deviates from some hypothetical value by more than can reasonably be attributed to the chance errors of sampling.

Statistic - a value computed entirely from a sample.

Standard deviation - a measure of dispersion of a set of numbers defined as the root-mean-square deviation of the observed numbers from their arithmetic mean. Expressed in algebraic terms, this is

$$\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \bar{x}) / n}$$

Variables inspection - inspection in which the characteristic of interest is measured rather than merely classified qualitatively as in sampling by attributes.

Verification - a combination of monitoring actions, inspection or both for the purpose of determining compliance of the contractor with the provision of the contract in regard to quality.

Weibull distribution - the Weibull distribution has a probability density function given by

$$f\left(x:\alpha,\beta,\gamma\right)=\beta\frac{[x-\gamma]}{\alpha^{\beta}}e^{-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}},\ x>\gamma$$

where $\alpha > o$ is a scale parameter in X units, $\beta > o$ is a shape parameter (dimensionless) and γ (any real value) is a location parameter. The Weibull distribution is gaining wide recognition at the present time in reliability studies because of its ease in approximating actual data. Nelson[11] and Plait[12] have described in published articles two methods of approximating data by the Weibull distribution and Berrettoni[13] has demonstrated in another article the wide spread applicability of the Weibull distribution in describing empirical data.

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[11] Lloyd S. Nelson, "Weibull Probability Paper," Industrial Quality Control, Vol. 23, No. 9 (March, 1967), pp. 452-453.

[12] Alan Plait, "The Weibull Distribution," Industrial Quality Control, Vol. XIX, No. 5 (November, 1962), pp. 17-26.

[13] J. N. Berrettoni, "Practical Applications of the Weibull Distributions," Industrial Quality Control, Vol. 21, No. 2 (August, 1964), pp. 71-79.

CHAPTER II

HISTORICAL BACKGROUND

In 1933 A. N. Kolmogorov[1] suggested a test of the null hypothesis H_o: $U(X) = F(X)$ for the goodness-of-fit problem based on the statistic

$$K_n = \sqrt{n} \sup_{-\infty < X < \infty} \left| F_n(X) - F(X) \right|$$

H_o is to be rejected if K_n is sufficiently large. The distribution of K_n is independent of $F(X)$ if H_o is true (i.e., <u>the test is distribution free</u>) and denoting its distribution by $\Phi_n K(X)$ Kolmogorov proved that

$$\lim_{n \to \infty} P \left\{ K_n < X \right\} = \lim \Phi_n(X) = \Phi(X) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2 x^2}$$

In 1939 N. V. Smirnov[2] developed a simpler proof and proved that the random variable

$$D_{mn} = \sqrt{\frac{mn}{m+n}} \sup_{-\infty < X < \infty} \left| F_n(X) - G_m(X) \right|$$

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[1] D. A. Darling, "The Kolmogorov-Smirnov, Cramer Von Mises Tests," *Annals of Mathematical Statistics*, Vol. 28 (1957), pp. 827-828.

[2] *Ibid*.

with distribution function $\Phi_{m,n}$ had if $o < a \leq \frac{m}{n} \leq b < \infty$, $m \longrightarrow \infty$, $n \longrightarrow \infty$ a limiting distribution Φ equal to Kolmogorov's Φ (X).

The properties of similarity and consistency together with a certain mathematical elegance gave the Kolmogorov-Smirnov test a wide appeal to mathematical statisticians. A considerable literature has developed over the years and the proposer of the test was awarded a Stalin prize by the U. S. S. R. (Kolmogorov and Hincin, 1951).[3]

The original proofs of both Kolmogorov and Smirnov were very intricate and two American statisticians, Feller[4] and Doob[5] set out to develop simplified proofs for the limiting forms of the difference between two cumulative distributions. There was nothing in either proof concerning the power of the test to detect varying differences in two distributions.

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[3]J. L. Hodges, Jr., "The Significance Probability of the Smirnov Two-Sample Test," Arkiv For Maternatik, Band 3, nr43 (June, 1957), p. 469.

[4]W. Feller, "On The Kolmogorov-Smirnov Limit Theorems For Empirical Distributions," Annals of Mathematical Statistics, Vol. 19 (1948), pp. 177-189.

[5]J. L. Doob, "Heuristic Approach To The Kolmogorov Smirnov Theorems," Annals of Mathematical Statistics, Vol. 20 (1949), pp. 393-403.

Following Feller's and Doob's work, Drion in 1952 described a method for obtaining an exact answer to the problem of the maximum difference between two empirical cumulative distributions of random samples from the same population, but only in the case if both samples are equal. His article explains some of the rationale behind the Kolmogorov-Smirnov test.[6]

In 1951 Massey became concerned with the power of the Kolmogorov-Smirnov test and in an article discussed its potential as a distribution free test and presented evidence that when applicable it may be better than the χ^2 test.[7]

In 1954 Dixon published a study he had made in which he campared the power of four nonparametric tests when comparing two samples drawn from normal populations with equal variance (equal standard deviations) but with different means. The study was interesting but covered only one test of the normal distribution for which there

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[6]E. F. Drion, "Some Distribution-Free Tests For the Difference Between Two Empirical Cumulative Distribution Functions," Annals of Mathematical Statistics, Vol. 23 (1952), pp. 563-574.

[7]Frank J. Massey, Jr., "The Kolmogorov-Smirnov Test For Goodness of Fit," Journal of the American Statistical Association, No. 51 (March, 1956), pp. 111-121.

is a very powerful test, the "t" test.[8]

In 1956 Miller became concerned with the extent of the then published tables of the Kolmogorov-Smirnov statistic and published tables which extended the percentage points in the distributions of D_n^+ and D_n for samples up to 100.[9]

In 1957 Hodges published a study in which he examined the probabilities associated with the Kolmogorov-Statistic. He found some significant errors when the sample sizes were not the same.[10]

In 1960 Birnbaum and Hall developed some tables for the Kolmogorov-Smirnov Statistic which went up to sample size 40 and included the case of 3 samples.[11]

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[8] W. J. Dixon, "Power Under Normality of Several Nonparametric Tests," Annals of Mathematical Statistics, Vol. 24 (1954), pp. 610-613.

[9] L. H. Miller, "Table of Percentage Points of Kolmogorov Statistics," Journal of the American Statistical Association No. 51 (March, 1956), pp. 111-121.

[10] Hodges, op. cit., pp. 469-486.

[11] Z. W. Birnbaum and R. A. Hall, "Small Sample Distribution For Multi-Sample Statistics of the Smirnov Type," Annals of Mathematical Statistics, Vol. 31 (1960) pp. 710-720.

Although there are many more instances of published articles on the Kolmogorov-Smirnov test there are no known positive statements concerning the power of the test except in the case of a normal distribution with equal variance and different means.

A search of the literature revealed a very high interest in non-parametric statistics and evidence of some inferences which do not seem able to meet empirical evidence.

For example, Clelland and Tate, in an article published in 1957, stated that the Kolmogorov-Smirnov two-sample test is sensitive to population differences in either location or shape and the test is intuitively simple. They also stated the Kolmogorov-Smirnov test appears to be more powerful than the χ^2 test for testing the significance of differences between two distributions.[12]

Lehmann stated that if normality is not assumed and two distributions are treated as continuous cumulative distribution functions F & G, it is possible to test for equality and the resulting tests are nearly as powerful as

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[12]Richard C. Clelland and Merle W. Tate, Non-parametric And Shortcut Statistics (Danville: Interstate Printers and Publishers, Inc., 1957), p. 93.

the standard normal tests.[13]

Siegel stated in his book on nonparametric statistics that the Kolmogorov-Smirnov two-tailed test is sensitive to any kind of difference in the distributions from which the two samples are drawn--differences in location (central tendency), in dispersion, in skewness, etc.[14] Siegel also stated that when compared with the "t" test, the Kolmogorov-Smirnov test has a high power efficiency (about 96%) for small samples. The Kolmogorov-Smirnov test seems to be more powerful in all cases than either the χ^2 or the median test.[15] He further stated that if the researcher is interested in determining whether his two samples are from populations which differ in any respect at all, i.e., in location or dispersion or skewness, etc., he should choose one of these tests: the χ^2 test, the Kolmogorov-Smirnov test (two-tailed), or the Wald-Wolfowitz runs test.[16]

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[13] E. L. Lehmann, Testing Statistical Hypothesis (New York: John Wiley and Sons, Inc., 1959), pp. 232-234.

[14] Sidney Siegel, Nonparametric Statistics for the Behavioral Sciences (New York: McGraw-Hill Book Company, 1956), p. 127.

[15] Ibid., p. 136.

[16] Ibid., p. 157.

Goodman claimed that one of the important topics in the field of non-parametric methods is the Kolmogorov-Smirnov Statistic. Recent results and tables on the topic have been prepared which contribute toward establishing the Kolmogorov-Smirnov Statistic as a standard non-parametric tool of statistical analysis.[17]

The Quality Control Department of Sandia Corporation became interested in the Kolmogorov-Smirnov test for data validation as a result of an internal study dated March 1966 by M. C. Carter and D. D. Sheldon,[18] then employed as statisticians in the Quality Control Department. Carter and Sheldon recommended the use of the Kolmogorov-Smirnov two-sample test because of its desirable mathematical properties, ease of application and freedom from restrictive assumptions. A procedure for applying the Kolmogorov-Smirnov test was developed and issued and many of Sandia's Field Inspection personnel have been trained in how to apply the test.

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[17] Leo A. Goodman, "Kolmogorov-Smirnov Tests for Psychological Research," Psychological Bulletin, Vol. 51, No. 2 (1954), p. 160.

[18] Memorandum to A. F. Cone prepared by M. C. Carter and D. D. Sheldon entitled, "Proposed Method of Validating Supplier Data," Quality Control Department, Sandia Corporation, Albuquerque, New Mexico, March 29, 1966 (in the files of the department).

Data validation is of interest to Sandia Corporation because of its potential cost savings possibilities. It is also of interest to the Department of Defense who have issued a policy statement requesting that maximum use be made of supplies data in determining the acceptability of product.[19] The U. S. Department of Defense has published a Quality Control and Reliability Handbook H 109, entitled "Statistical Procedures for Determining Validity of Suppliers Attributes Inspection," prepared by H. Elner and J. Mandelson of Edgewood Arsenal.[20] The same writers prepared a rough draft handbook entitled, "Statistical Procedures for Validating Results of Sampling Inspection by Variables."[21]

Mandelson indicated that the Department of Defense was having difficulty getting inspectors to use publication

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[19] Quality Control and Reliability Handbook (Interim) H 109, op. cit., p. 1.

[20] Quality Control and Reliability Handbook (Interim) H 109, op. cit., pp. 1-20.

[21] H. Elner and J. Mandelson, "Statistical Procedures for Validating Results of Sampling Inspection by Variables," Rough draft proposal for a Department of Defense Handbook, Edgewood Arsenal, 1964 (in the files of Quality Control Department, Sandia Corporation).

H 109 because it was more difficult to use than the standard sampling plans and required a higher level of technical competence than is generally available in government inspectors.[22] Mandelson reflected on this same problem in an article in Industrial Quality Control published in 1964.[23] He indicated that when the initial research was done on H 109 they did not have a computer available and most of the empirical tests were conducted with a bead box. He also indicated that he believed more studies designed to test actual applications should be performed to determine the practical limitations of the newer statistical tests. He further believed that by calling attention to the findings of this study investigation impetus will be given to the statisticians to reexamine some of their initial premises.

In the following chapter the method of applying the Kolmogorov-Smirnov test will be discussed.

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[22] Interview by telephone with J. Mandelson, Quality Assurance Engineer, Edgewood Arsenal, March 31, 1967.

[23] Joseph Mandelson, "Product Verification," Industrial Quality Control, Vol. XX, No. 10 (April, 1964) pp. 8-11.

CHAPTER III

METHOD OF APPLYING THE KOLMOGOROV-SMIRNOV TEST

The Kolmogorov-Smirnov two-sample test is a test of whether two independent samples have been drawn from the same population (or from populations with the same distribution). This test is concerned with the agreement between two cumulative distributions. If the two samples have in fact been drawn from the same population distribution, then the cumulative distribution of both samples may be expected to be fairly close to each other, inasmuch as they both should show only random deviations from the population distribution. If the two sample cumulative distributions are too divergent at any point this suggests that the samples came from different populations. Thus, a large enough deviation between the two sample cumulative distributions is evidence for rejecting the null hypothesis.

To apply the Kolmogorov-Smirnov two-sample test, a cumulative frequency distribution is made for each sample of observations, using the same intervals for both distributions. For each interval, then, subtract one step function from the other. The test focuses on the <u>largest</u> of these observed deviations.

Let $S_{n_1}(X)$ = the observed cumulative step function of one of the samples, that is, $S_{n_1}(X) = k/n_1$ where k = the number of scores equal to or less than X. And let $S_{n_2}(X)$ = the observed cumulative step function of the other sample, that is, $S_{n_2}(X) = k/n_2$. Now the Kolmogorov-Smirnov two-sample test focuses on

$$d = \text{maximum} \left| S_{n_1}(X) - S_{n_2}(X) \right|$$

for a two-tailed test which is the criterion adopted. Now the sampling distribution of d is known[1,2] and is tabulated in Appendix C for the particular sample sizes used. Note that the aforementioned expression is concerned with the maximum absolute deviation irrespective of direction or sign of the individual values in the cumulative distributions.

In the use of the Kolmogorov-Smirnov test on data,

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[1]N. Smirnov, "Table for Estimating the Goodness of Fit of Empirical Distributions," <u>Annals of Mathematical Statistics</u>, Vol. 19 (1948), pp. 279-281.

[2]F. J. Massey, Jr., "Distribution of the Maximum Deviation Between Two Sample Cumulative Step Functions," <u>Annals of Mathematical Statistics</u>, Vol. 22 (1951), pp. 125-128.

if the data are arranged in a cumulative frequency distribution with intervals similar to a histogram then it is well to use as many intervals as possible. When too few intervals are used the maximum deviation of two cumulative step functions may be obscured, thus leading to possible false conclusions.

In the detailed procedure issued to Sandia Corporation Field Inspectors on data validation (see Appendix B for complete procedure), the size and number of intervals are determined by dividing the range of data by the Sandia sample size or one-half the total of both sample sizes. The use of intervals for grouping data to construct a cumulative frequency histogram is used by Sandia Corporation because it was believed that it would be easier to use, easier to understand and less prone to error when used by people whose main interest is inspection or test and the technology that supports these functions.

When performing the Kolmogorov-Smirnov test on the computer, the data were rank ordered; then one cumulative distribution was evaluated against the other on the basic data. This procedure eliminates the possibility of obscuring the maximum deviation but is more difficult to use and more prone to human error when done by hand.

The procedure "Instruction For Data Validation Kolmogorov-Smirnov Test" included as Appendix B is believed to ba a satisfactory operating procedure that could be followed by almost any reasonably qualified inspector or test technician.

To summarize, the method of applying the Kolmogorov-Smirnov test utilizes the following steps:

1. Arrange each of the two samples of data in a cumulative frequency distribution, using the same intervals for both distributions. Use as many intervals as possible.

2. By subtraction, determine the difference between the two sample cumulative distributions at each interval.

3. By inspection, determine the largest of these differences d.

4. Compare d against the criteria supplied for accepting or rejecting the suppliers data.

The next chapter discusses the subject of testing a hypothesis with normally distributed data.

CHAPTER IV

TESTING HYPOTHESIS WITH DATA NORMALLY DISTRIBUTED

General Comments

To evaluate the effectiveness of the Kolmogorov-Smirnov test for use in data validation, the difference in per cent was calculated between the two normal distributions being compared by considering as the criteria, one of the distributions with a $\pm 3\sigma$ spread. Considering the extreme case where this distribution spread just equals the specification spread, then any values of the other distribution outside of these limits would represent defective product.

The calculations for differences between the series of two normal distributions chosen for comparison are included in Appendix D. The statements of difference are made as probability statements because individual cases will not always bear out the exact statements of the normal probability density function. Probability statements were converted to per cent figures rather than decimal figures to simplify comparison with the operating characteristic curves from Mil-Std-105D.[1] The curves of interest that were chosen for comparison were copied and placed in Figure 24 for the readers convenience. Standard tables covering the normal distribution were used in calculations and these have been referenced in Appendix D.

In the initial research plan it was planned to compare the probability of accepting a lot of material with a particular per cent defective utilizing data validation with sample sizes of twenty (20) against the probability of accepting this material under a single sampling plan which would require larger sample sizes for equivalent lot sizes. After the results from the simulation program became apparent it was decided to compare with an equal sample size in Mil-Std-105D[2] which appeared more meaningful under the circumstances.

Three levels of significance or α risks namely the 1%, 5%, and 10% were chosen. The 1% level was judged meaningless in this comparison because the customer is interested in detecting trouble and should certainly be

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[1]Mil-Std-105D, op. cit., p. 40.

[2]Mil-Std-105D, op. cit., pp. 40-51.

willing to make a decision and search for trouble 5% or even 10% of the time when none actually exists. The illustrations show the three (3) levels plotted and examination of the tables with the aid of Appendix C allow the reader to draw inferences about almost any level of significance. In all of the following discussions concerning the Kolmogorov-Smirnov tests, the comparisons will be made at the 5% level because this level is more commonly understood and used. The possibility of using higher α risks to make the concept work was considered if the Kolmogorov-Smirnov test had matched the advanced claims made for it. However, this turned out to be a fruitless consideration.

The levels of significance adopted and used in the illustrations for the 1%, 5%, and 10% levels are rounded figures taken from Appendix C. It is not possible to get even numbers for the Kolmogorov-Smirnov statistic because the k portion of the statistic is a whole number that does not convert to an even number for a particular level of significance (see Appendix C). In the case of the 5% and 10% level the figures are conservative, the actual α risks are 3.35% and 8.81%. In the case of the 1% level the α risk is understated, the actual α risk is 1.23%.

In attempting to get the program operating and establish the times required on the computer for running the tests, some differences evolved in the amount of tests for each comparison. These vary from a minimum of 317 tests to a high of 2500 with the norm finally established at 500. It seemed advisable to repeat some of the tests to determine if the random number generator was behaving properly, particularly as it was important that a new random octal number be furnished each time the program was run on the computer. This test is discussed later in this chapter.

No tests were planned comparing normal distributions with different means but same standard deviations. After the first results of comparisons of normal distributions with different standard deviations but same means it was decided to add these tests. The original concept for testing was to compare two distributions where only one parameter would be varied such as the standard deviation or the mean. However, as testing progressed a test where both the standard deviation and the mean were varied was added.

Tests for a Difference When One Mean is Varied

Four different comparisons of normal distributions with the same standard deviations were made. In one case two identical normal distributions were compared with the same means and standard deviations to check the α risks obtained by the simulation program with those stated in Appendix C.

In the other three cases, the means of one distribution were changed to obtain an estimate of the shape of the operating characteristic curves for α risks of 1%, 5%, and 10%. When the curves were constructed in Figure 20 for these comparisons, the curves with the risks were labeled in decimals and listed with the equivalent numerator k of the Kolmogorov-Smirnov statistic. This was done for all charts that show operating characteristic curves.

To better illustrate the actual comparisons which are shown in Table I and illustrated in Figure 20, graphs were constructed showing the distributions as they were compared. These are shown in Figures 1, 2, and 3.

Table 1 includes all of the actual results of these comparisons plus the calculated difference in per cent of the X or data values. Figure 20 shows the operating characteristic curves constructed to represent these calculated differences.

Examination of Table 1 and Figure 20 for the case where two identical distributions are compared, shows that at the 1% level the test would indicate no significant difference 99% of the time, at the 5% level no significant difference 96.6% of the time and at the 10% level no significant difference 90.8% of the time. This evidence verifies that the α risks are realistic.

Examination of Table 1 and Figure 20 for the remaining cases shows that at the 5% α risk ($d' \leq k = 8$ for acceptance) the Kolmogorov-Smirnov test will reject differences of 0.38%, 21.9% of the time; differences of 2.00%, 71.5% of the time; differences of 6.42%, 96.4% of the time. Comparing this with the OC curves for an equivalent example size for single sampling shown in Figure 24 shows that even with the tightest criteria given, an AQL of 0.65, that single sampling would only reject lots with defectiveness equivalent to a difference of 0.38%, about 7% of the time, defectiveness of 2% about 35% of the time; defectiveness of 6.42% about 73% of the time.

Under the circumstances stated in the aforementioned paragraph for the case of the two normal distributions with only a change in the mean, data validation would be superior to a single sampling plan.

The results of two repeat tests made to determine if the random number generator was behaving properly are shown in Table 2. Examination of the tabulated results show they are not identical but are very close and would logically be expected for two different random samples. The distributions of d' are not normal but skewed so no normal tests for significance were attempted. In lieu of any test for significance the means were compared. This was accomplished by calculating the means of the distribution d' for the initial test and retest and comparing them. Means were calculated from the data in Table 2 using the formula $\bar{d}' = \sum_{i=2}^{17} f\, d_i'/n$ with the following results:

1. $\bar{d}_1'$ (test $\mu_1 = 0,\ \mu_2 = 0.5) = 6.742$

   $\bar{d}_2'$ (retest $\mu_1 = 0,\ \mu_2 = 0.5) = 6.740$

2. $\bar{d}_1'$ (test $\mu_1 = 0,\ \mu_2 = 1.0) = 9.836$

   $\bar{d}_2'$ (retest $\mu_1 = 0,\ \mu_2 = 1.0) = 9.938$

The difference in means for the first retest was negligible and for the second retest was about 1% which appears well within the bounds of reasonable expectation.

Included in this section is a comparison where both the standard deviation and the mean were varied.

This test as mentioned earlier was added for information after running the tests where the means were the same and the standard deviation was varied. This comparison is illustrated in Fugure 4 and the results are shown in Table 3 together with the calculated difference in % of the X or data values (see Appendix D for calcualtions). As only one comparison was made no thought was given to an operating characteristic curve. Examination of the data using a k of 8 which corresponds to an α risk of 5% for the Kolmogorov-Smirnov test of $d' \leq k/n$ shows that the test would detect a difference between the two distributions of 14.40%, 54.6% of the time; this corresponds to a rejection of about 96% of the time with a single sampling plan where the AQL = 0.65. This comparison indicates the Kolmogorov-Smirnov test is not sensitive enough when both the mean and standard deviation vary.

Tests For a Difference When One Standard Deviation Is Varied

Eight comparison tests were run to test the difference between two normal distributions when the means were equal but the standard deviation of one distribution was varied. The graphs illustrating these comparisons are shown in Figures 5-11. The first comparison of two identical distributions is not shown in a graph and was put in Table 4 for information purposes. This comparison is identical to the comparison used in the previous section where two normal distributions with same standard deviations but with different means were compared.

The results of these comparisons are shown in Table 4 which includes the calculated differences in % of the X or data values (see Appendix D for calculations). The operating characteristic curves for this type of comparison are shown in Figure 21. These curves are constructed with an X axis showing the difference between the two distributions using the criteria established earlier in this chapter.

Examination of Table 4 and Figure 20 shows that at the 5% α level ($d' \leq k = 8$ for acceptance) the Kolmogorov-Smirnov test will reject differences of 4.30%, 6.3% of the time; 13.10%, 10.5% of the time; 22.76%, 18.1% of the time; 31.48%, 29.2% of the time; 45.06%, 46.2% of the time; 54.60%, 58.2% of the time; 61.44%, 72.8% of the time. Comparing these results with the operating characteristic curve for a single sampling plan with equivalent sample size and AQL = 0.65 shows that this plan would reject lots with defectiveness of 4.30%, about 59% of the time;

13.10%, about 94% of the time; 22.76% better than 99% of the time; 31.48% better than 99.9+% of the time; 45.06% better than 99.9+% of the time; 54.60%, better than 99.9+% of the time; 61.44%, better than 99.9+% of the time.

From the discussion in the aforementioned paragraph it becomes apparent that the Kolmogorov-Smirnov test has insufficient power for effectively discriminating between two normal distributions when the standard deviations are varied.

The next chapter discusses the subject of testing a hypothesis with data represented by a skewed distribution.

CHAPTER V

TESTING HYPOTHESIS WITH DATA REPRESENTED BY A SKEWED DISTRIBUTION

Weibull Distributions As a Match For Any Skewed Distribution

The Weibull distribution was proposed by Waloddi Weibull in an article in which he pointed out the great potential for his distribution to fit data covering a wide variety of phenomena such as yield strength of steel, size distribution of fly ash, fiber strength of cotton, life of electric bulbs, and similar phenomena.[1] His distribution has been widely used in life testing and reliability studies in the United States. Weibull pointed out that objection had been raised to his distribution function because it had no theoretical basis. He stated that with very few exceptions the same objections could be applied against all other distribution functions applied to real populations from natural or biological

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[1] Waloddi Weibull, "A Statistical Distribution Function of Wide Applicability," Journal of Applied Mechanics, Vol. 18 (September 1961), pp. 293-297.

fields, at least insofar as the theoretical basis has anything to do with the population in question. He stated his belief that in most cases the only practical way of progressing is to choose a simple function, test it empirically, and stick to it as long as none better has been found.[2]

The Weibull distribution was chosen to represent this study of skewed data because the distribution is widely used in Sandia Corporation and a computer program is available to fit a Weibull distribution to almost any data. Upon review of some final acceptance data with skewed distributions a shape was selected that was more extreme than any of the data reviewed to obtain a worst case condition. The distribution selected had a scale parameter $\alpha = 1.0$, shape parameter $\beta = 1.5$ and location parameter $\gamma = 0$. Examples of what this distribution looks like are shown in Figures 12, 13, and 14. In order to obtain the equivalence of a change in mean or standard deviation the parameters were changed to accomplish this as is explained in the following paragraphs.

To obtain the differences between two Weibull distributions representing a change in the mean or

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[2] Ibid., p. 293.

standard deviation, the differences were calculated from the equations for the Weibull distribution due to an inability to locate any tabled values similar to the normal distribution. In somewhat the same manner as used for the normal distribution, the extreme case where the $\mu + 3\sigma$ distribution spread of the Weibull just represents the specification spread was considered. The $\mu + 3\sigma$ distribution spread (slightly different than normal which is $\mu \pm 3\sigma$) represents 98.5+% of all the X or data values. These calculations are shown in Appendix D.

Tests For a Difference When One Mean Is Varied

Four comparison tests were run to test the difference between two Weibull distributions of the skewed shape characterized by a shape parameter $\beta = 1.5$ where the standard deviations are the same (controlled by the scale parameter $\alpha = 1.0$) and one mean was changed (controlled by a change in γ the location parameter). Again, as in the case of the normal distribution two identical Weibull distributions were compared to verify the risks. The results were almost identical with the results when comparing two identical normal distributions.

The graphs (drawn to scale) illustrating these comparisons are shown in Figures 12, 13, and 14.

The results of these comparisons are shown in Table 5 which includes the calculated differences in % of the X or data values (see Appendix D for calculations). The operating characteristic curves for this type of comparison are shown in Fugure 22. These curves are constructed with an X axis showing the difference between the two distributions using the criteria discussed earlier in the chapter.

Examination of Table 5 and Figure 22 for the case where two identical Weibull distributions are compared show that at the 1% level the test would indicate no significant difference 98.4% of the time (as stated earlier in Chapter IV the 1% α risk is understated), at the 5% level no significant difference 96.4% of the time and at the 10% level no significant difference 91.6% of the time. This evidence verifies that the α risks are realistic.

Examination of Table 5 and Figure 22 for the remaining cases shows that at the 5% α risk ($d' \leq k = 8$ for acceptance) the Kolmogorov-Smirnov test will reject differences of 4.80%, 8.6% of the time; differences of 14.61%, 24.1% of the time; differences of 26.66%, 42.4%

of the time. Comparing this with the operating characteristic curve for a single sampling plan with equivalent sample size and AQL = 0.65 shows that the plan would reject lots with defectiveness of 4.8%, about 65% of the time; 14.61%, about 96% of the time; 26.66% better than 99+% of the time.

From the discussion in the aforementioned paragraph it becomes apparent the Kolmogorov-Smirnov test has insufficient power for effectively discriminating between two skewed distributions represented by a Weibull shape parameter of $\beta = 1.5$ when the standard deviations are varied.

Tests for a Difference When One Standard Deviation is Varied

Four comparison tests were run to test the difference between two Weibull distributions of the skewed type characterized by a shape parameter $\beta = 1.5$ where the means are the same (controlled by the location parameter $\gamma = 0$) and one standard deviation was changed (controlled by a change in α, the scale parameter).

The graphs illustrating these comparisons are shown in Figures 15, 16, and 17. The first comparison of two identical distributions is not shown in a graph and was put in Table 6 for information purposes. This comparison is identical to the comparison used in the previous section where two Weibull distributions of shape parameter $\beta = 1.5$ with same standard deviations but with different means were compared.

The results of these comparisons are shown in Table 6 which includes the calculated differences in % of the X or data values (see Appendix D for calculations). The operating characteristic curves for this type of comparison are shown in Figure 23. These curves are constructed with an X axis showing the difference between the two distributions using the criteria discussed earlier in this chapter.

Examination of Table 6 and Figure 23 shows at the 5% α level ($d' \leq k = 8$ for acceptance) the Kolmogorov-Smirnov test will reject differences of 3.61%, 12.2% of the time; 14.20%, 51.8% of the time; 24.98%, 100% of the time. Comparing this with the operating characteristic curve for a single sampling plan with equivalent sample size and AQL = 0.65 shows that this plan would reject lots with defectiveness of 3.61%, about 54% of the time; 14.20%, about 96% of the time; 24.98%, better than 99+% of the time.

From the discussion in the aforementioned paragraph it again appears that the Kolmogorov-Smirnov test does not have adequate power to discriminate in the area below 20% defectiveness for this type of comparison.

As a follow-up investigation on the results of the aforementioned comparisons of Weibull distributions with shape parameters of $\beta = 1.5$, the effect of varying the shape parameter was checked. Two shapes were chosen, one with a shape parameter of $\beta = 2.0$ and the other with a shape parameter of $\beta = 1.0$. These two shapes straddle the skewed distribution of $\beta = 1.5$. The graphs illustrating these comparisons are shown in Figures 18 and 19.

Comparison tests of both of these distributions were made in which the standard deviations were the same but the mean was varied. The results of these comparisons are shown in Table 7 which includes the calculated differences in % of the X or data values (see Appendix D for calculations). As only one comparison was made for each distribution no thought was given to an operating characteristic curve.

Examination of Table 7 shows at the 5% α level ($d' \leq k = 8$ for acceptance) the Kolmogorov-Smirnov test will detect a difference between the two distributions when $\beta = 2.0$, $\alpha = 1$, $\gamma_1 = 0$, $\gamma_2 = 0.37$ of 11.07%, 46% of the time and detect a difference when $\beta = 1.0$, $\alpha = 1$, $\gamma_1 = 0$, $\gamma_2 = 0.25$ of 21.72%, 17.2% of the time. Comparing this with the operating characteristic curve for a single sampling plan with equivalent sample size and AQL = 0.65 shows that the plan would reject lots with defectiveness of 11.07%, about 92% of the time; 21.72%, 99+% of the time.

These tests indicated that for other skewed distributions the Kolmogorov-Smirnov test does not have adequate power to detect shifts in the mean which create substantial differences of X or data values between the two distributions. These tests also confirmed the belief that for distributions that have substantial amounts of the distribution at one end, as was the case when the shape parameter was $\beta = 1.0$, the Kolmogorov-Smirnov test is a very weak test.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Summary

The original objective of this study was to develop an empirical proof to support published evidence that the Kolmogorov-Smirnov test could be used to validate supplier inspection and/or test results. Published evidence seemed to indicate that this could be accomplished with smaller inspection and/or test samples than are required in the standard attributes acceptance plans. If the aforementioned objective had been accomplished, then worthwhile cost savings could be achieved particularly in the case of large purchases of supplies produced to rigid specifications.

Considerable evidence appeared to be available in the literature indicating that the newer nonparametric or distribution-free tests would be suitable for data validation. The Kolmogorov-Smirnov test appeared to be especially attractive because of ease of application and apparent power of discrimination.

Two Sandia Corporation statisticians recommended the Kolmogorov-Smirnov test for use as a method of data validation to Sandia Corporation, a prime contractor to the U. S. Atomic Energy Commission. Sandia personnel were interested in using data validation to replace attributes sampling plans for acceptance inspection. The interest in data validation appears to have originated in and to have been influenced by the Department of Defense policy statement requiring that optimum use be made of suppliers' data to determine the acceptability of supplies.

In order to test the validity of the assumption that the Kolmogorov-Smirnov test was a satisfactory method of data validation, a computer program was prepared for a Monte Carlo simulation of actual conditions to prove the formulated hypothesis or its alternative.

Comparisons were made (1) between two normal distributions with the same standard deviations but with different means, (2) between two normal distributions with same means but with different standard deviations, (3) between two skewed distributions with the same standard deviations but with different means and (4) between two skewed distributions with the same means but with different standard deviations.

A total of 15,911 comparison tests were run. As each test required two samples of twenty units, the generation of 636,440 random sample units was required. The number of tests performed was judged to have furnished ample evidence for the conclusion reported for this study.

Conclusions

This study was designed to establish or refute the following hypothesis H_I:

The sample n_1 required for the Kolmogorov-Smirnov test to validate suppliers' variables-type inspection and/or test data for acceptance of product with equivalent buyer protection is less than the sample n_2 required by sampling plans in Mil-Std-105D commonly used for most procurement inspection. The alternative hypothesis H_A is the converse of the above. Expressed in mathematical terms, this is:

$$H_I : n_1 < n_2$$

$$H_A : n_1 \geq n_2$$

The results of this study supported by evidence from 15,911 tests showed that for one case, the case of two normal distributions with the same standard deviations but with different means, the formulated hypothesis was true. For all other cases, the alternative hypothesis was true.

A good test for data validation should be one where it is not necessary to know the underlying distributions of data (be distribution free) and where the power to distinguish differences between two distributions is very high. The Kolmogorov-Smirnov test is not distribution free as originally claimed, at least in a practical sense, and it has been shown to possess a very low power to distinguish differences except in one isolated case. It may be safely concluded from the evidence of this study that the Kolmogorov-Smirnov test is not satisfactory for use for data validation. As a result, the proposed trial implementation of the Kolmogorov-Smirnov method of data validation at Sandia Corporation was cancelled.

Questions concerning the validatity of claims in the published literature concerning the Kolmogorov-Smirnov and other nonparametric tests may be raised as a result of this study. It is intended that the results will be published in a professional journal to draw attention to the limitations of the test.

Further study is needed to find some satisfactory method of data validation. When new proposals are developed, it is believed that the method illustrated in this study can provide a powerful and eoncomical means of testing proposals.

TABLES

TABLE 1

DISTRIBUTION OF THE MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO NORMAL DISTRIBUTIONS WHERE μ_2 IS VARIED, $\mu_1 = 0$ AND $\sigma_1 = \sigma_2 = 1.0$

Value	$\mu_2 = 0$ *0		$\mu_2 = 0.5$ *0.38		$\mu_2 = 1.0$ *2.00		$\mu_2 = 1.5$ *6.42	
of d′	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
2	8	1.6	3	0.3				
3	60	13.6	49	5.2	2	0.2		
4	121	37.8	133	18.5	5	0.7	1	0.2
5	126	63.0	151	33.6	26	3.3	1	0.4
6	91	81.2	164	50.0	47	8.0	2	0.8
7	48	90.8	142	64.2	82	16.2	3	1.4
8	29	96.6	139	78.1	123	28.5	11	3.6
9	12	99.0	88	86.9	146	43.1	22	8.0
10	2	99.4	61	93.0	161	59.2	36	15.2
11	3	100.0	41	97.1	162	75.4	67	28.6
12			22	99.3	108	86.2	77	44.0
13			5	99.8	71	93.3	88	61.6
14			0	99.8	38	97.1	83	78.2
15			2	100.0	21	99.2	64	91.0
16					6	99.8	28	96.6
17					2	100.0	14	99.4
18							2	99.8
19							0	99.8
20							1	100.0
Total	500		1000		1000		500	

*Probability in % of values of one distribution being outside $\mu \pm 3\sigma$ limits of other distribution.

TABLE 2

CHECK FOR REPEATABILITY OF DISTRIBUTION OF MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING KOLMOGOROV-SMIRNOV TEST THE SECOND TIME TO TESTS BETWEEN TWO NORMAL DISTRIBUTIONS WITH EQUAL STANDARD DEVIATIONS $\sigma_1 = \sigma_2 = 0$. RETESTS MADE ON COMPARISON OF $\mu_1 = 0$ VERSUS $\mu_2 = 0.5$ AND COMPARISON OF $\mu_1 = 0$ VERSUS $\mu_2 = 1.0$

Value	Test $\mu_1 = 0$ vs. $\mu_2 = 0.5$		Retest $\mu_1 = 0$ vs. $\mu_2 = 0.5$		Test $\mu_1 = 0$ vs. $\mu_2 = 1.0$		Retest $\mu_1 = 0$ vs. $\mu_2 = 1.0$	
of d′	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
2	2	0.4	1	0.2				
3	29	6.2	20	4.2	2	0.4		
4	62	18.6	71	18.4	2	0.8	3	0.6
5	75	33.6	76	33.6	18	4.4	8	2.2
6	84	50.4	80	49.6	18	8.0	29	8.0
7	72	64.8	70	63.6	48	17.6	34	14.8
8	65	77.8	74	78.4	66	30.8	57	26.2
9	41	86.0	47	87.8	68	44.4	78	41.8
10	30	92.0	31	94.0	79	60.2	82	58.2
11	24	96.8	17	97.4	73	74.8	89	76.0
12	13	99.4	9	99.2	50	84.8	58	87.6
13	2	99.8	3	99.8	38	92.4	33	94.2
14	1	100.0	0	99.8	20	96.4	18	97.8
15			1	100.0	15	99.4	6	99.0
16					2	99.8	4	99.8
17					1	100.0	1	100.0
Total	500		500		500		500	

TABLE 3

DISTRIBUTION OF THE MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO NORMAL DISTRIBUTIONS WHERE BOTH μ_2 AND σ_2 ARE VARIED

Value of d′	$\mu_1 = 0,\ \sigma_1 = 1.0$ vs. $\mu_2 = 1.0,\ \sigma_2 = 1.8$ *14.40		
	Frequency	% of Total	Cumulative % of Total
3	5	1.0	1.0
4	12	2.4	3.4
5	26	5.2	8.6
6	51	10.2	18.8
7	60	12.0	30.8
8	73	14.6	45.4
9	78	15.6	61.0
10	69	13.8	74.8
11	68	13.6	88.4
12	33	6.6	95.0
13	15	3.0	98.0
14	8	1.6	99.6
15	2	0.4	100.0
Total	500	100.0	

*Probability in % of values of one distribution being outside $\mu \pm 3\sigma$ limits of other distribution.

TABLE 4

DISTRIBUTION OF THE MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO NORMAL DISTRIBUTIONS WHERE σ_2 IS VARIED, $\sigma_1 = 1.0$ AND $\mu_1 = \mu_2 = 0$

Value of d′	$\sigma_2 = 1.0$ *0		$\sigma_2 = 1.5$ *4.30		$\sigma_2 = 2.0$ *13.10		$\sigma_2 = 2.5$ *22.76	
	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
2	8	1.6	1	0.3				
3	60	13.6	24	7.9	9	1.5	12	0.5
4	121	37.8	71	30.3	46	9.2	84	3.9
5	126	63.0	69	52.1	120	29.4	316	16.5
6	91	81.2	63	72.0	158	56.0	549	38.5
7	48	90.8	52	88.4	133	78.4	604	62.7
8	29	96.6	17	93.7	66	89.5	481	81.9
9	12	99.0	10	96.9	35	95.4	236	91.3
10	2	99.4	7	99.1	20	98.8	122	96.2
11	3	100.0	3	100.0	4	99.5	70	99.0
12					2	99.8	15	99.6
13					1	100.0	8	99.0
14							3	100.0
15								
Total	500		317		594		2500	

$\sigma_2 = 3.0$ *31.48		$\sigma_2 = 4.0$ *45.06		$\sigma_2 = 5.0$ *54.60		$\sigma_2 = 6.0$ *61.44	
Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
30	1.2	1	0.2				
172	8.1	7	1.6	4	0.8		
422	25.0	32	8.0	10	2.8	7	1.4
594	48.8	93	26.6	68	16.4	32	7.8
550	70.8	136	53.8	127	41.8	97	27.2
364	85.4	120	77.8	133	68.4	136	54.4
214	93.9	66	91.0	84	85.2	104	75.2
80	97.1	31	97.2	41	93.4	73	89.8
57	99.4	9	99.0	22	97.8	32	96.2
11	99.8	3	99.6	8	99.4	14	99.0
6	100.0	2	100.0	3	100.0	2	99.4
						3	100.0
2500		500		500		500	

*Probability in % of values of one distribution being outside $\mu \pm 3\sigma$ limits of other distribution.

TABLE 5

DISTRIBUTION OF THE MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO WEIBULL DISTRIBUTIONS WHERE THE LOCATION PARAMETER γ_2 IS VARIED, $\gamma_1 = 0$ AND SCALE PARAMETERS $\alpha_1 = \alpha_2 = 1.0$, SHAPE PARAMETERS $\beta_1 = \beta_2 = 1.5$

Value	$\gamma_2 = 0$		$\gamma_2 = 0.14$		$\gamma_2 = 0.30$		$\gamma_2 = 0.40$	
	*0		*4.80		*14.61		*26.66	
of d′	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
2	6	1.2	4	0.8	3	0.3		
3	77	16.6	46	10.0	33	3.6	5	1.0
4	141	44.8	88	27.6	83	11.9	19	4.8
5	107	66.2	109	49.4	130	24.9	33	11.4
6	89	84.0	79	65.2	164	41.3	62	23.8
7	38	91.6	78	80.8	190	60.3	81	40.0
8	24	96.4	53	91.4	156	75.9	88	57.6
9	10	98.4	22	95.8	92	85.1	76	72.8
10	7	99.8	13	98.4	74	92.5	57	84.2
11	1	100.0	6	99.6	44	96.9	38	91.8
12			1	99.8	21	99.0	25	96.8
13			1	100.0	9	99.9	11	99.0
14					0	99.9	3	99.6
15					1	100.0	1	99.8
16							1	100.0
Total	500		500		1000		500	

*Probability in % of values of one distribution being outside $\mu + 3\sigma$ limits of other distribution.

TABLE 6

DISTRIBUTION OF THE MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO WEIBULL DISTRIBUTIONS WHERE SCALE PARAMETER (CONTROLS STANDARD DEVIATION) α_2 IS VARIED, $\alpha_1 = 1.0$ AND SHAPE PARAMETERS $\beta_1 = \beta_2 = 1.5$, LOCATION PARAMETERS $\gamma_1 = \gamma_2 = 0$

Value of d′	$\alpha_2 = 1.0$ *0		$\alpha_2 = 1.3$ *3.61		$\alpha_2 = 1.8$ *14.20		$\alpha_2 = 2.25$ *24.98	
	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total	Frequency	Cumulative % of Total
2	6	1.2	4	0.8				
3	77	16.6	38	8.4	5	1.0		
4	141	44.8	98	28.0	15	4.0		
5	107	66.2	93	46.6	28	9.6		
6	89	84.0	99	66.4	57	21.0		
7	38	91.6	60	78.4	61	33.2		
8	24	96.4	47	87.8	75	48.2		
9	10	98.4	35	94.8	84	65.0		
10	7	99.8	14	97.6	70	79.0		
11	1	100.0	7	99.0	40	87.0		
12			2	99.4	41	95.2		
13			3	100.0	17	98.6		
14					4	99.4		
15					3	100.0		
16							4	0.8
17							13	3.4
18							65	16.4
19							183	53.0
20							235	100.0
Total	500		500		500		500	

*Probability in % of values of one distribution being outside $\mu + 3\sigma$ limits of other distribution.

TABLE 7

DISTRIBUTION OF MAXIMUM ABSOLUTE DIFFERENCE (d) WHEN APPLYING THE KOLMOGOROV-SMIRNOV TEST TO TWO WEIBULL DISTRIBUTIONS WHERE DIFFERENT SHAPE PARAMETERS THAN $\beta = 1.5$ ARE USED, LOCATION PARAMETER γ_2 IS VARIED, $\gamma_1 = 0$ AND SCALE PARAMETERS $\alpha_1 = \alpha_2 = 1.0$

Value of d′	$\beta_1 = \beta_2 = 2.0$, $\gamma_1 = 0$ vs. $\gamma_2 = 0.35$ *11.07			$\beta_1 = \beta_2 = 1.0$, $\gamma_1 = 0$ vs. $\gamma_2 = 0.25$, *21.72		
	Frequency	% of Total	Cumulative % of Total	Frequency	% of Total	Cumulative % of Total
2				2	0.4	0.4
3	6	1.2	1.2	23	4.6	5.0
4	17	3.4	4.6	54	10.8	15.8
5	42	8.4	13.0	101	20.2	36.0
6	55	11.0	24.0	93	18.6	54.6
7	77	15.4	39.4	90	18.0	72.6
8	73	14.6	54.0	51	10.2	82.8
9	79	15.8	69.8	41	8.2	91.0
10	64	12.8	82.6	29	5.8	96.8
11	47	9.4	92.0	13	2.6	99.4
12	21	4.2	96.2	3	0.6	100.0
13	15	3.0	99.2			
14	1	0.2	99.4			
15	2	0.4	99.8			
16	1	0.2	100.0			
Total	500	100.0		500	100.0	

*Probability in % of values of one distribution being outside $\mu + 3\sigma$ limits of other distribution.

FIGURES

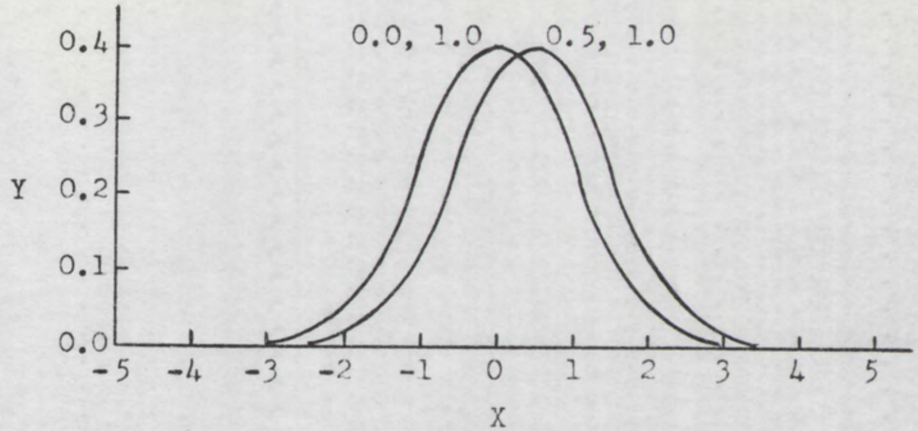

FIGURE 1. Comparison of Two Normal Distributions with Same Standard Deviation but with Different Means 0.0, 0.5

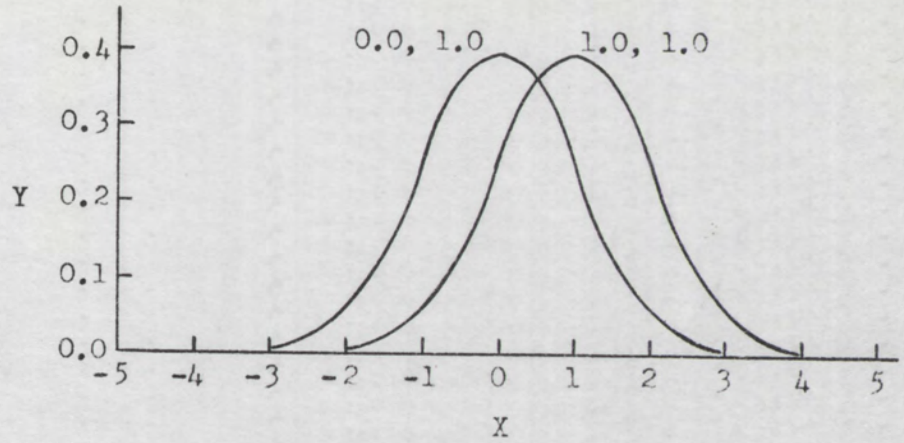

FIGURE 2. Comparison of Two Normal Distributions with Same Standard Deviation but with Different Means 0.0, 1.0

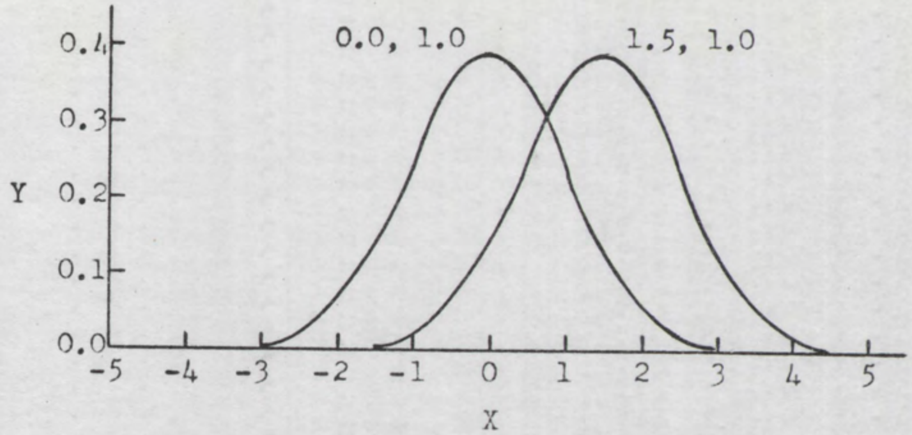

FIGURE 3. Comparison of Two Normal Distributions with Same Standard Deviation but with Different Means 0.0, 1.5

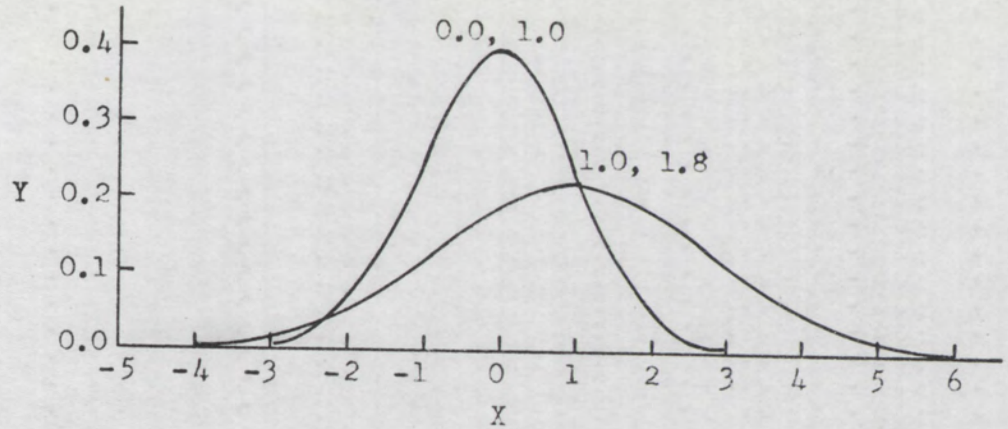

FIGURE 4. Comparison of Two Normal Distributions with Different Means 0.0, 1.0, and with Different Standard Deviations 1.0, 1.8

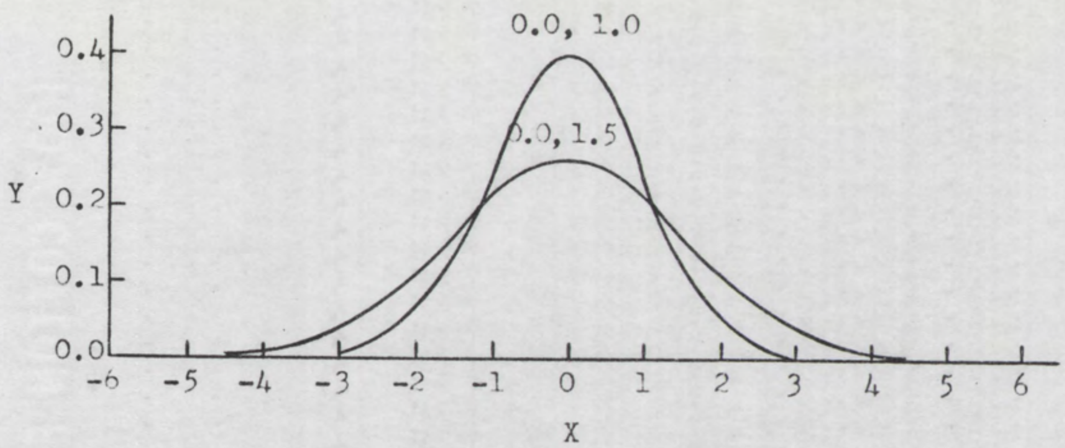

FIGURE 5. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 1.5

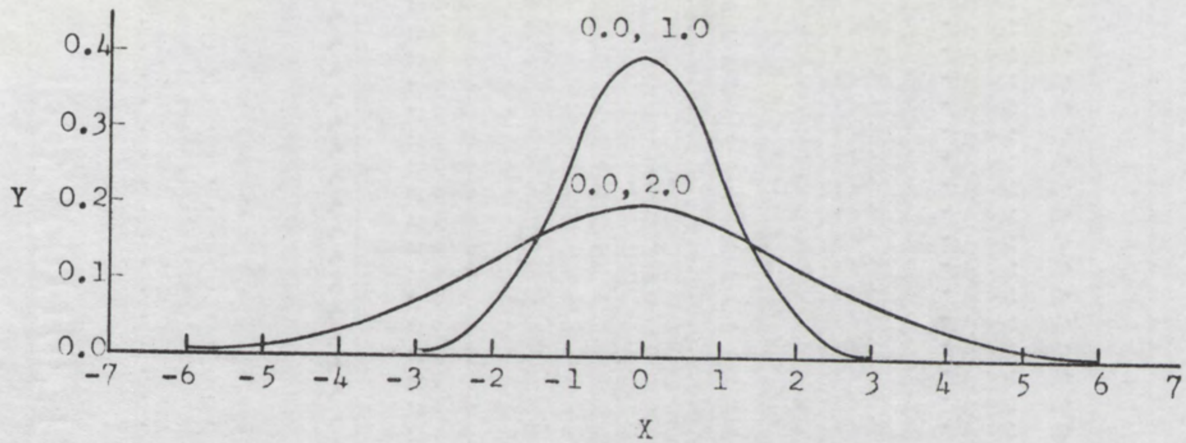

FIGURE 6. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 2.0

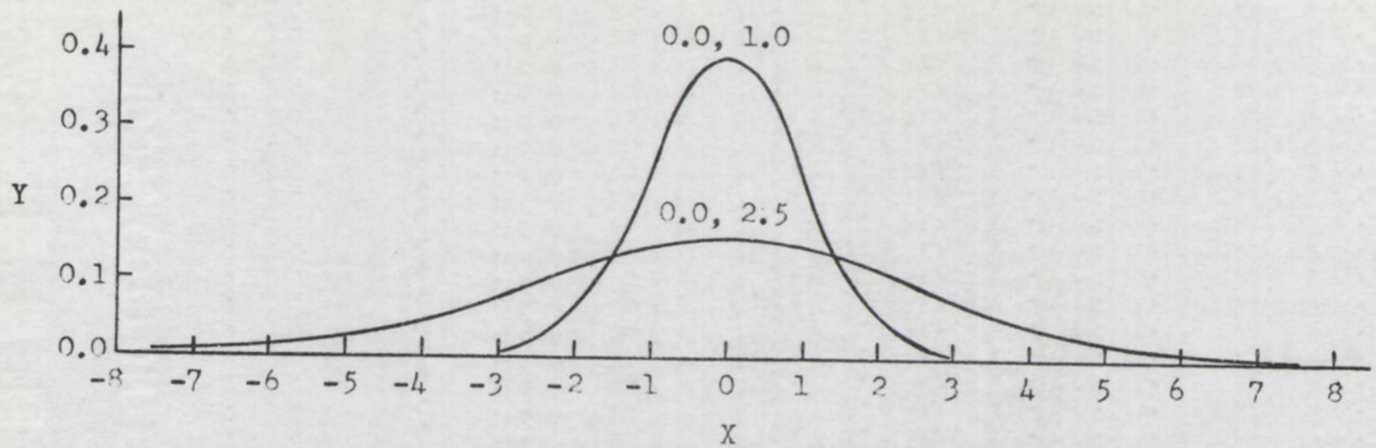

FIGURE 7. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 2.5

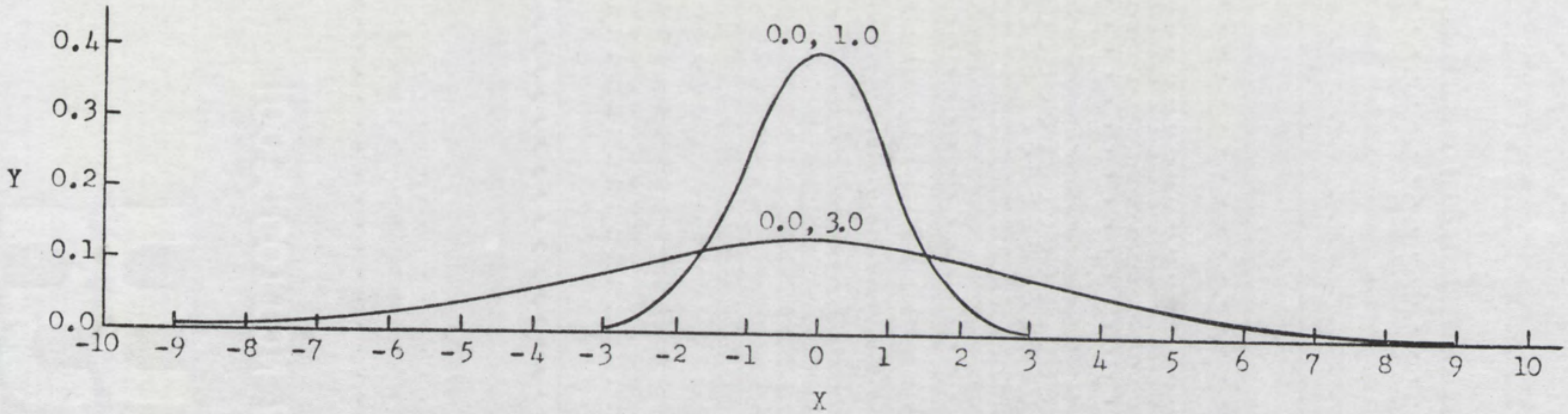

FIGURE 8. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 3.0

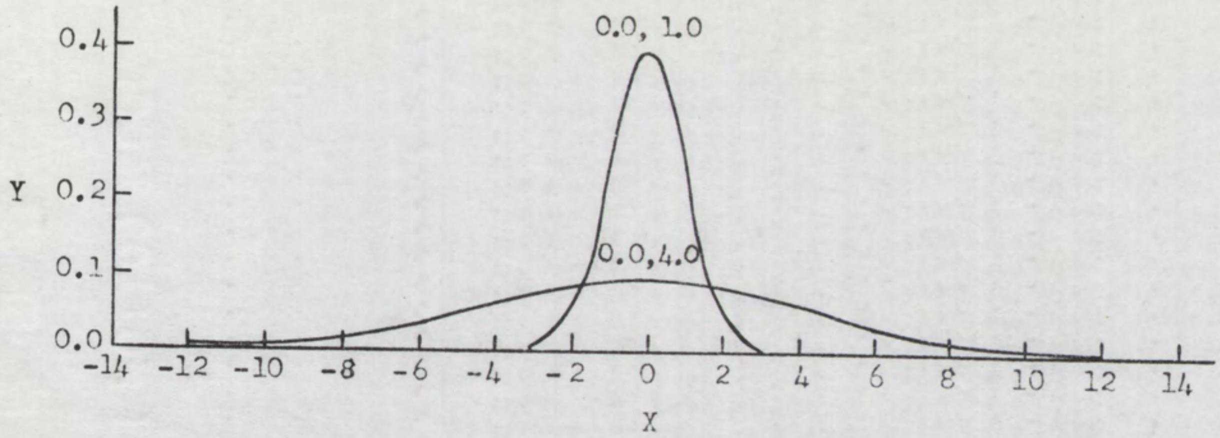

FIGURE 9. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 4.0

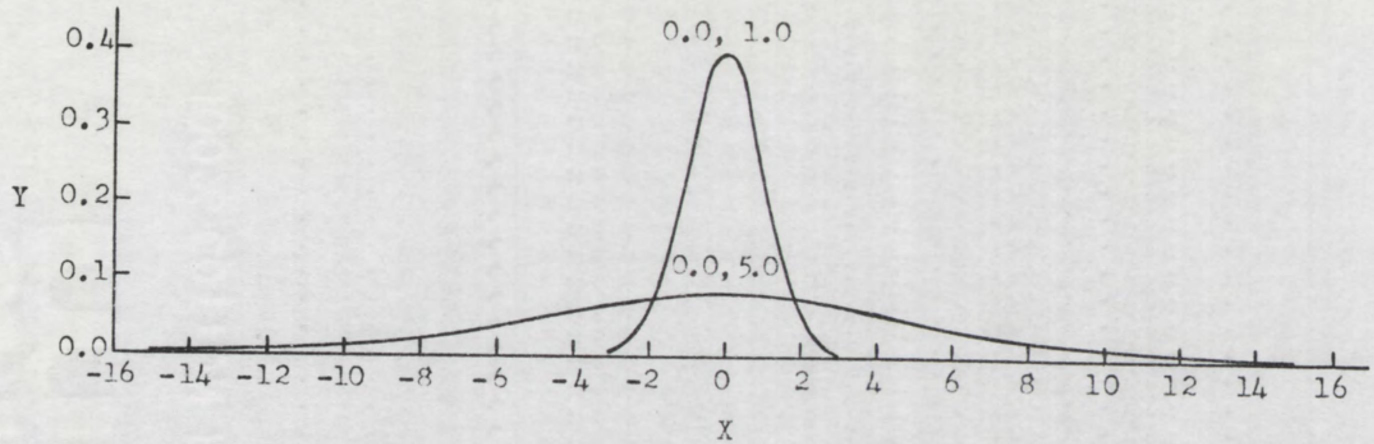

FIGURE 10. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 5.0

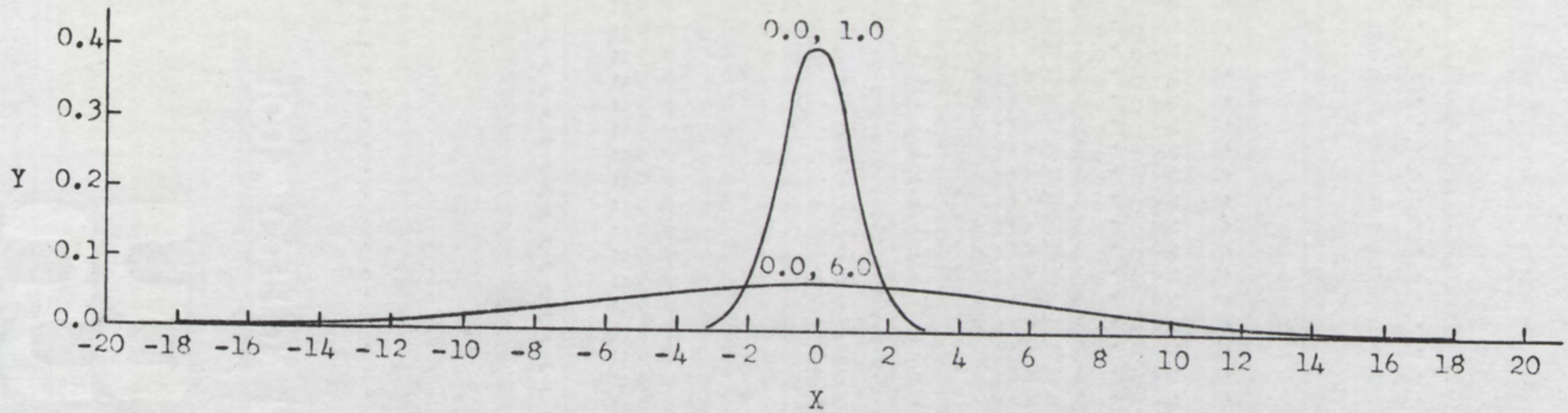

FIGURE 11. Comparison of Two Normal Distributions with Same Mean but with Different Standard Deviations 1.0, 6.0

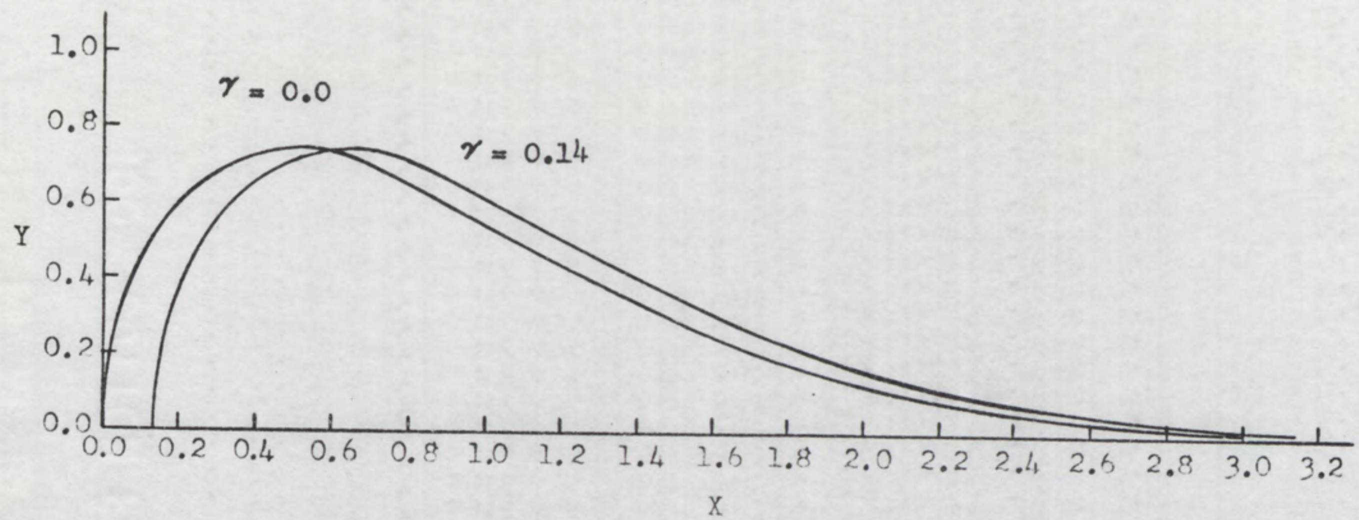

FIGURE 12. Comparison of Two Weibull Distributions with Same α and β but with Different γ's 0.0, 0.14

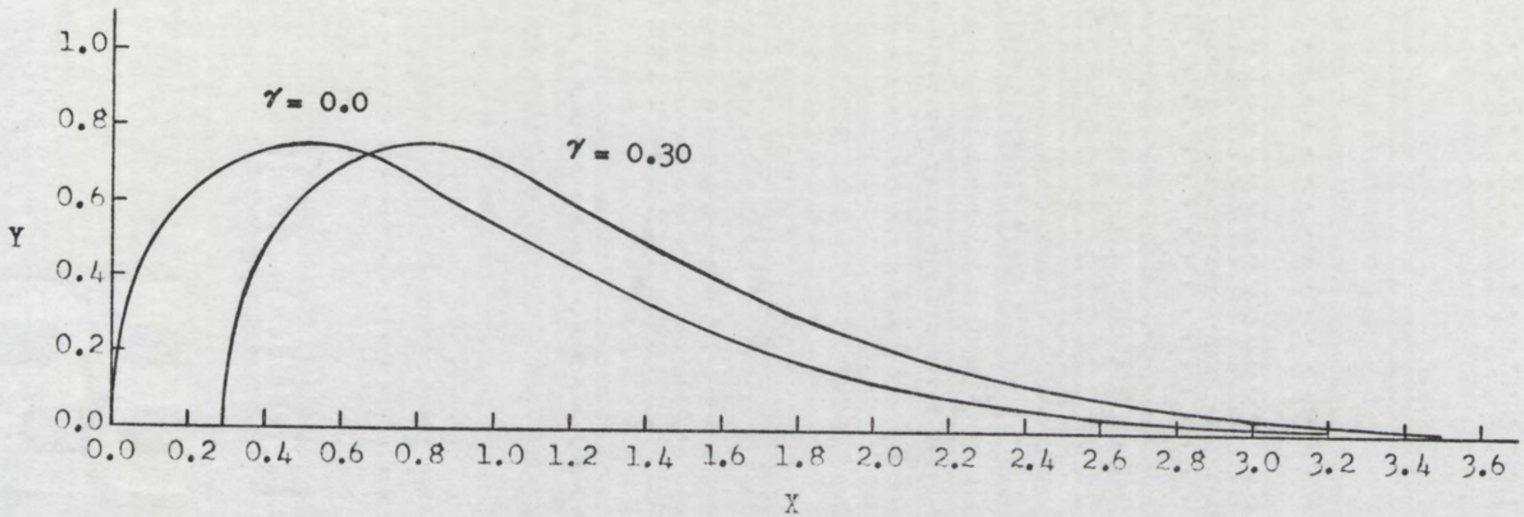

FIGURE 13. Comparison of Two Weibull Distributions with Same α and β but with Different γ's 0.0, 0.30

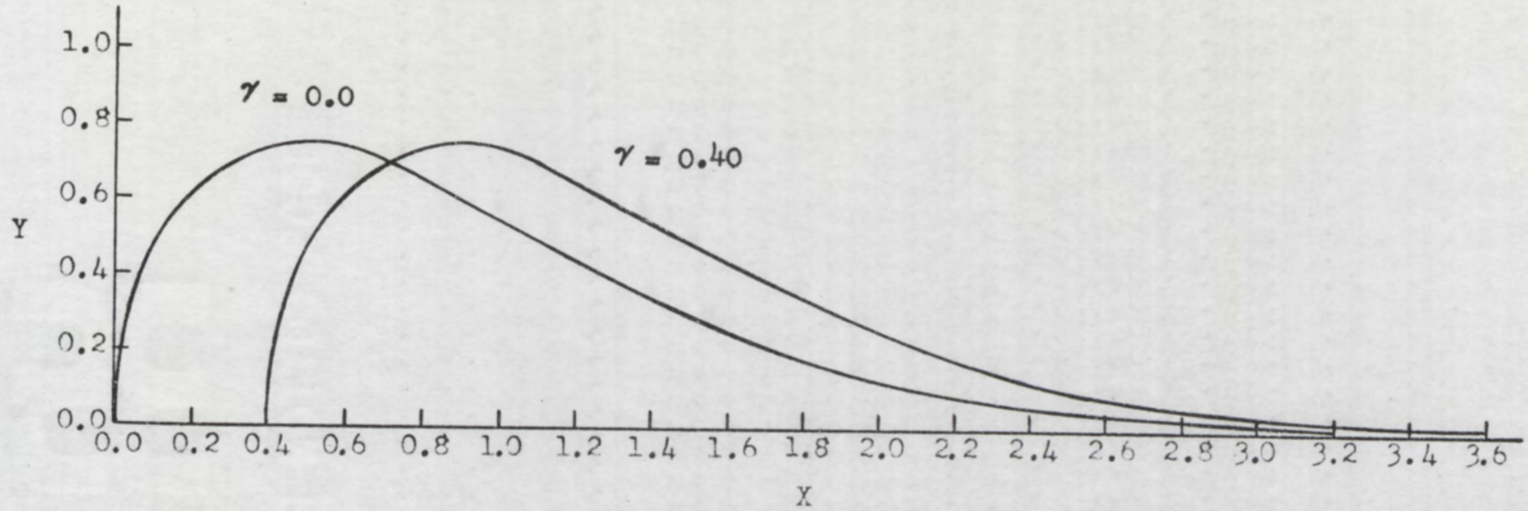

FIGURE 14. Comparison of Two Weibull Distributions with Same α and β but with Different γ's 0.0, 0.40

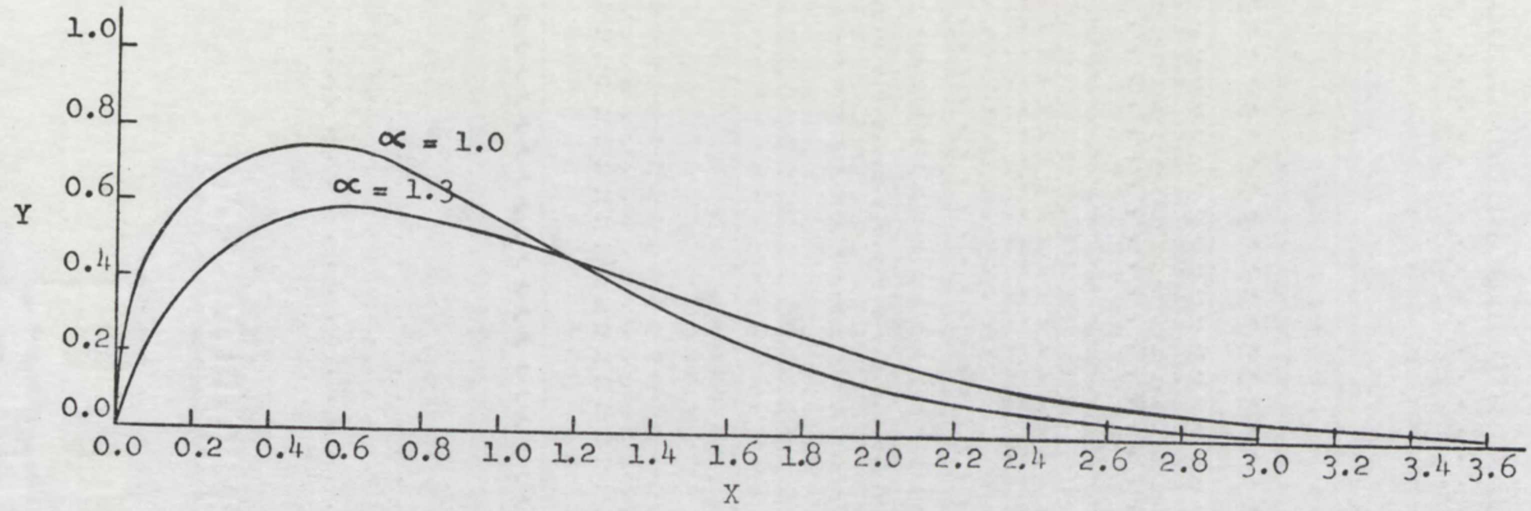

FIGURE 15. Comparison of Two Weibull Distributions with Same β's = 1.5, γ's = 0.0 but with Different α's 1.0, 1.3

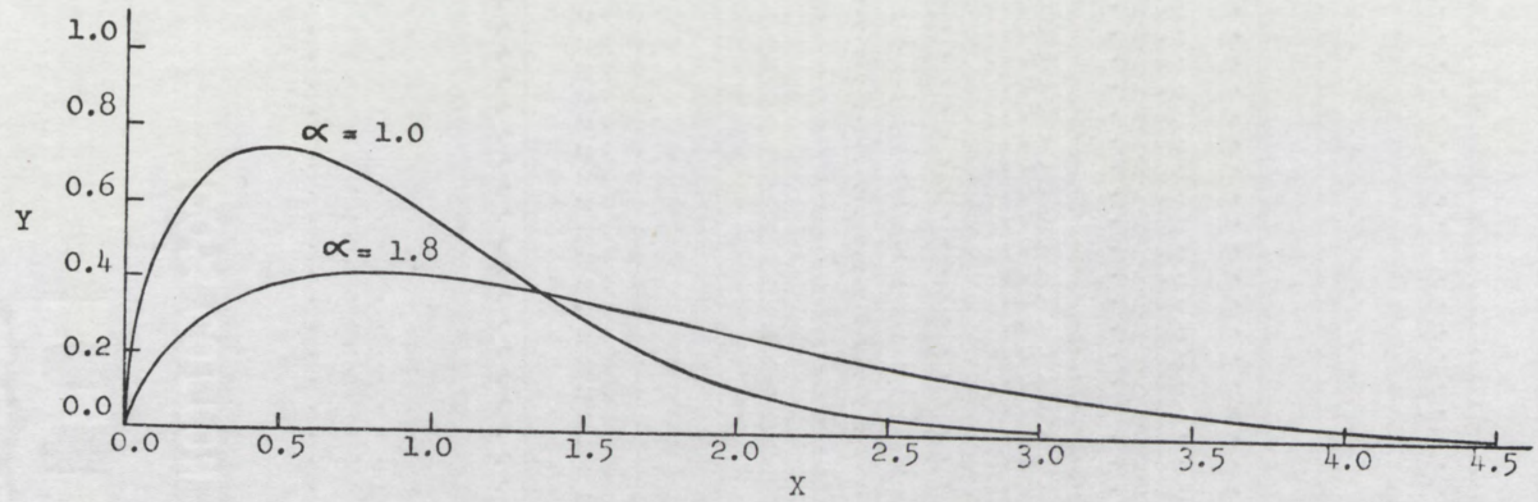

FIGURE 16. Comparison of Two Weibull Distributions with Same β's = 1.5, γ's = 0.0 but with Different α's 1.0, 1.8

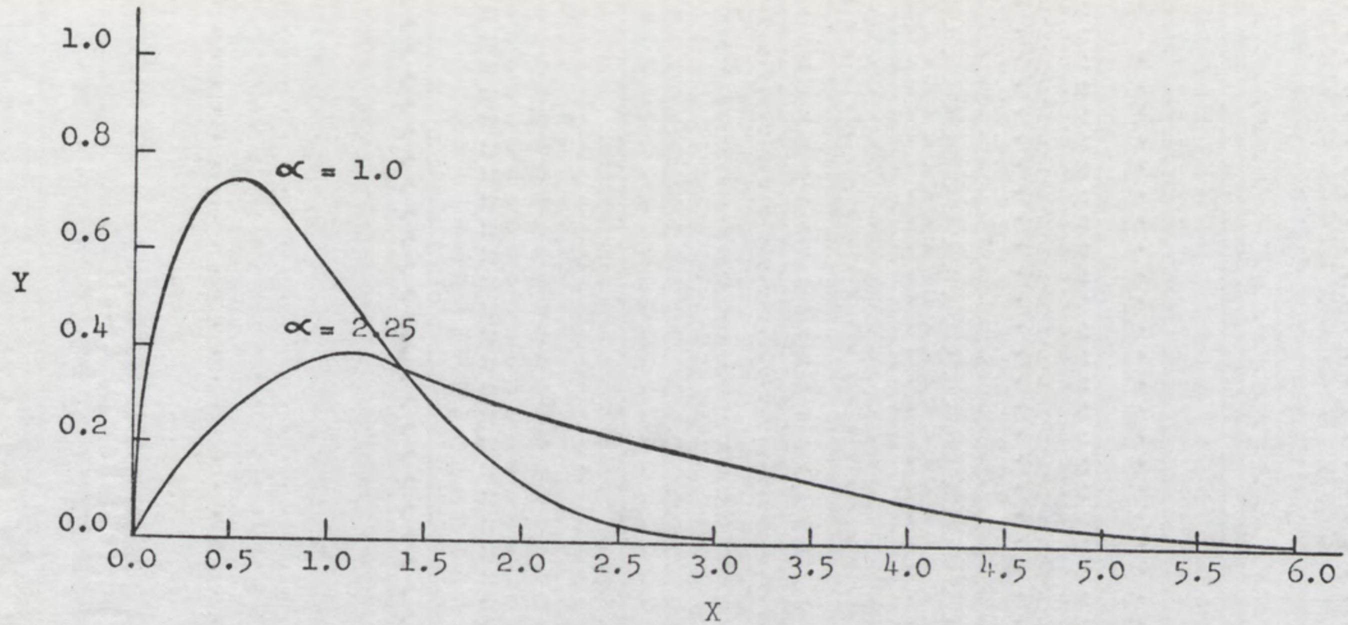

FIGURE 17. Comparison of Two Weibull Distributions with Same β's = 1.5, γ's = 0.0 but with Different α's 1.0, 2.25

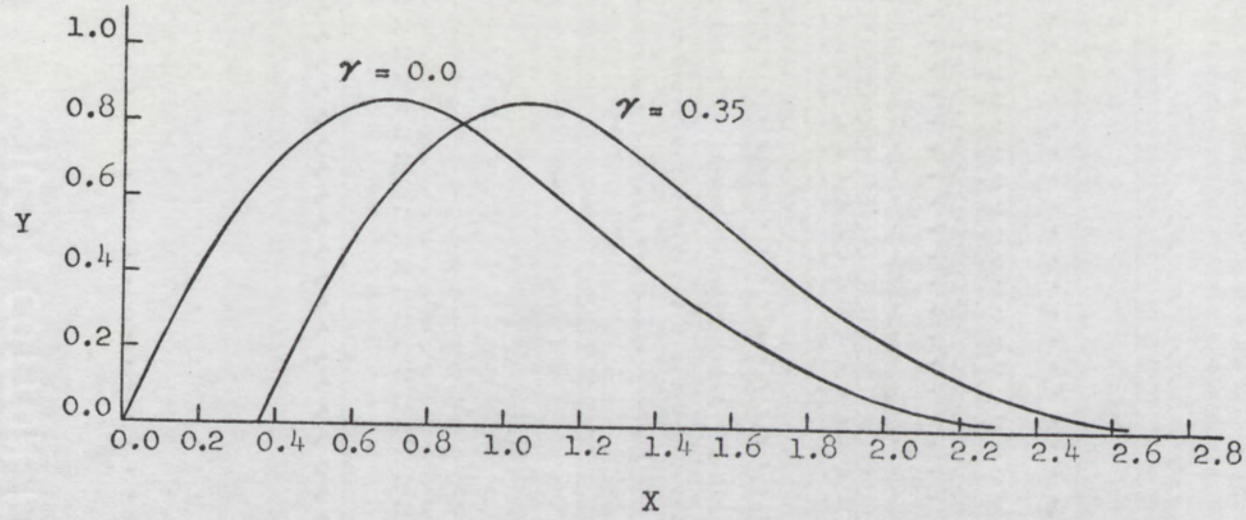

FIGURE 18. Comparisons of Two Weibull Distributions with Same α's = 1.0, β's = 2.0 but with Different γ's 0.0, 0.35

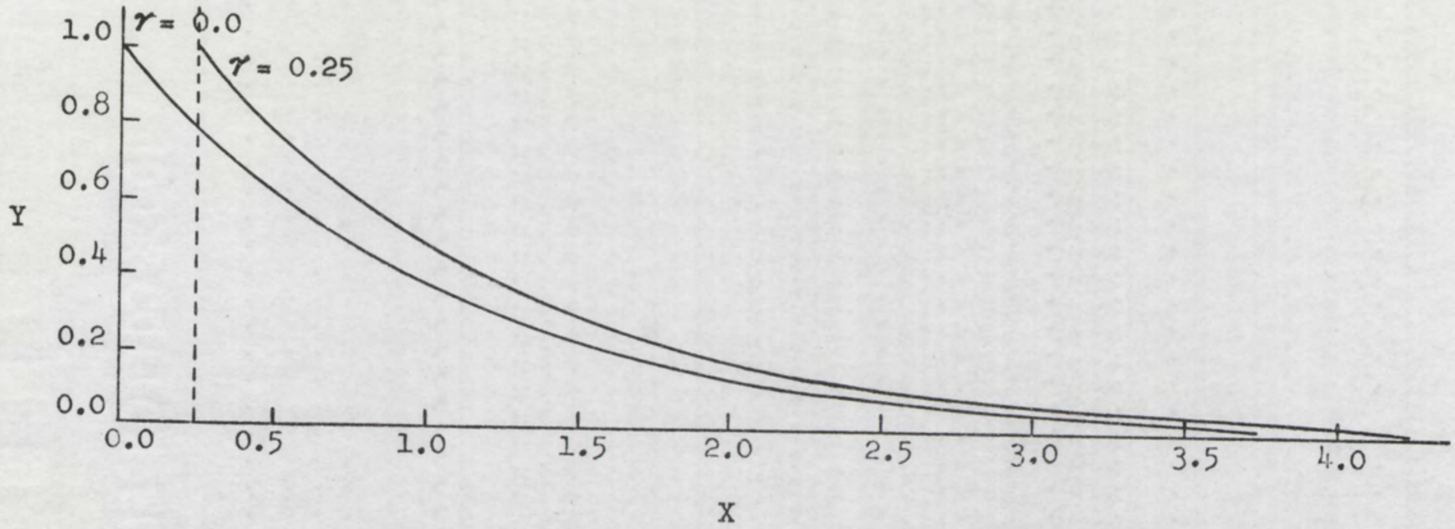

FIGURE 19. Comparison of Two Weibull Distributions with Same α's = 1.0, β's = 1.0 but with Different γ's 0.0, 0.25

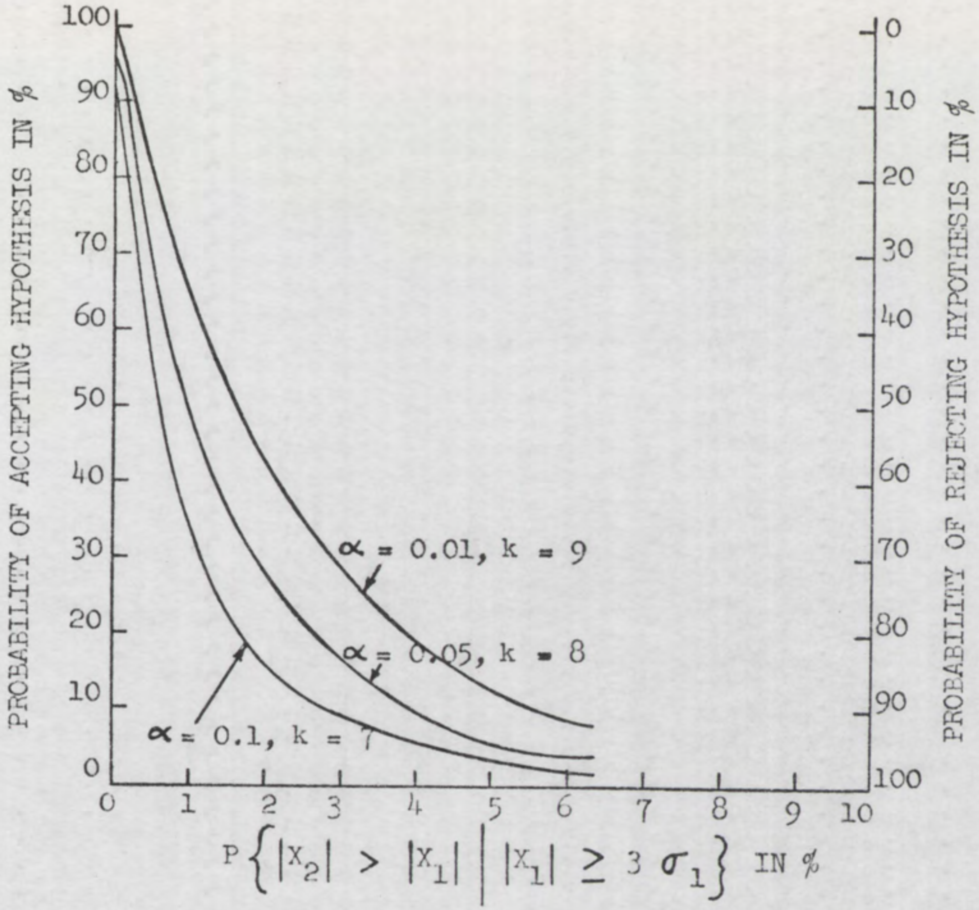

FIGURE 20. Probability of Acceptance or Rejection of $H_o: \mu_1 = \mu_2$ when Comparing Two Standard Normal Distributions with Same Standard Deviations but with Different Means

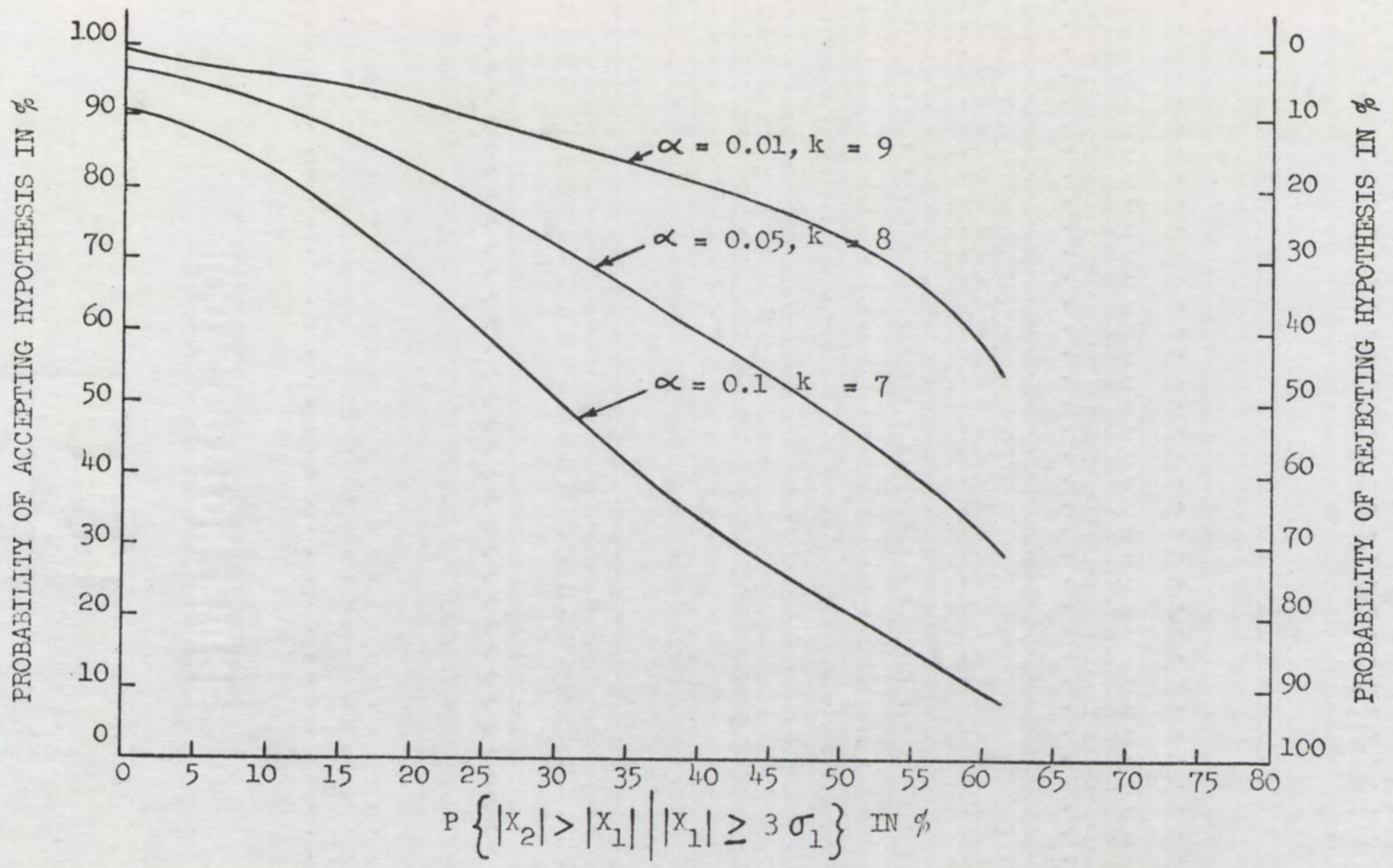

FIGURE 21. Probability of Acceptance or Rejection of $H_o : \sigma_1 = \sigma_2$ when Comparing Two Standard Normal Distributions with Same Means but with Different Standard Deviations

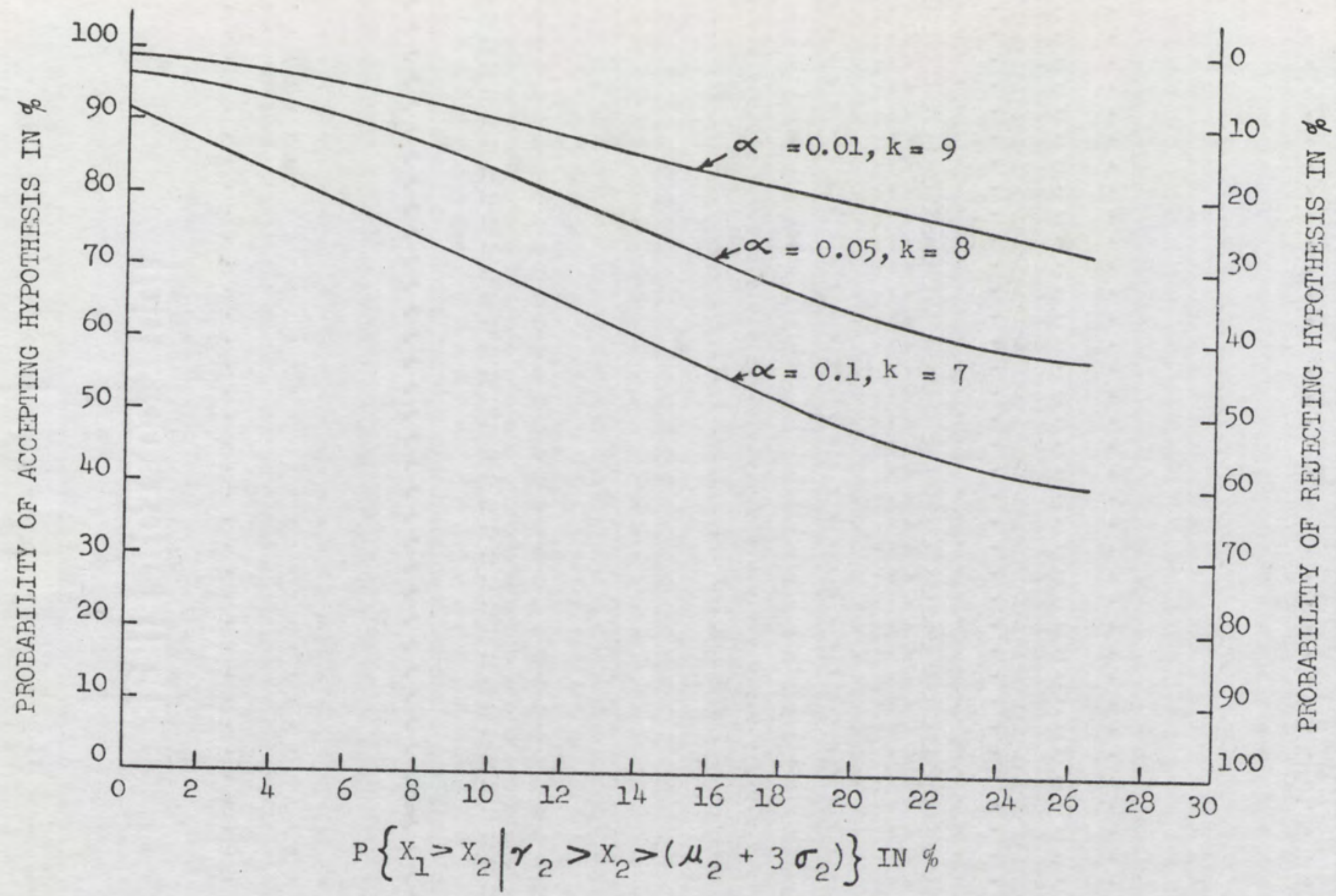

FIGURE 22. Probability of Acceptance or Rejection of $H_o: \mu_1 = \mu_2$ when Comparing Two Standard Weibull Distributions with Same α (1.0) and β (1.5) but with Different γ's

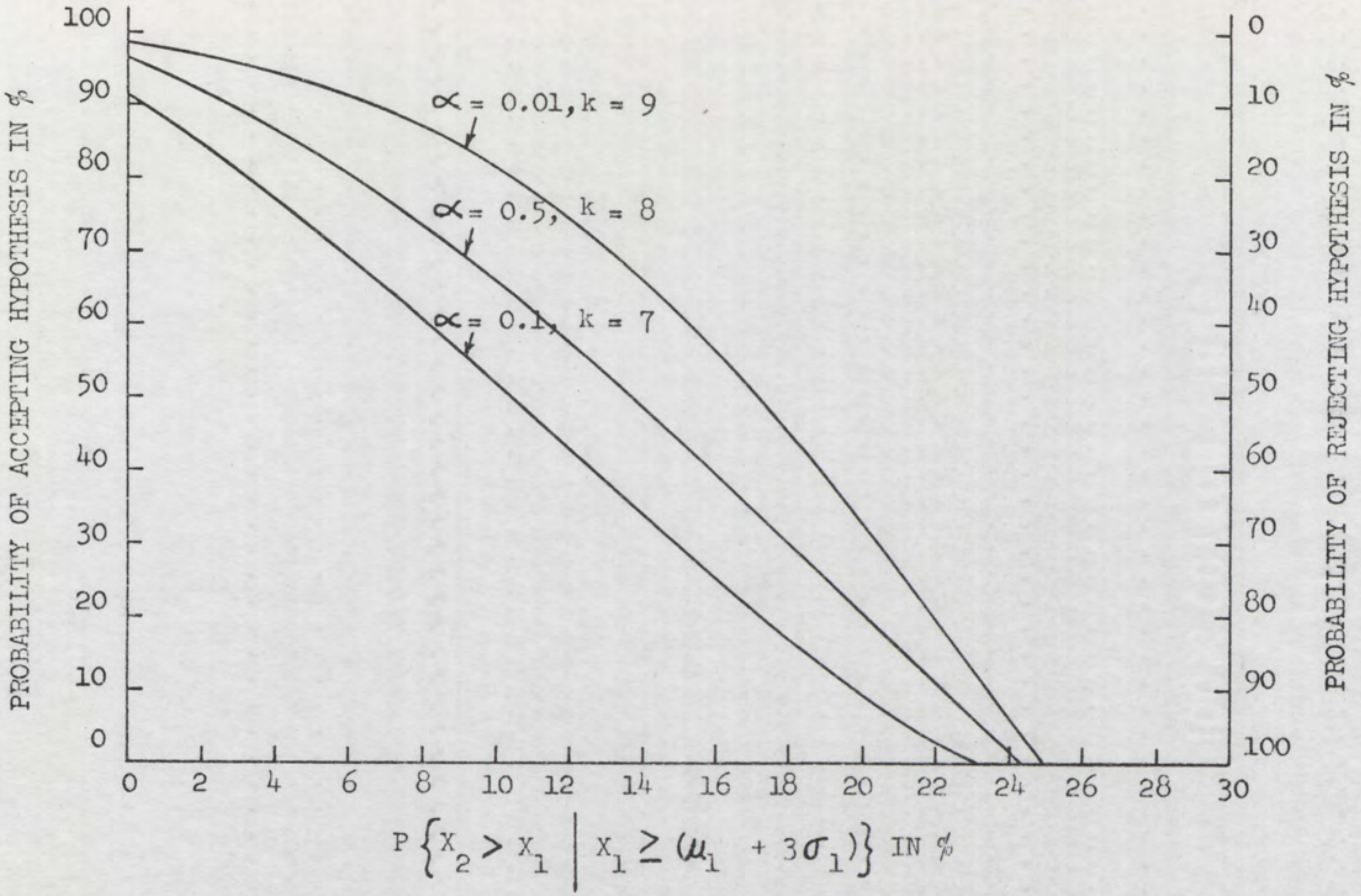

FIGURE 23. Probability of Acceptance or Rejection of $H_o: \sigma_1 = \sigma_2$ when Comparing Two Standard Weibull Distributions with Same β (1.5) and γ (0.0) but with Different α's

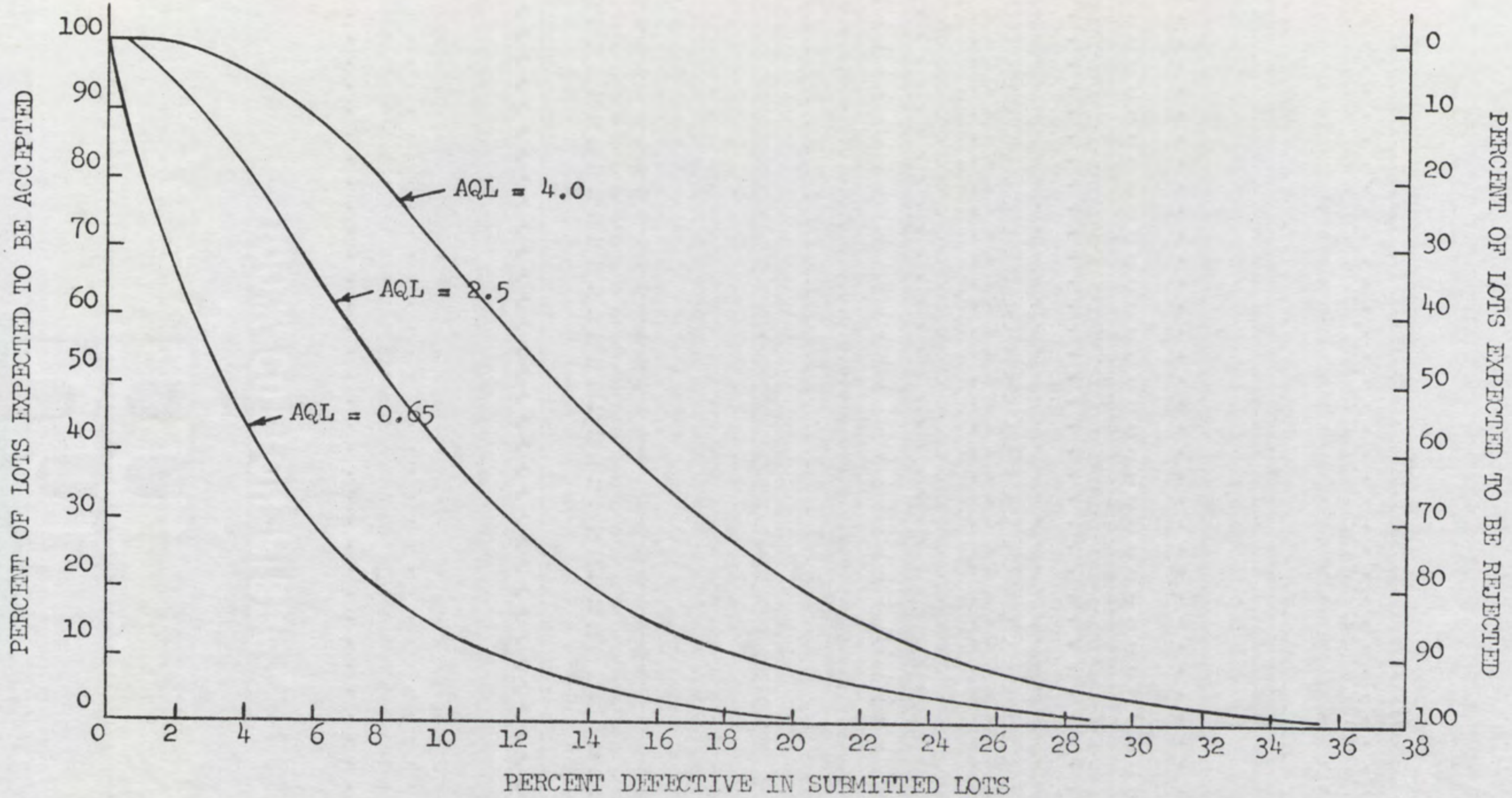

FIGURE 24. Operating Characteristic Curves for AQL's of 0.65, 2.5 and 4.0 for a Sample Size of 20 Taken from Mil-Std-105D*

*Mil-Std-105D, <u>op. cit.</u>, p. 40.

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APPENDIX A

COMPUTER PROGRAM FOR THE KOLMOGOROV-SMIRNOV (K-S) TEST

The program for selecting random samples from two different normal distributions or two different Weibull distributions and comparing them by the Kolmogorov-Smirnov method was written for the CDC-3600 in fortan language. The program was written so that it would print out the maximum absolute deviation of the cumulative step functions of the two samples. This allowed calculation of probabilities for each of the absolute deviations and calculation of cumulative probabilities. Cumulative probabilities of the absolute deviations were compared against the Kolmogorov-Smirnov statistic for various significance levels. From this comparison it was then possible to determine the probability of rejecting the null hypothesis that both distributions were equal when the mean or variance of the second distribution was changed such that the probability of non-common values for the two distributions would approach a given percentage. The program is as follows:

```
      PROGRAM KANDS
      DIMENSION W(40),X(20),Y(20),XANDY(40),KTSUMY(40),
     IKTSUMX(40),KDIF(40),PARM(6)
    1 READ(5,2)NP
    2 FORMAT(I5)
      IF(NP-99999)4,3,4
    3 CALL EXIT
    4 READ(5,5)NS,NT,NW,(PARM(I),I=1,6)
    5 FORMAT (2I4,I2,6F10.0)
      WRITE (6,8) NT,NS
    8 FORMAT (1H1,10X,I4,7HSAMPLES,1X,2HOF,1X,4HSIZE,1X,I4)
      READ (5,10)NSTART
   10 FORMAT (O16)
      IF (NW-1)11,12,12
   11 WRITE (6,16)
   16 FORMAT (10X,6HSAMPLE,1X,4HFROM,1X,6HNORMAL)
      CALL ANRSET (NSTART)
      GO TO 13
   12 WRITE (6,17)
   17 FORMAT(10X,6HSAMPLE,1X,4HFROM,1X,7HWEIBULL)
      CALL UDSET (NSTART)
   13 CONTINUE
      DO 500 II=1,NT
      IF(NW-1)15,20,20
```

```
15 CALL NORM (NS,PARM(1),PARM(2),X)
   CALL NORM (NS,PARM(3),PARM(4),Y)
   GO TO 25
20 CALL WEIBULL (NS,PARM(1),PARM(2),PARM(3),X)
   CALL WEIBULL (NS,PARM(4),PARM(5),PARM(6),Y)
25 M=NS
   MS=2*NS
   DO 30 I=1,NS
   M=M+1
   W(I)=X(I)
   W(M)=Y(I)
30 CONTINUE
   CALL SIFT (NS,X)
   CALL SIFT (NS,Y)
   CALL SIFT (MS,W)
   J=1
   D=W(1)
   DO 107 I=2,MS
   IF(D-W(I))105,110,110
105 XANDY(J)=D
   D=W(I)
   J=J+1
110 ED=W(I)
107 CONTINUE
   XANDY(J)=ED
```

```
      KCX=1
      KCY=1
      KTX=0
      KTY=0
      DO 142 I=1,J
  112 IF(XANDY(I)-Y(KCY))120,115,120
  115 KCY=KCY+1
      KTY=KTY+1
      GO TO 112
  120 KTSUMY(I)=KTY
  125 IF(XANDY(I)-X(JCX))135,130,135
  130 KCX=KCX+1
      KTX=KTX+1
      GO TO 125
  135 KTSUMX(I)KTX
  140 KDIF(I) =IABS(KTSUMX(I)-KTSUMY(I))
  142 CONTINUE
      DUM = KDIF(1)
      DO 150 I=2,J
      IF(DUM-KDIF(I) 145,145,150
  145 DUM=KDIF(I)
  150 CONTINUE
      MAXDIF=DUM
      WRITE(6,200)II,MAXDIF
```

```
200 FORMAT (1HO,6HSAMPLE,I4,7HMAXIMUM,1X,10HDIFFERENCE,I5)
500 CONTINUE
    GO TO 1
    END

    SUBROUTINE SIFT (N,X)
    DIMENSION X(40)
    M=N
  1 M=M/2
    IF(M) 3,2,3
  2 RETURN
  3 K=N-M
    J=1
  4 I=J
  5 L=I+M
    IF (X(I)-X(L))7,7,6
  6 A=X(I)
    X(I)=X(L)
    X(L)=A
    I=I-M
    IF(I)7,7,5
  7 J=J+1
    IF(J-K)4,4,1
    END
```

```
      SUBROUTINE NORM(N,X,Y,S)
      DIMENSION S(20)
      DO 10 I=1,N
      S(I) = ANRV(0)*Y+X
   10 CONTINUE
      RETURN
      END

      SUBROUTINE WEIBULL(N,X,Y,Z,S)
      DIMENSION S(20)
      DO 10 I=1,N
      S(I) = X* (-ALOG(UDGEN(0.)))**(1.0/Y)+Z
   10 CONTINUE
      RETURN
      END
```

The instructions for preparing cards to operate the program are as follows:

1. First card
   This card is a dummy card. Put the run number in the first five columns.

2. Second card - control card

Columns	Contents
1-4	Size of sample to be taken (I4), right justified
5-8	Number of samples to be taken (I4), right justified
9-10	Use 0 if samples are to be taken from a normal distribution Use a value that is not 0 if samples are to be taken from a Weibull distribution
11-20	If columns 9-10 are 0 then μ_1 If columns 9-10 are not 0 then α_1
21-30	If columns 9-10 are 0 then σ_1 If columns 9-10 are not 0 then β_1
31-40	If columns 9-10 are 0 then μ_2 If columns 9-10 are not 0 then γ_1
41-50	If columns 9-10 are 0 then σ_2 If columns 9-10 are not 0 then α_2
51-60	If columns 9-10 are 0 then blank If columns 9-10 are not 0 then β_2
61-70	If columns 9-10 are 0 then blank If columns 9-10 are not 0 then γ_2

3. Third card
   This card must contain a 16 digit octal number selected from a table of random numbers.

Each time the program is run a new random number must be selected to avoid repeating selections.

4. Last card

This card terminates the program and must be five 9's punched in the first five columns.

The first three cards can be repeated to obtain as many runs as are necessary.

APPENDIX B

INSTRUCTION FOR DATA VALIDATION

KOLMOGOROV-SMIRNOV TEST (Issue 2)

Purpose of Instruction

The purpose of this instruction is to provide the Sandia Field Quality Control Representative with a simple statistical procedure for comparing data taken by the Supplier with that taken by the SFQCR on the same lot or batch of product. The reasons for making this comparison are:

1. To verify that the Supplier is correctly observing and correctly recording the results of tests, measurements, etc.

2. To detect differences in operating procedures between personnel, differences in performance between two pieces of test equipment, and changes in the values of product parameters.

Description of Data Validation Technique

The Kolmogorov-Smirnov test as described here is designed to compare two sets of data on sample sizes from five to twenty. The same units need not be used to make this comparison.

Minor differences between two samples from the same homogeneous population are not unusual. The Kolmogorov-Smirnov test takes into account that these differences do exist and may vary within certain limits. The limits of variation become smaller as the number of items in the sample increase. If the sample contains only three items, the differences between two samples could be relatively large. As the sample size approaches infinity, the permissible difference becomes relatively smaller. The amount of permissible difference for any fixed sample size is a function of significance level. Table I, Attachment IV holds the significance level equal to or slightly less than 5%. This means that if application of the test reveals a statistically significant difference, the probability is about .95 $\left(\frac{95}{100}\right)$ that a difference exists. Conversely, there is about a 5% probability that no difference exists even though the test says yes.

Instructions for Applying Data Validation Technique

1. If the supplier has data on 21 or more items, select a random sample of 20 items. If the supplier has data on 20 or fewer items, use all items for the sample. Record the data for all tests on the data sheet (see Attachment 1 for an example). Determine in the usual manner that these data meet requirements.

2. Select a random sample of product equal in size to the sample selected in 1 above. Test these samples in accordance with the requirements of the applicable AIP. Record the data on the data sheet (see Attachment 2 for an example).

3. Complete the headings on the histogram form (see Attachment 3 for an example). Use a different sheet for each test.

   The following headings are explained to assure that what is wanted is clear:

   Item type -- MC, SA, PT, Module type, etc.

   Dwg. No. -- This is the top drawing number.

   Issue -- This is the issue of the drawing recorded above.

   P.O. No. -- This is the number of the applicable purchase contract.

   Sub. No. -- This is the SFQCR submittal number, or, the supplier lot number if no submittal has yet been made.

   Test -- This is the test condition i.e. $-65^{o}F$, vibration, etc.

   QS Dwg. -- This is the number of the applicable test specifications.

   Issue -- This is the issue of the test specification.

   Para. No. -- This is the paragraph in the test specification which describes the test.

Operator -- Print the operator's initials adjacent to "Supplier" or "SFQCR" as is appropriate.

Tester -- This is the designation of the tester used. If two testers are used, list both.

Units -- These are the units of measure for the recorded data, i.e., volts $X10^{-2}$, ohms, amps. $X10^{-4}$, etc.

Date of test -- This is the test date.

High reading -- This is the largest observation in both samples.

Low reading -- This is the smallest observation in both samples.

Cell Width -- This is the distance between the upper and lower boundaries of the cells in the histogram. This value is obtained by rounding $\frac{\text{Range}}{n_1}$ to a convenient value.

4. Plot histograms

   a. In the first cell, write down the lowest valued reading in the total sample.

   b. Add the cell width to this value to get the lower boundary of the second cell.

   c. Repeat this process until the entire range of readings has been bounded. If an observation falls on a cell boundary, place the observation in the cell which has the value of the observation as its lower boundary.

   d. Tally individual observations in appropriate cell.

   NOTE: Record supplier data in supplier column and Sandia data in Sandia column. Show dates on which data were taken.

5. Calculate cumulative frequencies

   a. In the "Freq" column record the number of readings in each cell.

   b. In the "Cumulative freq" column write the cumulative sums of the numbers of the "Freq" column.

   c. In the "Difference in Cumulative Freq" column record the differences between Supplier Cumulative Frequency and Sandia Cumulative Frequency for each cell.

6. Determine the largest number in the "Difference in Cumulative Freq" column. Record this number on the line labeled "D".

   NOTE: In the example this occurs on Lines 8-11, Attachment 3.

7. Find, in Table 1 (Attachment 4) the critical value, L, which corresponds to your sample size. Record this number on the line labeled "L".

   NOTE: In the example, for $n_1 = n_2 = 20$, $L = 8$

8. Compare D and L

   $D > L$ The test indicates significant difference.
   $D \leq L$ The test indicates no significant difference.

Supplier [X]
Sandia [ ]

DATA SHEET FOR SA-1966-2

ATTACHMENT I

Device Number	Resistance (10 Meg. ± 1%) Para. 4.3.3.1				
1	9.90				
2	9.94				
3	9.91				
4	10.00				
5	9.99				
6	10.02				
7	10.02				
8	9.98				
9	9.95				
10	9.96				
11	10.00				
12	10.01				
13	9.98				
14	10.04				
15	10.02				
16	10.00				
17	9.97				
18	9.99				
19	10.03				
20	10.05				

Supplier ☐
Sandia ☒

DATA SHEET FOR SA-1466-2

ATTACHMENT II

Device Number	Resistance (10 Meg. ± 1%) Para. 4.3.3.1				
1	9.90				
2	9.95				
3	9.96				
4	10.04				
5	9.98				
6	10.01				
7	10.00				
8	9.95				
9	9.90				
10	9.95				
11	10.01				
12	9.97				
13	9.96				
14	10.03				
15	10.03				
16	9.98				
17	9.95				
18	9.99				
19	10.02				
20	10.04				

HISTOGRAM FORM

Item SA-1466-2 Dwg. No. 357920 Date ______
P.O. No. 25-7874 Sub. No. 1A Issue A
QS Dwg. PS 357874 Issue A Test Resistance
Par. No. 4.3.3.1

High Reading 10.05 Range/n_1 = 0.0075
Low Reading 9.90 Cell width = 0.008
Range 0.15
Supplier sample size n_1 = 20 = SFQCR sample size n_2
TESTER PT 1431 UNITS OHMS

Operator	Supplier Data			SFQCR Data			
Lower Boundary of Cell	Date of Test			Date of Test			Diff. in Cum. Freq.
	Tally	Freq.	Cum. Freq.	Tally	Freq.	Cum. Freq.	
9.900	/	1	1	//	2	2	1
9.908	/	1	2			2	0
9.916			2			2	0
9.924			2			2	0
9.932			2			2	0
9.940	/	1	3			2	1
9.948	/	1	4	////	4	6	2
9.956	/	1	5	//	2	8	3
9.964	/	1	6	/	1	9	3
9.972			6			9	3
9.980	//	2	8	//	2	11	3
9.988	//	2	10	/	1	12	2
9.996	///	3	13	/	1	13	0
10.004	/	1	14	//	2	15	1
10.012			14			15	1
10.020	///	3	17	/	1	16	1
10.028	/	1	18	//	2	18	0
10.036	/	1	19	//	2	20	1
10.044	/	1	20			20	0
10.052							

ATTACHMENT III

Critical Value L from Table I = 8 Max. Diff. D = 3

D > L ☐ Test indicates significant difference

D ≤ L ☒ Test indicates no significant difference

TABLE I

KOLMOGOROV-SMIRNOV TEST

CRITICAL VALUES FOR THE MAXIMUM DIFFERENCE IN RELATIVE CUMULATIVE FREQUENCY

For $n_1 = n_2$

$\alpha \leq .05$

$n_1 = n_2$	Critical Value = L
5	4
6	4
7	5
8	5
9	5
10	6
11	6
12	6
13	6
14	7
15	7
16	7
17	7
18	8
19	8
20	8

ATTACHMENT IV

APPENDIX C

TABLE OF PROBABILITIES OF THE MAXIMUM DEVIATION OF THE

KOLMOGOROV-SMIRNOV STATISTIC FOR EQUAL SAMPLES OF 5, 10, 15 AND 20

Probability of $d \leq k/n$

n = m	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k = 10	k = 11	k = 12
5	.126984	.642857	.920635	.992063	1.0000							
10	.005542	.213070	.582476	.832179	.947552	.987659	.997943	.999783	.999989	1.00000		
15	.000211	.061669	.321862	.614453	.815584	.924536	.973752	.992344	.998163	.999647	.999948	.999994
20	$.0^{5}761$	.016863	.168030	.428664	.664409	.825467	.918942	.966458	.987701	.996033	.998884	.999729

*F. J. Massey, Jr., <u>op. cit</u>., pp. 126-127.

Note: When making comparison against the Kolmogorov-Smirnov criterion, d′ was used for making comparison against k. The quantity d′ is derived as follows: $d = d'/n \leq k/n \therefore d' \leq k$.

APPENDIX D

Probability Calculations

I. Normal Distribution — Probabilities were calculated utilizing Table A4, Cumulative Normal Distribution,[1] from Introduction to Statistical Analysis by Dixon and Massey.

A. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0.5,\ \sigma_2=1.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0.5}{1.0} = -3.5$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0.5}{1.0} = +2.5$$

Area of Distr. 2 from $-\infty$ to $-3\sigma(-z_2 = -3.5)$ = 0.0002

" " " 1 " " " " $(-z_1 = -3.0)$ = 0.0013

Diff. = -0.0011

" " " 2 " $+3\sigma$ to $+\infty(+z_2 = +2.5)$ = 0.0062

" " " 1 " " " " $(+z_1 = +3.0)$ = 0.0013

Diff. = +0.0049

Total Diff. = -0.0011 + 0.0049 = 0.0038

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{0.38\%}}$$

B. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=1.0,\ \sigma_2=1.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 1.0}{1.0} = -4.0$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 1.0}{1.0} = +2.0$$

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[1] Wilfred J. Dixon and Frank J. Massey, Jr., Introduction to Statistical Analysis (New York: McGraw-Hill Book Company, Inc., 1957), pp. 382-383.

Area of Distr. 2 from $-\infty$ to $-3\sigma(-z_2 = -4.5)$ = 0.0000

" " " 1 " " " " $(-z_1 = -3.0)$ = <u>0.0013</u>

Diff. = -0.0013

" " " 2 " $+3\sigma$ to $+\infty(+z_2 = 2.0)$ = 0.0228

" " " 1 " " " " $(+z_1 = 3.0)$ = <u>0.0013</u>

Diff. = +0.213

Total Diff. = -0.0013 + 0.0213 = 0.0200

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{2.00\%}}$$

C. Comparison of $\mu_1=0$, $\sigma_1=1.0$ versus $\mu_2=1.5$, $\sigma_2=1.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 1.5}{1.0} = -4.5$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 1.5}{1.0} = +1.5$$

Area of Distr. 2 from $-\infty$ to $-3\sigma(-z_2 = -4.5)$ = 0.0000

" " " 1 " " " " $(-z_1 = -3.0)$ = <u>0.0013</u>

Diff. = -0.0013

" " " 2 " $+3\sigma$ to $+\infty(+z_2 = 1.5)$ = 0.0668

" " " 1 " " " " $(+z_1 = 3.0)$ = <u>0.0013</u>

Diff. = +0.0655

Total Diff. = -0.0013 + 0.0655 = 0.0642

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{6.42\%}}$$

D. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=1.0,\ \sigma_2=1.8$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 1.0}{1.8} = -2.22$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 1.0}{1.8} = +1.11$$

Area of Distr. 2 from $-\infty$ to -3σ	$(-z_2 = -2.22)$	= 0.0132
" " " 1 " " " "	$(-z_1 = -3.0)$	= 0.0013
	Diff.	= +0.0119
" " " 2 " $+3\sigma$ to $+\infty$	$(+z_2 = 1.11)$	= 0.1334
" " " 1 " " "	$(+z_1 = 3.00)$	= 0.0013
	Diff.	= +0.1321

Total Diff. = 0.0119 + 0.1321 = 0.1440

$$\therefore P\left\{ |x_2|>|x_1| \;\middle|\; |x_2| \geq 3.0 \right\} = \underline{\underline{14.40\%}}$$

E. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0,\ \sigma_2=1.5$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{1.5} = -2.0$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{1.5} = +2.0$$

Area of Dist. 2 from $-\infty$ to -3σ	$(-z_2 = -2.0)$	= 0.0228
" " " 1 " " " "	$(-z_1 = -3.0)$	= 0.0013
	Diff.	= +0.0215
" " " 2 " $+3\sigma$ to $+\infty$	$(+z_2 = +2.0)$	= 0.0228
" " " 1 " " " "	$(+z_1 = +3.0)$	= 0.0013
	Diff.	= +0.0215

$$\text{Total Diff.} = 0.0215 + 0.0215 = 0.0430$$

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{4.30\%}}$$

F. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0,\ \sigma_2=2.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{2.0} = -1.5$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{2.0} = +1.5$$

Area of Dist. 2 from $-\infty$ to -3σ	$(-z_2 = -1.5)$	= 0.0668
" " " 1 " " " "	$(-z_1 = -3.0)$	= <u>0.0013</u>
	Diff.	= +0.0655
" " " 2 " $+3\sigma$ to $+\infty$	$(+z_2 = +1.5)$	= 0.0668
" " " 1 " " " "	$(+z_1 = +3.0)$	= <u>0.0013</u>
	Diff.	= +0.0655

$$\text{Total Diff.} = 0.0655 + 0.0655 = 0.1310$$

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{13.10\%}}$$

G. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0,\ \sigma_2=2.5$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{2.5} = -1.2$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{2.5} = +1.2$$

Area of Dist. 2 from $-\infty$ to $-3\sigma(-z_2 = -1.2)$ = 0.1151

" " " 1 " " " " $(-z_1 = -3.0)$ = 0.0013

Diff. = +0.1138

" " " 2 " $+3\sigma$ to $+\infty(+z_2 = +1.2)$ = 0.1151

" " " 1 " " " " $(+z_1 = +3.0)$ = 0.0013

Diff. = +0.1138

Total Diff. = 0.1138 + 0.1138 = 0.2276

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{22.76\%}}$$

H. Comparison of $\mu_1=0$, $\sigma_1=1.0$ versus $\mu_2=0$, $\sigma_2=3.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{3.0} = -1.0$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{3.0} = +1.0$$

Area of Dist. 2 from $-\infty$ to $-3\sigma(-z_2 = -1.0)$ = 0.1587

" " " 1 " " " " $(-z_1 = -3.0)$ = 0.0013

Diff. = +0.1574

" " " 2 " $+3\sigma$ " $+\infty(+z_2 = +1.0)$ = 0.1587

" " " 1 " " " " $(+z_1 = +3.0)$ = 0.0013

Diff. = +0.1574

Total Diff. = 0.1574 + 0.1574 = 0.3148

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{31.48\%}}$$

I. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0,\ \sigma_2=4.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{4.0} = -0.75$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{4.0} = +0.75$$

Area of Dist. 2	from	$-\infty$ to	$-3\sigma(-z_2 = -0.75)$	=	0.2266
" " " 1	"	" "	" $(-z_1 = -3.0)$	=	0.0013
			Diff.	=	+0.2253
" " " 2	"	$+3\sigma$ "	$+\infty(+z_2 = +0.75)$	=	0.2266
" " " 1	"	" "	" $(+z_1 = +3.0)$	=	0.0013
			Diff.	=	+0.2253

Total Diff. = 0.2253 + 0.2253 = 0.4516

$$\therefore P\left\{ |x_2| > |x_1| \;\middle|\; |x_1| \geq 3.0 \right\} = \underline{\underline{45.16\%}}$$

J. Comparison of $\mu_1=0,\ \sigma_1=1.0$ versus $\mu_2=0,\ \sigma_2=5.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{5.0} = -0.6$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{5.0} = +0.6$$

Area of Dist. 2	from	$-\infty$ to	$-3\sigma(-z_2 = -0.6)$	=	0.2743
" " " 1	"	" "	" $(-z_1 = -3.0)$	=	0.0013
			Diff.	=	+0.2730

Area of Dist. 2 from $+3\sigma$ to $+\infty(+z_2 = +0.6)$ = 0.2743

" " " 1 " " " " $(+z_1 = +3.0)$ = 0.0013

Diff. = +0.2730

Total Diff. = 0.2730 + 0.2730 = 0.5460

$$\therefore P\left\{ |x_2| > |x_1| \;\Big|\; |x_1| \geq 3.0 \right\} = \underline{\underline{54.60\%}}$$

K. Comparison of $\mu_1=0$, $\sigma_1=1.0$ versus $\mu_2=0$, $\sigma_2=6.0$

$$-z_1 = \frac{-3.0 - 0}{1.0} = -3.0 \qquad -z_2 = \frac{-3.0 - 0}{6.0} = -0.50$$

$$+z_1 = \frac{+3.0 - 0}{1.0} = +3.0 \qquad +z_2 = \frac{+3.0 - 0}{6.0} = +0.50$$

Area of Dist. 2 from $-\infty$ to $-3\sigma(-z_2 = -0.5)$ = 0.3085

" " " 1 " " " " $(-z_1 = -3.0)$ = 0.0013

Diff. = +0.3072

" " " 2 " $+3\sigma$ " $+\infty(+z_2 = +0.5)$ = 0.3085

" " " 1 " " " " $(+z_1 = +3.0)$ = 0.0013

Diff. = +0.3072

Total Diff. = 0.3072 + 0.3072 = 0.6144

$$\therefore P\left\{ |x_2| > |x_1| \;\Big|\; |x_1| \geq 3.0 \right\} = \underline{\underline{61.44\%}}$$

II. Weibull Distributions — Probabilities were calculated utilizing the following formulas taken from an article published by Lehman[2] in the IEEE Transactions On Reliability entitled, "Shapes, Moments, and Estimators Of The Weibull Distribution."

$$\text{MEAN} = \mu = \frac{1}{\beta}!\,\alpha + \gamma$$

$$\text{STANDARD DEVIATION} = \sigma = \alpha\sqrt{\frac{2}{\beta}! - \left(\frac{1}{\beta}!\right)^2}$$

$$\text{AREA} = A = F(x;\alpha,\beta,\gamma) = \int_{\gamma}^{x} f(x;\alpha,\beta,\gamma)\,dx$$

$$= 1 - e^{-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}}$$

A. Comparison of $\alpha_1=1.0$, $\beta_1=1.5$, $\gamma_1=0$ versus $\alpha_2=1.0$, $\beta_2=1.5$, $\gamma_2=0.14$

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 1.0 + 0.14 = 1.043$$

$$\sigma_1 = \sigma_2 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = \sqrt{0.37298} = 0.613$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 1.043 + 3 \times 0.613 = 2.882$$

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[2]Eugene H. Lehman, Jr., "Shapes, Moments and Estimators Of The Weibull Distribution," IEEE Transactions On Reliability, Vol. R12, No. 3 (September, 1963), pp. 32-38.

A_{1a} = Area from γ_1 to γ_2

A_{1b} = " " γ_1 to $(\mu_2 + 3\sigma_2)$

A_{1c} = " " $(\mu_2 + 3\sigma_2)$ to $+\infty = 1 - A_{1b}$

A_{2a} = " " γ_2 to $(\mu_2 + 3\sigma_2)$

A_{2b} = " " $(\mu_2 + 3\sigma_2)$ to $+\infty = 1 - A_{2a}$

$$A_{1a} = 1 - e^{-\left(\frac{0.14 - 0}{1}\right)^{1.5}} = 1 - e^{-0.0523} = 1 - 0.9498$$

$$= 0.0512$$

$$A_{1b} = 1 - e^{-\left(\frac{2.882 - 0}{1}\right)^{1.5}} = 1 - e^{-4.89} = 1 - 0.0075$$

$$= 0.9925$$

$$A_{1c} = 1 - 0.9925 = 0.0075$$

$$A_{2a} = 1 - e^{-\left(\frac{2.882 - 0.14}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{2b} = 1 - 0.9893 = 0.0107$$

Area of Dist. 1 from o to γ_2	=	0.0512
" " " 2 " " " "	=	0.
Diff.	=	+0.0512
" " " 1 " $(\mu_2 + 3\sigma_2)$ to $+\infty$	=	0.0075
" " " 2 " " " "	=	0.0107
Diff.	=	-0.0032

$$\text{Total Diff.} = 0.0512 - 0.0032 = 0.048$$

$$\therefore P\left\{ x_1 > x_2 \,\middle|\, \gamma_2 > x_2 > (u_2 + 3o_2) \right\} = \underline{\underline{4.8\%}}$$

B. Comparison of $\alpha_1=1.0$, $\beta_1=1.5$, $\gamma_1=0$ versus $\alpha_2=1.0$, $\beta_2=1.5$, $\gamma_2=0.30$.

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 1.0 + 0.30 = 1.203$$

$$\sigma_1 = \sigma_2 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.613$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 1.203 + 3 \times 0.613 = 3.042$$

$$A_{1a} = 1 - e^{-\left(\frac{0.30 - 0}{1}\right)^{1.5}} = 1 - e^{-0.164} = 1 - 0.8487$$

$$= 0.1513$$

$$A_{1b} = 1 - e^{-\left(\frac{3.042 - 0}{1}\right)^{1.5}} = 1 - e^{-5.21} = 1 - 0.0055$$

$$= 0.9945$$

$$A_{1c} = 1 - 0.9945 = 0.0055$$

$$A_{2a} = 1 - e^{-\left(\frac{3.042 - 0.30}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{2b} = 1 - 0.9893 = 0.0107$$

$$\text{Area of Dist. 1 from 0 to } \gamma_2 = 0.1513$$

$$\text{" " " 2 " " " "} = \underline{0}$$

$$\text{Diff.} = +0.1513$$

$$\text{" " " 1 " } (\mu_2 + 3\sigma_2) \text{ to } +\infty = 0.0055$$

$$\text{" " " 2 " " " "} = \underline{0.0107}$$

$$\text{Diff.} = -0.0052$$

$$\text{Total Diff.} = 0.1513 - 0.0052 = 0.1461$$

$$\therefore P\left\{ x_1 > x_2 \,\middle|\, \gamma_2 > x_2 > (\mu_2 + 3\sigma_2) \right\} = \underline{\underline{14.61\%}}$$

C. Comparison of $\alpha_1=1.0$, $\beta_1=1.5$, $\gamma_1=0$ versus $\alpha_2=1.0$, $\beta_2=1.5$, $\gamma_2=0.40$

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 1.0 + 0.40 = 1.303$$

$$\sigma_1 = \sigma_2 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.613$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 1.303 + 3 \times 0.613 = 3.142$$

$$A_{1a} = 1 - e^{-\left(\frac{0.40 - 0}{1}\right)^{1.5}} = 1 - e^{-0.253} = 1 - 0.7765$$

$$= 0.2235$$

$$A_{1b} = 1 - e^{-\left(\frac{3.142 - 0}{1}\right)^{1.5}} = 1 - e^{-5.57} = 1 - 0.0038$$

$$= 0.9962$$

$$A_{1c} = 1 - 0.9962 = 0.0038$$

$$A_{2a} = 1 - e^{-\left(\frac{3.142 - 0.40}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{2b} = 1 - 0.9893 = 0.0107$$

Area of Dist. 1 from 0 to γ_2	=	0.2235
" " " 2 " " " "	=	0
Diff.	=	+0.2235
" " " 1 " $(\mu_2 + 3\sigma_2)$ to $+\infty$	=	0.0038
" " " 2 " " " "	=	0.0107
Diff.	=	-0.0069
Total Diff. = 0.2735 - 0.0069	=	0.2666

$$\therefore P\left\{x_1 > x_2 \,\middle|\, \gamma_2 > x_2 > (\mu_2 + 3\sigma_2)\right\} = \underline{\underline{26.66\%}}$$

D. Comparison of $\alpha_1=1.0$, $\beta_1=1.5$, $\gamma_1=0$ versus $\alpha_2=1.3$, $\beta_2=1.5$, $\gamma_2=0$

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 1.3 + 0 = 0.903 \times 1.3 = 1.174$$

$$\sigma_1 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.613$$

$$\sigma_2 = 1.3\sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.797$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 1.174 + 3 \times 0.797 = 3.565$$

Note: Comparisons are slightly different from sections A, B and C so will redefine areas with new subscripts.

A_{1d} = Area from γ_1 to $(\mu_1 + 3\sigma_1)$

A_{1e} = " " $(\mu_1 + 3\sigma_1)$ to $+\infty$

A_{2c} = " " γ_2 to $(\mu_2 + 3\sigma_2)$

A_{2d} = " " $(\mu_2 + 3\sigma_2)$ to $+\infty$

$$A_{1d} = 1 - e^{-\left(\frac{2.742 - 0}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{1e} = 1 - 0.9893 = 0.0107$$

$$A_{2c} = 1 - e^{-\left(\frac{2.742 - 0}{1.3}\right)^{1.5}} = 1 - e^{-3.06} = 1 - 0.0468$$

$$= 0.9532$$

$$A_{2d} = 1 - 0.9532 = 0.0468$$

Area of Dist. 2 from $(\mu_1 + 3\sigma_1)$ to $+\infty$ = 0.0468

" " " 1 " " " " = 0.0107

Diff. = +0.0361

$$\therefore P\left\{ x_2 > x_1 \;\middle|\; x_1 \geq (\mu_1 + 3\sigma_1) \right\} = \underline{\underline{3.61\%}}$$

E. Comparison of $\alpha_1=1.0$, $\beta_1=1.5$, $\gamma_1=0$ versus $\alpha_2=1.8$, $\beta_2=1.5$, $\gamma_2=0$

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 1.8 + 0 = 1.625$$

$$\sigma_1 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.613$$

$$\sigma_2 = 1.8\sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 1.103$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 1.625 + 3 \times 1.103 = 4.934$$

$$A_{1d} = 1 - e^{-\left(\frac{2.742 - 0}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{1e} = 1 - 0.9893 = 0.0107$$

$$A_{2c} = 1 - e^{-\left(\frac{2.742 - 0}{1.8}\right)^{1.5}} = 1 - e^{-1.879} = 1 - 0.1527$$

$$= 0.8473$$

$$A_{2d} = 1 - 0.8473 = 0.1527$$

Area	of	Dist.	2	from	$(\mu_1 + 3\sigma_1)$	to	$+\infty$ =	0.1527
"	"	"	1	"	"	"	" =	0.0107
							Diff. =	+0.1420

$$\therefore P\left\{ x_2 > x_1 \;\middle|\; x_1 \geq (\mu_1 + 3\sigma_1) \right\} = \underline{\underline{14.20\%}}$$

F. Comparison of α_1=1.0, β_1=1.5, γ_1=0 versus α_2=2.25, β_2=1.5, γ_2=0

$$\mu_1 = \frac{1}{1.5}! \times 1.0 + 0 = 0.903$$

$$\mu_2 = \frac{1}{1.5}! \times 2.25 + 0 = 2.031$$

$$\sigma_1 = \sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 0.613$$

$$\sigma_2 = 2.25\sqrt{\frac{2}{1.5}! - \left(\frac{1}{1.5}!\right)^2} = 1.379$$

$$\mu_1 + 3\sigma_1 = 0.903 + 3 \times 0.613 = 2.742$$

$$\mu_2 + 3\sigma_2 = 2.031 + 3 \times 1.379 = 6.168$$

$$A_{1d} = 1 - e^{-\left(\frac{2.742 - 0}{1}\right)^{1.5}} = 1 - e^{-4.54} = 1 - 0.0107$$

$$= 0.9893$$

$$A_{1e} = 1 - 0.9893 = 0.0107$$

$$A_{2c} = 1 - e^{-\left(\frac{2.742 - 0}{2.25}\right)^{1.5}} = 1 - e^{-3.45} = 1 - 0.2605$$

$$= 0.7395$$

$$A_{2d} = 1 - 0.7395 = 0.2605$$

Area of Dist. 2 from $(\mu_1 + 3\sigma_1)$ to $+\infty$ =	0.2605
" " " 1 " " " " =	0.0107
Diff. =	+0.2498

$$\therefore P\left\{x_2 > x_1 \;\middle|\; x_1 \geq (\mu_1 + 3\sigma_2)\right\} = \underline{\underline{24.98\%}}$$

G. Comparison of $\alpha_1=1.0$, $\beta_1=2.0$, $\gamma_1=0$ versus

$\alpha_2=1.0$, $\beta_2=2.0$, $\gamma_2=0.35$

$$\mu_1 = \frac{1}{2.0}! \times 1.0 + 0 = 0.886$$

$$\mu_2 = \frac{1}{2.0}! \times 1.0 + 0.35 = 1.236$$

$$\sigma_1 = \sigma_2 = \sqrt{\frac{2}{2}! - \left(\frac{1}{2}!\right)^2} = 0.463$$

$$\mu_1 + 3\sigma_1 = 0.886 + 3 \times 0.463 = 2.276$$

$$\mu_2 + 3\sigma_2 = 1.236 + 3 \times 0.463 = 2.626$$

Note: Will use areas as defined in Section A.

$$A_{1a} = 1 - e^{-\left(\frac{0.35 - 0}{1}\right)^2} = 1 - 0.8847 = 0.1153$$

$$A_{1b} = 1 - e^{-\left(\frac{2.626 - 0}{1}\right)^2} = 1 - 0.0010 = 0.9990$$

$$A_{1c} = 1 - 0.9990 = 0.0010$$

$$A_{2a} = 1 - e^{-\left(\frac{2.626 - 0.35}{1}\right)^2} = 1 - e^{-5.181}$$

$$= 1 - 0.0056 = 0.9944$$

$$A_{2b} = 1 - 0.9944 = 0.0056$$

Area of Dist. 1 from 0 to γ_2	=	0.1153
" " " 2 " " "	=	0.
Diff.	=	+0.1153

$$\text{Area of Dist. 1 from } (\mu_2 + 3\sigma_2) \text{ to } +\infty = 0.0010$$

$$\text{" \quad " \quad " \quad 2 \quad " \qquad " } = \underline{0.0056}$$

$$\text{Diff.} = -0.0046$$

$$\text{Total Diff.} = 0.1153 - 0.0046 = 0.1107$$

$$\therefore P\left\{ x_1 > x_2 \;\middle|\; x_2 \geq (\mu_2 + 3\sigma_2) \right\} = \underline{\underline{11.07\%}}$$

H. Comparison of $\alpha_1=1.0$, $\beta_1=1.0$, $\gamma_1=0$ versus

$\alpha_2=1.0$, $\beta_2=1.0$, $\gamma_2=0.25$

$$\mu_1 = \frac{1}{1}! \times 1.0 + 0 = 1.0$$

$$\mu_2 = \frac{1}{1}! \times 1.0 + 0.25 = 1.25$$

$$\sigma_1 = \sigma_2 = \sqrt{\frac{2}{1}! - \frac{1}{1}!} = \sqrt{2 - 1} = 1.0$$

$$\mu_1 + 3\sigma_1 = 1.0 + 3 \times 1.0 = 4.0$$

$$\mu_2 = 3\sigma_2 = 1.25 + 3 \times 1.0 = 4.25$$

$$A_{1a} = 1 - e^{-\left(\frac{0.25 - 0}{1}\right)} = 1 - 0.7788 = 0.2212$$

$$A_{1b} = 1 - e^{-\left(\frac{4.25 - 0}{1}\right)} = 1 - 0.0143 = 0.9857$$

$$A_{1c} = 1 - 0.9857 = 0.0143$$

$$A_{2a} = 1 - e^{-\left(\frac{4.25 - 0.25}{1}\right)} = 1 - 0.0183 = 0.9817$$

$$A_{2b} = 1 - 0.9817 = 0.0183$$

Area of Dist. 1 from	0 to γ_2	=	0.2212
" " " 2 "	" " "	=	0.
	Diff.	=	+0.2212
" " " 1 "	$(\mu_2 + 3\sigma_2)$ to $+\infty$	=	0.0143
" " " 2 "	" " "	=	0.0183
	Diff.	=	-0.0040

Total Diff. = 0.2212 - 0.0040 = 0.2172

$$\therefore P\left\{x_1 > x_2 \,\middle|\, x_2 > (\mu_2 + 3\sigma_2)\right\} = \underline{\underline{21.72\%}}$$