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Interpolation with Bounded Real Rational Units with Applications to Simultaneous Stabilization

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ABSTRACT

In this paper we present sufficient conditions for the existence of a bounded real rational unit in H^∞ to exactly interpolate to points in the right half plane (RHP). It is shown that these sufficient conditions are equivalent to the necessary and sufficient conditions for the existence of a bounded real irrational unit in H^∞ to interpolate to points in the RHP, as initially described by Tannenbaum. The technique is then applied to the simultaneous stabilization problem.

1 Introduction

Bounded real interpolating units in H^∞ have applications in robust control and simultaneous stabilization. Tannenbaum [9] applied bounded real interpolating units to the robust stabilization of a single plant using a stable controller. Applications in simultaneous stabilization of SISO plants are discussed in [1] and [2].

In this paper, sufficient conditions to interpolate to points in the RHP with bounded real *rational* units are presented. These sufficient conditions are shown to be equivalent to the necessary and sufficient conditions to interpolate to the same points in the RHP with bounded real *irrational* units. An interpolation algorithm is applied to the construction of a controller for two simultaneous stabilization examples.

This paper is organized in the following manner. In Section 2, the background of the bounded unit interpolation theory is discussed. The bounded real rational unit interpolation theorem is presented in Section 3. The sufficient conditions of this theorem are equivalent to the necessary and sufficient conditions for existence of a bounded real irrational interpolating unit as indicated in Section 4. The new interpolation algorithm is applied to the simultaneous stabilization problem in Section 5. Finally, the conclusions are in Section 6.

2 Background

The problem of stabilizing n different plants is a longstanding problem in the robust control literature. In [1], it was shown that under some conditions, the simultaneous stabilization problems of 3 or more plants is reducible to that of interpolating a given set of points with bounded real units in H^∞. A similar interpolation problem is discussed by Tannenbaum in [8] and [9], for the purpose of stabilzing plants with uncertain gains. His approach uses the irrational mapping $U(s) = e^{-Z(s)}$ to convert the search from a bounded real interpolating unit, $U(s)$, to a positive real interpolating function, $Z(s)$. He then uses Youla-Saito's [12] positive real testing matrix to determine whether such a $Z(s)$ exists. Unfortunately, this leads to an irrational controller, which can only be realized using a rational approximation. Ganesh and Pearson [6] describe such a rational approximation through the truncation of the infinite Taylor series expansion. Although the truncated series comes arbitrarily close to the interpolation points, it does not exactly interpolate to the desired points with polynomials of finite order. It also provides no indication of what minimum order is necessary to be *sufficiently close.*

There are several interpolation mappings related to the mapping presented in this paper. This paper presents a mapping, which creates a bounded real rational unit exactly interpolating to values in the RHP. There are two obvious approaches, which could possibly achieve this result. One approach would be to start with an interpolation mapping, which produces units in H^∞ (see for example [5]), and then impose magnitude restrictions to create a bounded function. Another approach would be to start with an interpolation mapping, which produces bounded rational functions, and then impose numerator restrictions to create a unit. The latter is the approach adopted in this paper. It is interesting that this approach is related to the positive real mapping as discussed in Section 4.

The problem of finding a bounded rational function, which interpolates to a collection of points inside the unit disk is discussed in [3] and [11]. Only interpolations in the right half plane are addressed in this paper, but the procedure is similar for the unit circle. The problem of finding a bounded real rational polynomial function, $U_1(s)$, which interpolates to a collection of points in the right half plane, is discussed by Kimura in [7] and summarized below.

Let the interpolation points be as follows:

$$U_1(\alpha_i) = \beta_i^{(1)} \in \mathbb{C}, \quad \alpha_i \in \mathbb{C}^+$$

where complex interpolation points appear in conjugate pairs, i.e.

$$U_1(\alpha_i) = \beta_i^{(1)}, \Rightarrow U_1(\overline{\alpha}_i) = \overline{\beta}_i^{(1)} \tag{1}$$

and where real interpolation points are real. It is then known that

$$\|U_1(s)\|_\infty < 1 \Leftrightarrow \|U_2(s)\|_\infty < 1$$

where

$$U_1(s) = \frac{\beta_1^{(1)} + (\frac{s-\alpha_1}{s+\overline{\alpha}_1})U_2(s)}{1 + \overline{\beta}_1^{(1)}(\frac{s-\alpha_1}{s+\overline{\alpha}_1})U_2(s)} \tag{2}$$

The first interpolation condition, $U_1(\alpha_1) = \beta_1^{(1)}$, is satisfied by this mapping. So, the interpolation at $s = \alpha_1$ is no longer imposed on U_2. The resulting interpolatings imposed on U_2 are derived from the mapping (2).

$$\begin{aligned}
\beta_i^{(2)} &= U_2(\alpha_i) = \frac{\left[\beta_i^{(1)} - \beta_1^{(1)}\right](\alpha_i + \overline{\alpha}_1)}{\left[1 - \overline{\beta}_1^{(1)}\beta_i^{(1)}\right](\alpha_i - \alpha_1)}, \\
\beta_k^{(k)} &= U_k(\alpha_k) = \frac{\left[\beta_k^{(k-1)} - \beta_{k-1}^{(k-1)}\right](\alpha_k + \overline{\alpha}_{k-1})}{\left[1 - \overline{\beta}_{k-1}^{(k-1)}\beta_k^{(k-1)}\right](\alpha_k - \alpha_{k-1})}
\end{aligned}$$

This procedure continues iteratively until there is only one interpolation point. Any bounded rational polynomial function, which interpolates to the single interpolation condition of the last mapping, may be used to reconstruct the bounded function, $U_1(s)$, which interpolates to all of the desired points, by applying each of the successive inverse mappings in reverse order. The necessary and sufficient conditions for existence of a bounded function that interpolates to these points is that the magnitude of all the interpolation values in all of the mappings be bounded by 1, i.e.

$$|\beta_i^{(j)}| < 1, \quad \forall j = 1, 2, \ldots, k \\ \forall i = j, j+1, \ldots, k$$

This is equivalent to the Nevanlinna-Pick (Bounded Real) testing matrix, $\widehat{\mathbf{N}}_{\mathbf{BR}}(\beta^{(1)})$, being positive definite, where

$$\widehat{\mathbf{N}}_{\mathbf{BR}}(\beta^{(1)}) = \begin{bmatrix} \frac{1-\beta_1^{(1)}\overline{\beta}_1^{(1)}}{\alpha_1+\overline{\alpha}_1} & \cdots & \frac{1-\beta_1^{(1)}\overline{\beta}_k^{(1)}}{\alpha_1+\overline{\alpha}_k} \\ \frac{1-\beta_2^{(1)}\overline{\beta}_1^{(1)}}{\alpha_2+\overline{\alpha}_1} & \cdots & \frac{1-\beta_2^{(1)}\overline{\beta}_k^{(1)}}{\alpha_2+\overline{\alpha}_k} \\ \vdots & \ddots & \vdots \\ \frac{1-\beta_k^{(1)}\overline{\beta}_1^{(1)}}{\alpha_k+\overline{\alpha}_1} & \cdots & \frac{1-\beta_k^{(1)}\overline{\beta}_k^{(1)}}{\alpha_k+\overline{\alpha}_k} \end{bmatrix} \tag{3}$$

The fact that the interpolation values appear in complex conjugate pairs and are real when α_i is real ensures that a bounded real function can be formed when (3) is positive definite. Notice that this mapping ensures that the denominator of $U_1(s)$ in (2) is a strictly Hurwitz polynomial. *Our only task remaining is to find the conditions in this mapping that will ensure the numerator is also a strictly Hurwitz polynomial.* This will give us a unit in H^∞, with a norm bounded by 1.

3 Interpolation with Bounded Real Rational Units

In this section, sufficient conditions are derived for the existence of a finite order rational bounded real unit in H^∞ which exactly interpolates to prescribed complex values at finite points in the open right half s-plane. It is shown in [10] that the values of a real unit in H^∞ must have the same sign at all points on the real line in the right half plane. Values on the nonnegative real line must be either all positive or all negative.

For a *real* (i.e. real for real points) interpolating unit, the complex interpolation values must appear in complex conjugate pairs. The requirements, that the complex interpolations appear in complex conjugate pairs and that the nonnegative real interpolations all have the same sign, ensure that the resulting unit is real. Without this assumption, the testing matrices may still indicate the existence of a bounded unit, but the unit would contain complex coefficients.

Let $\gamma_i^{(1)}, \forall i = 1, 2, \ldots, k$ represent the desired interpolation values at the corresponding RHP points, $s = \alpha_i$. Let the subsequent mappings be defined as

$$\gamma_i^{(2)} = \frac{\left[\ln(\gamma_1^{(1)}) - \ln(\gamma_i^{(1)})\right](\alpha_i + \overline{\alpha}_1)}{\left[\ln(\overline{\gamma}_1^{(1)}) + \ln(\gamma_i^{(1)})\right](\alpha_i - \alpha_1)}, \forall i = 2, 3, \ldots, k$$

$$\gamma_i^{(j)} = \frac{\left[\gamma_i^{(j-1)} - \gamma_{j-1}^{(j-1)}\right](\alpha_i + \overline{\alpha}_{j-1})}{\left[1 - \overline{\gamma}_{j-1}^{(j-1)}\gamma_i^{(j-1)}\right](\alpha_i - \alpha_{j-1})}, \forall j = 3, 4, \cdots, k$$

$$\forall i = j, j+1, \cdots, k \tag{4}$$

In the theorems that follow, it is assumed that the simple interpolations appear in complex conjugate pairs. Without loss of generality, the interpolation values on the real line are assumed to have a positive sign. Otherwise, their negative values are used.

Theorem 1 (Bounded Real Rational Unit Interpolation Theorem - Simple Finite Interpolations)

Sufficient conditions to find a bounded real rational unit, $U(s)$, such that

$$U(\alpha_i) = \gamma_i^{(1)}, \quad \forall i = 1, 2, \ldots, k$$

are that, either

1.

$$0 < |\gamma_i^{(j)}| < 1, \quad \forall j = 1, 2, \ldots, k \quad \forall i = j, j+1, \ldots, k$$

or,

2.

$$|\gamma_1^{(1)}| < 1$$

and the Nevanlinna-Pick bounded real testing matrix, $\mathbf{N_{BR}}(\gamma^{(2)})$ defined above, is positive definite.

Proof: See [2]. ■

Comments: The main idea of this theorem and its proof can be easily seen from the basic Nevanlinna-Pick mapping.

$$U_1(s) = \frac{\gamma_1^{(1)} + \left(\frac{s-\alpha_1}{s+\bar{\alpha}_1}\right) U_2(s)}{1 + \bar{\gamma}_1^{(1)} \left(\frac{s-\alpha_1}{s+\bar{\alpha}_1}\right) U_2(s)} \tag{5}$$

If, in addition to the requirement that $\|U_2(s)\|_\infty < 1$, $\|U_2(s)\|_\infty$ is further restricted to be less than $|\gamma_1^{(1)}|$, then the second term in the numerator will also be bounded by $|\gamma_1^{(1)}|$ since the Blaschke product, $\left(\frac{s-\alpha_1}{s+\bar{\alpha}_1}\right)$, is bounded by 1. Therefore, under this condition, the numerator is a strictly Hurwitz polynomial from the small-gain theorem [4].

In the event that the interpolation requirements on $U_2(s)$ do not happen to satisfy this additional restriction using the original interpolation points, the more stringent requirements could be relaxed by using the n-th root of the original interpolation points. In the limit as $n \to \infty$, $|\sqrt[n]{\gamma_1^{(1)}}|$ increases to 1. The conditions of Theorem 1 reflect the sufficient conditions (and conjectured to be necessary conditions) for the interpolation requirements on $U_2(s)$ to be satisfied by a bounded function in the limit.

4 Equivalence with Bounded Real Irrational Units

In this section, the sufficient conditions of Theorem 1 are shown to be equivalent to the necessary and sufficient conditions for the existence of the irrational function $f(s) = e^{-Z(s)}$, which interpolates to the same points. The approach described by Tannenbaum in [9], imposes the interpolation requirements on $Z(s)$, which must be a positive real function for $e^{-Z(s)}$ to be a real function bounded by 1. Tannenbaum used Youla and Saito's [12] necessary and sufficient conditions for a positive real interpolating function to determine whether a bounded real unit of the form $f(s) = e^{-Z(s)}$ exists. By transforming the interpolation conditions from the original requirements on the bounded real unit, $f(s)$, to those of the positive real function, $Z(s)$, a fairly simple test can be performed to determine whether such a positive real $Z(s)$ exits. If the original interpolation points for $f(s)$ are $f(\alpha_i) = \beta_i$, $\forall i = 1, 2, \ldots, k$, then after the transformation, the interpolation conditions become $Z(\alpha_i) = -\ln(\beta_i)$, $\forall i = 1, 2, \ldots, k$. The necessary and sufficient conditions for the existence of a positive real function, $Z(s)$, which interpolates to these points, are that Youla-Saito's positive real testing matrix, $\widehat{\mathbf{N}}_{\mathbf{PR}}(\beta)$ shown below, be positive definite, and that the interpolation points appear in complex conjugate pairs and are positive real when s is positive real.

$$\widehat{\mathbf{N}}_{\mathbf{PR}}(\beta) = \begin{bmatrix} \frac{-\ln(\beta_1) - \ln(\bar{\beta}_1)}{(\alpha_1+\bar{\alpha}_1)} & \cdots & \frac{-\ln(\beta_1) - \ln(\bar{\beta}_k)}{(\alpha_1+\bar{\alpha}_k)} \\ \frac{-\ln(\beta_2) - \ln(\bar{\beta}_1)}{(\alpha_2+\bar{\alpha}_1)} & \cdots & \frac{-\ln(\beta_2) - \ln(\bar{\beta}_k)}{(\alpha_2+\bar{\alpha}_k)} \\ \vdots & \ddots & \vdots \\ \frac{-\ln(\beta_k) - \ln(\bar{\beta}_1)}{(\alpha_k+\bar{\alpha}_1)} & \cdots & \frac{-\ln(\beta_k) - \ln(\bar{\beta}_k)}{(\alpha_k+\bar{\alpha}_k)} \end{bmatrix}$$

It should also be noted that the requirement that the matrix, $\widehat{\mathbf{N}}_{\mathbf{PR}}(\beta)$, be positive definite is only a necessary and sufficient condition for the existence of a bounded real interpolating unit of the form $f(s) = e^{-Z(s)}$. When the interpolation points are entirely complex, the form $f(s) = -e^{-Z(s)}$ may also need to be checked, when determining the existence or the minimum order of a bounded real interpolating unit, as is illustrated in [2].

Theorem 2 (Rational-Irrational Bounded Unit Equivalence Theorem)
The Youla-Saito positive real testing matrix, $\widehat{\mathbf{N}}_{\mathbf{PR}}(\gamma^{(1)})$ defined above, is positive definite, if and only if

$$\left|\gamma_1^{(1)}\right| < 1$$

and the corresponding Nevanlinna-Pick bounded real testing matrix, $\mathbf{N_{BR}}(\gamma^{(2)})$ defined above, is positive definite.

Proof: See [2]. ■

Comments: The proof identifies an invertible matrix, X, such that

$$X^*\widehat{\mathbf{N}}_{\mathbf{PR}}(\gamma^{(1)})X = \begin{bmatrix} \frac{-\ln(\gamma_1^{(1)})-\ln(\overline{\gamma}_1^{(1)})}{(\alpha_1+\overline{\alpha}_1)} & 0 \\ 0 & \widehat{\mathbf{N}}_{\mathbf{BR}}(\gamma^{(2)}) \end{bmatrix}$$

5 Applications to Simultaneous Stabilization

In this section, a bounded unit interpolation algorithm, developed in [2], is used in conjunction with simultaneous stabilization problems. When the sufficient conditions of Theorem 1 are satisfied, there exists a bounded function, $U_2'(s)$, which satisfies the following interpolation requirements for some n sufficiently large.

$$U_2'(\alpha_i) = \frac{U_2(\alpha_i)}{\sqrt[n]{\gamma_1^{(1)}}} = \frac{\left[\sqrt[n]{\gamma_i^{(1)}} - \sqrt[n]{\gamma_1^{(1)}}\right](\alpha_i+\overline{\alpha}_1)}{\sqrt[n]{\gamma_1^{(1)}}\left[1-\sqrt[n]{\overline{\gamma}_1^{(1)}\gamma_i^{(1)}}\right](\alpha_i-\alpha_1)}, \quad \forall i = 2,3,\ldots,k$$

Using $U_2(s) = \sqrt[n]{\gamma_1^{(1)}} \cdot U_2'(s)$ and replacing $\gamma_1^{(1)}$ with its n-th root in (5), the desired real rational bounded interpolating unit is $[U_1(s)]^n$.

Example 1 *Find a stable controller that simultaneously stabilizes the two plants,*

$$P_1(s) = \frac{\frac{1}{81}(s-1)(s-2)}{(s-\frac{1}{9})^2}$$

$$P_2(s) = \frac{\frac{1}{81}(s-1)(s-2)}{s^2} \tag{6}$$

Solution: Let $h_1(s) = h_2(s) = (s+1)^2$, so that

$$N_1(s) = N_2(s) = \frac{\frac{1}{81}(s-1)(s-2)}{(s+1)^2}$$

$$D_1(s) = \frac{(s-\frac{1}{9})^2}{(s+1)^2},\ D_2(s) = \frac{s^2}{(s+1)^2}$$

Forming the function, $W_2(s)$, which must be bounded by a unit in accordance with the solution of this problem described in [1] and [2], then

$$W_2(s) = \frac{N_1^-(s)D_2(s) - N_2^-(s)D_1(s)}{N_2^-(s)} = \frac{\frac{1}{81}(18s-1)}{(s+1)^2}$$

A unit, $W(s)$, which bounds $W_2(s)$, is given by

$$W(s) = \frac{\frac{1}{81}(18s+1)(\frac{s}{100}+1)}{(s+1)^2}$$

so that

$$\gamma_1^{(1)} = \frac{W(1)}{D_1(1)} = 0.29984,\ \gamma_2^{(1)} = \frac{W(2)}{D_1(2)} = 0.130588$$

The positive real testing matrix is then given by

$$\begin{bmatrix} 1.204493773 & 1.080066641 \\ 1.080066641 & 1.017853074 \end{bmatrix}$$

The upper left nested principal minors are 1.2 and 0.059. Since this matrix is positive definite, a solution to the strong simultaneous stabilization problem exists for the two plants in (6).

Using the bounded unit interpolation algorithm in [2] it is easy to verify that $n = 5$ is the minimum root necessary to interpolate to these two points with a bounded unit using the Nevanlinna-Pick mapping. The 5th root of each of the original interpolation points becomes

$$\alpha_1 = 1, \quad \sqrt[5]{\gamma_1^{(1)}} = \sqrt[5]{0.29984375} = 0.785921193$$

$$\alpha_2 = 2, \quad \sqrt[5]{\gamma_2^{(1)}} = \sqrt[5]{0.130588235} = 0.665550188$$

Calculating the modified interpolation point in the first Nevanlinna-Pick mapping

$$\beta_2^{(2)} = \frac{\left[\sqrt[5]{\gamma_2^{(1)}} - \sqrt[5]{\gamma_1^{(1)}}\right](\alpha_2+\overline{\alpha}_1)}{\sqrt[5]{\gamma_1^{(1)}}\left[1-\sqrt[5]{\overline{\gamma}_1^{(1)}\gamma_2^{(1)}}\right](\alpha_2-\alpha_1)}$$

we get

$$\alpha_2 = 2, \quad \beta_2^{(2)} = -0.963406336$$

A function, $U_2'(s)$, bounded by 1 and interpolating to this point is

$$U_2'(s) = \beta_2^{(2)} = -0.963406336$$

Therefore, the function bounded by $\sqrt[5]{0.29984375}$ interpolating to the points in the first standard Nevanlinna-Pick mapping of the 5th root of the original interpolation points is

$$U_2(s) = \sqrt[5]{0.29984375} \cdot U_2'(s) = 0.757161457$$

The bounded unit, $U(s)$, interpolating to the original interpolation points is formed from $U_2(s)$.

$$U(s) = \left[\frac{\sqrt[5]{0.29984375} + (\frac{s-1}{s+1})U_2(s)}{1+\sqrt[5]{0.29984375}(\frac{s-1}{s+1})U_2(s)}\right]^5$$

$$U(s) = \left[\frac{0.071016147(s+53.66018683)}{(s+3.93911596)}\right]^5$$

The stable 5th order compensator, $C(s)$, is formed from the unit, $U_1(s)$, which interpolates to $D_1(s)$ at the RHP zeros of $N_1(s)$.

$$C(s) = \frac{U_1(s) - D_1(s)}{N_1(s)}, \; U_1(s) = \frac{W(s)}{U(s)}$$

$$C(s) = 99,571 \cdot \left[\frac{(s+100)(s+10.747)(s+0.00444)(s^2+11.8797s+85.011)}{(s+53.66)^5}\right]$$

Next, consider the problem of designing a stable compensator which stabilizes a pendulum, with delayed control effort, about three positions, $\theta = 0, \pi/2$, and π. The nonlinear dynamics of the pendulum are given by

$$\ddot{\theta} + l/g sin\theta = \frac{1}{l^2 m} T(t-\tau)$$

where θ is the angular position of the pendulum, T is the input torque, and τ is the control effort time delay. If the pure time delay term $e^{-\tau s}$ is approximated by the first order rational function $(1-s/2\tau)/(1+s/2\tau)$ and the pendulum dynamics are linearized about the three points above, one obtains the three plant transfer functions of the following example, using the parameter values $l/g = 1$ and $\tau = 1$.

Example 2 *Find a stable controller that simultaneously stabilizes the three plants,*

$$P_1(s) = \frac{-(s-2)}{(s^2-1)(s+2)}$$
$$P_2(s) = \frac{-(s-2)}{s^2(s+2)}$$
$$P_3(s) = \frac{-(s-2)}{(s^2+1)(s+2)} \qquad (7)$$

where $P_1(s), P_2(s)$ and $P_3(s)$ correspond to the linearizations about $\theta = \pi, \pi/2$, and 0 respectively.

Solution: If we let $h_1(s) = h_2(s) = h_3(s) = (s+2)^3$, so that

$$N_1(s) = N_2(s) = N_3(s) = \frac{-(s-2)}{(s+2)^3}$$
$$D_1(s) = \frac{(s^2-1)}{(s+2)^2}, \; D_2(s) = \frac{s^2}{(s+2)^2}, \; D_3(s) = \frac{(s^2+1)}{(s+2)^2}$$

then for a stable compensator, $C = \frac{U_1(s) - D_1(s)}{N_1(s)}$, we must satisfy the interpolation conditions

$$U_1(2) = D_1(2) = \frac{3}{16}$$
$$U_1(\infty) = D_1(\infty) = 1$$
$$\left.\frac{d}{dt}U_1(t)\right|_{t=0} = \left.\frac{d}{dt}D_1(t)\right|_{t=0} = -4 \qquad (8)$$

The last two conditions in (8) are required to ensure that the term $[U_1(s) - D_1(s)]$ has relative degree equal to 2 at infinity, as discussed in [2]. Computing the functions,

$$W_i(s) = \frac{N_1^-(s)D_i(s) - N_i^-(s)D_1(s)}{N_i^-(s)}, \; \forall i = 2,3 \quad (9)$$

which must be bounded by a unit for the solution of this problem described in [1] and [2], then

$$W_2(s) = \frac{1}{(s+2)^2}, \; W_3(s) = \frac{2}{(s+2)^2}$$

A Unit $W(s)$ which bounds $W_i(s)$ in this case is given by

$$W(s) = \frac{2(\frac{s}{100}+1)^2}{(s+2)^2}$$

Using the equation

$$U(s) = \frac{W(s)}{U_1(s)}$$

the interpolation conditions on $U(s)$ become

$$U(2) = \frac{10404}{15000}$$
$$U(\infty) = \frac{1}{5000}$$
$$\left.\frac{d}{dt}U(t)\right|_{t=0} = \frac{1}{25}$$

From [2], such a bounded real interpolating unit exists, and therefore, a solution to the strong simultaneous stabilization problem exists for the three plants in (7).

Multiplying the original interpolation points by a constant, $k > 1$, but small enough to still allow interpolations with a bounded real unit, $U_k(s)$, then the bounded unit $\frac{1}{k}U_k(s)$ interpolates to the original points. The maximum value of k can be computed using the procedure described in [2]. Subsequent products with $\frac{1}{k}U_k(s)$ must have an H^∞ norm less than k to remain bounded. Choosing the factor $k = 1.25$, then the new simple interpolation points become

$$U_k(2) = 0.867, \; U_k(\infty) = 0.00025$$

Using the techniques developed in [2], it is easy to verify that $n = 1$ is sufficient to construct a bounded real interpolating unit. From the standard Nevanlinna-Pick interpolation mapping, one such unit is

$$U_k(s) = \frac{0.00025(s+55851.31643)}{(s+14.10533968)}$$
$$\Rightarrow \quad U_0(s) = \frac{1}{k}U_k(s)$$
$$\Rightarrow \quad U_0(s) = \frac{0.0002(s+55851.31643)}{(s+14.10533968)}$$

To satisfy the interpolation condition on the first derivative at $s = \infty$, we follow the same procedure as in [2].

$$\begin{aligned}
U_1(s) &= U_0(s) \cdot \Omega_1(s) \\
\Omega_1(s) &= \left[1 + \frac{c_1}{(s+d_1)} \frac{(s-2)}{(s+2)}\right] \\
\frac{d}{dt} U_1(1/t)\bigg|_{t=0} &= \frac{d}{dt} U_0(1/t)\bigg|_{t=0} + \\
& \quad U_0(\infty) \cdot \frac{d}{dt} \Omega(1/t)\bigg|_{t=0} \\
\Rightarrow \ c_1 &= -55,637.21109
\end{aligned}$$

In order for $\|\Omega_1(s)\|_\infty$ to be less than $k = 1.25$,

$$\begin{aligned}
\left\| \frac{c_1}{(s+d_1)} \frac{(s-2)}{(s+2)} \right\|_\infty &< 0.25 \\
\Rightarrow \left| \frac{c_1}{d_1} \right| &< 0.25 \\
\Rightarrow d_1 &> 222,548.8444
\end{aligned}$$

Arbitrarily choosing $d_1 = 250,000.0$, we get a bounded real interpolating unit

$$U(s) = 0.0002 \cdot \left[\frac{(s+23.145)(s+55851.316)(s+194,361.644)}{(s+2)(s+14.10534)(s+250,000.0)}\right]$$

Solving for $U_1(s) = \frac{W(s)}{U(s)}$ and then for $C(s)$ results in a stable third order compensator

$$C(s) = 1.0802 \times 10^{10} \cdot \left[\frac{(s+1.4045)(s+2)(s+3.4493)}{(s+3.145)(s+55,851)(s+194,361.644)}\right]$$

6 Conclusions

In this paper, the necessary and sufficient conditions to interpolate to points in the RHP with a bounded real irrational unit in H^∞ were shown to be equivalent to new sufficient conditions to interpolate with a bounded real rational unit in H^∞. It is conjectured that the new sufficient conditions for existence of a bounded real rational unit in H^∞ are also necessary. An interpolation algorithm is used to achieve exact interpolation with a bounded real rational unit. Examples illustrate the bounded real rational unit interpolation algorithm as applied to the construction of simultaneously stabilizing controllers.

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